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A NOTE ON INTEGRAL REPRESENTATIONS OF THE SKOROKHOD MAP

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Abstract

We present a very short derivation of the integral representation of the two-sided Skorokhod reflection Z of a continuous function X of bounded variation, which is a generalization of the integral representation of the one-sided map featured in Anantharam and Konstantopoulos (2011) and Konstantopoulos *et al.* (1996). We also show that Z satisfies a simpler integral representation when additional conditions are imposed on X.

Keywords: Reflection map; regulator map; Skorokhod problem

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1. Introduction

It is known that the one-sided Skorokhod reflection map Z of a continuous real-valued function X of bounded variation satisfies a certain integral representation. This was first shown in [7] for X belonging to a broad class of continuous functions. Subsequently, in [6] the representation was extended to the case where X is allowed to have discontinuity points. These integral representations are also briefly discussed in [3, Chapter 3], and an interesting result addressing the uniqueness of functions satisfying such representations can be found in [1].

The purpose of this paper is to present a new, substantially shorter derivation of the above-mentioned representation when X is continuous, that can also be easily applied to obtain an analogous integral representation for the two-sided Skorokhod map as well. Our derivation shares some similarity to the derivation given in [3, p. 193–194] in that we make use of change-of-variable techniques; however, unlike [3] we do not make use of any specific representations of the reflection map.

We will also show how our new integral representation simplifies when X is represented as the difference of two nondecreasing continuous functions with disjoint supports. The latter, being always possible due to Hahn's decomposition of signed measures.

2. Notation and main result

Suppose that $X\colon [0, \infty) \to \mathbb{R}$ is continuous and of bounded variation on finite intervals with $X(0) = 0$. This is equivalent to assuming that for some $A, C\colon [0, \infty) \to \mathbb{R}$ nondecreasing and continuous with $A(0) = C(0) = 0$, it follows that $X(t) = A(t) - C(t)$ for each $t \geq 0$.

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Our main goal is to derive integral representations for the two-sided (Skorokhod) reflection map (Z, L, U) of X in the interval $[0, a]$, for some $a > 0$, which is defined (uniquely) as follows:

(i) $Z(t) = X(t) + L(t) - U(t) \in [0, a]$ for each $t \geq 0$;

(ii) L, U are nondecreasing and (in our case) continuous with $L(0) = U(0) = 0$ and

$$\int_0^\infty \mathbf{1}_{\{Z(t)>0\}} \, \mathrm{d}L(t) = \int_0^\infty \mathbf{1}_{\{Z(t)<a\}} \, \mathrm{d}U(t) = 0. \tag{2.1}$$

Proof of the existence and uniqueness of (Z, L, U) (as a functional of X) may be found in [11]. Closed-form expressions for Z in terms of X are also known. For more on this topic see, e.g. [2], [8], [9], and the references therein.

We now present our main result, which is an integral representation for Z.

Theorem 2.1. *For each $t \geq 0$, we have*

$$Z(t) = \int_0^t \mathbf{1}_{\{Z(s)>C(t)-C(s)+\int_s^t \mathbf{1}_{\{Z(u)=a\}} \, \mathrm{d}X(u)\}} \, \mathrm{d}A(s). \tag{2.2}$$

Observe that when $a = \infty$, the integral representation simplifies to

$$Z(t) = \int_0^t \mathbf{1}_{\{Z(s)>C(t)-C(s)\}} \, \mathrm{d}A(s),$$

which is the integral representation found in [1], [3], and [7].

Our proof of Theorem 2.1 will make use of the following simple lemmas. Note also that Z is both continuous on $[0, \infty)$ and of bounded variation on finite intervals. This is a consequence of [8, Proposition 1.3] and we will make use of this fact in our derivations more than once.

