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Positive solutions of nonlinear fourth-order boundary-value problems with local and non-local boundary conditions

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We establish new existence results for multiple positive solutions of fourth-order nonlinear equations which model deflections of an elastic beam. We consider the widely studied boundary conditions corresponding to clamped and hinged ends and many non-local boundary conditions, with a unified approach. Our method is to show that each boundary-value problem can be written as the same type of perturbed integral equation, in the space $C[0, 1]$, involving a linear functional $\alpha[u]$ but, although we seek positive solutions, the functional is *not* assumed to be positive for all positive u. The results are new even for the classic boundary conditions of clamped or hinged ends when $\alpha[u] = 0$, because we obtain sharp results for the existence of one positive solution; for multiple solutions we seek optimal values of some of the constants that occur in the theory, which allows us to impose weaker assumptions on the nonlinear term than in previous works. Our non-local boundary conditions contain multi-point problems as special cases and, for the first time in fourth-order problems, we allow coefficients of both signs.

1. Introduction

We shall study the existence of multiple positive solutions of the fourth-order differential equation

$$u^{(4)}(t) = g(t)f(t, u(t)) \quad \text{for almost every } t \in (0, 1), \tag{1.1}$$

subject to various boundary conditions (BCs); g and f are non-negative and g is allowed to have singularities. Equation (1.1) models the stationary states of the deflection of an elastic beam. The standard BCs that are often imposed are

$$u(0) = 0, \quad u(1) = 0, \quad u'(0) = 0, \quad u'(1) = 0, \tag{1.2}$$

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which correspond to both ends being clamped, and

$$u(0) = 0, \quad u(1) = 0, \quad u''(0) = 0, \quad u''(1) = 0, \tag{1.3}$$

which correspond to hinged ends, when there is no bending moment at the ends, also termed 'simply supported' (it also models a rotating shaft).

The existence of a solution for equation (1.1), or its generalizations, under the above conditions and other local linear BCs which cover different physical situations, has been considered extensively in the literature (see, for example, [2, 7, 8, 11, 16, 20–22, 25, 27, 30–32]).

Moreover, there are papers dealing with general BCs that include some of the usual linear ones. For example, nonlinear BCs are considered in [5, 6, 33]. Although each of these uses the technique involving upper and lower solutions, their results are not usually comparable because each considers a perturbation of a different linear problem.

Here, we shall discuss a general approach based on fixed-point index theory to deal with the existence of *positive* solutions for boundary-value problems (BVPs) with each of the BCs (1.2), (1.3), and also some non-local BCs which include these as special cases. In particular, we deal with each of the following non-local BCs:

$$u(0) = 0, \quad u(1) = \alpha[u], \quad u'(0) = 0, \quad u'(1) = 0, \tag{1.4}$$

$$u(0) = 0, \quad u(1) = 0, \quad u'(0) = 0, \quad u'(1) + \alpha[u] = 0, \tag{1.5}$$

$$u(0) = 0, \quad u(1) = \alpha[u], \quad u''(0) = 0, \quad u''(1) = 0, \tag{1.6}$$

$$u(0) = 0, \quad u(1) = 0, \quad u''(0) = 0, \quad u''(1) + \alpha[u] = 0. \tag{1.7}$$

In these equations $\alpha[u]$ denotes a linear functional on $C[0,1]$ given by

$$\alpha[u] = \int_0^1 u(s)\,\mathrm{d}A(s) \tag{1.8}$$

involving a Stieltjes integral. This includes the well-known multi-point BCs, where $\alpha[u] = \sum_{i=1}^{m} \alpha_i u(\eta_i)$, $\eta_i \in (0,1)$. It is clear that, for each of the BCs (1.4)–(1.6), for a positive solution u to exist we must have $\alpha[u] \geqslant 0$. However, in contrast to other work, we do not suppose that $\alpha[u] \geqslant 0$ for all $u \geqslant 0$ but we allow a *signed* measure, that is, A is a function of bounded variation. Thus, we consider rather general BCs; these include multi-point BCs as special cases and we do not insist that all coefficients in these are positive, which increases the versatility of the model.

The BC (1.4) can be thought of as having the end at 0 clamped, and having some mechanism at end 1 that controls the displacement according to feedback from devices measuring the displacements along parts of the beam. Similarly, for the BC (1.5), depending on the feedback, the angular attitude of the beam at end 1 is adjusted while maintaining a fixed displacement. The BC (1.7) corresponds to controlling the bending moment at 1 according to the feedback about the displacements along the beam.

We refer the reader to [13, 23] for situations where modelling the deflection of a beam with point loadings leads to multi-point boundary-value problems. It is also interesting to note that such types of condition arise naturally when constructing Floquet theory of the beam equation (see [28]).

Webb and Infante [35] gave a general method which was applied to second-order equations and showed that non-local BCs can be studied by a common method. The advantages of this method are: its simplicity; multi-point problems are included as special cases; the Green's function does not need to be calculated for each individual BC; and its ability to allow a signed measure, corresponding to allowing terms of both signs to occur in multi-point BCs, subject to some overall positivity condition. We show that the fourth-order problems also fit this common framework; thus, we can, and do, study many BVPs in a unified manner rather than on an ad hoc, case-by-case basis.

Some recent papers (see, for example, [9,24,37]) have established existence results for the beam equation under some multi-point BCs different from ours, but they do not use a general theory.

The method in [35] is to find solutions of each non-local BVP as solutions of a perturbed Hammerstein integral equation of the type

$$u(t) = \gamma(t)\alpha[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s, \tag{1.9}$$

where $k(t,s)$ is the Green's function for the unperturbed problem (see § 3.1, below, for further details on how this is done).

With this common framework available, we are able to give results for perturbed problems by verifying that the well-known Green's function for each of the BCs (1.2), (1.3) satisfies the required conditions. We remark that other BCs can be considered equally easily once it is shown that they can be written in the form of (1.9) and satisfy the relevant hypotheses.

Our method establishes the existence of arbitrary numbers of positive solutions under suitable conditions on the nonlinear term f (see theorem 3.1 and figure 1); some of these conditions involve the 'principal eigenvalue' of the related linear differential equation.

In the local BC situation, for both the clamped and hinged ends cases, Rynne [29] discusses solutions of both signs and proves the existence of infinitely many solutions having a large number of nodes. Korman [15] uses techniques of bifurcation theory to give exact multiplicity of positive solutions for the fourth-order clamped ends problem when f is convex, and also gives a uniqueness result. Our methods are different and we obtain different types of results which complement theirs.

Our results have four main features. Firstly, we improve on previous results, even in the unperturbed case. By making a more careful analysis of some of the constants that occur in the theory we are able to obtain values that are optimal for this method. We improve, for example, the recent results of [16, 38] for the clamped ends case, (1.2), and [2, 39] for the hinged ends case, (1.3). Secondly, we obtain sharp results for the existence of one positive solution. Thirdly, we obtain new results on the existence of arbitrary numbers of positive solutions for many non-local BCs in one general method, and under weak conditions. Fourthly, for the first time in fourth-order problems, we provide a systematic study of multi-point problems which allows coefficients of both signs rather than having all these coefficients positive. We have placed emphasis on giving results where constants that occur in the hypotheses can be computed.

