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Study of nonlinear four-wave interactions

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The nonlinear interaction between four electrostatic waves in a plasma media is investigated analytically in the presence of linear damping or growth and also frequency mismatch, using a nonlinear perturbation method. Depending on the various initial conditions of the wave amplitudes, solutions of different types are discussed.

1. Introduction

In a plasma medium, the main nonlinear effects may be classified as wave–particle and wave–wave interactions, which in general are included when considering the evolution of plasma instabilities. When wave–particle effects are weak, the evolution of the wave amplitudes may be governed by the interaction with other waves present in the system.

The study of nonlinear plasma-wave interaction is of fundamental importance in understanding the dynamics of a weakly turbulent plasma and also the energy exchange among different wave modes. As even weak nonlinear coupling between coherent waves can drastically affect the stability of the waves, more and more cases of such wave–wave interactions are being investigated, not only in plasmas but also in other media.

The consequence of nonlinear phenomena might be of importance for the development of future technology, especially in cases such as exploiting the energy resources of the hot plasmas that appear in new electro-optic devices, the penetration of radiation in laser fusion, and also in studies of space plasma physics and astrophysics.

In plasmas, and especially in high-temperature plasmas in magnetic fields, the linear dynamics involves a rich variety of waves. Nonlinear wave–wave interactions are of importance in ionospheric propagation (Fejer 1977), in the evolution of various plasma instabilities (Coppi et al. 1969; Tsytovich 1970; Hasegawa 1975; Cap 1976, 1978), and also in problems of plasma heating with high-power electromagnetic sources, for example with lasers for pellet fusion (Dubois 1974; Drake et al. 1974; Manheimer and Ott 1975; Watson and Bers 1977; Bers 1978; Porkolab 1978). The contribution to the interaction from the third-order current is of the four-wave interaction type. However, attention is now shifting to four-wave interactions – mainly for two reasons.

(i) It may be possible that the selection rule or resonance conditions cannot be satisfied for three waves. Then, a third-order process, giving four-wave interactions, is the significant nonlinear contribution (Sitenko 1973).

(ii) With the more intense power sources that are now becoming available studies of higher-order processes are feasible.

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Coupled mode equations describing four-wave interactions have been studied in many branches of physics – for example, gravity waves Benney (1962) and nonlinear optics (Armstrong et al. 1962). Inoue (1975) studied the bounded solutions of the coupled mode equations for four waves in a general dispersive medium. Walters and Lewak (1977) derived the equations for the interactions of four waves to second order, and showed that weaker triplet was stabilized by the stronger one against explosive instabilities, while the converse was not true. Verheest (1976) investigated the different resonance conditions that yield distinct sets of amplitude equations and the corresponding solutions for the coupling between two, three and four waves in plasma using a general formalism based on multiple time scales. Boyd and Turner (1977, 1978) used the Lagrangian formalism and Turner (1980) used the well-defined phase approximation to study four-wave interactions in plasmas; they also calculated the growth rate and threshold for the instability. Degenerate four-wave mixing in plasmas was studied by Steel and Lau (1979) using fluid equations.

There has been a resurgence of interest in the theory of phase conjugation by four-wave mixing in plasmas (Nebezahl 1988; Hellwarth et al. 1989; Williams et al. 1989a, b; Percival 1989). Some plasma experiments on four-wave mixing and/or phase conjugation have been performed (Kitagawa et al. 1989; Domier et al. 1989; Lehner 1989). The treatment of four-wave mixing has, however, been restricted to high-frequency waves in unmagnetized plasmas (Papen and Tataronis 1989). Goldman and Williams (1991) considered time-dependent four-wave mixing and phase conjugation in plasmas, taking into account resonant longitudinal plasma modes and the effects of spatial non-uniformity. A space–time formulation of the four-wave process in the ponderomotive force regime was derived, and was solved analytically as a combined initial/boundary-value problem in slab and inhomogeneous geometries.

More recently, resonant four-wave mixing of finite-amplitude Alfvén waves was studied by Rauf and Tataronis (1996). The evolution equations governing the amplitudes of the interacting waves and the conservation relations were derived from the basic equations. Those evolution equations were used to study parametric amplification and oscillation of two small-amplitude Alfvén waves due to two large-amplitude pump waves. It was shown that three pump waves can mix together to generate a low-frequency Alfvén wave in a dissipative medium. This suggests the existence of a number of novel low-frequency nonlinear phenomena in magnetized laboratory, astrophysical and upper-atmosphere plasmas, where Alfvén waves abound.

