

INFORMATION AND UNCERTAINTY IN A QUEUING SYSTEM

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This article deals with the effect of information and uncertainty on profits in an unobservable single-server queuing system. We consider scenarios in which the service rate, the service quality, or the waiting conditions are random variables that are known to the server but not to the customers. We ask whether the server is motivated to reveal these parameters. We investigate the structure of the profit function and its sensitivity to the variance of the random variable. We consider and compare variations of the model according to whether the server can modify the service price after observing the realization of the random variable.

1. INTRODUCTION

There is a growing interest in queuing models where servers and customers react to each other's action, such as prices and queue discipline set by the server, and an arrival process generated in a noncooperative way by the customers (see [9] for a survey). Much of the literature on this subject assumes that the system's parameters are fixed and known to the participating agents. In other cases, customers own private information on the individual parameters, and one asks whether an adequate pricing system can be used to motivate them to reveal this information to the queue operator. In contrast, this article deals with the common situation in which some parameters are random variables whose realization at a particular time is known to the queue operator but not to the potential customers. For example, the rate of service might vary due to various reasons, the quality of the service might be subject to changes, and even the conditions of waiting in the queue, which might be of importance to a customer when deciding on whether or not to join, might vary. The subject of this article is the effect that this uncertainty has on the decisions

made by the server and the customers, and, in particular, on the expected rate of profit. We ask whether the server is motivated to reveal the information, what the effect of increased uncertainty is, and what happens when the server is restricted to a fixed price independent of the realization of the random variable.

Several articles have dealt with the value of information in congested systems. Some have proved that information might reduce customers' welfare. This phenomenon can be broadly interpreted as a result of the noncooperative way in which customers behave.

Hassin [6] asked whether forcing a profit maximizing server to inform customers about the length of the queue is necessarily beneficial, given that with (or without) this information, customers' behavior is not necessarily optimal. It turns out that, depending on the input parameters, it might or might not be desired to reveal the length of the queue in front of a profit maximizing server to customers who consider whether to join it.

Guo and Zipkin [5] considered customers with time values uniformly distributed on $[0, 1]$ and three levels of delay information: no information, length of queue, and exact waiting time. They showed how to compute the performance measures in the three systems. Their model has no entry pricing, and welfare increases when more accurate delay information is available.

Debo, Parlour, and Rajan [4] considered a firm that knows the quality of the service it provides but cannot credibly communicate it to its potential customers. There are two types of customer, characterized by their private signal on the quality of the service. Customers update their beliefs after observing the length of the queue and decide whether to join. The authors show that, in general, when waiting in the queue is costly, the equilibrium behavior is not of the threshold type. Other results on the value of information in queuing systems are discussed in [1, 7, 8, 10].

Arnott, de Palma, and Lindsey [2, 3] analyzed effects of information on participation and time-of-use decisions in congestible systems, when capacity and demand fluctuate. Other articles dealing with related questions are mentioned in these references. They concluded that with appropriate price regulation of the system, added information can improve the efficiency of the system. Otherwise, the effect of information might be negative.

In this article we consider a structured model of a congested system—that of a Markovian queue managed by a single server who aims to maximize profits. To simplify the derivations, we assume that the random variable in question might only obtain two values. For this model, we are able to obtain explicit answers to the effect of information about the system's parameters on the servers profits and the system's overall welfare.

We consider an $M/M/1$ system with a large potential demand of risk-neutral customers, service rate μ , waiting cost of C per unit time, entry fee T , and value of service R . The queue length is unobservable to the customers while making their decision of whether to join it. We consider three types of uncertainty, defined by assuming that a random parameter takes on one of two values. In the first, the capacity of the server, measured by μ , is random. In the second, the cost of spending time in

the system, as reflected by C , is random. In the third, the quality of the service as reflected by R is a random variable. For example, the service rate and quality might depend on the particular server on duty, and the waiting cost might be affected by local time-variable conditions or quality of accommodation.

In each case we consider three submodels. In the first, the customers are not informed about the realized value of the random parameter. In particular, this means that the server sets a single price independent of the realized value of the parameter, because price differentiation might serve as a signal to the customers about the value of the parameter.¹ In the other two cases, the realized value is revealed to the customers. In the second case the server might set a different price for each realized value, charging a higher fee for better quality or for faster service. In the third case this is not possible, for example, because the server cannot observe the realized value or because it is technically impossible or too costly to modify the prices.

2. MAIN RESULTS

For a given set of parameters and information, customers join the system according to a (Nash) equilibrium rate. By our assumption of large potential demand, this rate is such that a customer is indifferent between joining and not joining. This means that both in the case of uninformed customers and in the case of informed customers and two prices, the expected net benefit of a customer, after deduction of the admission fee and waiting cost, is zero. It is well known (see, e.g., [9]) that since in this case the server extracts all the customer surplus, the server's objective coincides with the social objective; therefore, the server's decisions conform with welfare maximization. Note that with informed customers and a single price, customers might have positive surplus and the profit maximizing solution is, in general, not socially optimal.

Let Π^{un} , Π_1^{in} , and Π_2^{in} denote the server's maximum profit when customers are uninformed or informed, and in the latter case when a single price or two prices are set by the server, respectively. Since customers' arrival rate in equilibrium under the profit maximizing policy is socially optimal, additional information to the customers can only increase social welfare. Therefore, $\Pi_2^{in} \geq \Pi^{un}$. Obviously, $\Pi_2^{in} \geq \Pi_1^{in}$. However, when the server is restricted to a single price, there is no straightforward answer to whether informing the customers increases profits (and social welfare).

We find that with uncertain service rate or waiting cost, $\Pi_1^{in} \geq \Pi^{un}$. However, with uncertain value of service, this inequality does not hold in general. In particular, when the waiting cost is sufficiently small, we get the reverse inequality.

- For uncertain service rate, we prove (Theorem 4.3) a result that is stronger than the obvious $\Pi_2^{in} \geq \Pi^{un}$: Informing customers is desired even if the price is exogenously fixed and is not optimally set by the server. Moreover, the difference between the profits obtained with informed and uninformed customers increases with the amount of uncertainty. Note that in such a case, customers enjoy a positive consumer surplus and the server's objective differs from the social objective.

- With uncertain waiting cost, for prices that induce positive demand in the case with no information, the same level of social welfare is attained regardless of whether customers are informed or not, but informing customers might be beneficial when one of the possible values of the waiting cost is so high that without information, no customers will show up.
- With uncertain quality of service and a single price, when the difference between the possible values of service is large, the optimal price is so high that no customer will choose to arrive if the low value is realized. It follows that, for small values of C , the server is motivated to conceal the realized value of service from the customers. For large values of C , the opposite is true.

