B.A.J. **9,** II, 427-455 (2003)

OPERATION OF THE CAPITAL MARKET UNDER ENTITY SPECIFIC PRICING

By T. C. Jenkins

ABSTRACT

The paper introduces a setting in which each business in an economy faces considerable uncertainty about the cost of providing its service and therefore needs risk capital to give consumers confidence in its solvency. The paper then explores the operation of the capital market under conditions where businesses charge for this service dependability by setting an entity specific profit margin.

This entity specific profit margin is associated with an optimal, arbitrage-free investment portfolio for investors and, as they optimise their portfolios, generates a market equilibrium. Under these conditions it is shown that the entity specific risk premium for investors is the same as the premium for non-diversifiable risk relative to the market portfolio. This reconciles an entity specific approach with the capital asset pricing model, and in this setting would allow the use of the former without having to explicitly take the latter into consideration.

A consequence of this is that entity specific pricing and valuation are fair in such an setting and, for example, the fair value accounting of insurance liabilities can be performed on an entity specific basis.

KEYWORDS

Expected Utility; Entity Specific Pricing; Optimal Portfolio; Market Equilibrium; Capital Asset Pricing Model (CAPM); State Price Deflator; Fair Value

CONTACT ADDRESS

T. C. Jenkins, F.I.A., F.I.A.A., 6 Barrie Place, Davidson, NSW 2085, Australia. Tel: +61(2) 9451-0028; E-mail: tim.jenkins@au.pwc.com

1. Introduction

1.1 *Purpose of the Paper*

1.1.1 The purpose of this paper is to examine the operation of the capital market in a competitive economy in which there is considerable profit uncertainty (such as the uncertainty that an insurer has about the level of claims it will experience, or that a telecom provider has about the cost of providing a reliable network), and in which business entities charge on a basis that allows for the risk generated by this uncertainty. This charge is by way of a planned profit margin built into the price of the service. The paper is motivated by a spirit of enquiry into the fair value of business liabilities in such an environment.

https://doi.org/10.1017/S1357321700004232 Published online by Cambridge University Press

1.1.2 A service is taken to be purchased at the start of a period and performed at the end of it. A business entity is taken to be the provider of a single type of service, and to charge for the risk in its service on a basis that is *entity specific*. It is assumed that consumers are prepared to pay for entity specific risk as part of the overall cost of a service if, by doing so, they protect themselves from the cost to them of business failure, and thereby improve the expected utility of their future consumption. As part of their consideration of the service, consumers therefore require the provider to be dependably solvent. As a result, each business entity needs capital from investors in order to be confident of being in business to perform its service in the face of uncertain outcomes, such as variable claims for the insurer or network costs for the telecom provider. Investors provide capital support at the start of the period only to the extent that, by doing so, they can expect a payoff that improves the expected utility of their consumption at the end of it. Competitive forces function in the usual fashion, except that competition is focused, not only on reducing the underlying cost of a service, but also on demonstrating the dependability of the promise to perform it.

1.1.3 In offering to provide its service, a business entity is making a promise to perform it at a certain cost to the consumer, but an uncertain cost to it. In this paper the promise and the cost of meeting it is called an *obligation*. In the case of the insurer, the uncertain cost of meeting the outstanding claims liability is an obligation, as is the uncertain cost that the telecom provider will incur in running a reliable network. The cost of meeting an obligation is also referred to as a *liability*.

1.1.4 By introducing the obligations that a business entity takes on by offering a service, the paper goes a step further than is usual in a study of the capital market. It does this by tracing the connection between:

— the uncertain cost outcomes that a business entity faces in the course of fulfilling the obligations inherent in its service;
— the price that it charges consumers for the service, including the profit loading for the entity specific risk that springs from this uncertainty; and
— the resulting payoff for its investors.

1.2 *Approach*

1.2.1 To pursue its purpose, the paper addresses preferences between different consumption plans in the face of uncertainty. This is done using a von Neumann-Morgenstern *expected utility* representation (von Neumann & Morgenstern, 1947), in which one consumption plan is preferred over another if its expected utility under a given utility function is greater than that of the other. The shape of the utility function determines the consumer's attitude to risk, with a concave utility function indicating *risk aversion*. The accepted measure of concavity of a utility function, and hence of risk aversion, is known as the Arrow-Pratt coefficient of risk aversion (Pratt,

https://doi.org/10.1017/S1357321700004232 Published online by Cambridge University Press

1964; Arrow, 1965). The reciprocal of the coefficient of risk aversion is called the *risk tolerance.*

1.2.2 The paper begins by applying this framework of expected utility and the risk aversion of the associated utility function in a simplified context, in which a business entity is viewed in isolation and investors all have the same risk aversion. The business charges consumers of its service a profit margin in order to raise the capital needed for consumers to have confidence that it will perform. This charge or profit margin is set so as to allow investors to maximise the expected utility of their future consumption by investing the exact amount of capital that is required. In effect, businesses act as agents for their investors, with a common utility function that represents the weighted average of theirs. We call this charge or profit margin the *entity specific price* for risk. We derive this entity specific price on the assumption of normally distributed business obligations and a utility function, which is a variant of the negative exponential utility function, in which risk aversion is inversely proportional to existing wealth of an investor.

1.2.3 The paper then explores the entity specific price for risk in an economy with different types of business entity with different variability in investor return, and different types of investor with different levels of risk aversion. Consumers demand a supply of dependable services, despite the inherent uncertainty faced by providers, and investors are intent upon constructing optimal portfolios in accordance with their own individual risk aversions. Investors are motivated to support businesses because, by providing their capital, they are able to construct *optimal portfolios* of investments, namely portfolios that maximise the expected utility of their end of period consumption. As a result of the interaction between the demand by consumers for services, the supply of capital available from investors to support those services, and the general expectation of consistent pricing for entity specific risk, the capital market finds an *equilibrium* at a *uniform market-driven coefficient of risk aversion* across the economy. This, in turn, leads to an *entity specific pricing model* of share market returns, and to the result that the entity specific risk premium for investors is the same as the premium for non-diversifiable risk relative to the market equilibrium portfolio.

1.3 *Comparison with CAPM*

1.3.1 The best known model of capital market asset prices is the capital asset pricing model (Sharpe, 1964). The CAPM assumes a competitive and efficient capital market in which investors are mean-variance averse — that is that, when presented with two securities with the same expected return, investors prefer the one with the lower variance of returns. The CAPM makes no assumption as to how the overall market of securities comes about other than through competitive efficiency, but it does, in effect, assume that investors can hold securities in their portfolios in the same proportion as in the overall market, so that, when the effect on consumption of investing existing wealth is abstracted from other consumption effects, the efficient market portfolio is the starting point.

1.3.2 It was noted above that the CAPM obtains in equilibrium when investors have mean-variance preferences. Mean-variance preference is axiomatic when payoffs are normally distributed and when the utility function is von Neumann–Morgenstern in form, which admits a range of types of risk aversion. With mean-variance preferences, the equilibrium market return lies on the mean-variance frontier (LeRoy & Werner, 2001), which is the frontier formed by the lowest variance for any given price and expected payoff. This in turn leads (LeRoy & Werner, 2001) to security prices lying on the celebrated *security market line*, and to the fundamental result that investors require a reward in excess of that on the risk-free asset only to the extent of non-diversifiable risk, that is to the extent that the risk involved in investing in a particular security is not diversified away when it is part of an investment in the market portfolio.

