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COMPARATIVE DYNAMICS IN STOCHASTIC MODELS WITH RESPECT TO THE L^∞–L^∞ DUALITY: A DIFFERENTIAL APPROACH

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Many economic analyses are based on the property that the value of a commodity vector responds continuously to a change in economic environment. As is well known, however, many infinite-dimensional models, such as an infinite–time horizon stochastic growth model, lack this property. In the present paper, we investigate a stochastic growth model in which dual vectors lie in an L^∞ space. This result ensures that the value of a stock vector is jointly continuous with respect to the stock vector and its support price vector. The result is based on the differentiation method in Banach spaces that Yano [*Journal of Mathematical Economics* 18 (1989), 169–185] develops for stochastic growth models.

Keywords: Stochastic Optimal Growth Model, Comparative Dynamics,
Infinite-Dimensional Differential Approach

1. INTRODUCTION

It is one of the most important properties in economic analysis that the value of economic activities behaves continuously with respect to an exogenous parameter of the model. For example, the gross domestic product is often treated as continuous with respect to economic environments. This is because, usually, both the equilibrium economic activities and their prices can be assumed to be continuous. Although this is true in the finite-dimensional deterministic model, it does not hold in infinite-dimensional stochastic models such as the stochastic optimal growth model; even if both a state variable vector and its dual price vector are convergent in their respective spaces, their inner product (i.e., the value of a state variable) may not be convergent [Mas-Colell and Zame (1991)]. This difficulty has hampered various comparative dynamic analyses in stochastic models.

To overcome this difficulty, the present study constructs a stochastic optimal growth model in which the partial derivatives of the expected utility function lie in the same L^∞ space as that in which the state variables lie. In that case, the

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"inner product" between a dual vector (partial derivative) and a state variable may be continuous with respect to an exogenous parameter of the model.

For this purpose, we adopt the differential approach of Yano (1989), in which he transforms a stochastic optimal growth model into an infinite-dimensional deterministic model that is differentiable. Although his study is confined to local structure around the stochastic modified–golden rule state, we are concerned with the global property of a support price.

This study is related to Bewley (1972, 1981), Marimon (1989), and Evstigneev and Flåm (2002), in which the existence of a dual price vector in the L^1 space is demonstrated.[1] As is well known, however, the "inner product" (value) of a state-variable vector in L^∞ and a price vector in L^1 is not continuous on their product space, $L^\infty \times L^1$. In contrast, we demonstrate that the dual price of a state-variable vector lies in the L^∞ space, in which case the "inner product" of a state-variable vector and its price is continuous in $L^\infty \times L^\infty$, which makes it possible to perform comparative dynamics. This study is related also to Arkin and Evstigneev (1987), in which they derive a dual vector in L^1 by extending the Lagrangian method to an infinite-dimensional stochastic growth model.[2]

There has been an extensive literature on stochastic growth theory. Brock and Mirman (1972) show that there is a unique stationary distribution of allocations at each period. Mirman and Zilcha (1975) show the existence of the optimal policy function and the steady state stock using the theory of Markov processes. Stokey and Lucas (1989) and Hopenhayn and Prescott (1992), among others, prove similar existence theorems using stochastic monotonicity and monotone Markov processes.

The rest of the paper is organized as follows: In Section 2, we will provide mathematical preliminaries regarding to differentiation in Banach spaces. We review the stochastic growth model of Yano (1989) in Section 3. Section 4 is concerned with assumptions. We will investigate the validity of one of those assumptions in Section 5. After reviewing how the stochastic model is transformed into an equivalent deterministic model in Section 6, we will present the main theorem in Section 7. In Section 8, we give a brief discussion of our result.

2. MATHEMATICAL PRELIMINARIES AND NOTATION

Let $(H, \mathscr{H}, \pi)$ be a probability space and $\mathbf{R}^m$ the m-dimensional Euclidean space. We denote as $\mathscr{B}^m$ the usual Borel σ-algebra on $\mathbf{R}^m$. $L^\infty(H, \mathscr{H}; \mathbf{R}^m)$ denotes the set of $\mathscr{H}$-measurable functions that are bounded with probability 1. We treat two functions $f, g \in L^\infty(H, \mathscr{H}; \mathbf{R}^m)$ as equivalent if $f = g$ on some measurable set $A \in \mathscr{H}$ such that $\pi(A) = 1$.[3] $L^\infty(H, \mathscr{H}; \mathbf{R}^m)$ is a Banach space equipped with the norm

$$\|f\|_\infty := \inf_{A \in \mathscr{H} : \pi(A) = 1} \{\sup |f(x)| : x \in A\}.$$

Our approach is based on the fact that differentiation is defined for functions between Banach spaces $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$. We define the derivative of $f : D(f) \to W$ at the point $x \in D(f)$ as the bounded linear operator $\nabla f(x) : V \to W$ that satisfies

$$\lim_{y \to x,\ x \neq y} \frac{\|\{f(y) - f(x)\} - \nabla f(x)(y - x)\|_W}{\|y - x\|_V} = 0,$$

where $D(f) \subset V$ is the domain of f. The continuous differentiability of a derivative and the Lipschitz number are defined in a standard manner. We denote the partial derivatives of $f(x_1, x_2)$ with respect to the first and the second coordinates as $\nabla_1 f(x_1, x_2)$ and $\nabla_2 f(x_1, x_2)$, respectively.