The effect of increasing uncertainty on the three profit functions is different from case to case:

- With informed customers and two prices, increasing the variance has a positive effect on profits and welfare in all three types of uncertainty that we consider.
- With informed customers and a single price, the effect is positive except for when the variance is small and the random variable is the quality of service.
- With uninformed customers, profits are independent of the variance when the random variable is the waiting cost or quality of service. However, with random service rate, profits decrease as the variance increases.

The article is organized as follows. In Section 3 we describe the well-known outcome of the model when all parameters are known with certainty. Sections 4, 5, and 6 deal with uncertain service rate, waiting cost, and quality of service, respectively. Each section is divided into five subsections. The first three deal with the cases of uninformed customers, informed customers with two prices, and informed customers with a single price, respectively. The fourth subsection compares the results and draws conclusions on the effect of the assumptions on the server's expected profit. The last subsection analyzes the effect of uncertainty, as expressed by the variance of the random variable, in each of the three models.

3. BACKGROUND

For given values of the parameter R , C , μ , and T , if $R - T < (C/\mu)$, then the equilibrium arrival rate is zero. Otherwise, it satisfies $R - T = C/(\mu - \lambda)$ or $\lambda = \mu - (C/(R - T))$.

Suppose that R , C , and μ are given and that T is a decision variable. The rate of profit is $\Pi = \lambda T$. If $R < C/\mu$, then it is not possible to gain any profits. Otherwise,

Π is optimized by setting

$$T = R - \sqrt{\frac{RC}{\mu}} \tag{1}$$

Thus, the maximum rate of profit is

$$\Pi^* = \begin{cases} (\sqrt{R\mu} - \sqrt{C})^2, & R \geq \frac{C}{\mu} \\ 0, & R < \frac{C}{\mu} \end{cases} \tag{2}$$

(see, e.g., Table 3.2 in [9]).

In the following we measure uncertainty by the variance of the random parameter. Suppose that the uncertain parameter is q and it obtains the values q_1 and q_2 with probabilities p and $1 - p$, respectively. Denote its expected value by \bar{q} , then $\text{Var}(q) = p(q_1 - \bar{q})^2 + (1 - p)(\bar{q} - q_2)^2$. We consider two ways of controlling the variance. In one, we fix q_1 and obtain desired values of $\text{Var}(q)$ and \bar{q} is by adjusting p and q_2 . In the other, we fix p and adjust q_1 and q_2 . In both types, we maintain a constant value of $\bar{q} = pq_1 + (1 - p)q_2$. Thus, $q_2 = (\bar{q} - pq_1)/(1 - p)$ and

$$\text{Var}(q) = \frac{p}{1 - p}(q_1 - \bar{q})^2 \tag{3}$$

4. UNCERTAIN SERVICE RATE

Assume that $\mu = \mu_1$ with probability p and that $\mu = \mu_2$ with probability $1 - p$, where $\mu_1 > \mu_2$. Let

$$v = \frac{R - T}{C}$$

and

$$r = \frac{R}{C}.$$

Thus, v and r represent the values of service and net gain from service after deduction of service fee, respectively, normalized with respect to the time value.

4.1. Uniformed Customers

Without loss of generality,

$$R - T \geq C \left(\frac{p}{\mu_1} + \frac{1 - p}{\mu_2} \right), \tag{4}$$

since otherwise $\lambda = 0$ and $\Pi = 0$. The proof of the following lemma is given in the appendix.

LEMMA 4.1: *The equilibrium value of v is a root of*

$$\begin{aligned}
 & (\mu_1 - \mu_2)^2 [(\mu_1 + \mu_2)^2 - (\mu_1 - \mu_2)^2] v^4 \\
 & + 2(1 - 2p)(\mu_1 - \mu_2)[(\mu_1 + \mu_2)^2 - (\mu_1 - \mu_2)^2] v^3 \\
 & + [(\mu_1 + \mu_2)^2 - 2r(\mu_1 - \mu_2)^2 \\
 & \times [\mu_1 + \mu_2 + (1 - 2p)(\mu_1 - \mu_2)] - (1 - 2p)^2(\mu_1 - \mu_2)^2] v^2 \\
 & + 2r(\mu_1 - \mu_2)[-2(1 - 2p)(\mu_1 + \mu_2) - (1 - 2p)^2(\mu_1 - \mu_2) - (\mu_1 - \mu_2)] v \\
 & + r[r(\mu_1 - \mu_2)^2 - 2(\mu_1 + \mu_2) - r(1 - 2p)^2(\mu_1 - \mu_2)^2 \\
 & - 2(1 - 2p)(\mu_1 - \mu_2)] = 0.
 \end{aligned}$$

We solve the above polynomial for v and then compute $\Pi^{in} = \lambda T = \lambda(R - Cv)$.²

4.2. Informed Customers: Two Prices

Suppose that customers are informed about the service rate and the server charges different prices depending on the realization of μ . In this case, we use (2) to obtain that the server’s rate of profit is

$$\Pi_2^{in} = \begin{cases} pC(\sqrt{r\mu_1} - 1)^2 + (1 - p)C(\sqrt{r\mu_2} - 1)^2, & r \geq \frac{1}{\mu_2} \\ pC(\sqrt{r\mu_1} - 1)^2, & \frac{1}{\mu_1} < r < \frac{1}{\mu_2} \\ 0, & r \leq \frac{1}{\mu_1}. \end{cases} \quad (5)$$

4.3. Informed Customers: Single Price

Assume that the firm must set a fixed price and cannot change it when the value of μ is revealed. Moreover, customers are informed about the realized value of μ .

Assume first that the price T is not too large, so that $\lambda > 0$ for both values of μ . The arrival rate, given $\mu = \mu_i$, is $\lambda_i = \mu_i - (C/(R - T))$. The expected arrival rate, given T , is

$$\begin{aligned}
 \lambda &= p\left(\mu_1 - \frac{C}{R - T}\right) + (1 - p)\left(\mu_2 - \frac{C}{R - T}\right) \\
 &= p\mu_1 + (1 - p)\mu_2 - \frac{C}{R - T}.
 \end{aligned} \quad (6)$$

Define $\bar{\mu} = p\mu_1 + (1 - p)\mu_2$. The expected rate of profit, given T , is $\Pi(T) = T\lambda$. Its derivative with respect to T is

$$\bar{\mu} - \frac{C}{R - T} - \frac{TC}{(R - T)^2}.$$

The first-order optimality condition is $(\bar{\mu}/C)(R - T)^2 - R = 0$, which gives

$$T = R \pm \sqrt{\frac{RC}{\bar{\mu}}}.$$

Since $T < R$ is required, only the root associated with the minus is relevant, and $\lambda = \bar{\mu} - \sqrt{C\bar{\mu}/R}$. Therefore,

$$\Pi = \left(R - \sqrt{\frac{RC}{\bar{\mu}}} \right) \left(\bar{\mu} - \sqrt{\frac{C\bar{\mu}}{R}} \right) = C \left(\sqrt{\frac{\bar{\mu}R}{C}} - 1 \right)^2. \tag{7}$$

