BULL. AUSTRAL. MATH. SOC. 30C45, 30C55
VOL. 25 (1982), 387-396.

# ON ANALYTIC FUNCTIONS WITH REFERENCE TO THE BERNARDI INTEGRAL OPERATOR

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Bernardi has proved that if $f$ is starlike univalent in the unit disc $E$, then so is the function $g$ given by

$$g(z) = (c+1)z^{-c} \int_0^z t^{c-1} f(t)dt .$$

In the first part of the paper, we extend Bernardi's theorem to a certain class $S_p^*(A, B)$ of $p$-valent starlike functions in $E$. We prove that if $f \in S_p^*(A, B)$ then $g$, defined by

$$g(z) = (p+c)z^{-c} \int_0^z t^{c-1} f(t)dt ,$$

also belongs to $S_p^*(A, B)$. In the second part of the paper we examine the converse problem for functions with negative coefficients, satisfying certain conditions.

## 1. Introduction

Let $E = \{z : |z| < 1\}$ and let

$$H = \{w \text{ regular in } E : w(0) = 0, \ |w(z)| < 1, \ z \in E\} .$$

Let $P(A, B)$ denote the class of regular functions in $E$ which can be put in the form $(1+Aw(z))/(1+Bw(z))$, $-1 \le A < B \le 1$, $w \in H$. Let $S_p^*(A, B)$ denote functions $f$ of the form

Received 25 November 1981.



https://doi.org/10.1017/S0004972700005438 Published online by Cambridge University Press

$$f(z) = a_p z^p + \sum_{p+1}^{\infty} a_m z^m \quad (p \geq 1)$$

such that $(1/p)\left(zf'(z)/f(z)\right) \in P(A, B)$ . Bernardi [1] has proved that if $f$ is starlike univalent in $E$ , then so is the function $g$ given by

$$g(z) = (c+1)z^{-c} \int_0^z t^{c-1} f(t)dt .$$

In the first part of the paper we extend Bernardi's theorem to the class $S_p^*(A, B)$ . We prove that if $f \in S_p^*(A, B)$ then $g$ , defined by

$$g(z) = (p+c)z^{-c} \int_0^z t^{c-1} f(t)dt ,$$

also belongs to $S_p^*(A, B)$ . In the second part of the paper we examine the converse problem for functions with negative coefficients. Let $T_p$ denote the class of functions

$$f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{n+k} z^{n+k} , \quad p \geq 1 , \quad k \geq p ,$$

$a_{n+k} \geq 0$ , $a_p > 0$ regular in $E$ , satisfying $f\left(z_0\right) = z_0^p$ for a given real number $z_0$ , $-1 < z_0 < 1$ , $z_0 \neq 0$ . The class $T_1$ has been studied by Silverman [3]. Consider the subclass $S_p^*\left(z_0, A, B\right)$ defined as follows:

$$S_p^*\left(z_0, A, B\right) = \left\{f \in T_p : \frac{1}{p} \frac{zf'(z)}{f(z)} \in P(A, B)\right\} .$$

We observe that $S_p^*\left(z_0, A, B\right)$ is a subclass of $T_p$ consisting of $p$-valently starlike functions. The definition of $S_p^*\left(z_0, A, B\right)$ implies that functions $f$ in $S_p^*\left(z_0, A, B\right)$ satisfy $\operatorname{Re}\left(zf'(z)/f(z)\right) > 0$ , $z \in E$ . Further if $f \in S_p^*\left(z_0, A, B\right)$ ,

$$\frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re}\left\{\frac{re^{i\theta} f'\left(re^{i\theta}\right)}{f\left(re^{i\theta}\right)}\right\} d\theta = \frac{p}{2\pi} \int_0^{2\pi} \operatorname{Re} \frac{1+A\omega(z)}{1+B\omega(z)}\, d\theta = \frac{p}{2\pi} \cdot 2\pi = p ,$$

since $\operatorname{Re}\left(\left(1+A\omega(z)\right)/\left(1+B\omega(z)\right)\right)$ is a harmonic function in $E$ with $\omega(0) = 0$ . This proves $p$-valence of $f \in S_p^*\left(z_0, A, B\right)$ . If

$g \in S_p^*(z_0, A, B)$ and $f$ is defined by

$$g(z) = (c+p)z^{-c} \int_0^z t^{c-1} f(t)dt ,$$

we determine the radius of the largest disc in which $f$ is $p$-valently starlike of order $\alpha$, $0 \le \alpha < 1$.

