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Mean topological dimension for random bundle transformations

XIANFENG MA†, JUNQI YANG† and ERCAI CHEN‡§

† *Department of Mathematics, East China University of Science and Technology, Shanghai 200237, China*
(e-mail: xianfengma@ecust.edu.cn, y30140124@mail.ecust.edu.cn)
‡ *School of Mathematical Science, Nanjing Normal University, Nanjing 210097, China*
(e-mail: ecchen@njnu.edu.cn)
§ *Center of Nonlinear Science, Nanjing University, Nanjing 210093, China*

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Abstract. We introduce the mean topological dimension for random bundle transformations, and show that continuous bundle random dynamical systems with finite topological entropy or satisfying the small boundary property have zero mean topological dimensions.

1. *Introduction*

Topological entropy plays an important role in the theory of dynamical systems. It was first introduced by Adler, Konheim and McAndrew [**1**] as an invariant of topological conjugacy for studying dynamical systems in compact topological spaces. Later, in metric spaces, a different but equivalent definition was introduced by Bowen [**6**] and Dinaburg [**13**] independently.

Shub and Weiss [**43**] developed the notion of a small set, which plays a crucial role in the study of the small entropy factors of a topological dynamical system. The mean topological dimension, an analogue of the Lebesgue covering dimension, was introduced by Gromov [**20**] for studying the dynamical properties of certain spaces of holomorphic maps and complex varieties. Lindenstrauss and Weiss [**35**] systematically investigated the mean dimension of dynamical systems for $\mathbb{Z}$-actions and used it to answer in the negative an open question raised by Auslander [**4**], concerning whether every minimal system (X, T) can be embedded in $[0, 1]^{\mathbb{Z}}$. They also defined the metric mean dimension and the small boundary property (SBP) and connected these concepts to the mean dimension. In [**33**], Lindenstrauss introduced a topological Rokhlin lemma and provided a sufficient condition for an affirmative solution to the embedding problem for extensions of aperiodic minimal systems.

The theory of mean dimension has been developed in several related aspects. Lindenstrauss and Tsukamato [**34**] constructed a minimal system with mean dimension equal to $d/2$ that cannot be embedded into $([0, 1]^d)^{\mathbb{Z}}$ and raised the conjecture as to whether a system with periodic dimension and mean dimension strictly bounded by $d/2$ can be embedded into $([0, 1]^d)^{\mathbb{Z}}$. Gutman and Tsukamoto [**50**] proved that minimal systems of mean dimension less than $d/2$ can be embedded into $([0, 1]^d)^{\mathbb{Z}}$. Gutman [**21**, **49**] introduced the notion of a marker and verified the conjecture for finite-dimensional systems and a class of dynamical systems arising in physics. The embedding theorems for extensions of aperiodic finite-dimensional systems and aperiodic subshifts were also obtained in [**22**, **49**]. Coornaert and Krieger [**11**] constructed a closed subshift for actions of discrete amenable groups and showed that the mean topological dimension can take any value in $[0, \infty]$. Li [**31**] extended the mean dimension to continuous actions of countable sofic groups. Li and Liang [**32**] showed that the mean dimension, the mean rank and the von Neumann–Lück rank coincide with each other for the induced group action on the Pontryagin dual. Gournay [**19**] developed a Hölder covariant version of mean dimension for any infinite countable group G acting on $\ell^p(G)$. Elliott, Niu and Phillips *et al* [**16**, **41**, **42**] investigated the mean dimension for C*-algebras and AH-algebras. Tsukamoto and Matsuo [**39**, **45–47**] applied the mean dimension to moduli spaces of Brody curves, of anti-self-dual (ASD) connections over $\mathbb{S}^3 \times \mathbb{R}$ and coming from the Yang–Mills gauge theory.

Random dynamical systems (RDSs) evolve by the composition of different maps instead of the iterations of one self-map. The basic framework was established by Ulam and von Neumann [**48**] and later by Kakutani [**23**] in proofs of the random ergodic theorem. Bogenschütz [**5**] gave the definition of the topological entropy for random transformations acting on one space. Furstenberg [**17**, **18**] studied the products of random matrices and the average behavior of its norms. Topics of smooth RDSs were discussed in Liu and Qian [**36**, **38**]. Other related aspects were discussed in [**9**, **26**, **37**]. Kifer [**28**] systematically studied the systems generated by random transformations chosen independently with identical distributions and introduced the notions of topological entropy and topological pressure. More general models of random transformations are formed by skew-product maps restricted to random invariant sets and act between different spaces as a class of bundle RDSs. Kifer [**27**] showed that, in this situation, the corresponding topological pressure can be obtained by almost sure limits and gave a proof of the relativized variational principle.

In the present paper, we follow [**27**, **35**] and introduce the mean topological dimension for a continuous bundle RDS, which enables one to assign a quantity to systems with infinite-dimensional state spaces or infinite topological entropy. We also define the metric mean dimension and the SBP for a bundle RDS, including the deterministic case when the probability measure is supported on a single point. We show that RDSs with finite topological entropy or satisfying the SBP have zero mean dimension.

This paper is organized as follows. In §2, we recall some background on continuous bundle RDSs, covering dimension, and some basic results with respect to set-valued measurability, as well as introducing the notion of isomorphism for continuous bundle RDSs. In §3, we prove the measurability of $D(\alpha^n(\omega))$ and show that it can be reached by a refined random cover. In §4, we give the notions of the mean topological dimension and metric mean dimension, show that the former is an isomorphism invariant and that a finite topological entropy implies that the mean topological dimension is zero. We also estimate the mean dimension for a constructed example. In §5, we introduce the notion of a small set and the SBP for random bundle transformations and point out that the SBP implies zero mean topological dimension.