3. STOCHASTIC ECONOMY

Here, we review the definitions and concepts used in Yano (1989).

3.1. States of Nature

We describe a state of nature by a bilaterally infinite stream of exogenously driven "economic conditions." The formal definitions are given as follows:

A set S consists of all possible economic conditions in each period. $\mathscr{S}$ denotes a σ-algebra on S, and hence $(S, \mathscr{S})$ is a measurable space. The history of economic conditions is described by an S-valued two-sided sequence; that is, the set of histories is $H := \prod_{n \in \mathbf{Z}} S_n$, where $S_n = S$ for every $n \in \mathbf{Z}$,[4] and a history (or state) is an element of H. Let $\mathscr{H}_n$ denote the σ-algebra generated by sets of the form $(\ldots, A_{n-1}, A_n, S, S, \ldots)$, where $A_k \in \mathscr{S}$ for all $k = n, n-1, \ldots$. Note that $\{\mathscr{H}_n\}_{n \in \mathbf{Z}_+}$ constructs a filtration of the measurable space $(H, \mathscr{H})$, where $\mathscr{H}$ is the σ-algebra generated by $\bigcup_{n=0}^{\infty} \mathscr{H}_n$.

Let π be a probability measure on $(H, \mathscr{H})$. With this measure, a history may be thought of as a realization of a stochastic process.[5] Thus, we regard economic conditions as driven by a stochastic process $h \in H$. Moreover, we restrict the analysis to the case where the associated probability distribution is stationary. In order to state this rigorously, let us denote the backward-shift operator $\sigma : H \to H : \{h_n\}_{n=-\infty}^{\infty} \mapsto \{h_{n+1}\}_{n=-\infty}^{\infty}$. We say that the stochastic environment is stationary [6] if $\pi(A) = \pi(\sigma A)$ for all $A \in \mathscr{H}$.

3.2. Capital Stocks

We consider m capital stocks. A program is a stochastic process $\{\kappa_n\}_{n \in \mathbf{Z}_+}$ such that κ_n is independent of $h_{n+1}, h_{n+2}, \ldots$ Note that for each $n \in \mathbf{Z}_+$, κ_n is an $\mathscr{H}_n$-measurable function on H into $\mathbf{R}^m$. Furthermore, we assume that they are essentially bounded; i.e., $\kappa_n \in L^{\infty}(H, \mathscr{H}_n; \mathbf{R}^m)$. A stationary program $\{\kappa_n\}_{n \in \mathbf{Z}_+}$ generated by stock $\gamma \in L^{\infty}(H, \mathscr{H}_0; \mathbf{R}^m)$ is a program that satisfies $\kappa_n(h) = \gamma(\sigma^n h)$ for almost every $h \in H$.

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3.3. Utility Functions

In each period and in each state, the economy is subject to some resource and technological constraints. Let us assume that such constraints can be described by sets $\Delta^h \subset \mathbf{R}_+^m \times \mathbf{R}_+^m$. Rigorously, in period 0 and in the state $h \in H$, $(x, y) \in \Delta^h$ means that the stock level y in the end of the period is achievable with the beginning-of-period stock x. If we assume time invariance of the technology, $\Delta^{\sigma^n h}$ can be regarded as the period-n constraint.

Assume that in each period, n, and in each economic state, h, the stock level $(x, y) \in \Delta^{\sigma^n h}$ has utility $v_n^h(x, y)$, where $v_n^h : \Delta^{\sigma^n h} \to \mathbf{R}$ $(n \in \mathbf{Z}_+,\ h \in H)$ are $\mathscr{B}_+^m \otimes \mathscr{B}_+^m$-measurable [7] with respect to (x, y) and $\mathscr{H}_n$-measurable with respect to h.