However, the firm might choose a high price, which does not attract any customers if $\mu = \mu_2$. In this situation, the firm will set the price to gain maximum profit when the high rate is realized. Thus, $T = R - \sqrt{CR/\mu_1}$, as in (1), and

$$\Pi = pC \left(\sqrt{\frac{R\mu_1}{C}} - 1 \right)^2. \tag{8}$$

The firm will choose the solution giving the higher value between (7) and (8). The two values are equal if

$$\left(\sqrt{\frac{\bar{\mu}R}{C}} - 1 \right)^2 = p \left(\sqrt{\frac{R\mu_1}{C}} - 1 \right)^2,$$

or

$$\sqrt{\bar{\mu}r} - 1 = \sqrt{p}(\sqrt{r\mu_1} - 1).$$

Equivalently, $r = \eta$, where

$$\eta = \left(\frac{1 - \sqrt{p}}{\sqrt{\bar{\mu}} - \sqrt{\mu_1 p}} \right)^2.$$

Remark 4.2: By concavity of the square root function, $p\sqrt{\mu_1} + (1 - p)\sqrt{\mu_2} > \sqrt{\bar{\mu}}$. This inequality can be used to show that $\sqrt{\mu_1 p} + \sqrt{\mu_2} - \sqrt{\mu_2 p} > \sqrt{\bar{\mu}}$ or $\eta > 1/\mu_2$, for any $p \in (0, 1)$.

Denote the resulting profit by Π_1^{in} , then

$$\Pi_1^{in} = \begin{cases} C(\sqrt{r\bar{\mu}} - 1)^2, & r \geq \eta \\ pC(\sqrt{r\mu_1} - 1)^2, & \frac{1}{\mu_1} \leq r \leq \eta \\ 0, & r \leq \frac{1}{\mu_1}. \end{cases} \tag{9}$$

4.4. Discussion

Denote by $\lambda^{in}(T)$ the equilibrium expected arrival rate when customers are informed and there is a single price T . Similarly, let $\lambda^{un}(T)$ be the arrival rate when customers are not informed.

THEOREM 4.3: Consider fixed input values of p , μ_1 , and μ_2 . For every price T , $0 < T < R - C((p/\mu_1) + ((1 - p)/\mu_2))$, $\lambda^{in}(T) \geq \lambda^{un}(T)$.

PROOF: By (6) and (A2), we need to prove that $(\bar{\mu} - 1/v) - 1/2 (-1/v + \mu_1 + \mu_2 - S) \geq 0$ or that $S - [1/v + (\mu_1 - \mu_2)(1 - 2p)] \geq 0$. It is sufficient to show that

$$\left[\frac{1}{v^2} + (\mu_1 - \mu_2)^2 + \frac{2}{v}(1 - 2p)(\mu_1 - \mu_2) \right] - \left(\frac{1}{v} + (\mu_1 - \mu_2)(1 - 2p) \right)^2 \geq 0.$$

This is equivalent to $1 - (1 - 2p)^2 \geq 0$, and the claim follows. ■

Note that the inequality in the theorem is strict except for when $p \in \{0,1\}$, the case with no uncertainty. The difference in the expected arrival rates is concave and it is maximized when $p = 1/2$. Thus, we might say that *the difference in the expected arrival rates increases with the amount of uncertainty*.

Also note that the range of prices in the theorem is the allowed range for the uninformed customers case. For the informed customers case, the allowed range is wider, and in the added values, we have $\lambda^{in}(T) > 0$, whereas $\lambda^{un}(T) = 0$, conforming with the spirit of the theorem.

COROLLARY 4.4: $\Pi_2^{in} \geq \Pi_1^{in} \geq \Pi^{un}$. Therefore, the firm is motivated to reveal the realized value of μ to its customers.

PROOF: The first inequality is obvious, and the second follows from Theorem 4.3. ■

The relative behavior of the profit functions under the three models differs in different regions of r as follows (we use here the inequality $\eta > 1/\mu_2$ given in Remark 4.2):

- If $r \in (0, 1/\mu_1)$, then $\Pi^{un} = \Pi_1^{in} = \Pi_2^{in} = 0$.
- If $r \in (1/\mu_1, p/\mu_1 + (1 - p)/\mu_2)$, then $0 = \Pi^{un} < \Pi_1^{in} = \Pi_2^{in}$.
- If $r \in (p/\mu_1 + (1 - p)/\mu_2, 1/\mu_2)$, then $0 < \Pi^{un} < \Pi_1^{in} = \Pi_2^{in}$.
- If $r \in (1/\mu_2, \infty)$, then $0 < \Pi^{un} < \Pi_1^{in} < \Pi_2^{in}$. The difference $\Pi_2^{in} - \Pi_1^{in}$ is equal to $kC\sqrt{r}$, where $k = 2(\sqrt{\bar{\mu}} - p\sqrt{\mu_1} - (1 - p)\sqrt{\mu_2})$.

Figure 1 (left) gives the expected rate of profit for $p = 0.2$, $C = 30$, $\mu_1 = 1.5$, and $\mu_2 = 0.5$.³

Remark 4.5: Clearly, when $p = 1$, we have $\Pi^{un} = \Pi_1^{in} = \Pi_2^{in}$. However, $\lim_{p \rightarrow 1} \Pi^{un}(p) < \Pi^{un}(1)$, whereas $\lim_{p \rightarrow 1} \Pi_j^{in}(p) = \Pi_j^{in}(1)$, for $j = 1, 2$. The functions

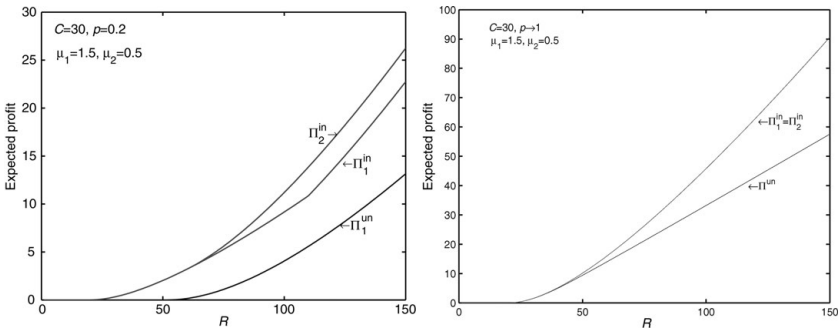


FIGURE 1. Profits when $p = 0.2$ (left) and when $p \rightarrow 1$ (right).

when $p \rightarrow 1$ are illustrated in Figure 1 (right). The reason is that when $p = 1$, we might have $\mu_2 < \lambda < \mu_1$, whereas for any $p < 1$, we must have $\lambda < \mu_2$ to guarantee a finite value for the expected waiting time.⁴ Actually, the value $\lambda = \mu_1 - \sqrt{c\mu_1/R}$ for $p = 1$ is given by the root corresponding to the plus sign in (A1).