2. *Preliminaries*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space together with a $\mathbb{P}$-preserving transformation ϑ and let X be a compact metric space with the distance function d and the Borel σ-algebra $\mathcal{B}_X$. Let $\mathcal{E}$ be a measurable subset of $\Omega \times X$ with respect to the product σ-algebra $\mathcal{F} \otimes \mathcal{B}_X$. The fibers $\mathcal{E}_\omega = \{x \in X : (\omega, x) \in \mathcal{E}\}$, $\omega \in \Omega$, are non-empty compact subsets of X. This means [**8**, Theorem III.2] that the mapping $\omega \mapsto \mathcal{E}_\omega$ is measurable with respect to the Borel σ-algebra induced by the Hausdorff topology on the space $2^X_\emptyset$ of compact subsets of X. A continuous bundle RDS T over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ is generated by the mappings $T_\omega : \mathcal{E}_\omega \to \mathcal{E}_{\vartheta\omega}$ so that the map $(\omega, x) \mapsto T_\omega x$ is measurable and the map $x \mapsto T_\omega x$ is continuous for $\mathbb{P}$-almost all (a.a.) ω. The family $\{T_\omega : \omega \in \Omega\}$ is called a random transformation and each T_ω maps the fiber $\mathcal{E}_\omega$ to $\mathcal{E}_{\vartheta\omega}$. The map $\Theta : \mathcal{E} \to \mathcal{E}$ defined by $\Theta(\omega, x) = (\vartheta\omega, T_\omega x)$ is called the skew-product transformation. Observe that $\Theta^n(\omega, x) = (\vartheta^n\omega, T^n_\omega x)$, where $T^n_\omega = T_{\vartheta^{n-1}\omega} \circ \cdots \circ T_{\vartheta\omega} \circ T_\omega$ for $n \geq 1$ and $T^0_\omega = \mathrm{id}$.

Definition 2.1. A closed random set A is a measurable set-valued map A from $(\Omega, \mathcal{F})$ to $(2^X_\emptyset, \mathcal{B}(2^X_\emptyset))$. An open random set U is a set-valued map U such that $\omega \mapsto X \backslash U(\omega)$ is a closed random set (see [**2**, Definition 1.6.1]).

Let $\mathrm{C}^o_\mathcal{E}$ be the set of all finite covers of $\mathcal{E}$ consisting of subsets of open random sets [**2**, Definition 1.6.1]. Similarly, we denote by C^o_K the set of finite open covers of $K \subset X$. Denote by $\mathfrak{C}^o_\mathcal{E}$ the set of all countable open random covers α of $\mathcal{E}$ with $\alpha(\omega) = \{A(\omega) : A \in \alpha\} \in \mathrm{C}^o_{\mathcal{E}_\omega}$ for $\mathbb{P}$-a.a. ω, that have an increasing sequence $\{\Omega_1 \subset \Omega_2 \subset \cdots\} \subset \mathcal{F}$ such that $\mathbb{P}(\Omega_n) \to 1$ and $\alpha \cap (\Omega_n \times X)$ is a finite family for each $n \geq 1$ [**14**, Definition 11.1].

Suppose that $\alpha = \{A^{(i)} : i \in I\} \in \mathfrak{C}^o_\mathcal{E}$. Let

$$\alpha^n(\omega) = \bigvee_{i=0}^{n-1} (T^i_\omega)^{-1} \alpha(\vartheta^i \omega).$$

Then $\alpha^n(\omega)$ is a finite open cover of $\mathcal{E}_\omega$ for $\mathbb{P}$-a.a. ω consisting of sets

$$A^{(j_0, j_1, \ldots, j_{n-1})}(\omega) = \bigcap_{i=0}^{n-1} (T^i_\omega)^{-1} A^{(j_i)}(\vartheta^i \omega), \qquad j_i \in I$$

(some of which may be empty). By the measurability of Θ, it is easy to prove that, for any $j = (j_0, j_1, \ldots, j_{n-1}) \in I^n$, the graph of $\omega \mapsto A^j(\omega)$, denoted by $A^j = \{(\omega, x) : x \in A^j(\omega)\}$, is measurable. In fact, $A^j = \bigcap_{i=0}^{n-1} (\Theta^i)^{-1} A^{(j_i)}$.

Let $K \subset X$ be compact and let α be a finite open cover of K. Write [**35**]

$$\mathrm{ord}(\alpha) = -1 + \sup_{x \in K} \sum_{U \in \alpha} 1_U(x) \quad \text{and} \quad D(\alpha) = \min_{\beta \succ \alpha} \mathrm{ord}(\beta),$$

where β runs over all finite open covers of K refining α.

For each $\omega \in \Omega$, write

$$D(\alpha^n(\omega)) = -1 + \min_{\beta} \sup_{x \in \mathcal{E}_\omega} \sum_{U \in \beta} 1_U(x),$$

where β runs over all finite open covers of $\mathcal{E}_\omega$ refining $\alpha^n(\omega)$. We will discuss the measurability and the subadditivity of the function $D(\alpha^n(\omega))$ in the next section.

We need the following standard facts, which will be used often in this paper.

LEMMA 2.1. *Let $\pi_\Omega : \Omega \times X \to \Omega$ be the natural projection and $G \in \mathcal{F} \otimes \mathcal{B}_X$. Then $\pi_\Omega G \in \mathcal{F}$ and there exists a measurable function $g : \pi_\Omega G \to X$ such that $(\omega, g(\omega)) \in G$ for each $\omega \in \pi_\Omega G$.*

LEMMA 2.2. *Let Γ be a set-valued map from a measurable space $(\mathcal{T}, \mathscr{C})$ to compact subsets of X. Consider the following properties.*

(i) $\{t \in \mathcal{T} : \Gamma(t) = \emptyset\} \in \mathscr{C}$ *and* $\{t \in \mathcal{T} : \Gamma(t) \cap U \neq \emptyset\} \in \mathscr{C}$ *for any open* $U \subset X$.

(ii) Γ *is a measurable map from* $(\mathcal{T}, \mathscr{C})$ *to* $(2^X_\emptyset, \mathcal{B}(2^X_\emptyset))$.

(iii) *The graph of* Γ *is* $\mathscr{C} \otimes \mathcal{B}_X$ *measurable.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii). Moreover, if $(\mathcal{T}, \mathscr{C}, m)$ is a complete probability space, then (iii) $\Rightarrow$ (i).

These results can be found in [**8**, Ch. III]. Lemma 2.1 is a combination of Theorems III.22 and III.23. Lemma 2.2 is from Definition 10, Proposition III.13 and a trivial application of Lemma 2.1; see also [**12**, Proposition 2.4]. We remark that the ω-section of G in Lemma 2.1 is not required to be a non-empty closed subset of X, which is a little different from the condition of the classical Kuratowski–Ryll–Nardzewski measurable selection theorem (see[**8**, Theorem III.8] or [**25**, Theorem 12.13]).