## 2. Effect of Bernardi operator on $S_p^*(A, B)$

LEMMA 1. *Let $N$ and $D$ be regular in $E$, $D$ map $E$ onto a many sheeted starlike region, $N(0) = D(0) = 0$, $N'(0)/D'(0) = p$, $(1/p)\big(N'(z)/D'(z)\big) \in P(A, B)$. Then $(1/p)\big(N(z)/D(z)\big) \in P(A, B)$, $p \ge 1$.*

Proof. We have

$$\frac{N'(z)}{pD'(z)} \in P(A, B) \Leftrightarrow \frac{N'(z)}{pD'(z)} = \frac{1+Aw(z)}{1+Bw(z)} < \frac{1+Az}{1+Bz} .$$

Also $(1+Az)/(1+Bz)$ maps $|z| < r$ onto the disc with centre $a(r) = \big(1-ABr^2\big)/\big(1-B^2r^2\big)$ and radius $b(r) = (B-A)r/\big(1-B^2r^2\big)$ . $N'(z)/pD'(z)$ takes values in this disc and therefore $\big|\big(N'(z)/pD'(z)\big)-a(r)\big| < b(r)$, $|z| < r$, $0 < r < 1$. Choose $A(z)$ so that

$$pD'(z)A(z) = N'(z) - pa(r)D'(z) .$$

Then $|A(z)| < b(r)$. Fix $z_0$ in $E$. Let $L$ be the segment joining $0$ and $D\big(z_0\big)$ which lies in one sheet of the starlike image of $E$ by $D$. Let $L^{-1}$ be the pre-image of $L$ under $D$. Then

$$\begin{aligned} \big|N\big(z_0\big)-pa(r)D\big(z_0\big)\big| &= \left|\int_0^{z_0} [N'(t)-pa(r)D'(t)]dt\right| \\ &= \left|\int_{L^{-1}} pD'(t)A(t)dt\right| \\ &< pb(r) \int_L |dD(t)| \\ &= pb(r)\big|D\big(z_0\big)\big| . \end{aligned}$$

That is $\big|\big(N\big(z_0\big)/pD\big(z_0\big)\big)-a(r)\big| < b(r)$; that is, $N(z)/pD(z) < (1+Az)/(1+Bz)$ or equivalently $N/pD \in P(A, B)$.

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LEMMA 2 [2]. *Let* $N$ *and* $D$ *be regular in* $E$ , $D$ *map* $E$ *onto a many sheeted starlike region,* $N(0) = D(0) = 0$ , $N'(0)/D'(0) = p$ , $\operatorname{Re}\left(N'(z)/D'(z)\right) > 0$ . *Then* $\operatorname{Re}\left(N(z)/D(z)\right) > 0$ .

Proof of Lemma 2 follows from Lemma 1 by taking $A = -1$ , $B = 1$ . Using Lemma 2, we can prove

LEMMA 3. *Let* $f \in S_p^*(A, B)$ ,

$$J(z) = \int_0^z t^{c-1}f(t)dt .$$

*Then* $j$ *is* $(p+c)$*-valent starlike for* $c = 1, 2, \ldots$ , *in* $E$ .

This is a generalization of Lemma C in [1] and the proof is analogous to the one given in [1] and hence is omitted.

THEOREM 1. *Let* $f \in S_p^*(A, B)$ *and* $g(z) = (c+p)z^{-c}J(z)$ , $c = 1, 2, \ldots$ , *where* $J(z)$ *is as in Lemma* 3. *Then* $g \in S_p^*(A, B)$ .

Proof. Let $g'(z) = (c+p)z^{-1}f(z) - cz^{-1}g(z)$ . Then

$$\frac{zg'(z)}{g(z)} = \frac{zg'(z)}{g(z)}\frac{z^c}{z^c} = \frac{z^cf(z)-cJ(z)}{J(z)} = \frac{N(z)}{D(z)}$$

where $N(z) = z^cf(z) - cJ(z)$ , $D(z) = J(z)$ . $D(z) = J(z)$ is $(p+c)$-valent starlike by Lemma 3, $N'(0)/D'(0) = zf'(z)/f(z)|_{z=0} = p$ and since $f \in S_p^*(A, B)$ , $(1/p)\, N'(z)/D'(z) = 1/p \times zf'(z)/f(z) \in P(A, B)$ . Lemma 1 now enables us to conclude $(1/p)(N/D) = (1/p)(zg'/g) \in P(A, B)$ . That is, $g \in S_p^*(A, B)$ .