4.5. The Effect of Uncertainty

We consider two types of change in uncertainty. In both types we fix $\bar{\mu}$.

In the first case, we also fix $\mu_1 > \bar{\mu}$ (recall that $\mu_1 > \mu_2$) and simultaneously change p and μ_2 so that $\bar{\mu}$ is preserved at the same level and $\text{Var}(\mu)$ varies. In Figure 2 (left) we assumed $\bar{\mu} = 2$ and $\mu_1 = 2.5$, so that by (3), $\text{Var}(\mu) = p/(4(1 - p))$. We note that this is an increasing function of p . In the second case, we fix p and modify μ_1 and μ_2 , again preserving $\bar{\mu}$. In Figure 2 (right) we set $p = 0.5$ and $\bar{\mu} = 2$, giving $\text{Var}(\mu) = (\mu_1 - 2)^2$.

As expected from (9), Π_1^{in} depends only on $\bar{\mu}$, not on the individual values of μ_1 and μ_2 , as long as $\eta \leq r$. For $\eta > r$, Π_1^{in} linearly increases with p in Figure 2 (left) (recall that μ_1 is fixed). We obtain that $\Pi_1^{in} = \Pi_2^{in}$ when p is large enough so that to maintain $\bar{\mu} = 2$, the value of μ_2 is smaller than C/R .⁵ Figure 2 (right) gives similar results, but this time, $p = 1/2$ is fixed, $\mu_1 \in (2, 3.4)$, and $\mu_2 = 4 - \mu_1 \in (C/R = 0.6, 2)$ so that, again, $\bar{\mu} = 2$ is preserved. The two figures are essentially identical except for a nonlinear change of scale.

The function $(\sqrt{\bar{\mu}} - 1)^2$ is convex. Suppose that $\text{Var}(\mu)$ is increased while maintaining $\bar{\mu}$. Then Π_2^{in} in the first two cases of (5) also increases. Therefore, Π_2^{in} , which is the maximum of these two expressions (and zero) is a nondecreasing function of the variance (and strictly increasing where the profit is positive).

Finally, Π^{un} decreases when the variance increases due to the convexity of the expected waiting time as a function of λ for any given μ .

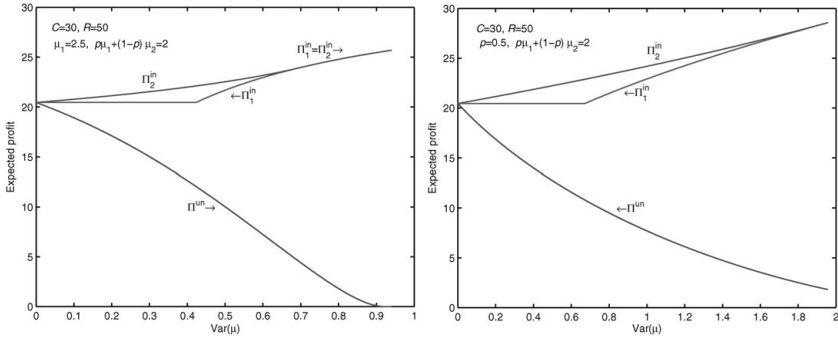


FIGURE 2. The effect of uncertain μ : changing (p, μ_2) (left) and (μ_1, μ_2) (right).

5. UNCERTAIN WAITING COST

Suppose that $C = C_1$ with probability p and $C = C_2$ with the complementary probability, where $C_1 < C_2$. Denote the expected value of C by $\bar{C} = pC_1 + (1 - p)C_2$.

5.1. Uniformed Customers

Given a price T , the equilibrium arrival rate is $\lambda = (\mu - \bar{C}/(R - T))^+$. By (2), the maximum rate of profit is

$$\Pi^{un} = \begin{cases} (\sqrt{R\mu} - \sqrt{\bar{C}})^2, & R\mu \geq \bar{C} \\ 0, & R\mu < \bar{C} \end{cases} \tag{10}$$

5.2. Informed Customers: Two Prices

Suppose that customers are informed about the service rate and that the server charges different prices depending on the realization of C . Then

$$\Pi_2^{in} = \begin{cases} p(\sqrt{R\mu} - \sqrt{C_1})^2 + (1 - p)(\sqrt{R\mu} - \sqrt{C_2})^2, & R\mu \geq C_2 \\ p(\sqrt{R\mu} - \sqrt{C_1})^2, & C_1 < R\mu < C_2 \\ 0, & R\mu \leq C_1. \end{cases} \tag{11}$$

5.3. Informed Customers: Single Price

Given the price T and the information that C_i is realized, the equilibrium arrival rate is $\lambda_i = (\mu - (C_i/(R - T)))^+$.

For $T < R - (C_1/\mu)$, we have $\lambda_1, \lambda_2 > 0$. The expected rate of profit is then

$$\begin{aligned} \Pi &= T[p\lambda_1 + (1 - p)\lambda_2] = T \left\{ p\mu - p \frac{C_1}{R - T} + (1 - p)\mu - (1 - p) \frac{C_2}{R - T} \right\} \\ &= T\mu - \frac{\bar{C}T}{R - T}. \end{aligned}$$

This is the same expression as in the uniformed customers case and the maximum profit is

$$\Pi = \left(\sqrt{R\mu} - \sqrt{\bar{C}} \right)^2. \tag{12}$$

Expression (12) is correct if the maximizing price satisfies $T < R - (C_1/\mu)$. Otherwise, it underestimates the profit by assuming a negative λ_2 , whereas the true value is zero. For $T \in (R - (C_2/\mu), R - (C_1/\mu))$, we have $\lambda_1 > 0$ and $\lambda_2 = 0$. In this range, the profit maximizing price is $R - \sqrt{RC_1/\mu}$ and the profit is

$$\Pi = p \left(\sqrt{R\mu} - \sqrt{C_1} \right)^2. \tag{13}$$

We compare (13) with (12) to compute the maximum possible profit given R and μ . Thus,

$$\Pi_1^{in} = \max \left\{ \left(\sqrt{R\mu} - \sqrt{\bar{C}} \right)^2, \quad p \left(\sqrt{R\mu} - \sqrt{C_1} \right)^2 \right\}.$$

The two values are equal if $\sqrt{R\mu} - \sqrt{\bar{C}} = \sqrt{p}(\sqrt{R\mu} - \sqrt{C_1})$, or $R\mu = \gamma$, where

$$\gamma = \left(\frac{\sqrt{\bar{C}} - \sqrt{pC_1}}{1 - \sqrt{p}} \right)^2.$$

Note that from $C_2 > C_1$ it follows that $\gamma > C_1$. Therefore,

$$\Pi_1^{in} = \begin{cases} \left(\sqrt{R\mu} - \sqrt{\bar{C}} \right)^2, & R\mu \geq \gamma \\ p \left(\sqrt{R\mu} - \sqrt{C_1} \right)^2, & C_1 < R\mu < \gamma \\ 0, & R\mu \leq C_1. \end{cases} \tag{14}$$