We say that a program $\{\kappa_n\}_{n \in \mathbf{Z}_+}$ is feasible from $\gamma \in L^\infty(H, \mathscr{H}_0; \mathbf{R}^m)$ if $\kappa_0 = \gamma$ and $(\kappa_{n-1}(h), \kappa_n(h)) \in \Delta^{\sigma^n h}$ for almost every $h \in H$ and for $n = 1, 2, \ldots$. A feasible program $\{\kappa_n\}_{n \in \mathbf{Z}_+}$ is said to be optimal if it maximizes the expected sum of utilities $E\left[\sum_{n=1}^\infty v_n^h(\zeta_{n-1}, \zeta_n)\right]$ over all feasible programs $\{\zeta_n\}_{n \in \mathbf{Z}_+}$ from γ. Here, $E[\cdot]$ is the expectation operator in terms of π.

3.4. Golden Rule States

The economy is said to be in a stochastic modified–golden rule state if it follows an optimal stationary program. If a stationary program maximizes periodwise expected utilities, we say that the economy is in a stochastic golden rule state.

In the rest of this paper, we extend the differential method of Yano (1989) into a global one. We will see that optimal paths are supported by dual vectors, which equips us with a tool for deriving a turnpike property [McKenzie (1976)]. Furthermore, the dual vectors are explicitly characterized by partial derivatives of the utility function. This property facilitates as application to various economic analyses.

4. ASSUMPTIONS

In this section we collect all the basic assumptions.

Assumption 1. The stochastic process that is the source of economic uncertainty is stationary on the probability space $(H, \mathscr{H}, \pi)$.

Assumption 2. For any $h \in H$, $\Delta^h \subset \mathbf{R}_+^m \times \mathbf{R}_+^m$ is closed and convex.

This assumption is standard and needs no explanation.

Assumption 3. The economy has quasi-stationary and strictly concave utility functions. That is, the utility $v_n^h(x, y) : \Delta^h \to \mathbf{R}$ in each period n and in each history h satisfies $v_n^h(x, y) = \rho^n v(x, y, h)$ for some $\rho \in (0, 1)$ and v, which is $\mathscr{B}^m \otimes \mathscr{B}^m \times \mathscr{H}_n$-measurable and strictly concave.

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Assumption 3 states that the utility function is additively separable. Its $\mathscr{H}_n$-measurability is a translation of the fact that the utility at period n is independent of the stock level at periods after n for an $\mathscr{H}_n$-measurable function that takes a value on $(\ldots, h_{n-1}, h_n, S, S, \ldots)$ for fixed $h_n, h_{n-1}, \ldots \in S$. The next assumption is rather technical, but it is required to ensure the existence of an expected utility.

Assumption 4. For any $(x, y) \in \mathbf{R}^m \times \mathbf{R}^m$ such that $(x, y) \in \Delta^h$ for almost every $h \in H$, $v(x, y, h)$ is integrable with respect to h.

Conditions similar to Assumptions 5–8 are usually imposed on a classical growth model. For a detailed discussion, see McKenzie (2002).

Assumption 5. For any $\alpha < \infty$, there is $\beta < \infty$ such that if $(x, y) \in \Delta^h$ and $|x| < \alpha$, then $|y| < \beta$.

Assumption 6. There are constants $\xi < \infty$ and $\gamma < 1$ such that if $|x| > \xi$ and $(x, y) \in \Delta^h$, then $|y| < \gamma |x|$.

Assumption 7. For any $(x, y) \in \Delta^h$, $x \leq z$ and $0 \leq w \leq y$ imply that $(z, w) \in \Delta^h$ and $v(x, y, h) \leq v(z, w, h)$.

Assumption 8. There exists a vector $(\tilde{x}, \tilde{y}) \in \mathbf{R}^m \times \mathbf{R}^m$ and $\tilde{\rho} \in (0, 1)$ such that $\tilde{\rho}\tilde{y} > \tilde{x}$ and $(\tilde{x}, \tilde{y}) \in \Delta^h$ for almost every h.

Smoothness assumptions are also imposed:

Assumption 9. The utility functions $v(\cdot, \cdot, h)$, $h \in H$ are continuous on Δ^h and continuously differentiable on the interior of Δ^h.

To derive existence and uniformity of derivatives of utility functions, we require an additional assumption. Let $\epsilon > 0$ and $v_\epsilon(\cdot, \cdot, h)$ be the restriction of $v(\cdot, \cdot, h)$ to the domain $\Delta^h_\epsilon := \{(x, y) | \exists \delta > \epsilon \text{ s.t. the } \delta\text{-neighborhood of } (x, y) \subset \Delta^h\}$.

Assumption 10. For every $\epsilon > 0$, $v_\epsilon(\cdot, \cdot, h)$ $(h \in H)$ are Lipschitz continuous. Furthermore, the associated Lipschitz numbers are uniformly bounded with respect to h.