We now give the definition of the isomorphism in the category of continuous bundle RDSs. For simplicity, we only consider the case of two continuous bundle RDSs over the same measure-preserving system. For the general case, see Arnold [**2**, §1.9].

Definition 2.2. Let T, S be two continuous bundle RDSs with state space X, Y over $(\Omega, \mathcal{F}, \mathbb{P}, \vartheta)$ on $\mathcal{E}$ and $\mathcal{G}$, respectively, where X, Y are compact metric spaces. T is called isomorphic to S if there exists a family of homeomorphisms $\psi_\omega : \mathcal{E}_\omega \to \mathcal{G}_\omega$ such that $(\omega, x) \mapsto \psi_\omega(x)$ is measurable and ψ_ω preserves the random transformation, i.e., $S_\omega \circ \psi_\omega = \psi_{\vartheta\omega} \circ T_\omega$. Let $\Psi : \mathcal{E} \to \mathcal{G}$ denote $\Psi(\omega, x) = (\omega, \psi_\omega x)$. Denote by Θ, Λ the skew-product transformations of T, S, respectively. We also say that Θ is isomorphic to Λ via Ψ.

In this definition, we do not require the measurability of $(\omega, y) \mapsto \psi_\omega^{-1}(y)$. In fact, Ψ is automatically bimeasurable by our assumption. Ψ is measurable, which can be easily obtained from the measurability of the map $(\omega, x) \mapsto \psi_\omega(x)$. In the other direction, set

$$G = \{(\omega, x, \psi_\omega(x)) \in \Omega \times X \times Y : (\omega, x) \in \mathcal{E}\}.$$

Then G is $\mathcal{F} \otimes \mathcal{B}_X \otimes \mathcal{B}_Y$ measurable by Lemma 2.2 as the graph of $(\omega, x) \mapsto \psi_\omega(x)$. Let π_Ω be the natural projection from $\Omega \times X \times Y$ onto Ω. For any $y \in Y$, $E \in \mathcal{B}_X$,

$$\{\omega \in \Omega : \psi_\omega^{-1}(y) \in E\} = \pi_\Omega(G \cap (\Omega \times E \times \{y\}))$$

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is $\mathcal{F}$ measurable by Lemma 2.1. So $\psi_\omega^{-1}(y)$ is measurable in ω. Note that ψ_ω is a homeomorphism, so $\psi_\omega^{-1}(y)$ is continuous in y. It follows that $(\omega, y) \mapsto \psi_\omega^{-1}(y)$ is a Carathéodory map, and hence is measurable [**3**, Definition 8.2.7, Lemma 8.2.6].

3. *The function* $D(\alpha^n(\omega))$

Recall that two families $(E_i)_{i\in I}$ and $(F_i)_{i\in I}$, with common index set I, are called combinatorially equivalent if

$$\bigcap_{i\in J} E_i \neq \emptyset \iff \bigcap_{i\in J} F_i \neq \emptyset$$

for every subset $J \subset I$. If α and β are families of subsets of a set K that are combinatorially equivalent, then $\mathrm{ord}(\alpha) = \mathrm{ord}(\beta)$.

The proof of the measurability of $D(\alpha^n(\omega))$ relies on the following lemmas [**10**, §1.6].

LEMMA 3.1. *Let K be a normal space. Let $(F_i)_{i\in I}$ be a finite family of closed subsets of K and $(U_i)_{i\in I}$ a family of open subsets of K such that $F_i \subset U_i$ for all $i \in I$. Then there exists a family $(V_i)_{i\in I}$ of open subsets of K satisfying the following conditions.*

(i) *$F_i \subset V_i \subset \overline{V_i} \subset U_i$ for all $i \in I$.*

(ii) *The families $(F_i)_{i\in I}$, $(V_i)_{i\in I}$ and $(\overline{V_i})_{i\in I}$ are combinatorially equivalent.*

LEMMA 3.2. *Let K be a normal space and let α be a finite open cover of K. Then*

$$D(\alpha) = \min_{\gamma} \mathrm{ord}(\gamma),$$

where γ runs over all finite closed covers of K that are finer than α.

THEOREM 3.1. *For every $n \in \mathbb{N}$ and $\alpha \in \mathfrak{C}^o_{\mathcal{E}}$, the function $\omega \mapsto D(\alpha^n(\omega))$ is measurable.*

Proof. Let $\alpha = \{A^{(i)}\}$, $|J| = |\alpha^n|$. Define $f : 2^X_\emptyset \times (2^X_\emptyset)^{|J|} \to \{-1\} \cup \mathbb{N}$ by

$$f(K, (F^j)_{j\in J}) = -1 + \min_{\beta} \sup_{x\in K} \sum_{B\in\beta} 1_B(x),$$

where β runs over all families of finite open subsets of K such that

$$\beta \succ (K \backslash F^j)_{j\in J} \quad \text{and} \quad K = \bigcup_{B\in\beta} B.$$

For $K = \emptyset$ or $\beta \in \emptyset$, we set $f = -1$. Obviously,

$$D(\alpha^n(\omega)) = f(\mathcal{E}_\omega, (\mathcal{E}_\omega \backslash A^j(\omega))_{j\in J}) \geq 0.$$

Note that $\omega \mapsto \mathcal{E}_\omega$ is measurable and, for every $j \in J$, $\{(\omega, x) : x \in \mathcal{E} \backslash A^j(\omega)\} = \mathcal{E} \backslash A^j$, the graph of the multifunction $\omega \mapsto \mathcal{E}_\omega \backslash A^j(\omega)$ is measurable. By Lemma 2.2, $\omega \mapsto \mathcal{E}_\omega \backslash A^j(\omega)$ is measurable for any $j \in J$. Recalling that the composition of two measurable functions is measurable, it suffices to prove that

$$\mathcal{Q}_q = \{(K, (F^j)_{j\in J}) : 0 \leq f(K, (F^j)_{j\in J}) \leq q\}$$

is measurable with respect to the product σ-algebra $\mathcal{B}(2^X_\emptyset) \otimes \mathcal{B}(2^X_\emptyset)^{\otimes|J|}$ for any $q \in \mathbb{N}$. In fact, $\mathcal{Q}_q$ is open in $2^X_\emptyset \times (2^X_\emptyset)^{|J|}$.