NOTE. By taking $p = 1$ , $A = -1$ , $B = 1$ in Theorem 1 we deduce the following theorem of Bernardi.

THEOREM A ([1], Theorem 1). *Let*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^* ,$$

*the class of starlike functions, and*

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$$g(z) = (c+1)z^{-c} \int_0^z t^{c-1} f(t)dt \ , \quad c = 1, 2, \ldots .$$

*Then* $g \in S^*$ .

## 3. The converse problem

Now we introduce the following notation for brevity. $n + k = m$ , $m(B+1) - p(A+1) = C_m$ , $p(B-A) = D$ , $\sum_{m=k+1}^{\infty} = \sum$ and $m(B+1)(1-\beta) = E$ . We need the following lemmas.

LEMMA 4. *Let* $f \in T_p$ . *Then* $f \in S_p^*(z_0, A, B)$ *if and only if*

(1) $$\sum \left[ \left(C_m/D\right) - z_0^{m-p} \right] a_m \leq 1 .$$

Proof. Suppose $f \in S_p^*(z_0, A, B)$ . Then

$$\frac{zf'(z)}{f(z)} = p \, \frac{1+Aw(z)}{1+Bw(z)} \ , \quad -1 \leq A < B \leq 1 \ , \quad w \in H \ , \quad z \in E .$$

That is, $w(z) = \left(p - zf'(z)/f(z)\right) / \left(Bzf'(z)/f(z) - Ap\right)$ , $w(0) = 0$ and

$$|w(z)| = \left|\left(zf'(z) - pf(z)\right) / \left(Bzf'(z) - Apf(z)\right)\right|$$
$$= \left| \left[\sum (m-p)a_m z^m\right] / \left[Da_p z^p - \sum (Bm-Ap)a_m z^m\right] \right| < 1 .$$

Thus

(2) $$\mathrm{Re}\left\{ \left[\sum (m-p)a_m z^m\right] / \left[Da_p z^p - \sum (Bm-Ap)a_m z^m\right] \right\} < 1 .$$

Take $z = r$ with $0 < r < 1$ . Then, for sufficiently small $r$ , the denominator of (2) is positive and so it is positive for all $r$ with $0 < r < 1$ , since $w(z)$ is regular for $|z| < 1$ . Since $f$ is in $T_p$ , $f\left(z_0\right)/z_0^p = 1 = a_p - \sum a_m z_0^{m-p}$ and we get

(3) $$a_p = 1 + \sum a_m z_0^{m-p} .$$

Then (2) gives $\sum (m-p)a_m r^m < Da_p r^p - \sum (Bm-Ap)a_m r^m$ , that is, $\sum C_m a_m r^m < Da_p r^p$ , and (1) follows on letting $r \to 1$ and using (3).

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Conversely suppose $f \in T_p$ and (1) holds. For $|z| = r$, $0 < r < 1$, we have, since $r^m < r^p$,

$$\sum [m(B+1)-p(A+1)]a_m r^m = \sum C_m a_m r^m < r^p \sum C_m a_m < Da_p r^p$$

by (1) and (3). So we have

$$\left|\sum (m-p)a_m z^p\right| \le \sum (m-p)a_m r^m < Da_p r^p - \sum (Bm-Ap)a_m r^m$$
$$\le \left|Da_p z^p - \sum (Bm-Ap)a_m z^m\right| .$$

This proves that $zf'(z)/f(z)$ is of the form $p\big((1+Aw(z))/(1+Bw(z))\big)$ with $w \in H$.

LEMMA 5. *Let* $g \in T_1$. *Then* $g'/a_1 \in P(A, B)$ *if and only if*

(4) $$\sum \left[m(B+1)-(B-A)z_0^{m-1}\right]a_m \le B - A .$$

Proof. $g \in T_1$ implies $g(z_0) = a_1 z_0 - \sum a_m z_0^m = z_0$. So we have

(5) $$a_1 = 1 + \sum a_m z_0^{m-1} .$$

Suppose $g'(z) = a_1\big((1+Aw(z))/(1+Bw(z))\big) = a_1 - \sum ma_m z^{m-1}$, $m \ge k+1$, $k \ge 1$. Then

$$|w(z)| = \left|(a_1-g'(z))/(Bg'(z)-a_1A)\right|$$
$$= \left|\left(\sum ma_m z^{m-1}\right)/\left(a_1(B-A)-B\sum ma_m z^{m-1}\right)\right| < 1 , \quad z \in E .$$