Pakter et al. (1997) analysed the transition from regular to chaotic states in the parametric four-wave interaction. The temporal evolution describing the coupling of two sets of three waves with quadratic nonlinearity was considered. Brillouin-enhanced four-wave mixing and phase conjugation of electromagnetic waves in weakly collisional fully ionized plasmas were considered by Domier and Luhmann (1993). It was found that the nonlinearity associated with the non-local electron heat transport may dominate over the ponderomotive force, and consequently there might appear an effective enhanced degenerate four-wave mixing and phase conjugation.

The nonlinear four-wave resonant interaction between two different type of waves – Alfvén and acoustic – was studied by Borcia and Ignat (1998). Their result may explain phenomena observed in the Earth's ionosphere.

Krasnosel'skikh et al. (1998) studied the dynamical properties of an ensemble of four interacting waves consisting of three Langmuir waves and one ion acoustic wave with exact resonance conditions. They described two types of instabilities – the decay and the modified decay instabilities – that could be due to the generation of weak and strong turbulence.

Under certain conditions when both positive- and negative-energy waves interact, the negative-energy waves can grow by transferring energy to the positive-energy waves, so that the amplitudes of all interacting waves increase with time, reaching unbounded values in finite time and producing an explosive instability. Explosive instability is of interest in high-power laser devices and also in astrophysical applications. The phenomena of explosive instability is also expected to play a significant role in the development of plasma turbulence and plasma heating. Explosive energy released during disruption of the Earth's plasma sheet can lead to magnetosphere substorms (Parker 1987).

The main difficulty in demonstrating that the instability does occur is its relatively weak interaction strength. Simultaneous growth of three frequencies was observed in one case where the nonlinear 'element' was a tunnel diode (Kiyasho et al. 1972). Oscillatory interchange of wave energy between wave modes has been observed in an experiment by Stenzel and Wong (1972). More recently, Chernos (1996) studied the excitation of Alfvén radiation by an explosive instability in a heated gyrotropic plasma penetrated by a mono-velocity flow of ionized gas. The amplitudes of the Alfvén-wave harmonics were found and the conditions for the stabilization of the explosion due to the nonlinear absorption associated with weakly damped Alfvén-wave harmonics were specified. Explosive energy release by disruption of current sheets was discussed recently by Lakhina and Tsurutani (1998). The roles of current-sheet disruptions in magnetospheric storms, solar flares and coronal transients were discussed.

In the present paper, an attempt is made to study analytically the linear interaction between four positive- and negative-energy electrostatic waves in a plasma in the presence of linear damping or growth and also including the effect of frequency mismatch. The amplitude equations are obtained from the requirement that the eventual solutions should exhibit no secular behaviour.

The perturbation technique used here is due to Coffey and Ford (1969), and has the distinct advantage of separating a given motion into a secular motion and a rapidly fluctuating motion of small amplitude. This method has been successfully applied to study explosive instability and its stabilization in the case of three-wave interactions (Khan et al. 1980; De et al. 1981), as well as to study the coupling of two three-wave systems in a plasma (Basu (née De) and Roychowdhury 1983).

The analytical solutions obtained here describe the time behaviour of the wave amplitudes. Depending on the various initial conditions, the solutions obtained may be classified into the following types:

(i) periodic solutions;
(ii) shock-like solutions;
(iii) soliton solutions.

The periodic solutions are obtained in terms of the Jacobi elliptic functions, showing that the waves either exchange energy in a periodic way or are infinite in a finite time corresponding to the explosive behaviour. The threshold value, explosion time and growth rate that characterize the instability have been calculated, and the effects of dissipation and frequency mismatch on these parameters have been shown.

2. Coupled mode equations

For the interaction between four electrostatic waves satisfying resonance conditions of the form

$$k_1 + k_2 = k_3 + k_4, \tag{2.1a}$$

$$\omega_1 + \omega_2 \approx \omega_3 + \omega_4, \tag{2.1b}$$

the coupled mode equations are (Boyd and Turner 1978; Turner 1980; Verheest 1980)

$$D_{1,2} F_{1,2} = V^*_{1,2} F^*_{2,1} F_3 F_4 \, e^{-it_2 \Delta\omega'} + F_{1,2} \sum_{i=1}^{4} g_{1,2i} F_i F^*_i, \tag{2.2}$$

$$D_{3,4} F_{3,4} = -V_{4,3} F^*_{4,3} F_1 F_2 \, e^{-it_2 \Delta\omega'} + F_{3,4} \sum_{i=1}^{4} g_{3,4i} F_i F^*_i, \tag{2.3}$$

where

$$D_i = \frac{\partial}{\partial t_2} + \mathbf{v}_{gi} \cdot \boldsymbol{\nabla} + \nu'_i, \tag{2.4a}$$