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For any $(K_0, (F_0^j)_{j\in J}) \in \mathscr{Q}_q$, by Lemma 3.2 there exists a family of finite closed subsets $(E^i)_{i\in I}$ of K_0 and a map $\phi : I \to J$ such that

$$\begin{cases} K_0 = \bigcup_{i\in I} E^i, & \\ E^i \subset K_0 \backslash F_0^{\phi(i)} & \text{for all } i \in I, \\ \sum_{i\in I} 1_{E^i}(x) \le q+1 & \text{for all } x \in K_0. \end{cases}$$

Applying Lemma 3.1 to $E^i \subset K_0\backslash F_0^{\phi(i)}$, $i \in I$, there exists a family of open subsets $(V^i)_{i\in I}$ of K_0 such that

$$E^i \subset V^i \subset \overline{V^i} \subset K_0\backslash F_0^{\phi(i)} \quad \text{for all } i \in I$$

and $(E^i)_{i\in I}$, $(V^i)_{i\in I}$ and $(\overline{V^i})_{i\in I}$ are combinatorially equivalent.

$$\sup_{x\in K_0} \sum_{i\in I} 1_{\overline{V^i}}(x) = \sup_{x\in K_0} \sum_{i\in I} 1_{V^i}(x) = \sup_{x\in K_0} \sum_{i\in I} 1_{E^i}(x) \le q+1.$$

Recall that, by Kuratowski ([**29**, §17], [**30**, §42]), the family of all sets $\{F : F \subset G\}$ and of all sets $\{F : F \cap G \neq \emptyset\}$, where G runs over all open subsets of X, is an open subbase for the Hausdorff metric topology, which coincides with the Vietoris topology on $2_\emptyset^X$ [**44**, Corollary 4.2.3]. Set

$$\mathscr{K}_1 = \bigg\{K \in 2_\emptyset^X : K \subset \bigcup_{i\in I} V^i\bigg\},$$

$$\mathscr{K}_2 = \bigcap_{\substack{\Gamma\subset I\\ |\Gamma|>q+1}} \bigg\{K \in 2_\emptyset^X : K \cap \bigcap_{i\in\Gamma} \overline{V^i} = \emptyset\bigg\}$$

and, for $j \in J$, set

$$\mathscr{F}_j = \bigg\{F \in 2_\emptyset^X : F \subset \bigcap_{i\in\phi^{-1}\{j\}} (X\backslash\overline{V^i})\bigg\}.$$

Hence $\mathscr{K}_1$, $\mathscr{K}_2$ and $\mathscr{F}_j$ are all open in $2_\emptyset^X$.

$$\mathscr{O} = (\mathscr{K}_1 \cap \mathscr{K}_2) \times \prod_{j\in J} \mathscr{F}_j$$

is open in $2_\emptyset^X \times (2_\emptyset^X)^{|J|}$, so that $(K_0, (F_0^j)_{j\in J}) \in \mathscr{O} \subset \mathscr{Q}_q$. Indeed, for any $(K, (F^j)_{j\in J}) \in \mathscr{O}$, the construction of $\mathscr{K}_1$ and $(\mathscr{F}_j)_{j\in J}$ implies that $(\overline{V^i})_{i\in I}$ is always a closed cover of K which is finer than $(K\backslash F^j)_{j\in J}$ and satisfies

$$\sup_{x\in K} \sum_{i\in I} 1_{\overline{V^i}}(x) \le q+1,$$

by the construction of $\mathscr{K}_2$. By Lemma 3.2 again this implies that $f(K, (F^j)_{j\in J}) \le q$. Since $(K_0, (F_0^j)_{j\in J})$ is arbitrary, $\mathscr{Q}_q$ is open in $2_\emptyset^X \times (2_\emptyset^X)^{|J|}$.

To prove that the open set $\mathscr{Q}_q$ is measurable, we simply observe that (X, d) is a compact metric space, so ([**30**, §42, V], [**40**]) $2_\emptyset^X$ is also a compact metric space (and hence separable) with the Hausdorff metric ($\emptyset$ is an isolated point in $2_\emptyset^X$).

Hence [**24**, Lemma 1.2] the product σ-algebra coincides with the σ-algebra generated by the product topology.

$$\mathcal{B}(2^X_\emptyset)\otimes\mathcal{B}(2^X_\emptyset)^{\otimes|J|}=\mathcal{B}(2^X_\emptyset\times(2^X_\emptyset)^{|J|}).$$

It follows that the open set $\mathcal{Q}_q$ is $\mathcal{B}(2^X_\emptyset)\otimes\mathcal{B}(2^X_\emptyset)^{\otimes|J|}$ measurable and the proof of this theorem is complete. □

COROLLARY 3.1. *For any finite random cover* $\alpha\in \mathrm{C}^o_{\mathcal{E}}$, $D(\alpha(\omega))$ *is integrable.*

Proof. Assume that $|\alpha|=l$.

$$D(\alpha(\omega))\le \operatorname{ord}(\alpha(\omega))\le l-1 \Longrightarrow \int D(\alpha(\omega))\,d\mathbb{P}(\omega)\le l-1. \qquad \square$$

By the measurability of $D(\alpha^n(\omega))$, the following result shows that the minimum in the definition of $D(\alpha^n(\omega))$ can be taken over 'measurable' in ω refinements of $\alpha^n(\omega)$.