5. AN EXAMPLE

Although Assumption 10 might appear stringent, it is relatively innocuous, as is shown here. Think of the economy summarized by the triple $(\{f^h\}_{h \in H}, u, \rho)$. To simplify the analysis, we confine ourselves to a one-good model. Thus, the state-contingent production functions $\{f^h\}$ $(h \in H)$ map $\mathbf{R}_+$ to $\mathbf{R}_+$ with $f^h(0) = 0$. They are strictly concave and satisfy Inada conditions. The (state-independent) utility function u is assumed to hold similar properties. Suppose also that f^h $(h \in H)$ and u are twice continuously differentiable.

The representative agent's problem is to maximize the expectation of the discounted sum of utilities from consumption under the production constraint. That

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is,

$$\max_{\{c_n\}_{n=0}^{\infty}} E\left[\sum_{n=0}^{\infty} \rho^{n-1} u(c_n(h))\right] \quad \text{subject to} \quad c_n(h) + k_n(h) \leq f^h\left(k_{n-1}(h)\right), \text{ a.e.,}$$

where $c_n(h)$ and $k_n(h)$ respectively denote the period-n consumption and the capital stock under the state h.

Assume that the production functions are bounded from above.

Assumption 11. There is a function g such that $g(x) \geq f^h(x)$ for all $x \in \mathbf{R}_+$ and $h \in H$.

The reduced-form utility functions are

$$v(x, y, h) := u(f^h(x) - y), \quad h \in H.$$

The feasible regions are $0 \leq y \leq f^h(x)$. Hence, the agent's utility maximization boils down to the following problem:

$$\max_{\{(k_n, k_{n+1})\}_{n=0}^{\infty}} E\left[\sum_{n=0}^{\infty} \rho^{n-1} v\left(k_n(h), k_{n+1}(h), h\right)\right]$$

$$\text{subject to} \quad (k_n(h), k_{n+1}(h)) \in \Delta^h, \text{ a.e.,}$$

where $\Delta^h := \{(x, y) \mid 0 \leq y \leq f^h(x)\}$.

The proposition we will show is the following:

PROPOSITION 1. *In this model, Assumption 11 implies Assumption 10.*

Note the following fact:

LEMMA 1. *Let f, g be continuously differentiable functions $\mathbf{R}_+ \to \mathbf{R}_+$ that satisfy $f \geq g$ on $\mathbf{R}_+$ and $f(0) = g(0)$. Then, for every $a > 0$, there exists some $\epsilon \in (0, a)$ such that $f'(\epsilon) \geq g'(\epsilon)$.*

Proof of Proposition 1. By the definition of $v(\cdot, \cdot, h)$, the slope of the utility function has the highest value on the edge of Δ^h that is determined by f^h. Let us define Δ^h_ϵ as the set of interior points in Δ^h that is apart from the edge with distance ϵ.

Pick any $(\tilde{x}, \tilde{y}), (\bar{x}, \bar{y}) \in \Delta^h_\epsilon$. We have

$$\begin{aligned}
|u(f^h(\tilde{x}) - \tilde{y}) - u(f^h(\bar{x}) - \bar{y})| &\leq u'(\epsilon)|(f^h(\tilde{x}) - \tilde{y}) - (f^h(\bar{x}) - \bar{y})| \\
&\leq u'(\epsilon)[|f^h(\tilde{x}) - f^h(\bar{x})| + |\tilde{y} - \bar{y}|] \\
&\leq u'(\epsilon)[(f^h)'(\epsilon)|\tilde{x} - \bar{x}| + |\tilde{y} - \bar{y}|].
\end{aligned}$$

The first and the last inequality follow from the fact that u and f^h are concave functions. By Lemma 1, there exists an $\epsilon' \in (0, \epsilon)$ such that $(f^h)'(\epsilon') \leq g'(\epsilon')$.

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Thus,

$$
\begin{aligned}
&u'(\epsilon')\{(f^h)'(\epsilon')\,|\tilde{x}-\bar{x}|+|\tilde{y}-\bar{y}|\} \\
&\quad \le u'(\epsilon')\cdot\max\{g'(\epsilon'),1\}\cdot(|\tilde{x}-\bar{x}|+|\tilde{y}-\bar{y}|) \\
&\quad \le \sqrt{2}u'(\epsilon')\cdot\max\{g'(\epsilon'),1\}\cdot|(\tilde{x},\tilde{y})-(\bar{x},\bar{y})|\,.
\end{aligned}
$$

■

As this lemma shows, Assumption 10 can be guaranteed by Assumption 11, which is economically interpretable in a straightforward fashion.