1.3.3 Under the entity specific pricing model in this paper, it is assumed that consumers, businesses and investors all expect that entity specific total risk should be rewarded on a consistent market-driven basis, and behave accordingly. This has the effect that, under the forces of supply of capital, demand for services and the construction of optimal portfolios by investors a specific equilibrium distribution of business is formed. The essential difference between the entity specific model and the CAPM, therefore, is that, by taking the extra step of looking at the price for risk in the obligations of a business and then relating it to the payoff for the investor, the paper has introduced an additional assumption (of entity specific pricing), and this has generated a specific market equilibrium portfolio. It is shown that, with this particular distribution of businesses in the economy, the price of entity specific total risk is the same as the price of non-diversifiable risk relative to the resulting market portfolio, thereby reconciling the model in the paper with the CAPM.

1.3.4 This happens because the entity specific model in the paper also has investors with mean-variance preferences, and the market return in equilibrium therefore likewise lies on the mean-variance frontier. Furthermore, in the entity specific model the mean return in excess of the risk-free rate for each security is proportional to the variance of the return (with the uniform market driven coefficient of risk aversion being the constant of proportionality), so that security returns and hence portfolio returns, including that of the market portfolio, all lie along a straight line, starting with the risk-free return. This line is, by definition, one and the same as the security market line. The entity specific model, therefore, is a special case of the CAPM, in which the market portfolio is determined by entity specific pricing rather than being simply taken as given, and in which the CAPM security market line can be described either in the CAPM way or in our entity specific way.

1.3.5 This reconciliation between the entity specific approach to pricing and valuation and the CAPM also leads to the conclusion that, in the assumed setting at least, fair pricing and fair value accounting of obligations, such as insurance liabilities, is entity specific. If the setting were an appropriate representation of the context for insurance obligations and the like, this result would have important implications for insurance premium rating and for fair value accounting under International Financial Reporting Standards.

2. The Setting

2.1 *The Economic Setting*

2.1.1 As outlined in the previous section, the paper assumes an economic setting in which all business activity is subject to considerable uncertainty

2.1.2 We assume that competing single service businesses of broadly similar scale provide consumers with a wide range of services, but that the eventual cost outcome of fulfilling the obligations associated with each service is uncertain, with business cost outcomes being assumed to be normally distributed. Because of this, capital support is required from investors in order that the service can be provided to consumers in a dependable fashion. Each investor's utility function is assumed to be a variant of the negative exponential utility function, in which an investor's risk aversion is inversely proportional to his or her initial wealth. Different investors may, however, have different intrinsic risk aversions and levels of initial wealth. Investors expect to be rewarded on a consistent basis for the risk in their investment and consumers expect to pay accordingly, provided that they can be confident that the promise to perform the service is dependable. Investors will not provide capital support unless, by doing so, they maximise the expected utility of their future consumption, and consumers will not buy the service unless it is dependable. Businesses act as agents for their investors, with a common utility function and a uniform risk aversion that represents theirs. Businesses compete, not only on the expected cost of their service, but also on reducing the uncertainty that they have to deal with in providing their service, thereby reducing capital support for the same level of dependability, and ultimately reducing the price of the service.

2.1.3 The aggregate capital requirement, the supply of capital (and hence of services) and the demand for services in this setting are assumed to be kept in equilibrium through variations in the uniform price of risk across the economy or in the extent of capital support (and hence dependability) required. Thus, for example, if there is sufficient capital in the economy to support aggregate activity, but it is not being subscribed by investors, the risk

https://doi.org/10.1017/S1357321700004232 Published online by Cambridge University Press

aversion allowed for in the price of risk is too low, and will rise until a balance is found. If there is a surplus supply of capital, the excess will be invested in the risk-free asset. If, on the other hand, the aggregate supply of capital is inadequate to meet the required extent of capital support, the standard of service dependability demanded by consumers will fall across the economy until a balance is found. Finally, if particular services would otherwise be under or over-supplied in the economy relative to the demand for them, the dependability required by consumers will adjust on a service-by-service basis, and prices will respond until supply and demand for each service are in balance.

2.1.4 Consumers are taken to have similar risk aversions as investors. They need certain services, such as insurance, that provide financial protection against contingent events that would otherwise have a severe impact on their future consumption. In the setting of the paper, it is also assumed that the failure of any service would be material, so that consumers need all services to be dependable and are prepared to pay accordingly.

2.1.5 Imputed tax credits are assumed to allow a choice between investing in the risk-free asset and investing in a business to be considered on a basis that is gross of taxation. Government debt is assumed to be the risk-free asset. There is assumed to be a single investment period. Supplies needed to perform a service are assumed to be paid for at the end of this period.

2.2 *Definitions*

2.2.1 In the course of the paper the general case of a number of businesses which, between them, need to raise capital from a number of investors, will be considered. Initially, however, the paper considers the case of one business and one investor.

2.2.2 Because the paper starts by considering the service provided by a business to consumers, there are a number of definitions that need to be expressed per unit expected obligation (using some monetary unit), where, as discussed earlier, an obligation is defined to be the actual cost of providing the service, where, at the outset, that cost is uncertain. Accordingly, let:

w be the risk capital required at the outset per unit expected obligation. This parameter connects a liability outcome to an investor payoff. As the paper progresses, it will allow us to develop a capital asset pricing model expressed in terms of underlying business outcomes.

L be the random variable representing the actual cost of providing the service at the end of the period or, in other words, of discharging the liability. L is assumed to be normally distributed with mean 1 and standard deviation σ.

w_1 be the payoff per unit expected obligation for an investor at the end of the period from investing w at the outset.

p be the planned profit margin per unit expected obligation. This is the charge that a business makes in order to attract the required risk capital w.

P be the price per unit expected obligation for the service, including the charge for risk.

2.2.3 Moving to an investor's perspective, let:

r be the rate of return from the risk-free asset over the period;

W be the initial wealth of the investor;

c be the end of period consumption arising from a unit of wealth (after abstracting this from other consumption effects); and

$v(W, c)$ be the utility function of the investor's consumption at the end of the period in each of the probability states that the actual cost of providing the service can take.

The paper assumes that the utility function is a variant of the negative exponential, namely we assume that $v(W, c) = -e^{-\alpha(c+W-1)}$ for $\alpha > 0$ (where α is constant). This function is a von Neumann-Morgenstern utility function, and while the utility of an investor with a unit of initial wealth and a payoff of W has in this representation been arbitrarily set to be the same as that of an investor with initial wealth W and a payoff of W, utility could be calibrated to any other point. The utility function can be expressed as $e^{-\alpha(W-1)}(-e^{-\alpha c})$, where the first term is not stochastic and does not affect the maximisation of the expected utility of future consumption, and the second term is the negative exponential utility function for a unit of initial wealth. In looking at an investor's preferences, the paper will therefore abstract the effect of investment decisions on the amount of wealth that could be consumed at the end of the period, and will work with a investor with a unit of wealth.

2.2.4 The Arrow-Pratt measure of risk aversion, referred to earlier, is the measure $-v''(y)/v'(y)$, where $v(y)$ is the utility of consumption of y. In this case the function is $v(W, c)$, as specified above, and differentiation is with respect to Wc. Performing the differentiations gives a coefficient of risk aversion of α/W, which is inversely proportional to initial wealth. The utility function has been chosen to achieve this feature, because it allows us readily to link the consumer's viewpoint, based on the cost of the service, to the investor's viewpoint, based on the return on the capital subscribed.

2.2.5 For an investor with a unit of initial wealth the risk aversion is α, and, because it is convenient to work with a unit of investment, where the context allows we will refer to α as the risk aversion and will take it as understood that this actually means a risk aversion of α/W for an investor whose initial wealth is W.

2.3 *Distribution of Investor Payoffs*

2.3.1 The paper takes it for granted that the price received and the capital raised are invested by the business in the risk-free asset until needed to pay for supplies and meet obligations, so that:

$$P = \frac{1+p}{1+r}$$

and

$$w_1 = (w+P)(1+r) - L.$$

2.3.2 By substituting for P, it can be seen that w_1 is normally distributed with mean $w(1+r)+p$ and standard deviation σ. The return has a mean of $(1+r)+\frac{p}{w}$ and a standard deviation of $\frac{\sigma}{w}$. The element $\frac{p}{w}$ is, of course, the risk premium per unit invested that is generated by charging the consumer the price p per unit expected obligation.