2. Positive solutions of perturbed integral equations

A standard approach to studying positive solutions of a BVP such as (1.1) with some BC is to find the corresponding Green's function k and seek solutions as fixed points of the integral operator

$$Su(t) := \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s$$

in the cone $P = \{u \in C[0,1] : u \geqslant 0\}$ of non-negative functions in the space $C[0,1]$ of continuous functions endowed with the usual supremum norm.

To obtain multiple positive solutions it has proved to be convenient to work in a smaller cone than P, namely, for some subinterval $[a,b]$ of $[0,1]$,

$$K_0 := \Big\{u \in P : \min_{t\in[a,b]} u(t) \geqslant c\|u\|\Big\},$$

where $c > 0$ is a constant. The cone K_0 is of a well-known type, apparently first used by Krasnosel'skiĭ, and may be found in [17, § 45.4], and by Guo (see, for example, [10]), and has been used by many other authors in the study of multiple solutions of BVPs.

Lan and Webb [19] gave a framework which fits the use of this cone rather well. However, for multi-point BCs in particular, the form of the Green's function can become very complicated, which leads to over-strong hypotheses being imposed on the coefficients because of the technical calculations. Webb and Infante [35] have refined this framework and shown that non-local BCs can be studied, in a unified way, without calculating a complicated Green's function, via a perturbed Hammerstein integral equation of the type

$$u(t) = \gamma(t)\alpha[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s := \gamma(t)\alpha[u] + Fu(t) := Tu(t), \quad (2.1)$$

where $\alpha[u]$ is as in (1.8) and k is the simpler Green's function of an unperturbed problem.

The rather weak conditions imposed on k, f and g in (2.1) are as follows.

(C_1) The kernel k is measurable, non-negative and, for every $\tau \in [0,1]$, satisfies

$$\lim_{t\to\tau} |k(t,s) - k(\tau,s)| = 0 \quad \text{for almost every } s \in [0,1].$$

(C_2) There exist a subinterval $[a,b] \subseteq [0,1]$, a measurable function Φ and a constant $c_1 \in (0,1]$ such that

$$\begin{aligned} k(t,s) &\leqslant \Phi(s) && \text{for } t \in [0,1] \text{ and almost every } s \in [0,1],\\ k(t,s) &\geqslant c_1\Phi(s) && \text{for } t \in [a,b] \text{ and almost every } s \in [0,1]. \end{aligned}$$

(C_3) $g\Phi \in L^1[0,1]$, $g \geqslant 0$ almost everywhere, and

$$\int_a^b \Phi(s)g(s)\,\mathrm{d}s > 0.$$

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(C_4) A is of bounded variation and

$$\mathcal{K}(s) := \int_0^1 k(t,s)\,\mathrm{d}A(t) \geqslant 0 \quad \text{for almost every } s.$$

(C_5) $\gamma \in C[0,1]$, $\gamma(t) \geqslant 0$, $0 \leqslant \alpha[\gamma] < 1$, and there exists $c_2 \in (0,1]$ such that

$$\gamma(t) \geqslant c_2\|\gamma\| \quad \text{for } t \in [a,b].$$

(C_6) $f : [0,1] \times [0,\infty) \to [0,\infty)$ satisfies Carathéodory conditions, that is, $f(\cdot,u)$ is measurable for each fixed $u \in [0,\infty)$ and $f(t,\cdot)$ is continuous for almost every $t \in [0,1]$ and, for each $r > 0$, there exists $\phi_r \in L^\infty[0,1]$ such that

$$0 \leqslant f(t,u) \leqslant \phi_r(t) \quad \text{for all } u \in [0,r] \text{ and almost all } t \in [0,1].$$

It is often convenient to establish the following type of inequality, which proves (C_2) when $c_1(t) \geqslant c_1 > 0$ on $[a,b]$:

(C_2') $c_1(t)\Phi(s) \leqslant k(t,s) \leqslant \Phi(s)$, for $0 \leqslant t,s \leqslant 1$.

We will do this in this paper.

The condition (C_3) means that we study weakly singular problems; g may have singularities at arbitrary points of $[0,1]$. The kernel k is often continuous and $k(t,s) > 0$ for $t \in (0,1)$ and $s \in [0,1]$, in which case (C_2) is satisfied for any $[a,b] \subset (0,1)$. However, careful selection of $[a,b]$ allows the use of weaker hypotheses on f in the fixed-point index calculations. The function Φ plays only a subsidiary role, but the value of c_1 enters explicitly into some of these calculations. Choosing c_1 as large as possible leads to a weaker condition to be satisfied by f in theorem 2.2, below. Note that the condition $\mathcal{K}(s) \geqslant 0$ in (C_4) is automatically satisfied for positive measures. Examples of sign-changing measures satisfying this condition in second-order problems are given explicitly in [35]. Example 3.5, below, shows how this condition can be satisfied in a fourth-order problem with a multi-point BVP having coefficients of both signs.

We use the classical theory of fixed-point index for compact maps (see, for example, [1] or [10] for details). Let $q : C[0,1] \to \mathbb{R}$ denote the continuous function

$$q(u) = \min\{u(t) : t \in [a,b]\}. \tag{2.2}$$

The above hypotheses allow us to work in the cone

$$K = \{u \in P,\ q(u) \geqslant c\|u\|,\ \alpha[u] \geqslant 0\}, \quad \text{where } c = \min\{c_1, c_2\}, \tag{2.3}$$

with c_1 as in (C_2) and c_2 as in (C_5). The cone defined in (2.3) was introduced in [35] to weaken the standard requirement that $\alpha[u] \geqslant 0$ for every $u \geqslant 0$ so that certain sign-changing measures can be dealt with. Note that $K = K_0 \cap \{u \in P : \alpha[u] \geqslant 0\}$, where K_0 is as mentioned above.

For $\rho > 0$ we define the following bounded open subsets of K:

$$K_\rho := \{u \in K : \|u\| < \rho\}, \qquad V_\rho := \{u \in K : q(u) < \rho\}.$$

The set V_ρ was so named in [12] and is equal to the set called $\Omega_{\rho/c}$ in [18].

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The theory of fixed-point index can be used, in an essentially routine way, to show the existence of arbitrary numbers of positive solutions under suitable conditions on f, once it is shown that the index is 1 on certain open subsets and 0 on certain others.

We give two types of index result: one exploits the behaviour of $f(t,u)/u$ near 0 and ∞ and gives some sharp results; the other concerns the behaviour on bounded intervals.