COROLLARY 3.2. *For every* $n\ge 1$ *and* $\alpha\in\mathfrak{C}^o_{\mathcal{E}}$, *there exists a random cover* $\beta\in\mathfrak{C}^o_{\mathcal{E}}$ *such that, for* $\mathbb{P}$*-a.a.* ω,

$$\beta(\omega)\succ\alpha^n(\omega)\quad \textit{and}\quad \operatorname{ord}(\beta(\omega))=D(\alpha^n(\omega)).$$

Proof. Let $p\ge 1$, $I_p=\{1,2,\ldots,p\}$. Define $h_p:2^X_\emptyset\times(2^X_\emptyset)^p\to\{-1\}\cup\mathbb{N}$ by

$$h_p(K,(F^i)_{i=1}^p)=-1+\sup_{x\in K}\sum_{i=1}^p 1_{K\setminus F^i}(x).$$

For $q\in\mathbb{N}$, let $\mathcal{R}_q=\{(K,(F^i)_{i=1}^p):h_p(K,(F^i)_{i=1}^p)\le q\}$. Then

$$\mathcal{R}_q=\bigcap_{\substack{\Gamma\subset I_p\\ |\Gamma|>q+1}}\left\{(K,(F^i)_{i=1}^p):K\cap\left(\bigcap_{i\in\Gamma}(K\setminus F^i)\right)=\emptyset\right\}$$
$$=\bigcap_{\substack{\Gamma\subset I_p\\ |\Gamma|>q+1}}\left\{(K,(F^i)_{i=1}^p):K\subset\bigcup_{i\in\Gamma}F^i\right\}.$$

Note that $(F^i)_{i\in\Gamma}\mapsto\bigcup_{i\in\Gamma}F^i$ is continuous [**29**, §17, III] and $\{(K,L):K\subset L\}$ is closed [**29**, §17, IV]. Consider the mapping $H_\Gamma:(2^X_\emptyset)^{|\Gamma|}\to(2^X_\emptyset)^2$ defined by $H_\Gamma(K,(F^i)_{i\in\Gamma})=(K,\bigcup_{i\in\Gamma}F^i)$. Then H_Γ is continuous and

$$\mathcal{R}_q=\bigcap_{\substack{\Gamma\subset I_p\\ |\Gamma|>q+1}}(H_\Gamma)^{-1}\{(K,L):K\subset L\}$$

is closed in $2^X_\emptyset\times(2^X_\emptyset)^p$. So h_p is measurable.

For the application of the measurable selection theorem to select an open cover $\beta(\omega)$ of $\mathcal{E}_\omega$, let

$$D_p=\left\{(K,(F^i)_{i=1}^p):K\ne\emptyset,\ \text{for all } i\in I_p,\ F^i\subset K,\bigcap_{i=1}^p F^i=\emptyset\right\}.$$

Then $D_p = L_p \cap M_p \cap N_p$, where

$$L_p = \{K : K \neq \emptyset\} \times (2^X_\emptyset)^p,$$
$$M_p = \bigcap_{k=1}^{p} \{(K, (F^i)_{i=1}^p) : F^k \subset K\},$$
$$N_p = 2^X_\emptyset \times \left\{(F^i)_{i=1}^p : \bigcap_{i=1}^{p} F^i = \emptyset\right\}.$$

We know that L_p, M_p, N_p is clopen, closed and open [**29**, §17, V] in $(2^X_\emptyset) \times (2^X_\emptyset)^p$. So D_p is measurable.

Define $h'_p : \Omega \times (2^X_\emptyset)^p \to \{-1\} \bigcup \mathbb{N}$ by

$$h'_p(\omega, (F^i)_{i=1}^p) = h_p(\mathcal{E}_\omega, (F^i)_{i=1}^p).$$

Then $h'_p(\omega, (F^i)_{i=1}^p) = \text{ord}((\mathcal{E}_\omega \backslash F^i)_{i=1}^p)$. Recall that $\omega \mapsto \mathcal{E}_\omega$ is measurable. By defining $\psi_p(\omega, (F^i)_{i=1}^p) = (\mathcal{E}_\omega, (F^i)_{i=1}^p)$, one can deduce that $h'_p = h_p \circ \psi_p$ is measurable. Let

$$B'_p = \{(\omega, (F^i)_{i=1}^p) : h'_p(\omega, (F^i)_{i=1}^p) \leq D(\alpha^n(\omega))\}.$$

By Theorem (3.1), we know that $D(\alpha^n(\omega))$ is measurable. So B'_p is $\mathcal{F} \otimes \mathcal{B}(2^X_\emptyset)^{\otimes p}$ measurable. To apply the selection theorem to select a refinement $\beta(\omega)$ of $\alpha^n(\omega)$, we need to restrict the domain of h'_p. Suppose that $\alpha = \{A^{(i)} : i \in I\}$ with $|I| \leq |\mathbb{N}|$. Let $J = I^n$ and let J^{I_p} denote the set of all mappings from I_p to J. Set

$$D'_p = \bigcup_{\phi \in J^{I_p}} \bigcap_{k=1}^{p} \{(\omega, (F^i)_{i=1}^p) : \mathcal{E}_\omega \backslash A^{\phi(k)}(\omega) \subset F^k\}.$$

Still, note that $\omega \mapsto \mathcal{E}_\omega \backslash A^{\phi(i)}(\omega)$ is measurable and $\{(K, F) : K \subset F\}$ is closed. So D'_p is $\mathcal{F} \otimes \mathcal{B}(2^X_\emptyset)^{\otimes p}$ measurable, since it is a countable union of measurable sets.

Let $D_p(\mathcal{E}_\omega) = \{(F^i)_{i=1}^p : (\mathcal{E}_\omega, (F^i)_{i=1}^p) \in D_p\}$ and let $B'_p(\omega)$, $D'_p(\omega)$ be the ω-section of B'_p, D'_p. Consider the multifunction $\omega \mapsto D_p(\mathcal{E}_\omega) \cap B'_p(\omega) \cap D'_p(\omega) = B_p(\omega)$. Now we can say that its graph

$$B_p = (\psi_p)^{-1} D_p \cap B'_p \cap D'_p \in \mathcal{F} \otimes \mathcal{B}(2^X_\emptyset)^{\otimes p} = \mathcal{F} \otimes \mathcal{B}((2^X_\emptyset)^p).$$