5.4. Discussion

The relative behavior of the profit functions under the three models differs in different regions of $R\mu$ defined by $0 \leq C_1 \leq \bar{C} \leq C_2 \leq \gamma < \infty$ as follows:

- If $R\mu \in (0, C_1)$, then $\Pi^{un} = \Pi_1^{in} = \Pi_2^{in} = 0$.
- If $R\mu \in (C_1, \bar{C})$, then $0 = \Pi^{un} < \Pi_1^{in} = \Pi_2^{in}$.
- If $R\mu \in (\bar{C}, C_2)$, then $0 < \Pi^{un} < \Pi_1^{in} = \Pi_2^{in}$.
- If $R\mu \in (C_2, \gamma)$, then $0 < \Pi^{un} < \Pi_1^{in} < \Pi_2^{in}$.

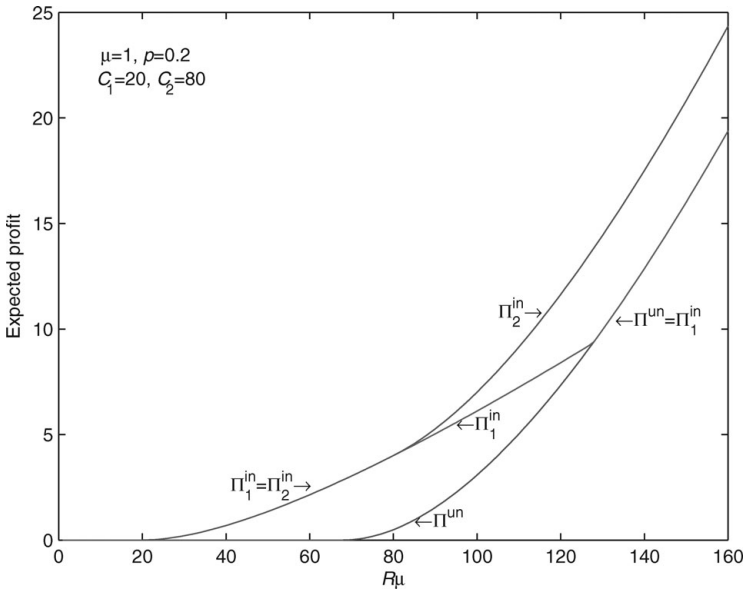


FIGURE 3. Random waiting cost example.

- If $R\mu \in (\gamma, \infty)$, then $0 < \Pi^{un} = \Pi_1^in < \Pi_2^in$. The difference $\Pi_2^in - \Pi_1^in$ is equal to $\kappa\sqrt{R\mu}$, where $\kappa = 2(\sqrt{C_2} - p\sqrt{C_1} - (1-p)\sqrt{C_2})$.

Figure 3 shows the profit functions for $p = 0.2$, $C_1 = 20$, and $C_2 = 80$. The variable corresponding to the x -axis is $R\mu$, which is the value produced by the server per unit of time of service.⁶

COROLLARY 5.1: $\Pi_2^in \geq \Pi_1^in \geq \Pi^{un}$. Therefore the firm is motivated to reveal the realized value of C to its customers.

5.5. The Effect of Uncertainty

We consider two types of change in uncertainty. In both types we fix \bar{C} . In the first case we also fix $C_1 \leq \bar{C}$ (recall that $C_1 \leq C_2$) and simultaneously change p and C_2 so that \bar{C} is preserved at the same level. In Figure 4 (left) we assumed $\bar{C} = 30$ and $C_1 = 20$, giving, by (3), $\text{Var}(\mu) = 100p/(1-p)$. In the second case we fix p and modify C_1 and C_2 , again preserving \bar{C} . An increase in C_1 comes with an adequate decrease of C_2 . In Figure 4 (right) we assumed $p = 0.5$ and $\bar{C} = 30$, giving, by (2), $\text{Var}(\mu) = (C_1 - 30)^2$.⁷

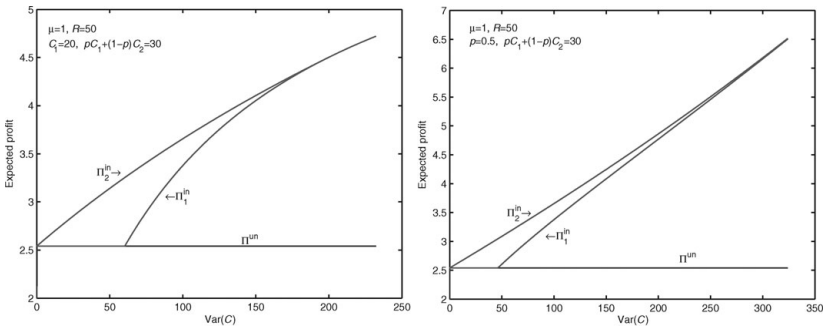


FIGURE 4. The effect of uncertain C : changing (p, C_2) (left) and (C_1, C_2) (right).

6. UNCERTAIN QUALITY OF SERVICE

Let $R = R_1$ with probability p , and $R = R_2$ with the complementary probability. Assume $R_1 > R_2$ and that μ and C are fixed. Denote $\bar{R} = pR_1 + (1 - p)R_2$.

6.1. Uninformed Customers

Given a price T , the equilibrium arrival rate is $\lambda = (\mu - C/(R - T))^+$. By (2), the maximum rate of profit is

$$\Pi^{un} = \begin{cases} (\sqrt{\bar{R}\mu} - \sqrt{C})^2, & \bar{R}\mu \geq C \\ 0, & \bar{R}\mu < C. \end{cases} \tag{15}$$

6.2. Informed Customers: Two Prices

Suppose that customers are informed about the service quality and that the server charges different prices depending on the realization of R . Then

$$\Pi_2^{in} = \begin{cases} p(\sqrt{R_1\mu} - \sqrt{C})^2 + (1 - p)(\sqrt{R_2\mu} - \sqrt{C})^2, & R_2 \geq \frac{C}{\mu} \\ p(\sqrt{R_1\mu} - \sqrt{C})^2, & R_2 < \frac{C}{\mu} < R_1 \\ 0, & R_1 \leq \frac{C}{\mu}. \end{cases} \tag{16}$$

6.3. Informed Customers: Single Price

For a given price T , the arrival rate is $\lambda_i = (\mu - C/(R_i - T))^+$ if R_i is realized.

To have positive gain, the price cannot exceed $R_1 - (C/\mu)$. We distinguish the two cases according to whether $T < R_2 - (C/\mu)$.