3. Entity Specific Pricing

3.1 *Purpose*

3.1.1 The purpose of this section is to introduce the relationship between uncertainty in obligations and the entity specific price for the service that gives rise to these obligations. This is done in an economy with one business and one type of investor, with risk aversion α_1.

3.1.2 The entity specific price for risk is the price that must be charged for the service in order to attract the required capital support. This price must therefore be such that an investor maximises the expected utility of his or her future consumption by subscribing the required capital, provided that the price is acceptable to the consumer.

3.2 *Entity Specific Pricing*

3.2.1 In the setting of the paper, investors and consumers are assumed to expect that businesses will adopt entity specific pricing and will provide a dependable service. This section considers one type of investor and one business in isolation. From the investor's viewpoint, the service must be priced so that the expected utility of his or her future consumption is maximised by subscribing exactly the fraction of existing wealth needed for the business to have sufficient capital support to meet consumer demand. If consumer demand is such that the fraction κ of investor wealth is not required, the service must be priced so that the investor leaves this fraction κ of initial wealth invested in the risk-free asset, but invests the balance of

$(1-\kappa)$ in the business. This will happen if the business prices its service so that the expected utility of the investor's future consumption is maximised by subscribing the required fraction $(1-\kappa)$ of his or her wealth.

3.2.2 The expected utility per unit wealth, when a proportion κ is invested risk-free and the balance is invested in the business (where the price for the risk in the service is p and the capital required is w, per unit expected obligation), is:

$$\frac{1}{\sqrt{2\pi}\frac{\sigma}{w}}\int_{-\infty}^{\infty} -e^{-\alpha_1\left(1+r+\frac{p}{w}(1-\kappa)+\varsigma(1-\kappa)\right)}e^{-\varsigma^2/2\left(\frac{\sigma}{w}\right)^2}d\varsigma$$
$$= -\exp\left(-\alpha_1\left(1+r+\frac{p}{w}(1-\kappa)-\alpha_1\frac{\sigma^2}{2w^2}(1-\kappa)^2\right)\right)$$

where ς is the random variable distributed $N(0, \left(\frac{\sigma}{w}\right)^2)$. Differentiating the expected utility with respect to κ and setting equal to zero gives:

$$\alpha_1\left(\frac{p}{w}-\alpha_1\frac{\sigma^2}{w^2}(1-\kappa)\right)\exp\left(-\alpha_1\left(1+r+\frac{p}{w}(1-\kappa)-\alpha_1\frac{\sigma^2}{2w^2}(1-\kappa)^2\right)\right)=0 \tag{3.1}$$

with this determining a maximum because of the shape of a utility function. Thus, the pricing method that will cause the investors to subscribe capital equal to the fraction $(1-\kappa)$ to the business is to set p so that it satisfies equation (3.1), giving:

$$\alpha_1\left(\frac{p}{w}-\alpha_1\frac{\sigma^2}{w^2}(1-\kappa)\right)=0$$

so that:

$$\frac{p}{w}=\alpha_1\frac{\sigma^2}{w^2}(1-\kappa). \tag{3.2}$$

If the risk premium for supporting the obligations were lower than this insufficient capital would be raised, while if the price p were higher consumer demand would fall.

3.2.3 It can be seen from equation (3.2) that the demand for capital reflected in the parameter $(1-\kappa)$ affects the fraction of wealth and the risk premium per unit of wealth invested. For simplicity, however, we conduct the rest of the analysis in the paper on the assumption that wealth is just sufficient and hence that $\kappa = 0$.

3.3 *The Consumer's Viewpoint*

3.3.1 The purpose of this section is to examine the impact of entity specific pricing on the expected utility of a consumer's future consumption.

3.3.2 Drawing upon equation (3.2), the charge p for risk per unit expected obligation is:

$$p = \alpha_1 \frac{\sigma^2}{w}. \tag{3.3}$$

It can be seen that p depends on the capital support w per unit expected obligation that is needed to satisfy consumers that the service is dependable, as well as on the coefficient α_1 of investor risk aversion and the variance σ^2 of actual obligations per unit expected obligation. Changing any one of these will change the price of the service, but this is subject to market forces. For example, while investors might seek to drive down w in order to raise the charge for risk, any such attempt would be countered by consumer demands that w be high enough to allow them to depend upon the service being performed.

3.3.3 Consider now the effect of paying the price in equation (3.3) from a consumer's viewpoint, again in the setting of one business and one type of investor and by reference to a consumer with a unit of wealth whose risk aversion is α_1. Suppose that risk-based capital support equates to a number z of standard deviations, such that the risk of failure is negligible. Thus $w = z\sigma$ and, using equation (3.2), the price of risk per unit wealth is α_1/z^2. Assume that the impact of service failure would cost the consumer a fraction F of wealth, and that, were it not for the capital support provided by investors, it would have a probability of occurring during the period of q. From the utility function, and abstracting just the effect of the risk charge per unit wealth, a consumer will be prepared to pay the charge for risk if:

$$-e^{-\alpha_1(-\alpha_1/z^2)} > -(1-q) - qe^{-\alpha_1(-F)}.$$

Expanding this in first order terms gives:

$$qF > \alpha_1/z^2 \tag{3.4}$$

which is the first order condition for a consumer to benefit from the financial strength of the service provider and the associated charge in a single business economy. For example, if $\alpha_1 = 0.5$ and $z = 3$, then qF must exceed the right hand side of inequality (3.4), which is the risk premium 0.056. This would be the case if, say, $q = 0.01$ and $F = 6$, as might apply if the consumer failed to insure a risk that could represent six times wealth, or if, say, $q = 0.5$ and $F = 0.125$, as might apply if the collapse of a telecom provider with no risk-based capital would, in one way or another, cost the consumer 12.5% of his or her wealth. The paper assumes that all services in this economy are material enough to satisfy inequality (3.4), after scaling back to the relative size of each in the economy. Thus, if this telecom provider represents 5% of the economy, its failure would be assumed to cost the consumer 5% of 12.5% of his or her wealth.

4. Market Operation under Entity Specific Pricing

4.1 *Purpose and Set-up of this Section*

4.1.1 The aim in this section is to demonstrate the operation of a market of many businesses and investors under entity specific pricing using a simplified set-up that is an extension of the single business entity and single investor case. In particular, the aim is to show that the market raises the required capital and has an equilibrium at which the portfolios of investors are optimal at their own risk aversions. This also demonstrates that this market is arbitrage-free, as the existence of an optimal portfolio and the existence of an arbitrage are mutually exclusive — if an arbitrage existed, it could be used to improve the result from the optimal portfolio, which is a contradiction.

4.1.2 It is assumed that consumers expect the risk element of the price charged by the provider of a service to reflect some weighted average of the risk aversion of investors that is consistent across all services, so that the market operates by entity specific pricing at some uniform risk aversion coefficient α^*.

4.1.3 Consider an economy with n businesses, where the capital required per unit obligation by business i is denoted by w_i ($i = 1$ to n). Between them they need to raise capital equal to a fraction $(1 - \kappa)$ of available wealth to meet consumer demand, leaving the residual fraction κ to be invested risk-free. However, as noted before, this parameter will be disregarded by setting κ to zero. The economy also has m investors in the market, where the wealth of investor η is denoted by W_η ($\eta = 1$ to m) and total wealth by W, where $W = \sum W_\eta$.