Let π_Ω be the projection of $\Omega \times (2^X_\emptyset)^p$ onto Ω. By Lemma 2.1, $\pi_\Omega B_p \in \mathcal{F}$. By the definition of B_p and $\mathcal{D}(\alpha^n(\omega))$, if $(\omega, (F^i)_{i=1}^p) \in B_p$, then $(\omega, ((F^i)_{i=1}^p, \mathcal{E}_\omega)) \in B_{p+1}$, and for $\mathbb{P}$-a.a. ω, there must exist a refinement $\mathrm{C}^o_{\mathcal{E}_\omega} \ni \gamma \succ \alpha^n(\omega)$ such that $\text{ord}(\gamma) = D(\alpha^n(\omega))$. So $\pi_\Omega B_p \subset \pi_\Omega B_{p+1}$ and $\mathbb{P}(\bigcup_{p=1}^\infty \pi_\Omega B_p) = 1$. Since $\mathcal{F}$ is complete and $2^X_\emptyset$ is a compact metric space, applying Lemma 2.1 to the multifunction $\omega \mapsto B_p(\omega)$ gives a measurable selector $\beta_p : \pi_\Omega B_p \to (2^X_\emptyset)^p$ such that $\beta_p(\omega) \in B_p(\omega)$. Set $B_0 = \emptyset$, $\widetilde{\Omega}_p = \pi_\Omega B_p \backslash \pi_\Omega B_{p-1}$ for $p \geq 1$. Let $\beta(\omega) = \mathcal{E}_\omega \backslash \beta_p(\omega)$ for $\omega \in \widetilde{\Omega}_p$. Then β clearly satisfies the required conditions by Lemma 2.2 with the increasing sequence $(\pi_\Omega B_p)_{p \geq 1} \subset \mathcal{F}$. □

4. *Mean topological dimension*

Before introducing the notion of mean topological dimension for a continuous bundle RDS, we first review some properties of $D(\alpha)$. These lemmas can be found in [**35**] and [**10**].

LEMMA 4.1. *Let X be a topological space. Let α and β be finite open covers of X such that $\alpha \succ \beta$. Then $D(\alpha) \geq D(\beta)$.*

LEMMA 4.2. *Let X and Y be topological spaces and let $f : X \to Y$ be a continuous map. Let β be a finite open cover of Y. Then*

$$D(f^{-1}(\beta)) \leq D(\beta).$$

LEMMA 4.3. *Let X be a normal space. Let α and β be finite open covers of X. Then*

$$D(\alpha \vee \beta) \leq D(\alpha) + D(\beta).$$

Let X and Y be topological spaces. Let α be a finite open cover of X. A continuous map $f : X \to Y$ is said to be α-compatible if there exists a finite open cover β of Y such that $f^{-1}(\beta) \succ \alpha$. We will use the notation $f \succ \alpha$ to denote that f is α-compatible.

LEMMA 4.4. *Let X be a compact space and Y be a topological space. Let $f : X \to Y$ be a continuous function such that for every $y \in Y$, $f^{-1}\{y\}$ is a subset of some $U \in \alpha$. Then f is α-compatible.*

LEMMA 4.5. *Let X be a topological space and let α be a finite open cover of X. Suppose that there is a topological space Y and an α-compatible continuous map $f : X \to Y$. Then $D(\alpha) \leq \dim Y$.*

The following lemma [**2**, Theorem 3.3.2] can be seen as a random version of Fekete's subadditive lemma.

LEMMA 4.6. (Kingman's subadditive ergodic theorem) *Let ϑ be a measure-preserving transformation on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $(f_n)_{n\geq 1}$ be a sequence of non-negative random variables such that*

$$f_{n+m}(\omega) \leq f_n(\omega) + f_m(\vartheta^n \omega).$$

If $f_n \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ for all n, then there exists an integrable function $f : \Omega \to \mathbb{R}$ such that, for $\mathbb{P}$-a.a. ω, $n^{-1} f_n(\omega) \to f(\omega)$. Moreover, the convergence also holds in $\mathbb{L}^1$ and satisfies

$$\int f \, d\mathbb{P} = \lim_{n\to\infty} \frac{1}{n} \int f_n \, d\mathbb{P} = \inf_{n\geq 1} \frac{1}{n} \int f_n \, d\mathbb{P} < \infty.$$

If, in addition, ϑ is ergodic, then $f(\omega) = \int f \, d\mathbb{P}$ for $\mathbb{P}$-a.a. ω.

Let $\alpha \in \mathfrak{C}^o_{\mathcal{E}}$. By Lemmas 4.2 and 4.3,

$$\begin{aligned} D(\alpha^{n+m}(\omega)) &\leq D(\alpha^n(\omega)) + D\bigg(\bigvee_{i=n}^{n+m-1} (T^i_\omega)^{-1}\alpha(\vartheta^i\omega)\bigg) \\ &= D(\alpha^n(\omega)) + D\bigg((T^n_\omega)^{-1}\bigvee_{i=0}^{m-1}(T^i_\omega)^{-1}\alpha(\vartheta^{n+i}\omega)\bigg) \\ &\leq D(\alpha^n(\omega)) + D\bigg(\bigvee_{i=0}^{m-1}(T^i_\omega)^{-1}\alpha(\vartheta^i\vartheta^n\omega)\bigg) \\ &= D(\alpha^n(\omega)) + D(\alpha^m(\vartheta^n\omega)). \end{aligned}$$

Hence $D(\alpha^n(\omega))$ is subadditive. Set

$$\mathcal{D} = \bigg\{\alpha \in \mathfrak{C}^o_{\mathcal{E}} : \int D(\alpha(\omega))\, d\mathbb{P}(\omega) < \infty\bigg\}.$$

By Corollary 3.1, $\mathrm{C}^o_{\mathcal{E}} \subset \mathcal{D}$. Note also that by Lemma 4.3, if $\alpha, \beta \in \mathcal{D}$, then $\alpha \vee \beta \in \mathcal{D}$. Thus $\mathcal{D}$ is a directed set.

Suppose $\alpha \in \mathcal{D}$. Then $D(\alpha^n(\omega)) \leq nD(\alpha(\omega))$ is integrable for every $n \geq 1$. Now write $D(\alpha, T, \omega)$ for the $\mathbb{P}$-almost surely (a.s.) limit of $(1/n)D(\alpha^n(\omega))$. Then, by Lemma 4.6,

$$\int D(\alpha, T, \omega)\, d\mathbb{P}(\omega) = \lim_{n\to\infty} \frac{1}{n}\int D(\alpha^n(\omega))\, d\mathbb{P}(\omega) = \inf_{n\geq 1}\frac{1}{n}\int D(\alpha^n(\omega))\, d\mathbb{P}(\omega).$$

Remark 4.1. $0 \leq \int D(\alpha, T, \omega)\, d\mathbb{P} \leq \int D(\alpha(\omega))d\mathbb{P}$.