For $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ and $u\in\mathbb{R}$ we define the following notation:

$$\begin{aligned}
\bar{f}(u) &= \sup_{0\leqslant t\leqslant 1} f(t,u), & \underline{f}(u) &= \inf_{0\leqslant t\leqslant 1} f(t,u),\\
f^0 &= \limsup_{u\to 0+} \bar{f}(u)/u, & f_0 &= \liminf_{u\to 0+} \underline{f}(u)/u,\\
f^\infty &= \limsup_{u\to\infty} \bar{f}(u)/u, & f_\infty &= \liminf_{u\to\infty} \underline{f}(u)/u,\\
f^{0,\rho} &= \sup_{0\leqslant u\leqslant\rho,\,0\leqslant t\leqslant 1} f(t,u)/\rho, & f_{\rho,\rho/c} &= \inf_{\rho\leqslant u\leqslant\rho/c,\,a\leqslant t\leqslant b} f(t,u)/\rho.
\end{aligned}$$

The following result uses the notion of the principal eigenvalue of an associated linear operator. Let μ_1 be the smallest positive number such that there exists $\varphi\in P\setminus\{0\}$ satisfying

$$\varphi=\gamma(t)\alpha[\varphi]+\mu_1 L_F\varphi, \tag{2.4}$$

where

$$L_F u=\int_0^1 k(t,s)g(s)u(s)\,\mathrm{d}s.$$

In [35] it is shown that $1/\mu_1$ is the radius of the spectrum of a compact linear operator which has an eigenfunction $\varphi\in P\setminus\{0\}$ under our hypotheses; μ_1 is often called the principal eigenvalue of the corresponding linear differential equation. In general, μ_1 is to be calculated from the integral equation (2.4) by some numerical method, but in some cases can be found from the differential equation.

The following index results from [35] were deduced from some results of [36].

Theorem 2.1.

(i) *If* $0\leqslant f^0<\mu_1$, *then there exists* $\rho_0>0$ *such that*

$$i_K(T,K_\rho)=1 \quad \textit{for each } \rho\in(0,\rho_0].$$

(ii) *If* $0\leqslant f^\infty<\mu_1$, *then there exists* R_0 *such that*

$$i_K(T,K_R)=1 \quad \textit{for each } R>R_0.$$

(iii) *If* $\mu_1<f_0\leqslant\infty$, *then there exists* $\rho_0>0$ *such that, for each* $\rho\in(0,\rho_0]$, *if* $u\neq Tu$ *for* $u\in\partial K_\rho$, *then*

$$i_K(T,K_\rho)=0.$$

(iv) *If* (C_2) *holds for an arbitrary* $[a,b]\subset(0,1)$ *and* $\mu_1<f_\infty\leqslant\infty$, *then there exists* R_1 *such that, for each* $R\geqslant R_1$, *if* $u\neq Tu$ *for* $u\in\partial K_R$, *then*

$$i_K(T,K_R)=0.$$

In [35, 36] part (iv) is only proved under an extra hypothesis, namely that the principal eigenvalue is the unique eigenvalue with a positive eigenvector. However, this condition can be removed because a result of Nussbaum (see [26, lemma 2, p. 226]) proves that [36, theorem 3.7] holds in every case. It has also been shown in [34] that the principal eigenvalue is in fact the unique eigenvalue with a positive eigenvector for the type of non-local BVPs that we are studying.

We now quote the result from [35] which takes account of the behaviour of the nonlinearity on bounded intervals. A stronger result is also given in [35]. We state only the weaker result here, which allows easy computations of all the constants involved in specific cases, as we show later in the paper with explicit examples. When the perturbation is 0 we recover a version of a result of [18]. We employ the following constants:

$$\frac{1}{m} := \sup_{t\in[0,1]} \int_0^1 k(t,s)g(s)\,\mathrm{d}s, \qquad \frac{1}{M(a,b)} := \inf_{t\in[a,b]} \int_a^b k(t,s)g(s)\,\mathrm{d}s. \tag{2.5}$$

THEOREM 2.2 (Webb and Infante [35]).

(i) *Suppose there exists $\rho > 0$ such that*

$$f^{0,\rho}\left(\frac{\|\gamma\|}{1-\alpha[\gamma]}\int_0^1 \mathcal{K}(s)g(s)\,\mathrm{d}s + \frac{1}{m}\right) < 1. \tag{2.6}$$

Then the fixed-point index, $i_K(T, K_\rho)$, is equal to 1.

(ii) *For $\rho > 0$, the fixed-point index $i_K(T, V_\rho)$ is equal to* 0 *if*

$$f_{\rho,\rho/c}\left(\frac{c_2\|\gamma\|}{1-\alpha[\gamma]}\int_a^b \mathcal{K}(s)g(s)\,\mathrm{d}s + \frac{1}{M(a,b)}\right) > 1. \tag{2.7}$$

Theorem 2.1 leads to the following sharp result: a positive solution exists if the nonlinearity ‘crosses’ the principal eigenvalue.

THEOREM 2.3. *Assume that* (C_1)–(C_6) *hold and that one of the following conditions holds:*

(H_1) $0 \leqslant f^0 < \mu_1$ *and* $\mu_1 < f_\infty \leqslant \infty$;

(H_2) $0 \leqslant f^\infty < \mu_1$ *and* $\mu_1 < f_0 \leqslant \infty$.

Then (2.1) has a positive solution $u \in K$, $r \leqslant \|u\| \leqslant R$ for some $0 < r < R$.

The proof is obtained by applying theorem 2.1 on K_r with r sufficiently small and on K_R with R sufficiently large and using the additivity property of the fixed-point index.

Theorem 2.2 and a standard application of fixed-point index theory (see, for example, [35]) yields the following theorem on the existence of multiple positive solutions for equation (2.1).

Theorem 2.4. *Equation (2.1) has at least one positive solution in K if either of the following conditions holds:*

(H_1) *there exist $\rho_1, \rho_2 \in (0, \infty)$ with $\rho_1 < \rho_2$ such that (2.6) holds for ρ_1 and (2.7) holds for ρ_2;*

(H_2) *there exist $\rho_1, \rho_2 \in (0, \infty)$ with $\rho_1 < c\rho_2$ such that (2.7) holds for ρ_1 and (2.6) holds for ρ_2.*

Equation (2.1) has at least two positive solutions in K if one of the following conditions holds:

(S_1) *there exist ρ_1, ρ_2, ρ_3 with $\rho_1 < \rho_2$ and $\rho_2 < c\rho_3$ such that (2.6) holds for ρ_1, (2.7) holds for ρ_2 and (2.6) holds for ρ_3;*

(S_2) *there exist ρ_1, ρ_2, ρ_3 with $\rho_1 < c\rho_2 < c\rho_3$ such that (2.7) holds for ρ_1, (2.6) holds for ρ_2 and (2.7) holds for ρ_3.*

Moreover, when (S_1) holds, (2.1) has a third solution, $u_0 \in K_{\rho_1}$ (possibly zero).