$$t_2 = \epsilon^2 t = \lambda t \quad \text{(say)}. \tag{2.4b}$$

g_{ji} and V_i are the coupling coefficients for the resonant and non-resonant parts of the interaction. g_{ji} are required to be purely imaginary if one wants to investigate what happens to an interacting quadruplet. Let

$$g_{ji} = ih_{ji} \quad (h_{ji} \text{ real}).$$

Only a temporal variation in the electric field amplitudes has been considered here:

$$F_i(t) = \rho_i(t) \exp[-ib_i(t)] \quad (i = 1, 2, 3, 4). \tag{2.5}$$

The classification scheme of Verheest (1980), characterized by $\mathrm{Re}\,\nu_i \neq 0$ and $\mathrm{Re}\,g_{ji} = 0$, has been used, which corresponds to a non-conservative medium. Then

$$\nu_i = |\nu_i| \exp(i\delta_i). \tag{2.6}$$

On using (2.4) and (2.5) and separating the real and imaginary parts of (2.2) and (2.3), one has the following sets of equations:

$$\frac{\partial \rho_1}{\partial t} + \nu_1 \rho_1 = \lambda |V_1| \rho_2 \rho_4 \rho_3 \cos(\psi + \delta_1), \tag{2.7a}$$

$$\frac{\partial \rho_2}{\partial t} + \nu_2 \rho_2 = \lambda |V_2| \rho_1 \rho_4 \rho_3 \cos(\psi + \delta_2), \tag{2.7b}$$

$$\frac{\partial \rho_3}{\partial t} + \nu_3 \rho_3 = -\lambda |V_3| \rho_2 \rho_4 \rho_1 \cos(\psi + \delta_3), \tag{2.7c}$$

$$\frac{\partial \rho_4}{\partial t} + \nu_4 \rho_4 = -\lambda |V_4| \rho_2 \rho_1 \rho_3 \cos(\psi + \delta_4), \tag{2.7d}$$

$$\begin{aligned}\frac{\partial \psi}{\partial \ t} = \Delta\omega + \lambda \Bigg[& |V_4| \frac{\rho_2 \rho_1 \rho_3}{\rho_4} \sin(\psi + \delta_4) \\ & + |V_3| \frac{\rho_2 \rho_1 \rho_4}{\rho_3} \sin(\psi + \delta_3) \\ & - |V_2| \frac{\rho_4 \rho_1 \rho_3}{\rho_2} \sin(\psi + \delta_2) \\ & - |V_1| \frac{\rho_2 \rho_1 \rho_4}{\rho_1} \sin(\psi + \delta_1) \Bigg] + \lambda \sum_{i=1}^{4} m_i \rho_i^2, \end{aligned} \tag{2.8}$$

where $\nu_i = \lambda \nu_i'$ denotes the damping or growth and $\Delta\omega = \lambda\, \Delta\omega'$ is the frequency mismatch;

$$\psi = b_4 + b_3 - b_2 - b_1 + \Delta\omega\, t,$$

$$m_i = h_{1i} + h_{2i} - h_{3i} - h_{4i}.$$

The system of equations (2.7) and (2.8) is of the form

$$\rho_i = \lambda A_i(\boldsymbol{\rho}, \psi) \quad (i = 1, 2, 3, 4), \tag{2.9a}$$

$$\dot{\psi} = \Delta\omega + \lambda B_i(\boldsymbol{\rho}, \psi) \quad (i = 1, 2, 3, 4), \tag{2.9b}$$

$$\boldsymbol{\rho} = (\rho_1, \rho_2, \rho_3, \rho_4). \tag{2.9c}$$

When λ is small but finite, the ρ_i will experience a slow secular growth on which is superimposed a small-amplitude rapid fluctuation. Similarly, ψ will experience rapid secular growth on which is superimposed a small-amplitude rapid fluctuation.

In order to solve the system of equations (2.7) and (2.8), the method of perturbation due to Coffey and Ford (1969), which is suitable for $\Delta\omega \neq 0$, is applied. To separate the secular motion from the rapidly fluctuating motion, a solution is introduced in the form

$$\rho_i = y_i + \sum_{n=1}^{\infty} \lambda^n F_i^n(\mathbf{y},\quad), \tag{2.10a}$$

$$\psi = \quad + \sum_{n=1}^{\infty} \lambda^n G^n(\mathbf{y},\quad), \tag{2.10b}$$

$$\dot{y}_i = \sum_{n=1}^{\infty} \lambda^n a_i^n(\mathbf{y}), \tag{2.11a}$$

$$\dot{\ } = \Delta\omega + \sum_{n=1}^{\infty} \lambda^n b^n(\mathbf{y}), \tag{2.11b}$$

$$\mathbf{y} = (y_1, y_2, y_3, y_4). \tag{2.11c}$$

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Inserting (2.10) into (2.7) and (2.8) and using (2.11), one can obtain a successive system of equations for each power of λ (for details, see Khan et al. 1980; De et al. 1981).