6. EQUIVALENT DETERMINISTIC ECONOMY

Our stochastic growth model can be regarded as an infinite-dimensional deterministic growth model. The domain of the latter's utility function is

$$
\begin{aligned}
\mathscr{D} := \{&(\xi,\eta)\in L^\infty(H,\mathscr{H}_0;\mathbf{R}^m) \\
&\times L^\infty(H,\mathscr{H}_0;\mathbf{R}^m)\mid \big(\xi(\sigma^{-1}h),\eta(h)\big)\in\Delta^h,\ \text{a.e. } h\}.
\end{aligned}
$$

The utility function $u:\mathscr{D}\to\mathbf{R}$ is defined by

$$
u(\xi,\eta) := E[v(\xi(\sigma^{-1}h),\eta(h),h)], \tag{1}
$$

where $E[\cdot]$ is the expectation operator with respect to $(H,\mathscr{H},\pi)$. Note here that u is well defined on $\mathscr{D}$. Because $\xi\in L^\infty(H,\mathscr{H}_0;\mathbf{R}^m)$, there is some $x\in\mathbf{R}^m$ such that $\xi(h)\le x$ a.e. h. Hence, $0\le v(\xi(\sigma^{-1}h),\eta(h),h)\le v(x,0,h)$, a.e. h by Assumption 7. By Assumption 4, the right-hand side of (1) is finite. Convexity of the feasible sets is also inherited to the infinite-dimensional model.

LEMMA 2. *$\mathscr{D}$ is a convex subset of $L^\infty(H,\mathscr{H}_0;\mathbf{R}^m)\times L^\infty(H,\mathscr{H}_0;\mathbf{R}^m)$.*

Our analysis is based on the facts that are shown in Yano (1989).

LEMMA 3 [Yano (1989, Lemma 1)]. *The deterministic model satisfies the following:*

1. *The utility function u is a concave function from $\mathscr{D}$ into $\mathbf{R}$.*
2. *For any $\alpha<\infty$, if $(\xi,\eta)\in\mathscr{D}$ and $\|\xi\|_\infty<\alpha$, there is $\beta<\infty$ such that $\|\eta\|_\infty<\beta$.*
3. *There are constants $\alpha<\infty$ and $\gamma<1$ such that $\|\xi\|_\infty>\alpha$ implies $\|\eta\|_\infty<\gamma\|\xi\|_\infty$ for any $(\xi,\eta)\in\mathscr{D}$.*
4. *If $(\xi,\eta)\in\mathscr{D}$, $\xi\le\zeta$ and $0\le\omega\le\eta$ imply $(\zeta,\omega)\in\mathscr{D}$ and $u(\xi,\eta)\le u(\zeta,\omega)$.*[8]
5. *There are $(\bar{\xi},\bar{\eta})\in\mathscr{D}$ and $\bar{\rho}<1$ such that $\bar{\rho}\bar{\eta}\ge\bar{\xi}+\epsilon 1_{L^\infty}$ for some $\epsilon>0$, where 1_{L^∞} is a function $h\mapsto(1,\ldots,1)\in\mathbf{R}^m$, which is in $L^\infty(H,\mathscr{H}_0;\mathbf{R}^m)$.*

This lemma implies that the infinite-dimensional deterministic model satisfies standard assumptions in a growth model.

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To make this relationship clear, it is convenient to represent the stock levels in terms of the present $t = 0$. So, for a given program κ, $\hat{\kappa}$ denotes the associated path in a deterministic model; i.e., $\hat{\kappa}_n(h) = \kappa(\sigma^{-n}h)$.

PROPOSITION 2 [Yano (1989, Theorem 1)]. *The stochastic model and the associated deterministic model are equivalent in the following sense:*

(1) $\kappa_n \in L^\infty(H, \mathscr{H}_n; \mathbf{R}^m)$ *for* $n \in \mathbf{Z}_+$ *iff* $\hat{\kappa}_n \in L^\infty(H, \mathscr{H}_0; \mathbf{R}^m)$ *for* $n \in \mathbf{Z}_+$*;*
(2) $\{\kappa_n\}$ *is feasible iff* $\{\hat{\kappa}_n\}$ *is feasible;*
(3) $\{\kappa_n\}$ *is stationary iff* $\{\hat{\kappa}_n\}$ *is stationary;*
(4) $\{\kappa_n\}$ *is optimal iff* $\{\hat{\kappa}_n\}$ *is optimal; and*
(5) *Stock* $k \in L^\infty(H, \mathscr{H}_0; \mathbf{R}^m)$ *generates a stochastic modified–golden rule state in the stochastic model iff it generates a deterministic modified–golden rule state in the deterministic model.*