It is routine to state results for the existence of three, four or an arbitrary number of positive solutions by expanding the lists in conditions (S_1) and (S_2). We illustrate the statement for three solutions in theorem 3.1, below, and indicate there (see figure 1) what the restrictions mean for the nonlinearity f. We leave these to the reader, who may consult [16, 18] to see such statements.

3. Clamped ends

We first consider equation (1.1) in the clamped ends case, that is with BCs

$$u(0) = 0, \quad u(1) = 0, \quad u'(0) = 0, \quad u'(1) = 0. \tag{3.1}$$

The Green's function associated with (1.1), (3.1) is (see, for example, [14])

$$k(t,s) = \begin{cases} \frac{1}{6}s^2(1-t)^2(3t - 2ts - s), & s \leqslant t, \\ \frac{1}{6}t^2(1-s)^2(3s - 2ts - t), & s > t. \end{cases} \tag{3.2}$$

We note that k has the following symmetry properties:

$$k(t,s) = k(s,t) = k(1-s, 1-t). \tag{3.3}$$

By a solution of the BVP (1.1) we shall mean a fixed-point of the integral equation

$$u(t) = \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s := Su(t), \tag{3.4}$$

with $k(t,s)$ as in (3.2), that is, a fixed point of S.

Here k is continuous so (C_1) holds. We have to verify the condition (C_2).

By the symmetry properties (3.3) it is sufficient to show the upper bounds in (C_2) for $0 \leqslant s \leqslant \frac{1}{2}$.

It is stated in [38] that

$$k(t,s) \leqslant \Phi(s) = \frac{2}{3}\frac{s^2(1-s)^3}{(3-2s)^2} \quad \text{for } 0 \leqslant s \leqslant \tfrac{1}{2},$$

and that

$$c_1(t) = \tfrac{2}{3}t^2 \quad \text{for } 0 \leqslant t \leqslant \tfrac{1}{2}.$$

Indeed, it is readily verified by calculus that $\Phi(s) = \max_{0\leqslant t\leqslant 1} k(t,s)$. By the symmetry properties of k we may choose

$$\Phi(s) = \begin{cases} \dfrac{2}{3}\dfrac{s^2(1-s)^3}{(3-2s)^2} & \text{for } 0 \leqslant s \leqslant \frac{1}{2}, \\[2ex] \dfrac{2}{3}\dfrac{s^3(1-s)^2}{(1+2s)^2} & \text{for } \frac{1}{2} < s \leqslant 1. \end{cases}$$

We will obtain a more precise estimate for $c_1(t)$ satisfying (C_2') than that from [38], above, which will allow us to give stronger results.

For $0 \leqslant t \leqslant \frac{1}{2}$ and $t < s \leqslant \frac{1}{2}$ we want to find $c_1(t)$ so that

$$\tfrac{1}{6}t^2(1-s)^2(3s-2ts-t) \geqslant c_1(t)\frac{2}{3}\frac{s^2(1-s)^3}{(3-2s)^2},$$

that is,

$$\frac{c_1(t)}{t^2} \leqslant \frac{1}{4}\frac{(3s-2ts-t)(3-2s)^2}{s^2(1-s)}.$$

The derivative of the right-hand side of this inequality with respect to s is negative for all $s \in [t, \frac{1}{2}]$. Therefore, the right-hand side is a decreasing function of s and so has its minimum value when $s = \frac{1}{2}$. Thus, it suffices to have

$$c_1(t) \leqslant 4t^2(3-4t).$$

For $0 \leqslant t \leqslant \frac{1}{2}$ and $s \leqslant t$ we need $c_1(t)$ such that

$$\tfrac{1}{6}s^2(1-t)^2(3t-2ts-s) \geqslant c_1(t)\frac{2}{3}\frac{s^2(1-s)^3}{(3-2s)^2},$$

that is,

$$\frac{c_1(t)}{(1-t)^2} \leqslant \frac{1}{4}\frac{(3t-2ts-s)(3-2s)^2}{(1-s)^3}.$$

The right-hand side, as a function of s, has precisely one critical point when $s = \frac{1}{2}(3-1/t)$, and this is a local maximum, so the minimum occurs either for $s = 0$ or for $s = t$. Thus, it suffices to have

$$c_1(t) \leqslant \min\{\tfrac{1}{2}t(3-2t)^2,\ \tfrac{27}{4}t(1-t)^2\}.$$

For $0 \leqslant t \leqslant \frac{1}{2}$ and $\frac{1}{2} \leqslant s \leqslant 1$ we require

$$\tfrac{1}{6}t^2(1-s)^2(3s-2ts-t) \geqslant c_1(t)\frac{2}{3}\frac{s^3(1-s)^2}{(1+2s)^2},$$

that is

$$\frac{c_1(t)}{t^2} \leqslant \frac{(1+2s)^2(3s-2ts-t)}{4s^3}.$$

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The sign of the derivative shows that the right-hand side is a decreasing function of $s \in [\frac{1}{2}, 1]$, and so its minimum is attained at $s = 1$. Hence, we need

$$c_1(t) \leqslant \tfrac{27}{4} t^2 (1-t).$$

The total requirement, for $0 \leqslant t \leqslant \frac{1}{2}$, is

$$c_1(t) = \min\{4t^2(3-4t),\ \tfrac{1}{2}t(3-2t)^2,\ \tfrac{27}{4}t(1-t)^2,\ \tfrac{27}{4}t^2(1-t)\} = \tfrac{27}{4}t^2(1-t).$$

From the symmetry properties of k, we therefore have

$$c_1(t) = \begin{cases} \frac{27}{4}t^2(1-t), & t \in [0, \frac{1}{2}], \\ \frac{27}{4}t(1-t)^2, & t \in (\frac{1}{2}, 1]. \end{cases}$$

We now compute the constants m and $M = M(a,b)$ for the special case $g \equiv 1$. We have

$$\frac{1}{m} = \sup_{t\in[0,1]} \int_0^1 k(t,s)\,\mathrm{d}s, \qquad \frac{1}{M(a,b)} := \inf_{t\in[a,b]} \int_a^b k(t,s)\,\mathrm{d}s,$$

and, by direct calculation, obtain $m = 384$. Kosmatov [16] has this value of the constant, but Yao [38] has a constant $\simeq 368$, which gives worse results. The 'optimal' $[a,b]$, the interval for which $M(a,b)$ is a minimum, is given (for example, using Maple) by the interval

$$[0.3037, 0.6963]. \tag{3.5}$$

This gives $M = 812.6995$ and $c_1 = 0.4336$. (Here and throughout the paper, constants have been rounded to four decimal places unless they are exact.) Yao [38] has an interval $[a,b]$ and gives expressions for the constants he uses. When specialized to $[a,b] = [\frac{1}{4}, \frac{3}{4}]$ he gives the values of $c_1 = \frac{1}{24} = 0.0417$ and a constant B which computes to $B \simeq 1839$, and where $B/c \simeq 44153$ replaces our M, so this is far from a good value. Kosmatov [16] uses $c_1(t) = \frac{2}{3}t^4$ (giving too small a value for c_1) and he computes $(M(a, 1-a))^{-1} = \frac{1}{24}a^2(1-2a)(1-2a^2)$. However, he does not show that the minimal value of $M(a,b)$ is $M(a,1-a)$. Nor does he compute the value of a that achieves this minimum, but this can easily be deduced from this formula as $a = 0.3037$. We found this value above by a different calculation.