4.2 *Analysis of Market Operation*

4.2.1 Begin by considering the fraction $k_{\eta i}$ of his or her wealth that would be subscribed by investor η to business i in circumstances where investors can invest or borrow at the risk-free rate and subscribe whatever business capital they choose. That is, in the absence of any constraints on available investments and endowed with wealth of W_η, investor η would subscribe $W_\eta k_{\eta i}$ of capital to business i, giving support to expected obligations of $W_\eta \frac{k_{\eta i}}{w_i}$.

https://doi.org/10.1017/S1357321700004232 Published online by Cambridge University Press

4.2.2 Let σ_i be the standard deviation of outcomes per unit expected obligation of the ith business in the market, with coefficients of correlation between businesses of ρ_{ij} ($i = 1$ to n, $j = 1$ to n); and let the risk aversions of the investors in the market be α_η ($\eta = 1$ to m).

4.2.3 A linear combination of normally distributed variables is also normally distributed. If we let ς_i ($i = 1$ to n) be the random variable associated with the investment payoffs generated around the mean per unit of capital contributed to business i, then the investment payoff per unit subscribed by investor η, namely $\sum_i k_{\eta i}\varsigma_i$, is distributed:

$$\mathrm{N}\left(0, \sum_i k_{\eta i}^2 \frac{\sigma_i^2}{w_i^2} + \sum_{j \neq i} \sum_i \rho_{ij} k_{\eta i} k_{\eta j} \frac{\sigma_i \sigma_j}{w_i w_j}\right)$$

where each of the cross-business terms appears in the summation twice. Using this and integrating the utility function, the expected utility per unit invested of the optimally priced portfolio held by investor η is:

$$\exp\left(-\alpha_\eta\left(1 + r + \alpha^* \sum_i k_{\eta i} \frac{\sigma_i^2}{w_i^2} - \frac{\alpha_\eta}{2}\left(\sum_i k_{\eta i}^2 \frac{\sigma_i^2}{w_i^2} + \sum_{j \neq i} \sum_i \rho_{ij} k_{\eta i} k_{\eta j} \frac{\sigma_i \sigma_j}{w_i w_j}\right)\right)\right). \tag{4.1}$$

By differentiating expected utility with respect to $k_{\eta i}$ and equating the results to zero for each business, it is established (using matrix algebra — refer to the Appendix) that maximum utility in the absence of any constraints occurs when the capital subscribed in respect of the ith business by the ηth investor (per unit invested) is:

$$k_{\eta i} = \frac{\alpha^*}{\alpha_\eta} \sum_j \psi_{ij} \frac{\sigma_j w_i}{\sigma_i w_j} \tag{4.2}$$

where, if the correlation matrix is denoted as $\boldsymbol{\rho}_{\mathbf{ij}}$, then ψ_{ij} is the element in row i and column j of $\boldsymbol{\rho}_{\mathbf{ij}}^{-1}$.

4.2.4 However, having set $\kappa = 0$, then in aggregate across all businesses and investors wealth must clear, so that:

$$\sum_\eta \sum_i k_{\eta i} W_\eta = W. \tag{4.3}$$

From this and equation (4.2) the uniform α^* that determines the price of uncertainty when the market has settled into equilibrium satisfies:

$$\alpha^* \sum_\eta \frac{W_\eta}{\alpha_\eta} \sum_i \sum_j \psi_{ij} \frac{\sigma_j w_i}{\sigma_i w_j} = W$$

giving:

$$\alpha^* = \frac{W}{\sum_\eta \frac{W_\eta}{\alpha_\eta} \sum_i \sum_j \psi_{ij} \frac{\sigma_j w_i}{\sigma_i w_j}} \tag{4.4}$$

which can be thought of as the product of a particular weighted average of investors' risk aversions $W \Big/ \sum_\eta \frac{W_\eta}{\alpha_\eta}$ and a diversification factor $1 \Big/ \sum_i \sum_j \psi_{ij} \frac{\sigma_j w_i}{\sigma_i w_j}$. It can be seen that, because of the second of these, α^* reflects the improved degree of diversification gained, as the market portfolio contains more investment opportunities. Putting result (4.4) into equation (4.2), we get the capital subscribed by investor η to business i, namely:

$$k_{\eta i} W_\eta = W \left(\frac{W_\eta / \alpha_\eta}{\sum_\eta W_\eta / \alpha_\eta} \right) \left(\frac{\sum_j \psi_{ij} \frac{\sigma_j w_i}{\sigma_i w_j}}{\sum_i \sum_j \psi_{ij} \frac{\sigma_j w_i}{\sigma_i w_j}} \right). \tag{4.5}$$

Using equation (4.5) we get the capital subscribed to business i by all investors in the market in equilibrium, namely:

$$\sum_\eta k_{\eta i} W_\eta = W \frac{\sum_j \psi_{ij} \frac{\sigma_j w_i}{\sigma_i w_j}}{\sum_i \sum_j \psi_{ij} \frac{\sigma_j w_i}{\sigma_i w_j}} \tag{4.6}$$

provided that this total stockholding is positive for all i (which will be the case in practice, as each w_i adjusts to give the distribution of businesses that consumers demand). The result (4.5) can obviously be re-expressed as investor η holding $\frac{W_\eta / \alpha_\eta}{\sum_\eta W_\eta / \alpha_\eta}$ of the total stock of business i.

4.2.5 From this, it can be seen that all investors hold capital stock distributed in the same proportions as in the market overall, as given by equation (4.6), with their absolute holdings being inversely proportional to their risk aversion, or equivalently, given that the reciprocal of the Arrow-Pratt measure of risk aversion is the risk tolerance, proportional to their risk tolerance. If this last result is inserted into the expected utility (4.1), it can be seen by inspection that the proportionate improvement in expected utility generated by the optimal portfolios of any two investors relative to the risk-free portfolio is the same, while at the same time reflecting their individual risk aversions.

4.3 *Description of Market Operation*

4.3.1 In the economic setting, there is a competitive market where there is an accepted price for entity specific risk, and the interaction between consumer demand and investor behaviour in constructing optimal portfolios reorganises the equilibrium as the influences that affect it undergo change. For example, if risk aversion changes, prices of services and returns to investors will change, and a new equilibrium will form. If a business wants to increase its role in the economy by reducing the price of its service, it will fail to raise the capital that it needs to support the new level of activity, and will return to its equilibrium position. If, to take another situation, consumer demand exceeds the scale of a business in the equilibrium market, it can assume more obligation per unit of capital to meet that demand at a new equilibrium (see example in ¶4.3.6).

4.3.2 From Section 4.2, the market has the following attributes as it settles into equilibrium:

— a uniform risk aversion coefficient (that is dictated by the market) for pricing assets and services on an entity specific basis;
— optimal portfolios constructed by each investor in accordance with their own particular risk aversion;
— freedom from arbitrage;
— an equilibrium to which the market will be driven by investor demand in constructing optimal portfolios that, because they correspond to those that they would construct if there were no constraints, are exactly market-clearing, involving neither shortfall in capital subscribed nor unsatisfied demand;
— a particular distribution of businesses of differing payoff variability at which the market will settle into equilibrium, accommodating the interaction between investor demand in constructing optimal portfolios and consumer demand in expecting a uniform basis for the charge made for risk by providers of services;
— optimal portfolios that have the same proportionate distribution as the market portfolio, differing only in the extent to which investors with different risk aversions support obligations — namely in proportion to each investor's risk tolerance; and
— optimal portfolios that give each investor's end of period consumption the same proportionate improvement in expected utility relative to investing a unit of wealth risk-free.

4.3.3 With entity specific pricing at a uniform risk aversion, therefore investors maximise the expected utility of their next period consumption by investing in the market portfolio proportionately to their risk tolerance. Investors with high risk tolerance will, if necessary, borrow directly or indirectly from investors with lower risk tolerance in order to do this. With the same utility function, all investors improve their expected utility (relative to investing a unit of wealth risk-free) by the same proportion, whatever their risk aversion.