Lemma 2.1. *For $0 \leq s \leq t$, we have*

$$\int_s^t \mathbf{1}_{\{Z(u)=a\}} \, \mathrm{d}X(u) = U(t) - U(s).$$

Proof. From (2.1), we have $\int_s^t \mathbf{1}_{\{Z(u)=a\}} \, \mathrm{d}L(u) = 0$ and $\int_s^t \mathbf{1}_{\{Z(u)=a\}} \, \mathrm{d}U(u) = \int_s^t \mathrm{d}U(u)$. Thus,

$$\begin{aligned}
\int_s^t \mathbf{1}_{\{Z(u)=a\}} \, \mathrm{d}X(u) &= \int_s^t \mathbf{1}_{\{Z(u)=a\}} \, \mathrm{d}(Z(u) - L(u) + U(u)) \\
&= \int_s^t \mathbf{1}_{\{Z(u)=a\}} \, \mathrm{d}Z(u) - 0 + \int_s^t \mathrm{d}U(u) \\
&= \int_{Z(s)}^{Z(t)} \mathbf{1}_{\{x=a\}} \, \mathrm{d}x + U(t) - U(s) = U(t) - U(s),
\end{aligned}$$

where the change of variables in the last line follows via bounded convergence from, e.g. the corollary in [10, p. 42] combined with the fact that $\mathbf{1}_{\{x=a\}}$ is a limit of uniformly bounded continuous functions. □

Lemma 2.2. *For each $t \geq 0$, we have*

$$\int_0^t \mathbf{1}_{\{Z(s)>C(t)-C(s)+U(t)-U(s)\}} \, \mathrm{d}L(s) = 0.$$

Proof. Since $Z(s)\mathrm{d}L(s) = 0$ and C and U are nondecreasing, then

$$\int_0^t \mathbf{1}_{\{Z(s)>C(t)-C(s)+U(t)-U(s)\}}\,\mathrm{d}L(s) = \int_0^t \mathbf{1}_{\{0>C(t)-C(s)+U(t)-U(s)\}}\,\mathrm{d}L(s) = 0$$

proving our claim. □

We are now ready to complete the proof of our main result.

Proof of Theorem 2.1. By applying first Lemma 2.1, then Lemma 2.2, and finally a change of variables, we have, for each $t \geq 0$,

$$\begin{aligned}
&\int_0^t \mathbf{1}_{\{Z(s)>C(t)-C(s)+\int_s^t \mathbf{1}_{\{Z(u)=a\}}\,\mathrm{d}X(u)\}}\,\mathrm{d}A(s) \\
&\quad= \int_0^t \mathbf{1}_{\{Z(s)>C(t)-C(s)+U(t)-U(s)\}}\,\mathrm{d}A(s) \\
&\quad= \int_0^t \mathbf{1}_{\{Z(s)+C(s)+U(s)>C(t)+U(t)\}}\,\mathrm{d}(Z(s)+C(s)+U(s)-L(s)) \\
&\quad= \int_0^t \mathbf{1}_{\{Z(s)+C(s)+U(s)>C(t)+U(t)\}}\,\mathrm{d}(Z(s)+C(s)+U(s)) \\
&\quad= \int_0^{Z(t)+C(t)+U(t)} \mathbf{1}_{\{x>C(t)+U(t)\}}\,\mathrm{d}x \\
&\quad= \int_{C(t)+U(t)}^{Z(t)+C(t)+U(t)} \mathrm{d}x \\
&\quad= Z(t),
\end{aligned}$$

which establishes (2.2). □

3. A simpler integral representation

The appearance of the integral term $\int_s^t \mathbf{1}_{\{Z(u)=a\}}\,\mathrm{d}X(u)$ within the indicator function found in (2.2) makes using this representation to derive useful versions of Little's law difficult, so we would like to know if the representation can be simplified in some way, in the hope of finding a representation that is more amenable to computation. In this section we show that simplification is possible, if for each $t \geq 0$, $A(t)$, and $C(t)$ coincide with the singular measures of $[0, t]$ obtained from the Hahn decomposition—see, e.g. [5, Theorem A, p. 121]—of the signed measure induced by X. In general, this is not the case, as A models cumulative input while C models cumulative potential or maximal output and there is no reason to assume that the input is blocked when a server is working or vice versa.

Therefore, throughout this section, we will assume that $[0, \infty) = S_A \cup S_C$, where $S_A \cap S_C = \varnothing$ and

$$\int_{S_A} \mathrm{d}C(s) = \int_{S_C} \mathrm{d}A(s) = 0. \tag{3.1}$$

Under this condition we have the following theorem.