- Consider first the case $T < R_2 - (C/\mu)$: The expected profit is

$$\begin{aligned} \Pi(T) &= T[p\lambda_1 + (1 - p)\lambda_2] \\ &= T\left(p\mu - \frac{pC}{R_1 - T} + (1 - p)\mu - \frac{(1 - p)C}{R_2 - T}\right) \\ &= T\mu - CT\left(\frac{p}{R_1 - T} + \frac{1 - p}{R_2 - T}\right). \end{aligned}$$

The first-order optimality conditions are

$$\Pi' = \mu - C\left(\frac{p}{R_1 - T} + \frac{1 - p}{R_2 - T}\right) - CT\left(\frac{p}{(R_1 - T)^2} + \frac{1 - p}{(R_2 - T)^2}\right) = 0.$$

Multiplication by $((R_1 - T)^2(R_2 - T)^2)/C$ gives

$$\begin{aligned} \frac{\mu}{C}(R_1 - T)^2(R_2 - T)^2 - (p(R_1 - T)(R_2 - T)^2 + (1 - p)(R_1 - T)^2(R_2 - T)) \\ - T(p(R_2 - T)^2 - (1 - p)(R_1 - T)^2) = 0, \end{aligned}$$

or

$$\frac{\mu}{C}(R_1 - T)^2(R_2 - T)^2 - pR_1(R_2 - T)^2 - (1 - p)(R_1 - T)^2R_2 = 0.$$

This gives the following polynomial:

$$\begin{aligned} \frac{\mu}{C}T^4 - 2\frac{\mu}{C}(R_1 + R_2)T^3 + \left(\frac{\mu}{C}(R_1^2 + R_2^2 + 4R_1R_2) - \bar{R}\right)T^2 \\ + 2R_1R_2\left(1 - \frac{\mu}{C}(R_1 + R_2)\right)T + R_1R_2\left(\frac{\mu}{C}R_1R_2 - pR_2 - (1 - p)R_1\right) = 0, \end{aligned} \tag{17}$$

where $\bar{R} = pR_1 + (1 - p)R_2$.

- Suppose now that $T \geq R_2 - (C/\mu)$. In this case $\lambda_2 = 0$ and the optimal price gives profit $p(\sqrt{R_1\mu} - \sqrt{C})^2$.

Figure 5 (left) shows the profit as a function of the price for $C = 20, p = 0.2, R_1 = 70, \mu = 1$, and $R_2 = 30$. With these parameters, $R_2 - (C/\mu) = 10$, and we see that the server gains more by raising the price to a level that induces no arrivals when R_2 is realized. Lowering R_1 to 60 and increasing R_2 to 40 (and then $R_2 - (C/\mu) = 20$), we obtain a new situation, shown in Figure 5 (right), in which it is worth lowering the price to attract customers also when R_2 is realized.

6.4. Discussion

When C is close to zero, $\Pi^{im} = \Pi_2^{im} = \mu\bar{R} > \mu[\max(R_2, pR_1)] = \Pi_1^{im}$. Thus, for small values of C , if the server is restricted to a single price, the server is motivated to

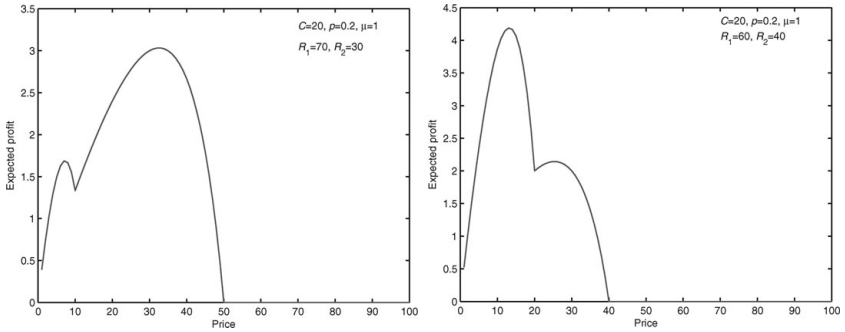


FIGURE 5. Optimal price is greater (left) or smaller (right) than $R_2 - C/\mu$.

conceal the realized value of R from the customers. However, for large values of C , the opposite is true.

Figure 6 gives the profits as a function of C when $R_1 = 100$, $R_2 = 30$, $p = 0.2$, and $\mu = 1$. Note that these are actually functions of C/μ .⁸

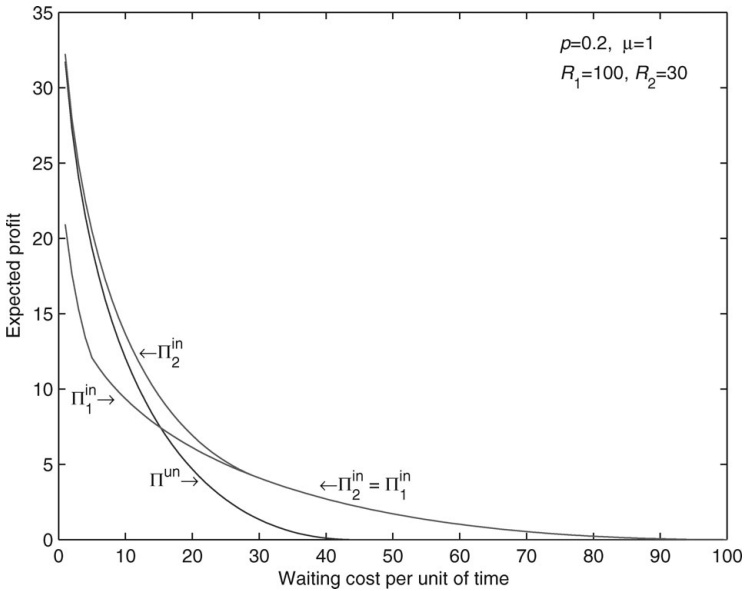


FIGURE 6. Random value of service.

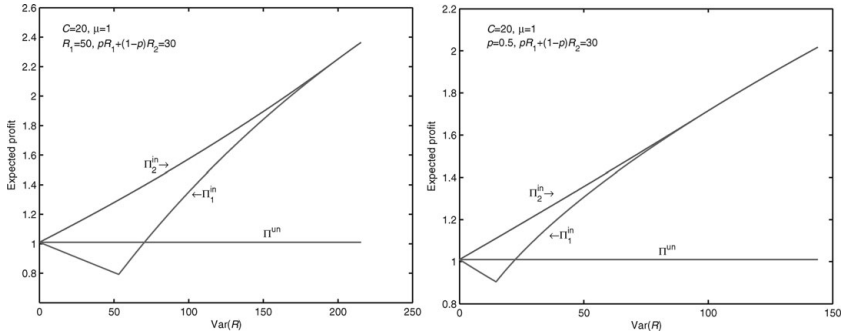


FIGURE 7. The effect of uncertain R : changing (p, R_2) (left) and (R_1, R_2) (right).