7. SUPPORT PRICES IN L^∞ SPACE

By Assumption 9, each $v(\cdot, \cdot, h)$ is continuously differentiable. So, for each $(\xi, \eta) \in L^\infty(H, \mathscr{H}_0; \mathbf{R}^m) \times L^\infty(H, \mathscr{H}_0; \mathbf{R}^m)$, let us define

$$u_1(\xi, \eta)(h) := E[v_1(\xi(h), \eta(\sigma h), \sigma h)|\mathscr{H}_0], \tag{2}$$

$$u_2(\xi, \eta)(h) := v_2(\xi(\sigma^{-1}h), \eta(h), h).$$

PROPOSITION 3. *For any* $(\xi, \eta) \in L^\infty(H, \mathscr{H}_0; \mathbf{R}^m) \times L^\infty(H, \mathscr{H}_0; \mathbf{R}^m)$, $u_1(\xi, \eta)$ *and* $u_2(\xi, \eta)$ *are well-defined on* H *into* $\mathbf{R}^m$ *and in* $L^\infty(H, \mathscr{H}_0; \mathbf{R}^m)$.

Proof. Let $i = 1$ or $i = 2$. By Assumptions 9 and 10, the norm of the partial derivative v_i is bounded by a Lipschitz number independent of h. Hence, $v_i(\xi \circ \sigma^{-1}(\cdot), \eta(\cdot), \cdot)$ is an $L^\infty(H, \mathscr{H}_0; \mathbf{R}^m)$ function, and this immediately implies that $u_2(\xi, \eta) \in L^\infty(H, \mathscr{H}_0; \mathbf{R}^m)$. Because $L^\infty \subset L^1$, v_1 is integrable. This implies that u_1 is well defined. Together with its uniform boundedness, we obtain $u_1(\xi, \eta) \in L^\infty(H, \mathscr{H}_0; \mathbf{R}^m)$. ■

By Proposition 3, $u_1(\xi, \eta)$ and $u_2(\xi, \eta)$ may be considered as bounded linear functionals on $L^\infty(H, \mathscr{H}_0; \mathbf{R}^m)$. For $\zeta \in L^\infty(H, \mathscr{H}_0; \mathbf{R}^m)$, let us define the bilinear forms

$$\langle \zeta, u_i(\xi, \eta) \rangle := \int_H u_i(\xi, \eta)(h) \cdot \zeta(h) d\pi(h), \quad i = 1, 2.$$

LEMMA 4. *For every* $(\xi, \eta) \in \operatorname{int} \mathscr{D}$*, there exists a neighborhood* $\mathscr{V} \subset \mathscr{D}$ *of* (ξ, η) *such that there is an* $\alpha > 0$ *that satisfies that*

$$\begin{aligned} &\left| v\left(\xi'(\sigma^{-1}h), \eta'(h), h\right) - v\left(\xi''(\sigma^{-1}h), \eta''(h), h\right) \right| \\ &\quad \leq \alpha \left| \left(\xi'(\sigma^{-1}h), \eta'(h)\right) - \left(\xi''(\sigma^{-1}h), \eta''(h)\right) \right| \end{aligned} \tag{3}$$

for almost every $h \in H$ *whenever* (ξ', η'), $(\xi'', \eta'') \in \mathscr{V}$.

Proof. Pick $(\xi, \eta) \in \operatorname{int} \mathscr{D}$. There is an $\epsilon > 0$ such that $\|(\xi, \eta) - (\xi', \eta')\|_\infty < \epsilon$ for every $(\xi', \eta') \in \operatorname{int} \mathscr{D}$. This implies that $|(\xi(\sigma^{-1}h), \eta(h)) - (\xi'(\sigma^{-1}h), \eta'(h))| < \epsilon$ for almost every $h \in H$, and hence $(\xi(\sigma^{-1}h), \eta(h)) \in \Delta_\epsilon^h$. Let $\mathscr{V} := \{(\xi', \eta') \in \mathscr{D} | (\xi'(\sigma^{-1}h), \eta'(h) \in \Delta_\epsilon^h\}$. This is a neighborhood of (ξ, η), and inequality (3) immediately follows from Assumption 10. ■

LEMMA 5. *Function* $u : \mathscr{D} \to \mathbf{R}$ *is differentiable on the interior of* $\mathscr{D}$. *The derivative of* $u(\xi, \eta)$, $\nabla u(\xi, \eta)$, *is a bounded linear functional on* $L^\infty(H, \mathscr{H}_0; \mathbf{R}^m) \times L^\infty(H, \mathscr{H}_0; \mathbf{R}^m)$ *satisfying*

$$\nabla u(\xi, \eta)(\zeta, \omega) = \langle \zeta, u_1(\xi, \eta) \rangle + \langle \omega, u_2(\xi, \eta) \rangle,$$

where $(\zeta, \omega) \in L^\infty(H, \mathscr{H}_0; \mathbf{R}^m) \times L^\infty(H, \mathscr{H}_0; \mathbf{R}^m)$.