Remark 4.2. $\alpha(\omega) \succ \beta(\omega)$, $\mathbb{P}$-a.a. $\omega \implies \int D(\alpha, T, \omega)\, d\mathbb{P} \geq \int D(\beta, T, \omega)\, d\mathbb{P}$.

PROPOSITION 4.1. *Let* $\beta \in \mathcal{D}$ *with an increasing sequence* $(\Omega_r)_{r\geq 1} \subset \mathcal{F}$ *satisfying* $\mathbb{P}(\Omega_r) \to 1$, *where each* $\beta \cap (\Omega_r \times X)$, $r \geq 1$, *is a finite family. Let*

$$\mathrm{C}^o_{\mathcal{E}} \ni \beta_r = [\beta \cap (\Omega_r \times X)] \bigcup [(\Omega^c_r \times X) \cap \mathcal{E}].$$

Then

$$\int D(\beta_r, T, \omega)\, d\mathbb{P} \to \int D(\beta, T, \omega)\, d\mathbb{P}.$$

Proof. It is not hard to check that $D(\beta_r, T, \omega)$ is increasing in r and that each member is pointwise less than $D(\beta, T, \omega)$ in ω by Lemma 4.1. It follows that

$$\lim_{r\to\infty}\int D(\beta_r, T, \omega)\, d\mathbb{P} \leq \int D(\beta, T, \omega)\, d\mathbb{P}. \tag{1}$$

In the other direction, set

$$I_{r,n}(\omega) = \{0 \leq k < n : \vartheta^k\omega \in \Omega_r\},$$

$$I^{r,n}(\omega) = \{0 \leq k < n : \vartheta^k\omega \notin \Omega_r\}$$

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and note that, by Lemmas 4.1 and 4.3,

$$\begin{aligned} D(\beta^n(\omega)) &\le D\bigg(\bigvee_{k\in I_{r,n}(\omega)} (T_\omega^k)^{-1}\beta(\vartheta^k\omega)\bigg) + D\bigg(\bigvee_{k\in I^{r,n}(\omega)} (T_\omega^k)^{-1}\beta(\vartheta^k\omega)\bigg) \\ &= D(\beta_r^n(\omega)) + D\bigg(\bigvee_{k\in I^{r,n}(\omega)} (T_\omega^k)^{-1}\beta(\vartheta^k\omega)\bigg) \\ &\le D(\beta_r^n(\omega)) + \sum_{k\in I^{r,n}(\omega)} D(\beta(\vartheta^k\omega)). \end{aligned}$$

It follows that

$$\begin{aligned} &\int D(\beta, T, \omega)\, d\mathbb{P}(\omega) \\ &\quad\le \int D(\beta_r, T, \omega)\, d\mathbb{P}(\omega) + \limsup_{n\to\infty} \frac{1}{n}\int \sum_{k=0}^{n-1} 1_{\Omega\setminus\Omega_r}(\vartheta^k\omega) D(\beta(\vartheta^k\omega))\, d\mathbb{P}(\omega) \\ &\quad\le \int D(\beta_r, T, \omega)\, d\mathbb{P}(\omega) + \int 1_{\Omega\setminus\Omega_r}(\omega) D(\beta(\omega))\, d\mathbb{P}(\omega). \end{aligned}$$

By the assumption that $\mathbb{P}(\Omega_r) \to 1$ and $\beta \in \mathcal{D}$,

$$\lim_{r\to\infty} \int 1_{\Omega\setminus\Omega_r}(\omega) D(\beta(\omega))\, d\mathbb{P}(\omega) = 0,$$

and hence

$$\int D(\beta, T, \omega)\, d\mathbb{P}(\omega) \le \lim_{r\to\infty} \int D(\beta_r, T, \omega)\, d\mathbb{P}(\omega). \tag{2}$$

Combining (1) with (2) we complete the proof. □

PROPOSITION 4.2. *For every $\beta \in \mathcal{D}$ and $\epsilon > 0$, there exists a finite open cover $\mathcal{V}$ of X such that*

$$\int D(\mathcal{V}, T, \omega)\, d\mathbb{P}(\omega) \ge \int D(\beta, T, \omega)\, d\mathbb{P}(\omega) - \epsilon,$$

where $D(\mathcal{V}, T, \omega) = \lim_{n\to\infty} (1/n) D(\mathcal{V}^n(\omega))$ with $\mathcal{V}(\omega) = \mathcal{V} \cap \mathcal{E}_\omega$.

Proof. Recall that the Lebesgue number $\lambda(\beta(\omega))$ of an open cover $\beta(\omega)$ on the fiber $\mathcal{E}_\omega$ is the largest λ such that, for every $x \in \mathcal{E}_\omega$, there is a $B(\omega) \in \beta(\omega)$ such that $d(x, \mathcal{E}_\omega\setminus B(\omega)) \ge \lambda$. Suppose that $\beta = \{B^{(1)}, B^{(2)}, \ldots, B^{(l_q)}\}$ on Ω_q, where $(\Omega_q)_{q\ge1} \subset \mathcal{F}$ is the family of increasing sets in the definition of $\mathfrak{C}_{\mathcal{E}}^o$.

$$\lambda(\beta(\omega)) = \frac{1}{l_q} \min_{x\in\mathcal{E}_\omega} \sum_{i=1}^{l_q} d(x, \mathcal{E}_\omega\setminus B^{(i)}(\omega)) \quad \text{on } \Omega_q.$$

Set $d(x, \emptyset) = \infty$, for convenience.

Note that $(\omega, x) \mapsto d(x, \mathcal{E}_\omega\setminus B^{(i)}(\omega))$ is a Carathéodory function and hence $\mathcal{F}\otimes\mathcal{B}_X$ measurable [**8**, Lemma III.14]. By Lemma III.39 from [**8**], $\lambda(\beta(\omega))$ is measurable.