The zeroth-order terms give

$$a_i^0 + \nu_i y_i = 0 \quad (i = 1, 2, 3, 4). \tag{2.12}$$

The first-order terms give

$$a_i^1 = 0, \qquad b^1 = m_1 y_1^2 + m_2 y_2^2 + m_3 y_3^2 + m_4 y_4^2, \tag{2.13a}$$

$$F_1^1 = \frac{|V_1|}{\Delta\omega_1} y_2 y_3 y_4 \sin(\phi + \delta_1 + \eta_1), \tag{2.13b}$$

$$F_2^1 = \frac{|V_2|}{\Delta\omega_2} y_1 y_3 y_4 \sin(\phi + \delta_2 + \eta_2), \tag{2.13c}$$

$$F_3^1 = -\frac{|V_3|}{\Delta\omega_3} y_1 y_2 y_4 \sin(\phi + \delta_3 + \eta_3), \tag{2.13d}$$

$$F_4^1 = -\frac{|V_4|}{\Delta\omega_4} y_1 y_2 y_3 \sin(\phi + \delta_4 + \eta_4), \tag{2.13e}$$

$$\begin{aligned} G^1 = \frac{1}{\Delta\omega}\bigg[& |V_1| \frac{y_4 y_2 y_3}{y_1} \cos(\phi + \delta_4) \\ & + |V_2| \frac{y_1 y_4 y_3}{y_2} \cos(\phi + \delta_2) \\ & - |V_3| \frac{y_1 y_2 y_4}{y_3} \cos(\phi + \delta_3) \\ & - |V_4| \frac{y_1 y_2 y_3}{y_4} \cos(\phi + \delta_4) \bigg], \end{aligned} \tag{2.14}$$

where

$$\tan \eta_i = \frac{\nu_i}{\Delta\omega_i}, \tag{2.15a}$$

$$\frac{1}{\Delta\omega_i} = \frac{1}{(1 + \nu_i^2/\Delta\omega^2)^{1/2}}. \tag{2.15b}$$

The second-order terms give

$$\begin{aligned} a_1^2 = \frac{1}{2}\bigg\{ & |V_2|\,|V_1|\, y_1 y_3^2 y_4^2 \left[\frac{\sin(\delta_2 - \delta_1 + \eta_2)}{\Delta\omega_1} + \frac{\sin(\delta_2 - \delta_1)}{\Delta\omega} \right] \\ & - |V_1|\,|V_3|\, y_1 y_2^2 y_4^2 \left[\frac{\sin(\delta_3 - \delta_1 + \eta_3)}{\Delta\omega_3} + \frac{\sin(\delta_3 - \delta_1)}{\Delta\omega} \right] \\ & - |V_1|\,|V_4|\, y_1 y_2^2 y_3^2 \left[\frac{\sin(\delta_4 - \delta_1 + \eta_4)}{\Delta\omega_4} + \frac{\sin(\delta_4 - \delta_1)}{\Delta\omega} \right] \bigg\}, \end{aligned} \tag{2.16a}$$

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$$a_2^2 = \frac{1}{2}\left\{|V_2|\,|V_1|\,y_2\,y_3^2\,y_4^2\left[\frac{\sin(\delta_1-\delta_2+\eta_1)}{\Delta\omega_1}+\frac{\sin(\delta_1-\delta_2)}{\Delta\omega}\right]\right.$$

$$-|V_2|\,|V_3|\,y_2\,y_1^2\,y_4^2\left[\frac{\sin(\delta_3-\delta_2+\eta_3)}{\Delta\omega_3}+\frac{\sin(\delta_3-\delta_2)}{\Delta\omega}\right]$$

$$\left.-|V_2|\,|V_4|\,y_2\,y_1^2\,y_3^2\left[\frac{\sin(\delta_4-\delta_2+\eta_4)}{\Delta\omega_4}+\frac{\sin(\delta_4-\delta_2)}{\Delta\omega}\right]\right\}, \tag{2.16b}$$

$$a_3^2 = \frac{1}{2}\left\{|V_3|\,|V_4|\,y_3\,y_1^2\,y_2^2\left[\frac{\sin(\delta_4-\delta_3+\eta_4)}{\Delta\omega_4}+\frac{\sin(\delta_4-\delta_3)}{\Delta\omega}\right]\right.$$