Theorem 3.1. *For each $t \geq 0$,*

$$Z(t) = \int_0^t \mathbf{1}_{\{Z(s)>C(t)-C(s)\}}\,\mathbf{1}_{\{Z(s)<a\}}\,\mathrm{d}A(s). \tag{3.2}$$

We derive this result by making use of a number of simple lemmas.

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Lemma 3.1. *For each* $t \geq 0$, *we have*

$$\int_0^t \mathbf{1}_{\{Z(s)=a\}} \, \mathrm{d}C(s) = 0.$$

Proof. We have

$$\begin{aligned}\int_0^t \mathbf{1}_{\{Z(s)=a\}} \, \mathrm{d}C(s) &= \int_{S_C \cap [0,t]} \mathbf{1}_{\{Z(s)=a\}} \, \mathrm{d}C(s) \\ &= \int_{S_C \cap [0,t]} \mathbf{1}_{\{Z(s)=a\}} \, \mathrm{d}(A(s) - Z(s) + L(s) - U(s)) \\ &= -\int_{S_C \cap [0,t]} \mathbf{1}_{\{Z(s)=a\}} \, \mathrm{d}U(s) \\ &\leq 0.\end{aligned}$$

Observe that the first equality is a consequence of (3.1), while the third equality follows from

$$\int_{S_C \cap [0,t]} \mathbf{1}_{\{Z(s)=a\}} \, \mathrm{d}A(s) \leq \int_{S_C} \mathrm{d}A(s) = 0,$$

$$\int_{S_C \cap [0,t]} \mathbf{1}_{\{Z(s)=a\}} \, \mathrm{d}Z(s) \leq \int_0^t \mathbf{1}_{\{Z(s)=a\}} \, \mathrm{d}Z(s) = \int_{Z(0)}^{Z(t)} \mathbf{1}_{\{x=a\}} \, \mathrm{d}x = 0,$$

$$\int_{S_C \cap [0,t]} \mathbf{1}_{\{Z(s)=a\}} \, \mathrm{d}L(s) \leq \int_0^t \mathbf{1}_{\{Z(s)>0\}} \, \mathrm{d}L(s) = 0.$$

The proof is complete. □

Lemma 3.2. *For* $0 \leq s \leq t$,

$$\int_0^t \mathbf{1}_{\{Z(s)>C(t)-C(s)\}} \mathbf{1}_{\{Z(s)=a\}} \, \mathrm{d}(Z(s) + C(s)) = 0.$$

Proof. Let $s(t) = \sup\{s \in [0,t] \colon C(t) - C(s) \geq a\}$, where $s(t) = 0$ when $C(t) < a$. Then

$$\begin{aligned}\int_0^t \mathbf{1}_{\{Z(s)>C(t)-C(s)\}} \mathbf{1}_{\{Z(s)=a\}} \, \mathrm{d}(Z(s) + C(s)) &= \int_{s(t)}^t \mathbf{1}_{\{Z(s)=a\}} \, \mathrm{d}(Z(s) + C(s)) \\ &= \int_{s(t)}^t \mathbf{1}_{\{Z(s)=a\}} \, \mathrm{d}Z(s) + \int_{s(t)}^t \mathbf{1}_{\{Z(s)=a\}} \, \mathrm{d}C(s) \\ &= 0.\end{aligned}$$

By a change of variables (once again) for the first term and Lemma 3.1 for the second, the right-hand side is 0 and the proof is complete. □

Our final lemma is analogous to Lemma 2.2.

Lemma 3.3. *For each* $t \geq 0$, *we have*

$$\int_0^t \mathbf{1}_{\{Z(s)>C(t)-C(s)\}} \mathbf{1}_{\{Z(s)<a\}} \, \mathrm{d}L(s) = 0.$$

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Proof. Again, since $Z(s)\mathrm{d}L(s) = 0$ and C is nondecreasing, we obtain

$$\int_0^t \mathbf{1}_{\{Z(s)>C(t)-C(s)\}}\,\mathbf{1}_{\{Z(s)<a\}}\,\mathrm{d}L(s) = \int_0^t \mathbf{1}_{\{0>C(t)-C(s)\}}\,\mathrm{d}L(s) = 0. \qquad \square$$

We complete this section with the following proof.