Proof.

$$\begin{aligned}
&O(\Delta\xi, \Delta\eta)\\
&\quad := |u(\xi + \Delta\xi, \eta + \Delta\eta) - u(\xi, \eta) - \{\langle \Delta\xi, u_1(\xi, \eta) \rangle + \langle \Delta\eta, u_2(\xi, \eta) \rangle\}|\\
&\quad = |E[v(\xi(\sigma^{-1}h) + \Delta\xi(\sigma^{-1}h), \eta(h) + \Delta\eta(h), h)]\\
&\quad - E[v(\xi(\sigma^{-1}h), \eta(h), h)]\\
&\quad - E[E\,[v_1(\xi(h), \eta(\sigma h), \sigma h)|\mathscr{H}_0]\,(h) \cdot \Delta\xi(h)]\\
&\quad - E[v_2(\xi(\sigma^{-1}h), \eta(h), h) \cdot \Delta\eta(h)]|\\
&\quad = |E[v(\xi(\sigma^{-1}h) + \Delta\xi(\sigma^{-1}h), \eta(h) + \Delta\eta(h), h) - v(\xi(\sigma^{-1}h), \eta(h), h)\\
&\quad - v_1(\xi(\sigma^{-1}h), \eta(h), h) \cdot \Delta\xi(\sigma^{-1}h) - v_2(\xi(\sigma^{-1}h), \eta(h), h) \cdot \Delta\eta(h)]|\\
&\quad \le E[|v(\xi(\sigma^{-1}h) + \Delta\xi(\sigma^{-1}h), \eta(h) + \Delta\eta(h), h) - v(\xi(\sigma^{-1}h), \eta(h), h)\\
&\quad - v_1(\xi(\sigma^{-1}h), \eta(h), h) \cdot \Delta\xi(\sigma^{-1}h) - v_2(\xi(\sigma^{-1}h), \eta(h), h) \cdot \Delta\eta(h)|]\\
&\quad =: E[O^h(\Delta\xi, \Delta\eta)].
\end{aligned}$$

Hence, we obtain

$$0 \le \frac{O(\Delta\xi, \Delta\eta)}{\|(\Delta\xi, \Delta\eta)\|_\infty} \le \frac{E[O^h(\Delta\xi, \Delta\eta)]}{\|(\Delta\xi, \Delta\eta)\|_\infty} \le E\left[\frac{O^h(\Delta\xi, \Delta\eta)}{|(\Delta\xi(\sigma^{-1}h), \Delta\eta(h))|}\right].$$

If $(\Delta\xi, \Delta\eta)$ lies in a sufficiently small neighborhood of $(0, 0)$, then $O^h/|(\Delta\xi(\sigma^{-1}h), \Delta\eta(h))|$ $(h \in H)$ are uniformly bounded by Lemma 4. By Assumption 9, $O^h/|(\Delta\xi(\sigma^{-1}h), \Delta\eta(h))| \to 0$ as $|(\Delta\xi(\sigma^{-1}h), \Delta\eta(h))| \to 0$. Applying the bounded convergence theorem, we have

$$\lim_{(\Delta\xi, \Delta\eta)\to 0} \frac{O(\Delta\xi, \Delta\eta)}{\|(\Delta\xi, \Delta\eta)\|_\infty} \le E\left[\lim_{(\Delta\xi, \Delta\eta)\to 0} \frac{O^h(\Delta\xi, \Delta\eta)}{|(\Delta\xi(\sigma^{-1}h), \Delta\eta(h))|}\right] = 0.$$

■

The last assumption we impose is as follows:

Assumption 12. For any optimal path $k = \{k_n\}_{n=0}^{\infty}$, k_n lies in the interior of $\mathscr{D}$ for $n = 0, 1, \ldots$.

The following theorem is the main result.

THEOREM 1. *An optimal path* $k = \{k_n\}_{n=0}^{\infty}$ *is supported by a sequence of dual vectors that are in* $L^{\infty}(H, \mathscr{H}_0; \mathbf{R}^m)$.

Proof. By Lemma 5 and the concavity of the problem, we have

$$u(k_n, k_{n+1}) + \langle k_n, u_1(k_n, k_{n+1})\rangle + \langle k_{n+1}, u_2(k_n, k_{n+1})\rangle$$
$$\geq u(\xi, \eta) + \langle \xi, u_1(k_n, k_{n+1})\rangle + \langle \eta, u_2(k_n, k_{n+1})\rangle, \quad n = 1, 2, \ldots,$$

for any $(\xi, \eta) \in \mathscr{D}$. Considering the Euler equations [9]

$$u_2(k_{n-1}, k_n) + \rho u_1(k_n, k_{n+1}) = 0, \quad n = 1, 2, \ldots,$$

define $q_n^k := u_2(k_{n-1}, k_n), \quad n = 1, 2, \ldots$. Then we obtain the desired result:

$$u(k_n, k_{n+1}) + \langle k_{n+1}, q_{n+1}^k\rangle - \rho^{-1}\langle k_n, q_n^k\rangle \geq u(\xi, \eta) + \langle \eta, q_{n+1}^k\rangle - \rho^{-1}\langle \xi, q_n^k\rangle.$$