Theorem 3.1. *Let m, M, c_1 be as in the previous paragraph. Then the clamped-ends BVP (1.1), (3.1) with $g \equiv 1$ has at least three positive solutions if either (T_1) or (T_2) holds.*

(T_1) *There exist $0 < \rho_0 < \rho_0/c_1 < \rho_1 < \rho_2 < \rho_2/c_1 < \rho_3 < \infty$, such that*

$$f_{\rho_0,\rho_0/c_1} > M, \quad f^{0,\rho_1} < m, \quad f_{\rho_2,\rho_2/c_1} > M, \quad f^{0,\rho_3} < m.$$

(T_2) *There exist $0 < \rho_0 < \rho_1 < \rho_1/c_1 < \rho_2 < \rho_3 < \infty$, such that*

$$f^{0,\rho_0} < m, \quad f_{\rho_1,\rho_1/c_1} > M, \quad f^{0,\rho_2} < m, \quad f_{\rho_3,\rho_3/c_1} > M.$$

Remark 3.2. When $g \neq 1$ and (C_3) is satisfied, the same result holds but the constants m and M have to be computed for that particular g. In condition (T_1)

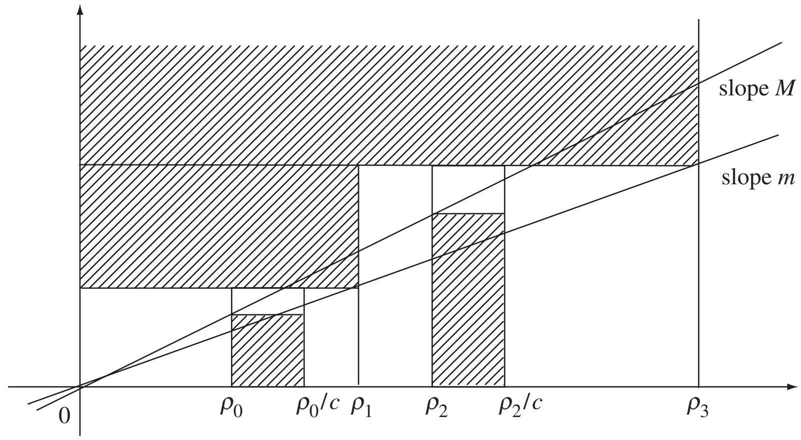

Figure 1. The graph of f cannot lie in the dashed region.

we may replace $f^{0,\rho_3} < m$ by $f^{\infty} < \mu_1$ and replace $f_{\rho_0,\rho_0/c_1} > M$ by $f_0 > \mu_1$. In condition (T_2) we may replace $f^{0,\rho_0} < m$ by $f^0 < \mu_1$ and replace $f_{\rho_3,\rho_3/c_1} > M$ by $f_\infty > \mu_1$. This gives stronger results when ρ_0 is very small and ρ_3 is very large, since it has been shown in [36] that $m \leqslant \mu_1 \leqslant M$ always holds, with strict inequality unless eigenfunctions are constants. Some other estimates of μ_1 may be found in [34].

REMARK 3.3. Note that the values of ρ_i are not independent. For example, in (T_1) it is *necessary* that $M\rho_0 < m\rho_1$ and $M\rho_2 < m\rho_3$. Figure 1 (not to scale) illustrates how, when f depends only on u, the graph of f is restricted in order to satisfy (T_1), and makes it clear that the conditions can readily be satisfied. The figure is easily modified when the conditions involving μ_1 are employed.

The eigenvalue μ_1 can be calculated directly from the differential equation (or numerically from the integral equation) when $g \equiv 1$. An eigenfunction is of the form

$$\varphi(t) = \sin(\omega t) - \sinh(\omega t) + C(\cos(\omega t) - \cosh(\omega t)),$$

where C is a constant, and a small calculation shows that ω is a root of the equation $\cos(\omega)\cosh(\omega) = 0$. The smallest positive root of this equation is $\omega = 4.7300$ and $\mu_1 = \omega^4 = 500.5639$. Comparing this with $m = 384$, and the optimal $M = 812.6995$ found above, shows that using μ_1 in place of m and M, whenever possible as in remark 3.2 above, gives better results, and gives a sharp result for the existence of one positive solution.

3.1. The boundary conditions (1.4)

We now consider the equation (1.1) with the non-local BCs

$$u(0) = 0, \quad u(1) = \alpha[u], \quad u'(0) = 0, \quad u'(1) = 0, \tag{3.6}$$

where

$$\alpha[u] = \int_0^1 u(s)\,\mathrm{d}A(s)$$

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for a *signed* measure $\mathrm{d}A$. By a solution of the BVP (1.1), (3.6) we shall mean a solution of the perturbed integral equation (see [35])

$$u(t) = \gamma(t)\alpha[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s,$$

where γ satisfies the BVP

$$\gamma^{(4)}(t) = 0, \quad \gamma(0) = 0, \quad \gamma(1) = 1, \quad \gamma'(0) = 0, \quad \gamma'(1) = 0.$$

Thus, the perturbed integral equation is

$$u(t) = t^2(3-2t)\alpha[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s, \tag{3.7}$$

with $k(t,s)$ as in (3.2). We have $\|\gamma\| = 1$ and $\min_{t\in[a,b]}\gamma(t) = \gamma(a) = 0.2207$ ($[a,b]$ as in (3.5)), so we need $c_2 = 0.2207$; therefore, $c = 0.2207$. We could immediately write down a theorem on existence of multiple positive solutions using theorem 2.4. Since the statements are now routine, similar to but more complicated than those of theorem 3.1, we omit them here; see § 4.1, below, for an example of an explicit calculation.

Example 3.4 (a discrete (four-point) problem). Consider the BC

$$u(0) = 0, \quad u(1) = \alpha_1 u(\eta_1) + \alpha_2 u(\eta_2), \quad u'(0) = 0, \quad u'(1) = 0, \quad 0 < \eta_1 < \eta_2 < 1.$$

We will now determine the restrictions to be placed on α_1, α_2 so that our hypotheses are satisfied.