Thus

(6) $$\operatorname{Re}\left\{\left(\sum ma_m z^{m-1}\right)/\left(a_1(B-A) - B\sum ma_m z^{m-1}\right)\right\} < 1 .$$

Take $z = r$ with $0 < r < 1$. Then, for sufficiently small $r$, $a_1(B-A) - B\sum ma_m r^{m-1} > 0$ and hence it is positive for all $r$ with $0 < r < 1$, since $w(z)$ is regular for $|z| < 1$. Inequality (6) then gives $\sum ma_m r^{m-1} < a_1(B-A) - B\sum ma_m r^{m-1}$ and (4) follows on letting $r \to 1$ and using (5). Conversely, for $|z| = r$, $0 < r < 1$, we have,

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since $r^{m-1} < 1$,

$$\sum m(B+1)a_m r^{m-1} < \sum m(B+1)a_m < a_1(B-A)$$

by (4) and (5) and

$$\left|\sum m a_m z^{m-1}\right| \le \sum m a_m r^{m-1} < a_1(B-A) - \sum B m a_m r^{m-1}$$
$$\le \left|a_1(B-A) - \sum B m a_m z^{m-1}\right| .$$

This proves that $g' \in P(A, B)$.

THEOREM 2. *Let* $g \in S_p^*(z_0, A, B)$ *and* $f(z) = \left(z^{1-c}/(p+c)\right)\left[z^c g(z)\right]'$. *Then* $f$ *is p-valently starlike of order* $\alpha$, $0 \le \alpha < 1$, *in the disc*

$$|z| < r = r(\alpha, A, B) = \inf_m \left[\frac{(p-\alpha)(p+c)}{(m-\alpha)(m+c)} \frac{C_m}{D}\right]^{1/(m-p)}, \quad c = 1, 2, \ldots .$$

Proof. From Lemma 4 we see that, since $g \in S_p^*(z_0, A, B)$, (1) holds.

Since $f(z) = \left(z^{1-c}/(p+c)\right)\left[z^c g(z)\right]' = a_p z^p - \sum \left((m+c)/(p+c)\right)a_m z^m$, it is enough to show that $\left|\left(zf'(z)/f(z)\right)-p\right| \le p - \alpha$ for $|z| < r(\alpha, A, B)$.

Now

(7) $\left|\left(zf'(z)/f(z)\right)-p\right|$

$$= \left|\left[\sum \left((m+c)(m-p)/(p+c)\right)a_m z^{m-p}\right] \Big/ \left[a_p - \sum \left((m+c)/(p+c)\right)a_m z^{m-p}\right]\right|$$
$$\le \left[\sum \left((m+c)(m-p)/(p+c)\right)a_m |z|^{m-p}\right] \Big/ \left|a_p - \sum \left((m+c)/(p+c)\right)a_m |z|^{m-p}\right| .$$

Consider the values of $z$ for which

$$|z| \le \inf_m \left[\frac{(p-\alpha)(p+c)}{(m-\alpha)(m+c)} \frac{C_m}{D}\right]^{1/(m-p)},$$

so that $|z|^{m-p} \le \left((p-\alpha)(p+c)/(m-\alpha)(m+c)\right)\left(C_m/D\right)$ holds for all $m \ge k+1$. Then

$$\sum \frac{m+c}{p+c} a_m |z|^{m-p} \le \sum \frac{(p-\alpha)}{m-\alpha} \frac{C_m}{D} a_m < \sum \frac{C_m}{D} a_m ,$$

since $p < m$. Now $\sum \left((m+c)/(p+c)\right)a_m |z|^{m-p} < a_p$ provided

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$\sum (C_m/D)a_m \le a_p$ . Since $f(z_0) = z_0^p$ , this condition is equivalent to $\sum (C_m/D)a_m \le a_p = 1 + \sum a_m z_0^{m-p}$ , that is, $\sum \left[(C_m/D) - z_0^{m-p}\right]a_m \le 1$ which is true by (1). Hence we can rewrite the denominator of the right hand side of inequality (7) for the considered values of $z$ , using the fact that $a_p > \sum \{(m+c)/(p+c)\}a_m|z|^{m-p}$ . Thus