4.3.4 An important attribute of the entity specific pricing approach developed in the paper is that the covariance characteristics of the businesses in the market, namely the standard deviations of payoffs per unit invested and the correlations between them, determine their relative size in the market at equilibrium. It is this distribution that accommodates the construction of optimal portfolios at entity specific prices. In Section 5 we will see that at this market equilibrium the entity specific pricing approach and the CAPM give identical risk premiums.

4.3.5 Table 4.1 illustrates the equilibrium in a market consisting of six businesses and three investors operating under entity specific pricing at a uniform, market driven risk aversion. Results have been rounded. It is straightforward to confirm independently the results in Table 4.1 by setting up a spreadsheet with formula (4.1) for each investor's expected utility, given variable portfolio mixes and the equilibrium α^*, and then using the solver function without any constraints to find the mix that maximises utility for each investor.

4.3.6 It is informative to illustrate the reorganisation of an equilibrium using this example. Suppose that consumer demand exceeds the equilibrium scale of business 1 in case 2 in Table 4.1. From the table it can be seen that this business raises 0.270 of capital out of a total wealth of 12, allowing it to support expected obligations of $0.270/0.1 = 2.70$ at its capital requirement of 0.100 per unit expected obligation. Suppose, however, that the economy requires support of expected obligations of 5.0 by this business. A recalculation shows that this is achieved if the business reduces its capital requirement to 0.09705 per unit expected obligation, thereby attracting capital because of the higher expected return to investors, as it supplies its service on a slightly less capital intensive and hence less dependable basis, and at a slightly higher price. With this change and from equation (4.6), the capital allocated to business 1 increases from 0.270 to 0.485 (at the expense of the capital allocated to each of the other businesses in the economy), and it supports an expected obligation for business 1 of $0.485/0.09725 = 5.0$, as required. In the course of this happening, the equilibrium-generating α^* changes from 0.525 to 0.522.

https://doi.org/10.1017/S1357321700004232 Published online by Cambridge University Press

Table 4.1. Illustration of selected results in respect of a six business and three investor market (where wealth is exactly sufficient, i.e.$\kappa = 0$)

Standard deviations of payoffs in the six businesses	[0.025, 0.050, 0.075, 0.100, 0.125, 0.150]
Capital requirement per unit expected obligation	[0.100, 0.175, 0.250, 0.325, 0.400, 0.475]
Standard deviation of returns	[0.250, 0.286, 0.300, 0.308, 0.313, 0.316]
Investor wealth (totalling 12)	[7, 3, 2]
Investor risk aversion	[1, 2, 8]
Weighted average risk aversion	1.371
	Case 1
Correlation matrix	$\begin{bmatrix} 1, 0, 0, 0, 0, 0 \\ 1, 0, 0, 0, 0 \\ 1, 0, 0, 0 \\ 1, 0, 0 \\ 1, 0 \\ 1 \end{bmatrix}$
Capital raised by the businesses (totalling investor wealth of 12)	[2, 2, 2, 2, 2, 2]
Standard deviation of this mix per unit invested	0.121
'Equilibrium-generating' α^*	$\alpha^* = 0.229$
Risk premium	[0.0143, 0.0189, 0.0206, 0.0216, 0.0223, 0.0228]
Total holdings of each investor (distributed in the same proportion as the total capital raised)	[9.6, 2.057, 0.343]
	Case 2
Correlation matrix	$\begin{bmatrix} 1, 0.5, 0.5, 0.5, 0.5, 0 \\ 1, 0.5, 0.5, 0.5, 0 \\ 1, 0.5, 0.5, 0 \\ 1, 0.5, 0 \\ 1, 0 \\ 1 \end{bmatrix}$
Capital raised by the businesses (totally investor wealth of 12)	[0.270, 1.385, 1.756, 1.942, 2.053, 4.594]
Standard deviation of this mix per unit invested	0.190
'Equilibrium-generating' α^*	$\alpha^* = 0.525$
Risk premium	[0.0328, 0.0429, 0.0472, 0.0497, 0.0513, 0.0524]
Total holdings of each investor (distributed in the same proportion as the total capital raised)	[9.6, 2.057, 0.343]

5. The Entity Specific Pricing Model and the CAPM

5.1 *Basis of the Model*

5.1.1 The paper has now reached the point where a simple share market pricing model can be proposed and compared with the CAPM.

5.1.2 Three ideas provide the foundation, namely that:

(1) The behaviour of investors, and hence of the market, can be approached by looking at the performance of businesses in meeting the obligations associated with delivering their service. In particular, the distribution of business outcomes conceptually represents the modelling and appraisal done by equity analysts and investors, so that what is being taken into account when shares change hands in an informed market is just such a model of a business entity based on its obligations, margins and prospects.

(2) When a consumer buys a service from a limited liability company a pair of implied theoretical transactions occurs. One part involves consumers and creditors giving the shareholders an option allowing them to walk away if the company is unable to meet its obligations, while the other part involves the price of the service and the terms of supply being correspondingly reduced to allow for the value of this option. In the economic setting, however, consumers expect to be able to depend upon the service that they have purchased actually being performed, which means that a business entity must be sufficiently well capitalised for the value of the option to be negligible. This allows the use of a normal distribution of returns as a reasonable representation of the value of a shareholding.

(3) The distribution of businesses settles into an arbitrage-free, market-clearing equilibrium as investors, businesses and consumers alike place an entity specific price on uncertainty at a market driven uniform risk aversion, and as investors construct optimal portfolios to suit their own risk aversions.

5.2 *The Entity Specific Pricing Model*

5.2.1 Let the expected total rate of return on the shares in a business entity be R, which, based on the entity specific pricing approach, is, as shown before, given by:

$$R = 1 + r + \alpha^* \frac{\sigma^2}{w^2}. \tag{5.1}$$

We call this result the entity specific pricing model.

5.2.2 When a market has settled into equilibrium, α^* applies uniformly across it. The market will therefore value every business in it using result (5.1), with parameters r, α^* being market-wide, albeit in a constant state of flux, and with σ, w being entity specific. The risk premium component of the return is $\alpha^* \frac{\sigma^2}{w^2}$. Using the second case in Table 4.1, where $\alpha^* = 0.525$ and $\kappa = 0$, and considering the business where $\sigma = 0.075$ and $w = 0.25$ (giving a

https://doi.org/10.1017/S1357321700004232 Published online by Cambridge University Press

standard deviation per unit invested of $\left(\frac{0.075}{0.25} = 0.3\right)$, the risk premium would, for example, be 0.525 $(0.3)^2$, or 4.725%.

5.3 *Comparison with the CAPM*

5.3.1 It is instructive to draw a comparison of this entity specific pricing model with the CAPM. The CAPM approach leads to the well-known result (Sharpe, 1964) that the excess of the risk discount rate (over the risk-free rate) for a particular investment equals the excess expected return for the market times beta, where beta is the correlation between the company and market returns times the ratio of the standard deviation of the company return to that of the market return. Beta measures risk as the product of: (i) a relative risk measure, namely the standard deviation of the particular return as a fraction of the standard deviation of the market return; and (ii) a measure of the extent to which risk is non-diversifiable, namely the correlation of the company return with the market return.

5.3.2 The entity specific approach is complementary to the CAPM. To illustrate this, suppose that the market is described by case 2 in Table 4.1. A calculation (using Table 4.1) of the overall risk premium for this distribution of businesses under entity specific pricing of services gives a market risk premium of 4.9455%. Using this, and after calculating the betas needed, Table 5.1 demonstrates that the CAPM gives exactly the same risk premium for each business in the equilibrium market as does entity specific pricing. In short, the entity specific approach to pricing is consistent with the CAPM provided that it uses a risk premium equal to the market-driven equilibrium risk aversion times the variance of returns, and provided that the market settles at the posited equilibrium mix of businesses.