We have $\alpha[u] = \alpha_1 u(\eta_1) + \alpha_2 u(\eta_2)$. Then $\alpha[\gamma] = \alpha_1\eta_1^2(3-2\eta_1) + \alpha_2\eta_2^2(3-2\eta_2)$, so we require

$$0 \leqslant \alpha_1\eta_1^2(3-2\eta_1) + \alpha_2\eta_2^2(3-2\eta_2) < 1. \tag{3.8}$$

We also have $\mathcal{K}(s) = \alpha_1 k(\eta_1,s) + \alpha_2 k(\eta_2,s)$. To satisfy $\mathcal{K}(s) \geqslant 0$ we need

$$\begin{aligned}
\alpha_1 s^2(1-\eta_1)^2(3\eta_1 - 2\eta_1 s - s) + \alpha_2 s^2(1-\eta_2)^2(3\eta_2 - 2\eta_2 s - s) \geqslant 0, &\quad 0 \leqslant s \leqslant \eta_1,\\
\alpha_1\eta_1^2(1-s)^2(3s - 2\eta_1 s - \eta_1) + \alpha_2 s^2(1-\eta_2)^2(3\eta_2 - 2\eta_2 s - s) \geqslant 0, &\quad \eta_1 < s \leqslant \eta_2,\\
\alpha_1\eta_1^2(1-s)^2(3s - 2\eta_1 s - \eta_1) + \alpha_2\eta_2^2(1-s)^2(3s - 2\eta_2 s - \eta_2) \geqslant 0, &\quad s > \eta_2.
\end{aligned}$$

Note that the condition $\mathcal{K}(s) \geqslant 0$ is automatically satisfied for positive coefficients.

Example 3.5. We give a numerical example to illustrate the type of restriction placed on the coefficients in the above four-point problem and to show that some negative coefficients are allowed. Taking $\eta_1 = \frac{1}{4}$, $\eta_2 = \frac{3}{4}$, the conditions become

$$0 \leqslant 5\alpha_1 + 27\alpha_2 < 32, \tag{3.9}$$

and

$$27\alpha_1(1-2s) + \alpha_2(9-10s) \geqslant 0 \quad \text{for } 0 \leqslant s \leqslant \tfrac{1}{4}, \tag{3.10}$$

$$\alpha_1(1-s^2)(10s-1) + \alpha_2 s^2(9-10s) \geqslant 0 \quad \text{for } \tfrac{1}{4} < s \leqslant \tfrac{3}{4}, \tag{3.11}$$

$$\alpha_1(10s-1) + 27\alpha_2(2s-1) \geqslant 0 \quad \text{for } \tfrac{3}{4} < s \leqslant 1. \tag{3.12}$$

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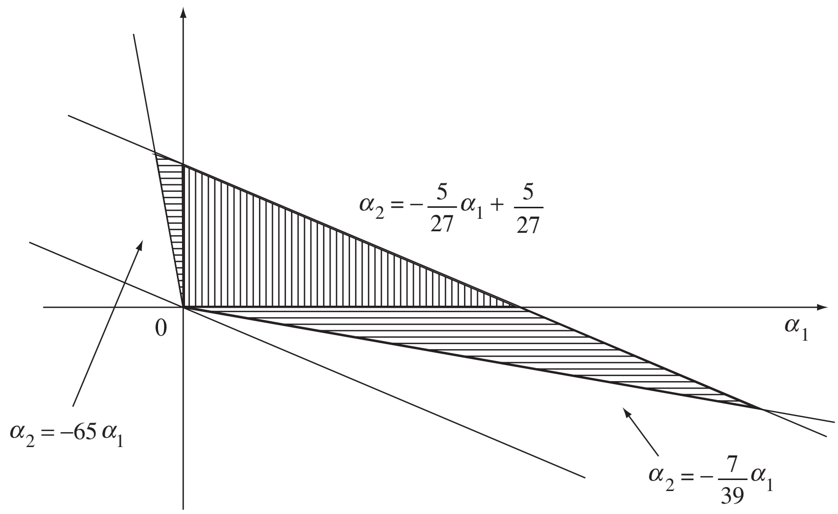

Figure 2. Part of the region for positive solutions.

This defines a region in the (α_1, α_2)-plane which includes the one shown in figure 2 (not to scale) and is much larger than the 'obvious' region, obtained when all coefficients are assumed to be non-negative and defined by

$$\alpha_1 \geqslant 0, \quad \alpha_2 \geqslant 0, \quad 0 \leqslant 5\alpha_1 + 27\alpha_2 < 32. \tag{3.13}$$

3.2. The boundary conditions (1.5)

Solutions of equation (1.1) with the non-local BCs

$$u(0) = 0, \quad u(1) = 0, \quad u'(0) = 0, \quad u'(1) + \alpha[u] = 0, \tag{3.14}$$

can be found as solutions of

$$u(t) = \gamma(t)\alpha[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s,$$

where γ satisfies the BVP

$$\gamma^{(4)}(t) = 0, \quad \gamma(0) = 0, \quad \gamma(1) = 0, \quad \gamma'(0) = 0, \quad \gamma'(1) + 1 = 0,$$

that is,

$$u(t) = t^2(1-t)\alpha[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s, \tag{3.15}$$

with $k(t,s)$ as in (3.2). Here γ has one critical point in $(0,1)$ at $t = \frac{2}{3}$, which is a local maximum. Hence, we have $\|\gamma\| = \frac{4}{27} = 0.1481$ and, with $[a,b]$ as in (3.5), $\min_{t\in[a,b]} \gamma(t) = \gamma(a) = 0.0642$, so we need $c_2 = 0.4336$; therefore, $c = 0.4336$.

4. Hinged ends

We now turn our attention to equation (1.1) with the hinged ends BCs

$$u(0) = 0, \quad u(1) = 0, \quad u''(0) = 0, \quad u''(1) = 0. \tag{4.1}$$

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The Green's function associated with the BVP (1.1), (4.1) is (see, for example, [14]),

$$k(t,s)=\begin{cases}\frac{1}{6}s(1-t)(2t-s^2-t^2), & s\leqslant t,\\ \frac{1}{6}t(1-s)(2s-t^2-s^2), & s>t.\end{cases} \tag{4.2}$$

We note that k has the same symmetries as above, that is

$$k(t,s)=k(s,t)=k(1-s,1-t). \tag{4.3}$$

The BVP (1.1) can be rewritten in integral form as

$$u(t)=\int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s, \tag{4.4}$$

with $k(t,s)$ as in (4.2).

We again have to verify (C_2). By the symmetry properties (4.3), it is sufficient to show the upper bounds in (C_2) for $0\leqslant s\leqslant\frac{1}{2}$.