6.5. The Effect of Uncertainty

We fix \bar{R} and thus $R_2 = (\bar{R} - R_2 p)/(1 - p)$. In the first case, we also fix $R_1 \geq \bar{R}$ (recall that $R_1 > R_2$) and simultaneously change p and R_2 so that \bar{R} is preserved. In Figure 7 (left) we assumed $\bar{R} = 30$ and $R_1 = 50$, giving (by (3)) $\text{Var}(\mu) = 400p/(1 - p)$. In the second case, we fix p and modify R_1 and R_2 , again preserving \bar{R} . In Figure 7 (right) we assumed $p = 0.5$ and $\bar{R} = 30$, giving $\text{Var}(\mu) = (R_1 - 30)^2$.

We see that Π^m only depends on \bar{R} and is not affected by the change in uncertainty, as expected from (15). We also see that in the case of informed customers and two prices, profits increase with uncertainty. However, in the case of a single price, profits initially decrease as a function of the variance, and only for larger variance is there an increase.⁹

Acknowledgment

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Notes

1. See [11] for how price might serve as a signal for quality.
2. In our numerical solution, we obtained at most two real roots that satisfy (4), only one of which corresponds to an equilibrium; see Remark A1.
3. In this example, $\eta \approx 110$, corresponding to $R \approx 110$. The R values corresponding to $1/\mu_1$, $p/\mu_1 + (1 - p)/\mu_2$, and $1/\mu_2$, are 20, 52, and 60, respectively.
4. In the other cases that we consider later, where C or R are random, we naturally have $\Pi^m = \Pi_1^m = \Pi_2^m$ in the limit, when $p \rightarrow \infty$.
5. Specifically, for the parameters used for Figure 2 (left), $\mu_2 \leq 30/50$ is obtained when $p \geq 14/19$, or $\text{Var}(\mu) \geq 0.7$.
6. With these data, $\bar{C} = 68$, $\gamma \approx 127.7$, and $\kappa \approx 0.39$.
7. $\Pi_1^m = \Pi_2^m$ when $R\mu \leq C_2$, see (11) and (14). With the parameters used for Figure 4 (right), this means $C_2 > 50$, $p \geq 2/3$, and, equivalently, $\text{Var}(C) > 200$. In Figure 4 (left), the analogous condition is $C_1 \leq 10$ or $\text{Var}(C) \leq 400$. Similarly, from (10) and (14), $\Pi_1^m = \Pi^m$ if $R\mu \geq p$. In Figure 4 (left), this means

$C_1 \leq (\sqrt{60} + \sqrt{50} - 10)^2$, or approximately $\text{Var}(C) \leq 60$. In Figure 4 (right), this means $C_1 \leq (\sqrt{60} + \sqrt{50} - 10)^2$, giving approximately $\text{Var}(C) \leq 46$.

8. For this instance, $\Pi^{im} > 0$ for $C < \mu\bar{R} < 44$, and the other curves are positive for $C < \mu R_1 = 100$.

9. We have $\Pi_1^{in} = \Pi_2^{in}$ if $R_2 \leq C/\mu$. With the parameters used to create Figure 7, this means $R_2 \leq 20$. In Figure 7 (left) this means $p \geq 1/3$ and, thus, $\text{Var}(R) \geq 200$. In Figure 7 (right) it means $R_1 \geq 40$ and, thus, $\text{Var}(R) \geq 100$.

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APPENDIX
Proof of Lemma 4.1

In equilibrium,

$$v = \frac{p}{\mu_1 - \lambda} + \frac{1 - p}{\mu_2 - \lambda},$$

or

$$v(\mu_1 - \lambda)(\mu_2 - \lambda) = p\mu_2 + (1 - p)\mu_1 - \lambda,$$

and

$$v\lambda^2 + \lambda(1 - v\mu_1 - v\mu_2) + v\mu_1\mu_2 - p\mu_2 - (1 - p)\mu_1 = 0,$$

and, finally,

$$\lambda = \frac{-1 + v(\mu_1 + \mu_2)}{2v} \pm \frac{\sqrt{(1 - v\mu_1 - v\mu_2)^2 - 4v[v\mu_1\mu_2 - p\mu_2 - (1 - p)\mu_1]}}{2v}$$

$$= \frac{1}{2} \left\{ -\frac{1}{v} + (\mu_1 + \mu_2) \pm \sqrt{\frac{1}{v^2} + (\mu_1 - \mu_2)^2 + \frac{2}{v}(1 - 2p)(\mu_1 - \mu_2)} \right\}. \tag{A.1}$$

Assuming $\mu_1 > \mu_2$, the square root is minimized when $p = 1$, and then it is equal to $v - (\mu_1 - \mu_2)$. Therefore, the root corresponding to the plus sign is greater than μ_2 . We conclude that the equilibrium is defined by the root corresponding to the minus sign. Hence,

$$\lambda = \frac{1}{2} \left\{ -\frac{1}{v} + (\mu_1 + \mu_2) - S \right\}, \tag{A.2}$$

where

$$S = \sqrt{\frac{1}{v^2} - (\mu_1 - \mu_2)^2 + \frac{2(1 - 2p)(\mu_1 - \mu_2)}{v}}$$

$$= \sqrt{\left(\frac{1}{v} - (\mu_1 - \mu_2)\right)^2 + 4(1 - p)\frac{\mu_1 - \mu_2}{v}}.$$

Note that by our assumptions that $p \in (0, 1)$ and $\mu_1 > \mu_2$, it follows that S is strictly positive. The derivative of λ with respect to v is

$$\lambda'_v = \frac{1}{2v^2} \left\{ 1 - \frac{-\frac{1}{v} - (1 - 2p)(\mu_1 - \mu_2)}{S} \right\},$$

and with respect to T , it is

$$\lambda'_T = \lambda'_v \frac{dv}{dT} = -\frac{1}{2Cv^2} \left\{ 1 - \frac{-\frac{1}{v} - (1 - 2p)(\mu_1 - \mu_2)}{S} \right\}.$$

The profit Π^{im} is equal to λT . The first-order optimality condition is $T\lambda'_T + \lambda = 0$, or

$$-\frac{T}{2Cv^2} \left(1 + \frac{\frac{1}{v} + (1 - 2p)(\mu_1 - \mu_2)}{S} \right) + \frac{1}{2} \left(-\frac{1}{v} + (\mu_1 + \mu_2) - S \right) = 0.$$

Multiply by $2S$ (allowed since $S > 0$),

$$S \left(-\frac{T}{Cv^2} - \frac{1}{v} + \mu_1 + \mu_2 \right) = \frac{T}{Cv^3} + T(1 - 2p)\frac{\mu_1 - \mu_2}{Cv^2} + \frac{1}{v^2}$$

$$+ (\mu_1 - \mu_2)^2 + 2(1 - 2p)\frac{\mu_1 - \mu_2}{v}.$$

Substitute $T = R - vC$:

$$S\left(-\frac{R}{Cv^2} + \mu_1 + \mu_2\right) = \frac{R}{Cv^3} + R(1 - 2p)\frac{\mu_1 - \mu_2}{Cv^2} + (1 - 2p)\frac{\mu_1 - \mu_2}{v} + (\mu_1 - \mu_2)^2. \tag{A.3}$$