5.3.3 The difference between the entity specific approach and the CAPM is that the former introduces an additional postulate, namely that entity specific total risk is priced consistently across the market using a uniform and equilibrium-generating risk aversion α^*. As Table 5.1 shows,

Table 5.1. Comparison of entity specific pricing and the CAPM (using a market described by case 2 in Table 4.1)

Business	1	2	3	4	5	6	Market
Capital raised per unit invested	2.3%	11.5%	14.6%	16.2%	17.1%	38.3%	100%
Beta	0.663	0.867	0.955	1.005	1.037	1.059	1
CAPM risk premium (beta × 0.049455)	3.28%	4.29%	4.72%	4.97%	5.13%	5.24%	4.95%
Entity specific pricing risk premium	3.28%	4.29%	4.72%	4.97%	5.13%	5.24%	4.95%

https://doi.org/10.1017/S1357321700004232 Published online by Cambridge University Press

with this further postulate and the equilibrium market that it implies, the two approaches give the same risk premium for a security in the market, despite the fact that the CAPM allows only for non-diversifiable risk, and that, at first sight, the two models appear to be contradictory.

5.3.4 The explanation of this is straightforward. Under the postulates of the CAPM, investors are mean-variance averse, and, as a result, there is an efficient market portfolio which is on the mean-variance frontier (LeRoy & Werner, 2001). In turn this leads to the security market line, which is a linear relationship between the return on the market portfolio r_m, the return on the risk-free asset of r and the return r_i on a particular stock i. This relationship is given by $(r_i - r) = \beta(r_m - r)$, where $\beta = \rho_{i,m}\frac{\sigma_i}{\sigma_m} = \frac{\mathrm{cov}(r_i, r_m)}{\mathrm{var}(r_m)}$. Under our entity specific pricing approach investors are also mean-variance averse and, while entity specific pricing of services has led to a particular and identifiable market equilibrium, this market equilibrium is optimal for investors, and it too lies on the frontier. The resulting entity specific stock pricing model proposed above is similarly a linear relationship between the parameters, namely $(r_i - r) = \alpha^* \mathrm{var}(r_i)$, with the risk-free return, security returns and portfolio returns, including the market return, all lying on a straight line. This line is, by definition, simply the security market line in a different setting, and described in a way that simply reflects the additional assumption of entity specific prices and the resulting equilibrium market.

5.3.5 To confirm this, consider business i in Section 4. Under entity specific pricing, we have from equation (4.6) the proportion θ_i that this business represents of the total market in equilibrium, namely:

$$\theta_i = \frac{\sum_j \psi_{ij}\frac{\sigma_j w_i}{\sigma_i w_j}}{\sum_i\sum_j \psi_{ij}\frac{\sigma_j w_i}{\sigma_i w_j}} = \frac{\alpha^*\sum_\eta \frac{W_\eta}{\alpha_\eta}}{W}\sum_j \psi_{ij}\frac{\sigma_j w_i}{\sigma_i w_j} \tag{5.2}$$

and its risk premium is $\alpha^*\frac{\sigma_i^2}{w_i^2}$. The overall market risk premium $(r_m - r)$ is, of course, given by:

$$(r_m - r) = \sum_i \alpha^*\frac{\sigma_i^2}{w_i^2}\theta_i. \tag{5.3}$$

From (5.2) and (5.3), after some matrix algebra (refer to the Appendix), we then get the following results for the covariance between the return from

https://doi.org/10.1017/S1357321700004232 Published online by Cambridge University Press

investing in the particular business and the market return, and for the variance of the market return:

$$\operatorname{cov}(r_i, r_m) = \frac{\sum_{\eta} \frac{W_\eta}{\alpha_\eta}}{W} \left(\alpha^* \frac{\sigma_i^2}{w_i^2} \right) \tag{5.4}$$

and

$$\operatorname{var}(r_m) = \frac{\sum_{\eta} \frac{W_\eta}{\alpha_\eta}}{W} \left(\sum_i \alpha^* \frac{\sigma_i^2}{w_i^2} \theta_i \right) \tag{5.5}$$

leading to:

$$\beta = \frac{\frac{\sigma_i^2}{w_i^2}}{\sum_i \frac{\sigma_i^2}{w_i^2} \theta_i} \tag{5.6}$$

which gives, drawing upon the CAPM and the market risk premium in (5.3):

$$(r_i - r) = \alpha^* \frac{\sigma_i^2}{w_i^2} \tag{5.7}$$

a result that, for the reasons already explained, accords with entity specific pricing, even though the latter has not explicitly taken the CAPM into account.

5.4 *A Note on Empirical Matters*

The entity specific approach provides a new line of enquiry for investigators. While still satisfying the CAPM, the entity specific approach predicts a risk premium given by equation (5.7), where α^* is uniform (though possibly in a state of flux). Furthermore, the variance of returns per unit of capital may also tend towards being uniform if businesses set their capital requirement to give more or less the same level of financial strength (or its converse, a uniformly low probability of failure) across the market. To the extent that this is the case, these factors taken together may lead to the risk premium observed in the market being more uniform than predicted by the standard form of the CAPM.

6. Fair Value of Liabilities

6.1 *Purpose of this Section*

The purpose of this section is to observe that a state price deflator follows from entity-specific pricing of obligations, and that it can be used in the setting of this paper to arrive at the fair value of all or any part of the liability arising from them.

6.2 *An Entity Specific State Price Deflator*

6.2.1 State price deflators are discount factors that allow the present value of a distribution of uncertain payoffs (or some part of it) to be calculated. In effect, deflators are discount factors for a stochastic valuation, and their use is well established. (See Jarvis, Southall & Varnell (2001) for a review of their application in actuarial work; and Duffie (1996) or LeRoy & Werner (2001) for a financial economics treatment.)

6.2.2 In the setting in the paper, there is a deflator associated with the optimal, market equilibrium portfolio that can be used to value investment payoffs in each of the probability states of that portfolio. This is the usual deflator associated with the CAPM, with the only special feature being that in the entity specific model, with its additional postulate of entity specific pricing, the composition of this portfolio is known.

6.2.3 What is unusual in this setting, however, is that there is, in addition, a set of entity specific deflators associated with entity specific pricing of each security. This happens because, as described in Section 3.2, entity specific pricing involves pricing obligations, and hence the return to an investor, in such a way that when investment in each business is viewed in isolation the investment is optimal. In other words, viewed in isolation, investing the capital required by each business in the risk-free asset maximises the expected utility of an investor's future consumption. That the usual CAPM deflator and each of the entity specific deflators are part of the same framework is established by the result derived in Section 5.3, namely that, in this setting, the CAPM gives the same asset pricing as the entity specific model.

6.2.4 A deflator has two important properties, in that:

(1) the deflator's expectation equals the risk-free discount factor; and
(2) the expectation of the product of the deflator in each probability state and the corresponding state payoff equals the amount originally invested.

Only when these properties both apply do we have a state price deflator. When we have a state price deflator we are able to perform a stochastic valuation. In our setting, where we have entity specific deflators that are compatible with the CAPM, we are able to perform entity specific stochastic valuations that represent fair value, at least to the extent that the CAPM captures fair value.