$$-|V_1|\,|V_3|\,y_3\,y_2^2\,y_4^2\left[\frac{\sin(\delta_1-\delta_3+\eta_1)}{\Delta\omega_1}+\frac{\sin(\delta_1-\delta_3)}{\Delta\omega}\right]$$

$$\left.-|V_2|\,|V_3|\,y_3\,y_1^2\,y_4^2\left[\frac{\sin(\delta_2-\delta_3+\eta_2)}{\Delta\omega_2}+\frac{\sin(\delta_2-\delta_3)}{\Delta\omega}\right]\right\}, \tag{2.16c}$$

$$a_4^2 = \frac{1}{2}\left\{|V_3|\,|V_4|\,y_4\,y_1^2\,y_2^2\left[\frac{\sin(\delta_3-\delta_4+\eta_3)}{\Delta\omega_3}+\frac{\sin(\delta_3-\delta_4)}{\Delta\omega}\right]\right.$$

$$-|V_1|\,|V_4|\,y_4\,y_2^2\,y_3^2\left[\frac{\sin(\delta_1-\delta_4+\eta_1)}{\Delta\omega_1}+\frac{\sin(\delta_1-\delta_4)}{\Delta\omega}\right]$$

$$\left.-|V_2|\,|V_4|\,y_4\,y_1^2\,y_3^2\left[\frac{\sin(\delta_2-\delta_4+\eta_2)}{\Delta\omega_2}+\frac{\sin(\delta_2-\delta_4)}{\Delta\omega}\right]\right\}. \tag{2.16d}$$

To make the nonlinear equations tractable, we consider the special case where all the damping or growth terms and the coupling-constant terms V_i are equal. Using the relation

$$\dot{y}_i = a_i^0 + \lambda a_i^1 + \lambda^2 a_i^2 + \dots,$$

from (2.16) one has the following nonlinear system of equations:

$$\frac{dX_1}{dt} = V^2\left\{X_1\,X_3\,X_4\left[\frac{\sin(\delta_2-\delta_1+\eta)}{\Delta\omega_k}+\frac{\sin(\delta_2-\delta_1)}{\Delta\omega}\right]\right.$$

$$-X_1\,X_2\,X_4\left[\frac{\sin(\delta_3-\delta_1+\eta)}{\Delta\omega_k}+\frac{\sin(\delta_3-\delta_1)}{\Delta\omega}\right]$$

$$\left.-X_1\,X_2\,X_3\left[\frac{\sin(\delta_4-\delta_1+\eta)}{\Delta\omega_k}+\frac{\sin(\delta_4-\delta_1)}{\Delta\omega}\right]\right\}, \tag{2.17a}$$

$$\frac{dX_2}{dt} = V^2\left\{X_2\,X_3\,X_4\left[\frac{\sin(\delta_1-\delta_2+\eta)}{\Delta\omega_k}+\frac{\sin(\delta_1-\delta_2)}{\Delta\omega}\right]\right.$$

$$-X_1\,X_2\,X_4\left[\frac{\sin(\delta_3-\delta_2+\eta)}{\Delta\omega_k}+\frac{\sin(\delta_3-\delta_2)}{\Delta\omega}\right]$$

$$-X_1 X_2 X_3 \left[\frac{\sin(\delta_4-\delta_2+\eta)}{\Delta\omega_k}+\frac{\sin(\delta_4-\delta_2)}{\Delta\omega}\right]\Bigg\}, \tag{2.17b}$$

$$\frac{dX_3}{dt} = V^2 \Bigg\{ X_1 X_3 X_2 \left[\frac{\sin(\delta_4-\delta_3+\eta)}{\Delta\omega_k}+\frac{\sin(\delta_4-\delta_3)}{\Delta\omega}\right]$$

$$-X_3 X_2 X_4 \left[\frac{\sin(\delta_1-\delta_3+\eta)}{\Delta\omega_k}+\frac{\sin(\delta_1-\delta_3)}{\Delta\omega}\right]$$

$$-X_1 X_4 X_3 \left[\frac{\sin(\delta_2-\delta_3+\eta)}{\Delta\omega_k}+\frac{\sin(\delta_2-\delta_3)}{\Delta\omega}\right]\Bigg\}, \tag{2.17c}$$

$$\frac{dX_4}{dt} = V^2 \Bigg\{ X_1 X_2 X_4 \left[\frac{\sin(\delta_3-\delta_4+\eta)}{\Delta\omega_k}+\frac{\sin(\delta_3-\delta_4)}{\Delta\omega}\right]$$

$$-X_3 X_2 X_4 \left[\frac{\sin(\delta_1-\delta_4+\eta)}{\Delta\omega_k}+\frac{\sin(\delta_1-\delta_4)}{\Delta\omega}\right]$$

$$-X_1 X_4 X_3 \left[\frac{\sin(\delta_2-\delta_4+\eta)}{\Delta\omega_k}+\frac{\sin(\delta_2-\delta_4)}{\Delta\omega}\right]\Bigg\}, \tag{2.17d}$$

where

$$x_i = X_i \exp(-2\nu t), \qquad \tau = \frac{1}{4\nu}(1-e^{-4\nu t}). \tag{2.18}$$