Recall that $r \equiv R/C$. Multiply the left-hand side of (20) by v^3 and square:

$$\begin{aligned} LS &= \left(\frac{1}{v^2} + (\mu_1 - \mu_2)^2 + 2(1 - 2p)\frac{\mu_1 - \mu_2}{v}\right)(r^2v^2 + v^6(\mu_1 + \mu_2)^2 \\ &\quad - 2r(\mu_1 + \mu_2)v^4) \\ &= r^2 + (\mu_1 + \mu_2)^2v^4 - 2r(\mu_1 + \mu_2)v^2 \\ &\quad + 2(1 - 2p)r^2(\mu_1 - \mu_2)v + (\mu_1 - \mu_2)^2r^2v^2 + (\mu_1 - \mu_2)^2(\mu_1 + \mu_2)^2v^6 \\ &\quad - 2r(\mu_1 - \mu_2)^2(\mu_1 + \mu_2)v^4 + 2(1 - 2p)(\mu_1 - \mu_2)(\mu_1 + \mu_2)^2v^5 \\ &\quad - 4r(1 - 2p)(\mu_1 - \mu_2)(\mu_1 + \mu_2)v^3. \end{aligned}$$

Multiply the right-hand side of (20) by v^3 and square:

$$\begin{aligned} RS &= r^2 + r^2(1 - 2p)^2(\mu_1 - \mu_2)^2v^2 + (1 - 2p)^2(\mu_1 - \mu_2)^2v^4 + (\mu_1 - \mu_2)^4v^6 \\ &\quad + 2r^2(1 - 2p)(\mu_1 - \mu_2)v + 2r(1 - 2p)(\mu_1 - \mu_2)v^2 + 2r(\mu_1 - \mu_2)^2v^3 \\ &\quad + 2r(1 - 2p)^2(\mu_1 - \mu_2)^2v^3 + 2r(1 - 2p)(\mu_1 - \mu_2)^3v^4 \\ &\quad + 2(1 - 2p)(\mu_1 - \mu_2)^3v^5. \end{aligned}$$

By (20), $LS - RS = 0$. Division by v^2 gives

$$\begin{aligned} &(\mu_1 - \mu_2)^2[(\mu_1 + \mu_2)^2 - (\mu_1 - \mu_2)^2]v^4 + 2(1 - 2p)(\mu_1 - \mu_2) \\ &\quad \times [(\mu_1 + \mu_2)^2 - (\mu_1 - \mu_2)^2]v^3 \\ &\quad + [(\mu_1 + \mu_2)^2 - 2r(\mu_1 - \mu_2)^2] \\ &\quad \times [\mu_1 + \mu_2 + (1 - 2p)(\mu_1 - \mu_2)] - (1 - 2p)^2(\mu_1 - \mu_2)^2v^2 \\ &\quad + 2r(\mu_1 - \mu_2)[-2(1 - 2p)(\mu_1 + \mu_2) - (1 - 2p)^2(\mu_1 - \mu_2) - (\mu_1 - \mu_2)]v \\ &\quad + r[r(\mu_1 - \mu_2)^2 - 2(\mu_1 + \mu_2) - r(1 - 2p)^2(\mu_1 - \mu_2)^2 \\ &\quad - 2(1 - 2p)(\mu_1 - \mu_2)] = 0. \end{aligned}$$

(With $\mu_1 = \mu_2 = \mu$, the polynomial reduces to $(2\mu)^2v^2 = 4r\mu$, or $v = \sqrt{r/\mu}$. This gives T as in (1). ■

Remark A.1: Figure A1 shows the roots as a function of R , with $p = 0.2$, $\mu_1 = 1.5$, and $\mu_2 = 0.5$. The figure shows the real roots that satisfy $0 \leq T \leq R - C((p/\mu_1) + ((1 - p)/\mu_2))$. (Recall that by (4), T must be bounded this way, ie, $\Pi = 0$ if $R < C((p/\mu_1) + ((1 - p)/\mu_2)) = 44$. For large values of R , we get two such roots, marked Π_1^{int} and Π_2^{int} . For example, with $R = 3000$, the values of λ , v , T , and Π^{int} are (0.44, 12.82, 2615.4, 1142.1) and (0.20, 2.80, 2916.1, 575.49),

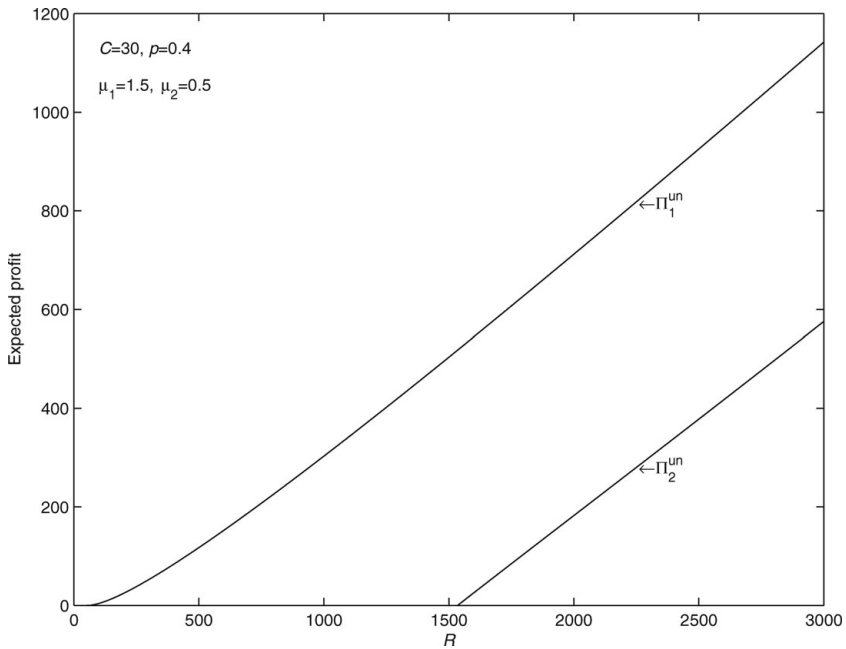


FIGURE A1. Profits corresponding to the two real roots.

respectively. However, the new root, denoted Π_2^{in} in Figure A1, does not correspond to a new equilibrium. Substituting these values in (A.3), we obtain that the left-hand side is negative, whereas the right-hand side is positive (and equal to the left-hand side in its absolute value). Thus, the root Π_2^{in} results from the squaring of the two sides of (A.3).