6.2.5 Drawing upon the mechanics of a deflator (Jarvis, Southell & Varnell, 2001), the state price deflator π_ς in the probability state that corresponds to the payoff $\left(1 + r + \alpha^* \frac{\sigma^2}{w^2} + \varsigma\right)$ in the entity specific pricing model is equal to the marginal utility of the payoff times a normalising constant λ, that is:

$$\pi_\varsigma = \lambda v'\left(1 + r + \alpha^* \frac{\sigma^2}{w^2} + \varsigma\right) \tag{6.1}$$

where the normalising constant is defined by:

$$\lambda = 1 \Big/ \mathrm{E}\left[\left(1 + r + \alpha^* \frac{\sigma^2}{w^2} + \varsigma\right) v'\left(1 + r + \alpha^* \frac{\sigma^2}{w^2} + \varsigma\right)\right] \tag{6.2}$$

and where $v(c) = -e^{-\alpha^* c}$. With this utility function, the entity specific deflator is therefore given by:

$$\pi_\varsigma = \lambda \alpha^* \exp\left(-\alpha^*\left(1 + r + \frac{\alpha^* \sigma^2}{w^2} + \varsigma\right)\right)$$

where ς is distributed $\mathrm{N}\left(0, \left(\frac{\sigma}{w}\right)^2\right)$, and the normalising constant λ is given by:

$$\lambda = 1 \Big/ \alpha^*(1 + r) \exp\left(-\alpha^*\left(1 + r + \frac{\alpha^* \sigma^2}{2w^2}\right)\right)$$

so that the entity specific deflator for an investment payoff in the probability state ς is:

$$\pi_\varsigma = \frac{\exp\left(-\alpha^*\left(\frac{\alpha^* \sigma^2}{2w^2} + \varsigma\right)\right)}{(1 + r)} \tag{6.3}$$

and this deflator can be used to value all or any part of the distribution of payoffs from an investment.

6.3 *Fair Value of the Liability*

6.3.1 We are often interested in the fair value of the liability L per unit of expected obligation. As discussed in Section 5.1, in our model the distribution of business payoffs conceptually represents the modelling and

https://doi.org/10.1017/S1357321700004232 Published online by Cambridge University Press

appraisal done by equity analysts and investors, so that what is being taken into account when securities change hands in an informed market is just such a model of a business entity based on its obligations, margins and prospects. The fair value of the liability is therefore inherent in the share price in our entity specific pricing model.

6.3.2 We can therefore derive a liability deflator, by expressing the random variable ς in the investment payoff in terms of the random variable L (defined in ¶2.2.2) for the liability outcome. This relationship is, of course:

$$\varsigma = (1 - L)/w \tag{6.4}$$

and from the investment deflator (and property 2 of a deflator referred to in ¶6.2.4) we have:

$$\mathrm{E}\left[\pi_{\varsigma}\left(1 + r + \frac{\alpha^{*}\sigma^{2}}{w^{2}} + \varsigma\right)\right] = 1$$

so that:

$$\mathrm{E}\left[\pi_{\varsigma}\left(1 + r + \frac{\alpha^{*}\sigma^{2}}{w^{2}} + \frac{1 - L}{w}\right)\right] = 1$$

and rearranging this by drawing on property 1 in ¶6.2.4 gives:

$$\frac{1}{1+r}\left(1 + r + \frac{1}{w} + \frac{\alpha^{*}\sigma^{2}}{w^{2}}\right) + \mathrm{E}\left[\pi_{\varsigma}\left(\frac{-L}{w}\right)\right] = 1.$$

Because of the symmetry of the normal distribution we have:

$$\mathrm{E}[\pi_{\varsigma} - L] = -\mathrm{E}[\pi_{-\varsigma}L]$$

indicating that when the liability outcome is adverse so too is the investor payoff, and the deflator reflects this. Putting this in the previous result gives the following for the fair value at the outset of a unit of expected liability:

$$\mathrm{E}[\pi_{-\varsigma}L] = \frac{1 + \dfrac{\alpha^{*}\sigma^{2}}{w}}{1 + r} \tag{6.5}$$

which, from ¶2.3.1, corresponds to the price P of the service, as would be expected. It can also be seen from equation (6.5) that, again as would be expected, uncertainty *increases* the value of a liability, which is the price that a business would have to pay someone else to take it off their hands, while it *reduces* the value of an asset.

6.3.3 From equations (6.3) and (6.4) we also have the following result for the state price deflator for liability outcomes:

$$\pi_{-\varsigma} = \frac{\exp\left(-\alpha^*\left(\frac{\alpha^*\sigma^2}{2w^2} - \varsigma\right)\right)}{(1+r)} \tag{6.6}$$

where, from equation (6.4), the liability outcome state corresponding to the investor payoff state $-\varsigma$ is $1 + w\varsigma$.

6.3.4 Spelt out in full, the left hand side of equation (6.5) is therefore the integration from $-\infty$ to ∞ of the product of:

— the liability outcome state, $1 + w\varsigma$; and

— the liability deflator, $\frac{1}{(1+r)}\exp\left(-\alpha^*\left(\frac{\alpha^*\sigma^2}{2w^2} - \varsigma\right)\right)$; and

— the normal probability density, $\frac{1}{\sqrt{2\pi}\frac{\sigma}{w}}\exp\left(-\varsigma^2 \Big/ 2\left(\frac{\sigma}{w}\right)^2\right)$;

with respect to ς.

6.4 *Example of an Actuarial Application*

6.4.1 We have derived a deflator that (in our setting at least) can be used to place a fair value on all or any part of the distribution of outcomes of a liability, despite being entity specific. To illustrate its use, consider an example of a class of insurance business that has a distribution of claims costs at the end of a year with a standard deviation of 0.1 of expected claims, and where the insurer is required to support its obligations with solvency capital of 0.3 of expected claims. The market-driven coefficient of risk aversion needs to be consistent with the observed market risk premium, which we will assume to be 5.5%, and the risk-free rate to be 5% per annum. The example in ¶3.3.3 suggests that, for this risk premium and level of financial strength, the market-driven coefficient of risk aversion would be about 0.5, which we will adopt for the purpose of this example. From equation (6.6), the fair value deflator $\pi_{-\varsigma}$ for the claims liability for the liability outcome state $1 + 0.3\varsigma$ is then:

$$\pi_{-\varsigma} = \frac{\exp\left(-0.5\left(\frac{0.5 \times 0.1^2}{2 \times 0.3^2} - \varsigma\right)\right)}{1.05}$$

where ς is distributed $N\left(0, \left(\frac{0.1}{0.3}\right)^2\right)$. By using this deflator, or more simply

https://doi.org/10.1017/S1357321700004232 Published online by Cambridge University Press

from equation (6.5), the fair value at the outset per unit expected liability is $\frac{1}{1.05}\left(1+\frac{0.5\times 0.1^2}{0.3}\right)=0.968.$

6.4.2 Suppose that we want to calculate the fair value per unit expected liability of just those outcomes that exceed the mean. From equation (6.4), $\varsigma = 0$ when $L = 1$, so that, following the procedure in ¶6.3.4, the fair value in question is:

$$\frac{1}{1.05\sqrt{2\pi}\left(\frac{0.1}{0.3}\right)}\int_0^\infty (1+0.3\varsigma)\exp\left(-0.5\left(\frac{0.5\times 0.1^2}{2\times 0.3^2}-\varsigma\right)\right)\exp\left(-\varsigma^2 \Big/ 2\left(\frac{0.1}{0.3}\right)^2\right)d\varsigma$$

$$= 0.586.$$

so that, for example, if expected claims are £100 million, the fair value of claims that exceed this amount would be £58.6 million.

7. Conclusion

7.1 *What has been Covered*

7.1.1 The paper set out to examine the operation of the capital market in a competitive economy, in which business must be conducted in a climate of considerable profit uncertainty, and in which consumers expect to be able to depend on service obligations being fulfilled. In this setting, each business builds in an entity specific profit margin, calculated to ensure that, when the business is viewed in isolation, investors would contribute the risk capital needed to ensure that the business can face the uncertain climate with a high degree of confidence. In our model of this economy, business obligations have been assumed to produce outcomes that are normally distributed, and investors are assumed to become less risk averse, or more risk tolerant, the more wealth they have when making their investment decisions.