3. Solutions of the coupled mode equations

3.1. Case 1: equal amplitudes

In the particular case where all four amplitudes are equal and

$$\delta_1 - \delta_2 = \delta, \delta_3 = \delta_4 = \delta',$$

with

$$\delta' - \delta = (2m+1)\pi \quad (m = 0, 1, 2, \ldots),$$

one gets

$$\frac{dx}{dt} + 2\nu x = V^2 \frac{\sin\eta}{\Delta\omega_k} x^3. \tag{3.1}$$

The solution for the wave amplitude $x(t)$ is

$$x(t) = \frac{1}{\{a^2 + b^2[1+2\nu(t-t_0)^2]\}^{1/2}}, \tag{3.2}$$

where

$$a^2 = \frac{V^2 \sin\eta}{2\nu\,\Delta\omega_k}, \qquad b^2 = \frac{1}{x_0^2} - \frac{\sin\eta}{2\nu\,\Delta\omega_k},$$

provided that the initial amplitudes are chosen so that

$$\frac{1}{x_0^2} > \frac{\sin\eta}{2\nu\,\Delta\omega_k}.$$

The solution (3.2) is a soliton with one maximum, and the amplitudes tend to zero for large times. The decrease of the amplitudes after the maximum corresponds to a collapse of four waves.

3.2. Case 2: unequal amplitudes

By suitable choice of $\nu, \Delta\omega$ and δ_i, one can solve the system of equations (2.17). Let ν and $\Delta\omega$ be chosen so that $\nu \ll \Delta\omega$. Assuming that

$$\delta_1 - \delta_2 = (2k+1)\,\pi, \qquad \delta_3 - \delta_4 = 2l\pi$$

(where l and k are integers, including zero), the constants of motion are

$$X_1(\tau) + X_2(\tau) = X_1(0) + X_2(0) = P, \tag{3.3}$$

$$X_3(\tau) - X_4(\tau) = X_3(0) - X_4(0) = R, \tag{3.4}$$

$$X_1 X_2 X_3 X_4 = X_1(0)\, X_2(0)\, X_3(0)\, X_4(0) = Q. \tag{3.5}$$

Equations (3.3) and (3.4) can be considered as the Manley–Rowe relations that imply the conservation of wave energy of the interacting quadruplet. Using (3.3)–(3.5) to eliminate X_2, X_3 and X_4 in favour of X_1 leads to

$$\frac{dX_1}{dt} = AR[\pi(X_1)]^{1/2}, \tag{3.6}$$

where

$$A = V^2 \sin(\delta_1 - \delta_4)\, \frac{\Delta\omega^2 + \Delta\omega_k^2}{\Delta\omega\, \Delta\omega_k^2},$$

$$\pi(X_1) = [(X_1 - \tfrac{1}{2}P)^2 - \tfrac{1}{4}P^2]\,[(X_1 - \tfrac{1}{2}P)^2 - \tfrac{1}{4}S^2], \tag{3.7}$$

with

$$S^2 = P^2 + \frac{16Qab}{R^2} > P^2.$$

Equation (3.6) is formally equivalent to the equation for a nonlinear oscillator subject to a potential

$$A^2R^2[(X_1 - \tfrac{1}{2}P)^2 - \tfrac{1}{4}P^2]\,[(X_1 - \tfrac{1}{2}P)^2 - \tfrac{1}{4}S^2].$$

From (3.6), the solution for $X_1(\tau)$ can only exist in the region where $\pi(X_1) \geqslant 0$. The character of the solution is greatly dependent on the nature of the roots and their ordering in magnitude.

The potential $\pi(X_1)$ has four real roots

$$\alpha_1 = \tfrac{1}{2}(P - S), \qquad \alpha_2 = P, \qquad \alpha_3 = \tfrac{1}{2}(P + S), \qquad \alpha_4 = 0.$$

From (3.6),

$$\frac{dX_1}{dt} = ARX_1(X_1 + \alpha_1)\,(X_1 - \alpha_2)\,(X_1 - \alpha_3). \tag{3.8}$$

In Fig. 1, the bounded solution of the coupled mode equation corresponds to X_1 oscillating between B and C, and the explosive instability corresponds to X_1 lying on the portion D∞ of the curve (X_1 being non-negative for all τ).