Proof of Theorem 3.1. Starting with the right-hand side of (3.2), we obtain

$$\begin{aligned}
&\int_0^t \mathbf{1}_{\{Z(s)>C(t)-C(s)\}}\,\mathbf{1}_{\{Z(s)<a\}}\,\mathrm{d}A(s)\\
&\quad = \int_0^t \mathbf{1}_{\{Z(s)>C(t)-C(s)\}}\,\mathbf{1}_{\{Z(s)<a\}}\,\mathrm{d}(Z(s)+C(s)-L(s)+U(s))\\
&\quad = \int_0^t \mathbf{1}_{\{Z(s)>C(t)-C(s)\}}\,\mathbf{1}_{\{Z(s)<a\}}\,\mathrm{d}(Z(s)+C(s))\\
&\quad = \int_0^t \mathbf{1}_{\{Z(s)>C(t)-C(s)\}}\,\mathrm{d}(Z(s)+C(s))\\
&\qquad - \int_0^t \mathbf{1}_{\{Z(s)>C(t)-C(s)\}}\,\mathbf{1}_{\{Z(s)=a\}}\,\mathrm{d}(Z(s)+C(s))\\
&\quad = \int_0^t \mathbf{1}_{\{Z(s)+C(s)>C(t)\}}\,\mathrm{d}(Z(s)+C(s))\\
&\quad = \int_0^{Z(t)+C(t)} \mathbf{1}_{\{x>C(t)\}}\,\mathrm{d}x\\
&\quad = \int_{C(t)}^{Z(t)+C(t)} \mathrm{d}x\\
&\quad = Z(t),
\end{aligned}$$

where the second equality follows from both the definition of U and Lemma 3.3, with the final two equalities following from applying both Lemma 3.2 and another change-of-variable argument. $\square$

We close by briefly explaining how our integral representations can be used to derive Little-type formulas for fluid queues. Suppose that A and C are stationary and ergodic random measures on $\mathbb{R}$ that are atomless with probability 1. Using these measures, we define another random measure X given by

$$X(s,t] = A(s,t] - C(s,t], \qquad s<t$$

and we let $\{Q(t);\, t\in\mathbb{R}\}$ be a process that satisfies, for each $t\in\mathbb{R}$ (see [4, p. 245]),

$$Q(t) = \sup_{u\le t}\Big(X(u,t] \wedge (a + \inf_{v\in(u,t]} X(v,t])\Big).$$

As shown in [4], this process coincides with the two-sided reflection of X in $[0,a]$. Letting $\lambda_A = \mathbb{E}[A(0,1]]$ and $\lambda_C = \mathbb{E}[C(0,1]]$, if we further assume that $\lambda_A < \lambda_C$, then by [4, Theorem 14, p. 248], $Q(t)$ reaches state 0 infinitely often as $t\to-\infty$ or $t\to\infty$. This leads, by Theorem 2.1, to the equality

$$Q(0) = \int_{-\infty}^0 \mathbf{1}_{\{Q(s)>C(s,0]+U(s,0]\}}\,A(\mathrm{d}s). \tag{3.3}$$

This equality can be used to derive a fluid version of Little's law, in a manner analogous to that given in [6] and [7] for fluid queues having an infinite buffer. Applying [3, Equation (1.2.22), p. 19] to (3.3), we obtain

$$\mathbb{E}[Q(0)] = \lambda_A \mathbb{E}_A\bigg[\int_0^\infty \mathbf{1}_{\{Q(0)>C(0,s]+U(0,s]\}}\,\mathrm{d}s\bigg] = \lambda_A \mathbb{E}_A[(C+U)^{-1}(Q(0))],$$

where $\mathbb{P}_A$ is the Palm measure induced by the random measure A, and where $(C+U)^{-1}$ denotes the inverse of $(C+U)$. For each $u > 0$,

$$(C+U)^{-1}(u) = \inf\{t \geq 0 \colon C(0,t] + U(0,t] \geq u\}.$$

Another Little-type formula can be derived in an analogous manner, starting with Theorem 3.1, assuming A and C also have disjoint support with probability 1: this formula is simply

$$\mathbb{E}[Q(0)] = \lambda_A \mathbb{E}_A[C^{-1}(Q(0))\,\mathbf{1}_{\{Q(0)<a\}}].$$

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