7.1.2 The paper examined the connection between the uncertain cost outcomes that a business entity faces in the course of fulfilling the obligations inherent in its service, the profit margin that it charges consumers for its service, including an entity specific price or profit margin for the risk that springs from this uncertainty, and the resulting payoff for its investors.

7.1.3 This connection was traced through the actions of investors in maximising the expected utility of their next period consumption, leading to:
— a particular equilibrium distribution of business activity in the economy;
— a resulting market portfolio that is optimal for investors; and
— investment in the market portfolio in proportion to each investors risk tolerance (and of course their financial capacity).

7.1.4 In this market, we found that entity specific pricing and CAPM give identical risk premiums. That is, we found that the price of entity specific total risk in isolation is the same as the price of non-diversifiable risk relative to the market equilibrium portfolio, and that this reconciliation happens as a matter of course, without entity specific pricing having to take the CAPM into account.

7.1.5 This result comes about because in both models the market return lies on the mean-variance frontier. This, in turn, leads to security prices that, in both cases, lie on the security market line. The difference between the models is that entity specific pricing determines the market portfolio, whereas under the CAPM it is taken as a given. The entity specific pricing model, therefore, transpires to be a special case of the CAPM.

7.2 *Practical Import of the Paper*

By tracing the connection between obligations and share market pricing, we also found that, in our setting, an entity specific valuation of the liability created by obligations is also a fair value. This has allowed us to propose an entity specific state price deflator for calculating the fair value of all or any part of the distribution of outcomes from an obligation. Given that the setting captures the claims variability and need for solvency in the insurance industry in particular, this finding may be relevant to such practical issues as determining premium rates on regulated classes of business and the fair value of insurance liabilities under International Financial Reporting Standards. It may well be that, with an appropriate choice of parameters, entity specific pricing and valuation can be fair.

Acknowledgement

I am indebted to Conor O'Dowd, Dan Tess, Robert Clark, Robert Wild, Mark Tomlins and Mark Johnston for the contribution that they have made to the evolution of the paper through discussion, debate and challenge. I am also most grateful to Tony Coleman for his interest and encouragement. While acknowledging the great benefit that I have gained from this dialogue, any errors or misadventures as I have explored this subject are entirely my own doing.

References

Arrow, K.J. (1965). Aspects of the theory of risk bearing. Yrjo Jahnsson lectures, Helsinki. Reprinted in *Essays in the theory of risk bearing* (1971), Markham, Chicago.

Duffie, D. (1996). *Dynamic asset pricing theory.* Princeton University Press, Princeton.

Jarvis, F., Southall, F. & Varnell, E. (2001). Modern valuation techniques. Paper presented to the Staple Inn Actuarial Society. www.sias.org.uk/papers/mut.pdf

LeRoy, S.F. & Werner, J. (2001). *Principles of financial economics.* Cambridge University Press.

PRATT, J.W. (1964). Risk aversion in the small and the large. *Econometrica*, **32**.

SHARPE, W.F. (1964). Capital asset prices: a theory of market equilibrium under conditions of risk. *Journal of Finance*, **19**.

VON NEUMANN, J. & MORGENSTERN, O. (1947). *Theory of games and economic behaviour*. Princeton University Press, Princeton.

https://doi.org/10.1017/S1357321700004232 Published online by Cambridge University Press

APPENDIX

DERIVATION OF RESULTS REQUIRING MATRIX ALGEBRA

Derivation of Equation (4.2)

Equation (4.2) gives the proportion of wealth invested by investor η in business i in the optimal portfolio, namely that which maximises the expected utility given in equation (4.1). Differentiation of the expected utility in (4.1) with respect to each $k_{\eta i}$ ($i = 1$ to n), and setting equal to zero to find the maximum, gives the following system of equations:

$$\alpha_\eta \boldsymbol{\rho}_{\mathbf{ij}} \left[k_{\eta i} \frac{\sigma_i}{w_i} \right]^{\mathbf{T}} = \alpha^* \left[\frac{\sigma_i}{w_i} \right]^{\mathbf{T}}$$

where α_η, α^* are scalars and the notation follows that in Section 4. Front multiplying both sides by $\boldsymbol{\rho}_{\mathbf{ij}}^{-1}$ and rearranging gives:

$$\left[k_{\eta i} \frac{\sigma_i}{w_i} \right]^{\mathbf{T}} = \frac{\alpha^*}{\alpha_\eta} \boldsymbol{\rho}_{\mathbf{ij}}^{-1} \left[\frac{\sigma_i}{w_i} \right]^{\mathbf{T}} \qquad \text{(A.1)}$$

which, after expanding and rearranging, gives equation (4.2), namely:

$$k_{\eta i} = \frac{\alpha^*}{\alpha_\eta} \sum_j \psi_{ij} \frac{\sigma_j w_i}{\sigma_i w_j}$$

where ψ_{ij} is the element in row i and column j of $\boldsymbol{\rho}_{\mathbf{ij}}^{-1}$.

Derivation of Covariance and Variance in Equations (5.4) and (5.5)

Under entity specific pricing, the proportion θ_i that business i represents of the total market in equilibrium is given by equation (5.2). From equations (5.2), (4.4) and (A.1) we have:

$$[\theta_i]^{\mathbf{T}} = \frac{\alpha^* \sum_\eta \frac{W_\eta}{\alpha_\eta}}{W} \mathrm{Diag}\left[\frac{w_i}{\sigma_i} \right] \boldsymbol{\rho}_{\mathbf{ij}}^{-1} \left[\frac{\sigma_i}{w_i} \right]^{\mathbf{T}} \qquad \text{(A.2)}$$

where the first terms on the right-hand side are scalars. The covariance between the return on the business r_i and that from the market r_m is then the product of:

(i) the vector comprising the standard deviation of the return from business i alone;

(ii) the correlation matrix; and

(iii) the transpose vector of standard deviations of business returns (at market weights), which is the product of:
  (a) the diagonal matrix of standard deviations of business returns; and
  (b) the transpose vector (A.2) of market weights.

This is the product of:

(i) $\left[0, 0, ..., \frac{\sigma_i}{w_i}, ..., 0\right]$;

(ii) $\boldsymbol{\rho}_{\mathbf{ij}}$;

(iii)
  (a) $\text{Diag}\left[\frac{\sigma_i}{w_i}\right]$; and

  (b) $\alpha^* \frac{\sum_\eta \frac{W_\eta}{\alpha_\eta}}{W} \text{Diag}\left[\frac{w_i}{\sigma_i}\right] \boldsymbol{\rho}_{\mathbf{ij}}^{-1} \left[\frac{\sigma_i}{w_i}\right]^{\mathbf{T}}$

which immediately gives equation (5.4), namely:

$$\text{cov}(r_i, r_m) = \frac{\sum_\eta \frac{W_\eta}{\alpha_\eta}}{W} \left(\alpha^* \frac{\sigma_i^2}{w_i^2}\right). \tag{A.3}$$

Similarly, the variance of the market return is the product of the vector of business returns at market weights, the correlation matrix and the transpose vector of business returns at market weights. This is the same as the preceding product, except that the first term is the vector:

(i) $\left[\frac{\sigma_i}{w_i} \theta_i\right]$.

It will be seen by inspection that the product yields equation (5.5), namely:

$$\text{var}(r_m) = \frac{\sum_\eta \frac{W_\eta}{\alpha_\eta}}{W} \left(\sum_i \alpha^* \frac{\sigma_i^2}{w_i^2} \theta_i\right). \tag{A.4}$$