For $0\leqslant s\leqslant\frac{1}{2}$ and $s\leqslant t$ we have

$$\tfrac{1}{6}s(1-t)(2t-s^2-t^2)\leqslant k(\tau(s),s)=\tfrac{\sqrt{3}}{27}s(1-s^2)^{3/2},$$

where

$$\tau(s):=1-\tfrac{\sqrt{3}}{3}\sqrt{1-s^2}.$$

For $0\leqslant s\leqslant\frac{1}{2}$ and $s>t$ we have

$$\tfrac{1}{6}t(1-s)(2s-t^2-s^2)\leqslant k(s,s)=\tfrac{1}{3}s(1-s)(s-s^2).$$

Since, for $0\leqslant s\leqslant\frac{1}{2}$, we have

$$\max\{\tfrac{\sqrt{3}}{27}s(1-s^2)^{3/2},\ \tfrac{1}{3}s(1-s)(s-s^2)\}=\tfrac{\sqrt{3}}{27}s(1-s^2)^{3/2},$$

using the symmetry properties of k we may choose

$$\varPhi(s)=\begin{cases}\frac{\sqrt{3}}{27}s(1-s^2)^{3/2} & \text{for } 0\leqslant s\leqslant\frac{1}{2},\\ \frac{\sqrt{3}}{27}(1-s)s^{3/2}(2-s)^{3/2} & \text{for } \frac{1}{2}<s\leqslant 1.\end{cases}$$

We now determine $c_1(t)$ so that (C_2') holds, that is $k(t,s)\geqslant c_1(t)\varPhi(s)$. Using (4.3), it suffices to consider only the case $0\leqslant t\leqslant\frac{1}{2}$. For $0\leqslant t\leqslant\frac{1}{2}$ and $s\leqslant t$ we want

$$\tfrac{1}{6}s(1-t)(2t-s^2-t^2)\geqslant c_1(t)\tfrac{\sqrt{3}}{27}s(1-s^2)^{3/2},$$

that is,

$$\frac{c_1(t)}{(1-t)}\leqslant\frac{3\sqrt{3}}{2}\frac{(2t-s^2-t^2)}{(1-s^2)^{3/2}}.$$

The right-hand side, considered as a function of s, has a critical point at $s=\sqrt{6t-3t^2-2}$ but this is in the correct range only when $0.422\simeq 1-\frac{\sqrt{3}}{3}\leqslant t\leqslant\frac{1}{2}$. This value corresponds to a maximum so the minimum is for $s=t$ or $s=0$. Thus, we need

$$c_1(t)\leqslant\min\left\{3\sqrt{3}\frac{t(1-t)^2}{(1-t^2)^{3/2}},\frac{3\sqrt{3}}{2}t(1-t)(2-t)\right\}.$$

The first case occurs when $0\leqslant t\leqslant 0.45668$, the second when $0.45668<t\leqslant\frac{1}{2}$.

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For $0 \leqslant t \leqslant \frac{1}{2}$ and $t < s \leqslant \frac{1}{2}$ we require

$$\tfrac{1}{6}t(1-s)(2s-t^2-s^2) \geqslant c_1(t)\tfrac{\sqrt{3}}{27}s(1-s^2)^{3/2},$$

that is

$$\frac{c_1(t)}{t} \leqslant \frac{3\sqrt{3}(1-s)(2s-t^2-s^2)}{2s(1-s^2)^{3/2}}.$$

The derivative of the right-hand side with respect to s is negative for all $s \in [t, \frac{1}{2}]$. Thus, the right-hand side is a decreasing function of s with its minimum at $s = \frac{1}{2}$. Therefore, we need

$$c_1(t) \leqslant 4t(\tfrac{3}{4} - t^2).$$

For $0 \leqslant t \leqslant \frac{1}{2}$ and $\frac{1}{2} \leqslant s \leqslant 1$ we want to show that

$$\tfrac{1}{6}t(1-s)(2s-t^2-s^2) \geqslant c_1(t)\tfrac{\sqrt{3}}{27}(1-s)(s(2-s))^{3/2}$$

or

$$\frac{c_1(t)}{t} \leqslant \frac{3\sqrt{3}(2s-t^2-s^2)}{2(s(2-s))^{3/2}}.$$

The sign of the derivative shows that the right-hand side is decreasing, so its minimum occurs when $s = 1$. Therefore, we need to have

$$c_1(t) \leqslant \tfrac{3}{2}\sqrt{3}t(1-t^2).$$

Hence, we may take, for $0 \leqslant t \leqslant \frac{1}{2}$,

$$\begin{aligned} c_1(t) &= \min\left\{3\sqrt{3}\frac{t(1-t)^2}{(1-t^2)^{3/2}},\ 4t(\tfrac{3}{4}-t^2),\ \tfrac{3}{2}\sqrt{3}t(1-t^2), \tfrac{3}{2}\sqrt{3}t(1-t)(2-t)\right\} \\ &= \tfrac{3}{2}\sqrt{3}t(1-t^2). \end{aligned}$$

By the symmetry properties of k, this yields

$$c_1(t) = \begin{cases} \frac{3\sqrt{3}}{2}t(1-t^2) & \text{for } t \in [0, \frac{1}{2}], \\ \frac{3\sqrt{3}}{2}t(1-t)(2-t) & \text{for } t \in (\frac{1}{2}, 1]. \end{cases}$$

Note that one can write the Green's function in this case in the form

$$k(t,s) = \int_0^1 k_0(t,\tau)k_0(\tau,s)\,\mathrm{d}\tau,$$

where

$$k_0(t,s) = \begin{cases} s(1-t) & \text{if } s \leqslant t, \\ t(1-s) & \text{if } s > t, \end{cases}$$

is the Green's function for the BVP

$$-u'' = 0, \quad u(0) = u(1) = 0,$$

because the BVP (1.1) can be written as the product of the two second-order operators. This has been done by a number of authors (see, for example, [2,11,39]).

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A simple method of obtaining some $\Phi(s)$ and some $c_1(t)$ is then to use the easily proved, well-known fact that

$$c_0(t)\Psi(s) \leqslant k_0(t,s) \leqslant \Psi(s)$$

for $\Psi(s) = s(1-s)$ and $c_0(t) = \min\{t, 1-t\}$. One then has

$$k(t,s) \leqslant \int_0^1 \Psi(\tau)k_0(\tau,s)\,\mathrm{d}\tau$$

and

$$k(t,s) \geqslant c_0(t)\int_0^1 \Psi(\tau)k_0(\tau,s)\,\mathrm{d}\tau.$$

This yields $c_1(t) = c_0(t)$, which is smaller than the constant we obtain; hence, we obtain stronger results. This simple factorization is not possible for the clamped ends case.

The constants m, M for the case $g \equiv 1$ are as follows. We find that $m = \frac{384}{5} = 76.8$, and the optimal $[a,b]$ is found to be $[\frac{1}{4}, \frac{3}{4}]$ (for example, by using MAPLE). This yields $M = \frac{768}{5} = 153.6$ and $c_1 = \frac{45\sqrt{3}}{128} = 0.6089$.

The eigenvalue μ_1 when $g \equiv 1$ is $\pi^4 = 97.4091$ (being a product of the two well-known second-order operators), which again shows that using μ_1 in place of m, M as in remark 3.2 leads to better results.