$$|(zf'(z)-pf(z))/f(z)|$$

$$\le \left\{\sum (m-p)(m+c)a_m|z|^{m-p}\right\}/\left\{(p+c)a_p - \sum (m+c)a_m|z|^{m-p}\right\} \le p - \alpha$$

if

$$\sum (m-p)(m+c)a_m|z|^{m-p} \le (p-\alpha)\left[(p+c)a_p - \sum (m+c)a_m|z|^{m-p}\right] ,$$

that is, if

$$\sum (m+c)(m-\alpha)a_m|z|^{m-p} \le (p-\alpha)(p+c)a_p = (p-\alpha)(p+c)\left(1 + \sum a_m z_0^{m-p}\right) ,$$

that is, if

$$\sum \left[\frac{(m-\alpha)(m+c)}{(p-\alpha)(p+c)} |z|^{m-p} - z_0^{m-p}\right]a_m \le 1 . \tag{8}$$

But in view of (1), (8) is satisfied if

$$\frac{(m-\alpha)(m+c)}{(p-\alpha)(p+c)} |z|^{m-p} \le \frac{C_m}{D} ,$$

that is, if

$$|z| \le \left[\frac{(p-\alpha)(p+c)}{(m-\alpha)(m+c)} \frac{C_m}{D}\right]^{1/(m-p)} .$$

The bound is sharp for the choice of function $g$ given by

$$g(z) = \left(C_m z^p - Dz^m\right)/\left(C_m - Dz_0^{m-p}\right)$$

and the corresponding $f$ . Hence the theorem.

NOTE. The conclusion of Theorem 2, is independent of the point $z_0$ .

THEOREM 3. *Let* $g \in T_1$ *and get* $g'/a_1 \in P(A, B)$ . *Define*

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$f(z) = \left(z^{1-c}/(c+1)\right)\left(z^c g(z)\right)'$ . *Then* Re $f'(z) > \beta$ , $0 \le \beta < 1$ , *for*

$$|z| < r = r(\beta, D, E) = \inf_m \left[\left((c+1)/m(m+c)\right)\left((pE/D)+\beta z_0^{m-1}\right)\right]^{1/(m-1)} ,$$

$c = 1, 2, \ldots$ .

Proof. $f(z) = \left(z^{1-c}/(c+1)\right)\left(z^c g(z)\right)' = a_1 z - \sum \left((c+m)/(c+1)\right)a_m z^m$ . It is enough to show that

$$|f'(z)-a_1| \le a_1 - \beta \quad \text{for} \quad |z| \le r(\beta, D, E) .$$

Now $|f'(z)-a_1| = \left|\sum \left(m(m+c)/(c+1)\right)a_m z^{m-1}\right| \le \sum \left(m(m+c)/(c+1)\right)a_m|z|^{m-1}$ .

Thus $|f'(z)-a_1| \le a_1 - \beta$ provided $\sum \left(m(m+c)/(c+1)\right)a_m|z|^{m-1} \le a_1 - \beta$ . As $g \in T_1$ implies (5), the above condition is equivalent to

$$\sum \left(\frac{m(m+c)}{c+1}|z|^{m-1} - z_0^{m-1}\right)a_m \le 1 - \beta . \tag{9}$$

Since $g'/a_1 \in P(A, B)$ we have, using Lemma 5,

$$\sum \left(\frac{m(B+1)}{B-A} - z_0^{m-1}\right)a_m \le 1 .$$

Hence (9) is true if

$$\frac{m(m+c)}{(c+1)(1-\beta)}|z|^{m-1} - \frac{1}{1-\beta}z_0^{m-1} \le \frac{m(B+1)}{B-A} - z_0^{m-1} = \frac{mp(B+1)}{D} - z_0^{m-1} ;$$

that is, if

$$|z| \le \left[\frac{c+1}{m(m+c)}\left(\frac{pE}{D} + \beta z_0^{m-1}\right)\right]^{1/(m-1)} .$$

The bound is sharp for the choice of function $g$ given by

$$g(z) = \frac{m(B+1)z-(B-A)z^m}{m(B+1)-(B-A)z_0^{m-1}} .$$

## References

[1] S.D. Bernardi, "Convex and starlike univalent functions", *Trans. Amer. Math. Soc.* 135 (1969), 429-446.

[2] R.J. Libera, "Some classes of regular univalent functions", *Proc. Amer. Math. Soc.* **16** (1965), 755-758.

[3] Herb Silverman, "Extreme points of univalent functions with two fixed points", *Trans. Amer. Math. Soc.* **219** (1976), 387-395.

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