From Fig. 1, for the instability to occur, one requires $X_1(0) > \alpha_3$, so that a threshold exists for the onset of instability.

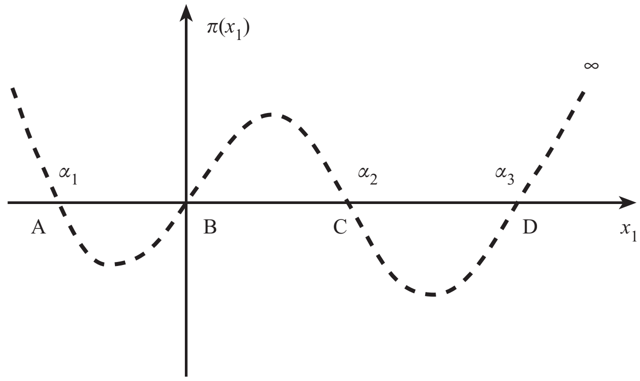

Figure 1. $\pi(X_1)$ corresponding to (3.7) in the case of four real roots of $\pi(X_1) = 0$.

4. Classification of solutions

According to the variety of initial states, one can classify the solutions of the four-wave interactions into the following types (Inoue 1975).

4.1. Periodic solutions

From (3.6), one gets

$$\int \{[(X_1 - \tfrac{1}{2}P)^2 - \tfrac{1}{4}P^2]\,[(X_1 - \tfrac{1}{2}P)^2 - \tfrac{1}{4}S^2]\}^{-1/2}\,dX_1 = AR\int d\tau. \tag{4.1}$$

Depending on the assignment of the roots $\frac{1}{4}S^2$ and $\frac{1}{4}P^2$ to the values a and b (where $a > b > 0$) (Byrd and Friedman 1954, p. 53), the solution for $X_1(\tau)$ is

$$(X_1 - \tfrac{1}{2}P)^2 = \frac{S^2 - P^2\,\mathrm{sn}^2\,(\alpha(\tau - \tau_0), k)}{4[1 - \mathrm{sn}^2\,(\alpha(\tau - \tau_0), k)]}, \tag{4.2}$$

where $\alpha = \frac{1}{2}ARS$, $k = P/S$ is the modulus of the Jacobi elliptic function sn, and τ_0 is a constant defined by

$$\tau_0 = \frac{2}{ARS}\,\mathrm{sn}^{-1}\left(\frac{[X_1(0) - \frac{1}{2}P]^2 - \frac{1}{4}S^2}{[X_1(0) - \frac{1}{2}P]^2 - \frac{1}{4}P^2}, k\right). \tag{4.3}$$

The solution (4.2) is generally periodic, with period of oscillation $2k/\lambda$, where

$$k = \int_0^{\pi/2} (1 - k^2 \sin^2\theta)^{-1/2}\,d\theta$$

is the complete elliptic integral of the first kind. The solutions for the wave amplitudes X_2, X_3 and X_4 can be obtained from (4.2) and the conservation laws. They are periodic, with the same period.

The amplitudes of all four waves become infinite when

$$\mathrm{sn}^2\,(\alpha(\tau - \tau_0), k) = 1,$$

and the explosion time τ_∞ is given by

$$\tau_\infty = \tau_0 + \frac{2}{ARS}\,\mathrm{sn}^{-1}\,(1, k) \tag{4.4}$$

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The growth rate is given by the reciprocal of the explosion time:

$$\Gamma_{\text{growth}} = \frac{1}{\tau_\infty}.$$

The effect of $\Delta\omega$ and ν is to introduce threshold values of the initial amplitudes for the explosion to occur and to increase the time of explosion.

4.2. Shock-like solutions

Let the maximum value of any of the amplitudes be zero. Then $Q = 0$, and from (3.7) one gets a solution given by

$$X_1(\tau) = \frac{P}{1-[1-P/X_1(0)]\,e^{APR(\tau-\tau_0)}}, \tag{4.5}$$

which represents a shock-like solution. The values of $X_1(\tau)$ vary between zero and P. The solution of the type (4.5) is quite different from those of nonlinear three-wave interactions.

5. Discussion

As an illustration of the general theory, we consider the interaction between four longitudinal waves in a warm isotropic plasma. The synchronism conditions

$$k_1+k_2 = k_3+k_4, \qquad \omega_1+\omega_2 \approx \omega_3+\omega_4$$

permit different distinct nonlinear processes, for example,

(i) the interaction between four electron plasma oscillations;
(ii) the interaction between four ion acoustic waves;
(iii) the interaction between two Langmuir waves and two ion acoustic waves;
(iv) the interaction between two Alfvén waves and two acoustic waves.