Hao and Debnath [11] prove the existence of one or two positive solutions using stronger conditions than we impose. They do not find an explicit constant m and assume that $f(t,u)/u \to \infty$ as $u \to 0+$ or as $u \to \infty$. Graef and Yang [7] prove the existence of one positive solution using constants similar to m and M. Bai and Wang [2] study multiple solutions; they use the constant $m = \frac{384}{5}$, take $c = \frac{1}{4}$ and have a constant B with B/c in place of M, where $B/c = 641.2308$. Yao [39] deals with a more general equation which includes this one as a special case. In our case, Yao also has worse constants. In place of 'm' he has $A' = 36$. He has $c = \frac{1}{4}$, but in place of M he has $B'/c = 1312$; these are approximations of the exact values he can use. He writes 'the constants A, B are not easy to compute explicitly' and he replaces A by A' and B by B'. For our special case we see that it is not difficult to compute the optimal constants for this method.

A theorem, with the same wording as theorem 3.1, holds here; we do not repeat the statement.

4.1. The boundary conditions (1.6)

Equation (1.1), subject to the non-local BCs

$$u(0) = 0, \quad u(1) = \alpha[u], \quad u''(0) = 0, \quad u''(1) = 0, \tag{4.5}$$

can be written, as before, in integral form as

$$u(t) = t\alpha[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s, \tag{4.6}$$

with $k(t,s)$ as in (4.2). Here $\|\gamma\| = 1$ and $\min_{t\in[1/4,3/4]}\gamma(t) = \frac{1}{4}$, so we need $c_2 = \frac{1}{4}$; therefore, $c = \frac{1}{4}$.

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To illustrate what the existence theorem similar to theorem 3.1 would say for these BCs when $g \equiv 1$, we consider the simple special case when $\alpha[u] = \alpha u(\frac{1}{2})$ with α a constant. We have $\alpha[\gamma] = \frac{1}{2}\alpha$ so we need $0 \leqslant \alpha < 2$. To obtain the index results of theorem 2.2 we need to compute

$$\int_0^1 \mathcal{K}(s)\,\mathrm{d}s \quad \text{and} \quad \int_{1/4}^{3/4} \mathcal{K}(s)\,\mathrm{d}s.$$

We have

$$\int_0^1 \mathcal{K}(s)\,\mathrm{d}s = 2\int_0^{1/2} \alpha k(\tfrac{1}{2}, s)\,\mathrm{d}s = 2\int_0^{1/2} \alpha \frac{s(\frac{3}{4} - s^2)}{12}\,\mathrm{d}s = \frac{5\alpha}{384},$$

and

$$\int_{1/4}^{3/4} \mathcal{K}(s)\,\mathrm{d}s = 2\int_{1/4}^{1/2} \alpha \frac{s(\frac{3}{4} - s^2)}{12}\,\mathrm{d}s = \frac{38\alpha}{4096} = \frac{19\alpha}{2048}.$$

The conditions become

$$f^{0,\rho}\left(\frac{2}{2-\alpha}\frac{5\alpha}{384} + \frac{5}{384}\right) < 1 \implies i_K(T, K_\rho) = 1,$$

and

$$f_{\rho,4\rho}\left(\frac{1/2}{2-\alpha}\frac{19\alpha}{2048} + \frac{5}{768}\right) > 1 \implies i_K(T, V_\rho) = 0.$$

For example, when $\alpha = 1$, these become

$$f^{0,\rho} < 25.6 \implies i_K(T, K_\rho) = 1, \qquad f_{\rho,4\rho} > 89.6934 \implies i_K(T, V_\rho) = 0.$$

These numbers 25.6 and 89.6934 replace m and M in theorem 3.1 and give an explicit result for the existence of multiple positive solutions of the three-point BVP.

The eigenvalue μ_1 can be calculated for this example. An eigenfunction is given by

$$\varphi(t) = \sinh(\omega)\sin(\omega t) + \sin(\omega)\sinh(\omega t)$$

and a small calculation shows that ω is a root of the equation

$$\cosh(\omega) + \cos(\omega) = 4\cosh(\omega)\cos(\omega).$$

The smallest positive root of this equation is $\omega = 2.5593$ and $\mu_1 = \omega^4 = 42.9023$, once more illustrating that using the eigenvalue rather than m, M, as in remark 3.2, gives stronger results.

4.2. The boundary conditions (1.7)

Similarly, as above, equation (1.1) with the BCs

$$u(0) = 0, \quad u(1) = 0, \quad u''(0) = 0, \quad u''(1) + \alpha[u] = 0, \tag{4.7}$$

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can be written in integral form as

$$u(t) = \tfrac{1}{6}t(1-t^2)\alpha[u] + \int_0^1 k(t,s)g(s)f(s,u(s))\,\mathrm{d}s, \tag{4.8}$$

with $k(t,s)$ as in (4.2). We have $\|\gamma\| = \gamma(\frac{1}{\sqrt{3}}) = \frac{\sqrt{3}}{27}$ and $\min_{t\in[1/4,3/4]}\gamma(t) = \frac{5}{128}$, so we need $c_2 = 0.6089 = c_1 = c$.

Remark 4.1. Results for each of the BCs

$$\begin{array}{llrr} u(0)=\alpha[u], & u(1)=0, & u'(0)=0, & u'(1)=0,\\ u(0)=0, & u(1)=0, & u'(0)=\alpha[u], & u'(1)=0,\\ u(0)=\alpha[u], & u(1)=0, & u''(0)=0, & u''(1)=0,\\ u(0)=0, & u(1)=0, & u''(0)+\alpha[u]=0, & u''(1)=0, \end{array}$$

are easily obtained by simply changing the variable from t to $\tau = 1-t$, which converts them into the types above, with the obvious modifications. We therefore omit the details.

Remark 4.2. Dalmasso [3, 4] has proved a uniqueness result for $u^{(4)}(t) = f(u(t))$ for both the clamped ends case and the hinged ends case under the assumptions $f \in C^1$ and

$$0 < f(u) < uf'(u) \quad \text{for } u > 0.$$

This is equivalent to $f \in C^1$ and $f(u)/u$ being strictly increasing for $u > 0$. This is consistent with our work, since it is impossible to have a figure such as figure 1 for two or more positive solutions when $f(u)/u$ is strictly increasing. For the hinged end case only, [2, theorem 5.1] claims a uniqueness result, which allows f to have explicit t dependence, under the conditions that, for each t, $f(t,u)$ is strictly positive and increasing, and $f(t,u)/u$ is either increasing or decreasing. However, the claimed result is incorrect. In the case when $f(t,u)/u$ is decreasing, the following example shows that there can be infinitely many solutions. For a constant $\theta > 0$, the BVP

$$u^{(4)}(t) = \pi^4 u(t) + \theta, \quad u(0)=0, \quad u(1)=0, \quad u''(0)=0, \quad u''(1)=0$$

has a positive solution $c\sin(\pi t) + \frac{1}{12}\theta(t^4 - 2t^3 + t)$ for each $c > 0$.

A similar counterexample, replacing π^4 by the eigenvalue, shows that the same happens in the clamped ends case. When f depends explicitly on t, as far as the authors are aware, the uniqueness question remains open.

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