The fact that $q_n^k \in L^{\infty}(H, \mathscr{H}_0; \mathbf{R}^m)$ is the result of Proposition 3. ■

8. DISCUSSION

Theorem 1 guarantees the joint continuity of a value with respect to a quantity vector and its support price vector. Many stochastic models do not satisfy this joint continuity, on which the standard economic analysis is based. Theorem 1 makes it possible to overcome this difficulty.

Under our assumptions, the pricing formula is given by the following duality between L^{∞} and itself:

$$\langle k, q\rangle = \int q(h)k(h)d\pi,$$

where the convergence in the spaces is defined by

$$k^s \to k :\Longleftrightarrow \langle k^s, q\rangle \to \langle k, q\rangle \quad \text{for all } q \in L^{\infty},$$
$$q^s \to q :\Longleftrightarrow \langle k, q^s\rangle \to \langle k, q\rangle \quad \text{for all } k \in L^{\infty}.$$

In this case, the joint continuity of the bilinear form $\langle k, q\rangle$ is readily proved as follows. Given the convergence of equilibrium capital $k_n^s \to k_n$, by (2), we obtain the convergence of price

$$q_n^s = u_2\left(k_{n-1}^s, k_n^s\right) \to u_2(k_{n-1}, k_n) =: q_n.$$

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By Theorem 1, we have

$$\begin{aligned}\left|\left\langle k_n^s, q_n^s\right\rangle - \langle k_n, q_n\rangle\right| &\leq \left|\left\langle k_n^s, q_n^s\right\rangle - \left\langle k_n^s, q_n\right\rangle\right| + \left|\left\langle k_n^s, q_n\right\rangle - \langle k_n, q_n\rangle\right| \\ &\leq \left\|k_n^s\right\| \cdot \left|\left\langle 1, q_n^s\right\rangle - \langle 1, q_n\rangle\right| + \left|\left\langle k_n^s, q_n\right\rangle - \langle k_n, q_n\rangle\right| \\ &\to 0.\end{aligned}$$

Unlike the standard construction in infinite-dimensional economic models, the structure of our price space ensures the continuity of a value with respect to parameters such as the discount rate, which makes it possible to conduct comparative dynamics with respect to a value of economic activities. A potential application of this result includes a proof of turnpike properties in a stochastic dynamic economy.

NOTES

1. As Bewley (1972) points out, the mathematically natural space for price vectors is the dual space of L^∞. The studies mentioned provide sufficient conditions for price vectors to lie in L^1, which is a subspace of the dual space of L^∞.

2. In deterministic optimal growth theory, the existence of a dual vector is proved in Weitzman (1973) and McKenzie (1976, 1983) for the discrete-time case, and in Benveniste and Scheinkman (1982) for the continuous-time case. Because their models deal with finite-dimensional commodity spaces, the discontinuity problem in the value of a commodity vector such as that of this study does not appear.

3. Sometimes, we just say $f = g$ instead of $f = g$ almost everywhere (a.e.) or almost surely (a.s.).

4. $\mathbf{Z}$ is the set of all integers and $\mathbf{Z}_+$ the set of all nonnegative integers.

5. More precisely, if we identify the coordinate process $W_n : \{h_k\}_{k=-\infty}^{\infty} \mapsto h_n$ $(n \in \mathbf{Z})$ with h itself, we can consider the history as a realization of a stochastic process on $(H, \mathscr{H}, \pi)$.

6. In a normal sense, stationarity is defined as $\pi(W^{-1}(A)) = \pi(W^{-1}(\sigma A))$. In our case, because $W^{-1}(A) = A$, the definition is valid.

7. $\mathscr{B}_+^m$ is the Borel σ-algebra of $\mathbf{R}_+^m$ and $\mathscr{B}_+^m \otimes \mathscr{B}_+^m$ is the σ-algebra generated by the product $\mathscr{B}_+^m \times \mathscr{B}_+^m$.

8. For $\xi, \eta \in L^\infty(H, \mathscr{H}_0; \mathbf{R}^m)$, $\xi \geq \eta \Leftrightarrow \xi(h) \geq \eta(h)$ a.e.

9. An optimal path satisfies the Euler equations even when we apply Banach-space differential analysis to models with an infinite-dimensional commodity space.

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