The growth rate, explosion time and threshold values of the amplitudes of the excited waves for such systems can be obtained by direct application of the theory described here, and the effect of dissipation and frequency mismatch on these parameters can be calculated.

The sets of constants of motion derived here from the direct coupled mode equations in presence of dissipation and frequency mismatch are different from those obtained in their absence, as in the cases of Boyd and Turner (1978) and Turner (1980). In the absence of dissipation and frequency mismatch, Boyd and Turner (1978) have shown that stable four-wave interactions (bounded) result from the continuous interchange of action (energy) between the four interacting waves (Fig. 2).

From the description of the wave interaction as motion in a potential, one may expect that dissipation will decrease the kinetic energy so that the particle will fall to the bottom of the potential well. This means that the oscillations of the amplitudes continuously decrease and approach constant asymptotic values in the presence of dissipation. The temporal behaviour of the amplitudes corresponding to the periodic solutions in the presence of dissipation and

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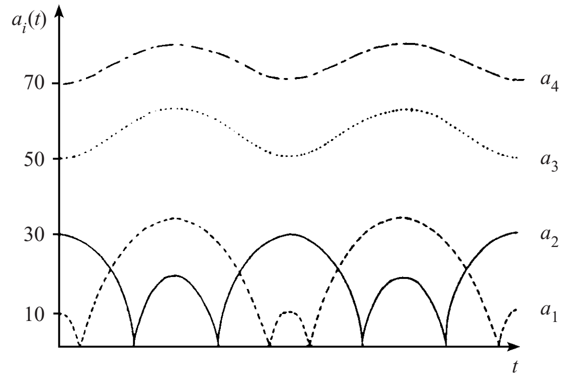

Figure 2. Plots of individual wave amplitudes in the absence of dissipation and frequency mismatch (Boyd and Turner 1978) for $a_1(0) = 10$, $a_2(0) = 30$, $a_3(0) = 50$ and $a_4(0) = 70.7$.

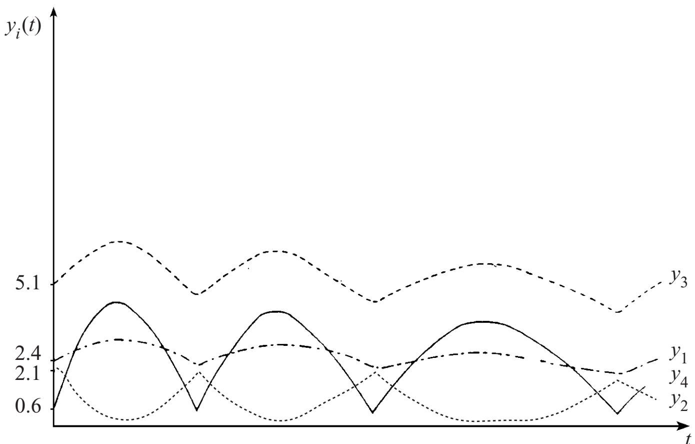

Figure 3. Plots of the individual wave amplitudes in the presence of dissipation and frequency mismatch for $y_1(0) = 2.4$, $y_2(0) = 2.1$, $y_3(0) = 5.1$ and $y_4(0) = 0.6$.

frequency mismatch (the present problem) are shown in Fig. 3. In this case, the amplitudes decay exponentially and the periodic oscillations ultimately tend to constant values.

6. Conclusions

In this paper, an attempt has been made to study analytically the nonlinear four-wave interactions between positive- and negative-energy waves in the presence of linear damping or growth and frequency mismatch. A perturbation method has been used to separate the secular motion from the slow evolution of the whole system.

Depending on the various initial conditions of the amplitudes, the solutions obtained may be classified as

(i) periodic solutions;
(ii) shock-like solutions;
(iii) soliton solutions.

The analytical approach used here can be applied successfully to obtain explicit solutions for different types of nonlinear four-wave interactions, even in complicated cases with frequency mismatch and dissipation included. For example, it can be used to verify the experimental results of Domier and Luhamann (1993), where four-wave mixing and phase conjugation of electromagnetic waves were considered. It can be applied to obtain analytical solutions for four-wave mixing of Alfvén waves, even in the presence of frequency mismatch, without the need for a numerical solution under the frequency-matching condition as in, for example, Rauf and Tataronis (1996).

Since with the use of the present theory it is possible to analytically determine the condition for explosive instability, its stabilization and the explosion time, it should be possible to predict plasma turbulence, to study the excitation of Alfvén radiation (as in Chernos 1996), and to find the explosive energy release during disruption of the Earth's plasma sheet, which can lead to magnetosphere substorms (Lakhina and Tsurutani 1998).

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