By the compactness of $\mathcal{E}_\omega$ we see that $\lambda(\beta(\omega)) > 0$ for all ω. Set

$$\Omega_{p,q} = \bigg\{\omega \in \Omega_q : \lambda(\beta(\omega)) \ge \frac{1}{p}\bigg\}.$$

Then $\bigcup_{p=1}^{\infty} \Omega_{p,q} = \Omega_q$. Let $\Omega_{0,q} = \emptyset$, $\widetilde{\Omega}_{p,q} = \Omega_{p,q}\setminus\Omega_{p-1,q}$ and $\widetilde{\Omega}_p = \bigcup_{q\ge1} \widetilde{\Omega}_{p,q}$.

Now we can choose a maximal set $S_p \subset X$ of n_p points in X such that $d(x, y) \geq 1/(3p)$ for all $x, y \in S_p$, $x \neq y$, and let $\mathcal{U}_p$ be the cover of X by open balls of radius $2/(3p)$ centered at points of S_p. Define an open random cover α by

$$\alpha(\omega) = \{(\widetilde{\Omega}_p \times U) \cap \mathcal{E} : U \in \mathcal{U}_p\} \quad \text{on } \widetilde{\Omega}_p \subset \Omega.$$

Then, for $\omega \in \widetilde{\Omega}_p$, $\operatorname{diam} \alpha(\omega) \leq \operatorname{diam} \mathcal{U}_p < (1/p) \leq \lambda(\beta(\omega))$, which implies that $\alpha(\omega) \succ \beta(\omega)$. Now $\lambda(\alpha(\omega)) \geq 1/(3p)$ by the construction of S_p.

Set $\Omega_r = \bigcup_{p=1}^{r} \widetilde{\Omega}_p$ and let $\alpha_r = [\alpha \cap (\Omega_r \times X)] \bigcup [(\Omega_r^c \times X) \cap \mathcal{E}]$. It is not hard to check that the sequence of non-negative random variables $D(\alpha_r^n(\omega))$ increases to $D(\alpha^n(\omega))$ for every $n \geq 1$, $\mathbb{P}$-a.a. ω, as $r \to \infty$. Again, choose a sequence $(\mathcal{V}_r)_{r \geq 1}$ of finite open covers of X such that $\operatorname{diam} \mathcal{V}_r < 1/(3r)$. Recall that $\mathcal{V}_r(\omega) = \mathcal{V}_r \cap \mathcal{E}_\omega$, so $\mathcal{V}_r(\omega) \succ \alpha_r(\omega)$ for $\mathbb{P}$-a.a. ω and

$$D(\mathcal{V}_r^n(\omega)) \geq D(\alpha_r^n(\omega)) \to D(\alpha^n(\omega)) \geq D(\beta^n(\omega)).$$

It follows from Proposition 4.1 that

$$\int D(\mathcal{V}_r, T, \omega)\, d\mathbb{P}(\omega) \geq \int D(\alpha_r, T, \omega)\, d\mathbb{P}(\omega) \to \int D(\alpha, T, \omega)\, d\mathbb{P}(\omega) \geq \int D(\beta, T, \omega)\, d\mathbb{P}(\omega).$$

Hence, for a sufficiently large value of r, $\mathcal{V}_r$ will satisfy our requirements. □

Definition 4.1. The mean topological dimension of a bundle RDS T on $\mathcal{E}$, denoted by $\mathbb{E}\mathrm{mdim}(T)$, is defined by

$$\mathbb{E}\mathrm{mdim}(T) = \mathbb{E}\mathrm{mdim}(\mathcal{E}, T) = \sup_{\alpha} \int D(\alpha, T, \omega)\, d\mathbb{P}(\omega),$$

where α runs over all countable random covers in $\mathcal{D}$.

Remark 4.3. $0 \leq \mathbb{E}\mathrm{mdim}(T) \leq \infty$.

Remark 4.4. By Proposition 4.2, one can obtain the same quantity $\mathbb{E}\mathrm{mdim}(T)$ in Definition 4.1 when α runs over all finite open random covers of $\mathcal{E}$, or all finite open covers $\mathcal{V}$ of X.

PROPOSITION 4.3. *Let T, S be two continuous bundle RDSs on $\mathcal{E}$ and $\mathcal{G}$, respectively. If T and S are isomorphic, then $\mathbb{E}\mathrm{mdim}\, T = \mathbb{E}\mathrm{mdim}\, S$.*

Proof. Suppose T and S are isomorphic via ψ_ω and let $\Psi : \mathcal{E} \to \mathcal{G}$ be given by $\Psi(\omega, x) = (\omega, \psi_\omega x)$. Let $\alpha \in \mathrm{C}_{\mathcal{E}}^{o}$. Then $\beta = \Psi(\alpha) \in \mathrm{C}_{\mathcal{G}}^{o}$. As $S_\omega \circ \psi_\omega = \psi_{\vartheta\omega} \circ T_\omega$, the homeomorphism ψ_ω sends $D(\alpha^n(\omega))$ to $D(\beta^n(\omega))$ for $n \geq 1$. It follows that $D(\alpha^n(\omega)) = D(\beta^n(\omega))$ and hence $D(\alpha, T, \omega) = D(\beta, S, \omega)$. Since $\alpha \mapsto \Psi(\alpha)$ provides a bijective correspondence between the finite open random covers of $\mathcal{E}$ and those of $\mathcal{G}$, by Remark 4.4 we deduce that $\mathbb{E}\mathrm{mdim}(T) = \mathbb{E}\mathrm{mdim}(S)$. □

PROPOSITION 4.4. *If X has finite topological dimension, then* $\mathbb{E}\mathrm{mdim}(T) = 0$.

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Proof. Let $\alpha \in \mathfrak{C}^o_{\mathcal{E}}$. Then

$$D(\alpha^n(\omega)) \leq \dim \mathcal{E}_\omega \leq \dim X < \infty$$

and hence $\mathbb{E}\text{mdim}(T) = 0$. □

PROPOSITION 4.5. *If a closed random set $\mathcal{C} \subset \mathcal{E}$ is forward invariant, i.e.,*

$$\mathcal{C}_\omega \subset (T_\omega)^{-1}\mathcal{C}_{\vartheta\omega} \quad \mathbb{P}\text{-a.a. } \omega,$$

then $\mathbb{E}\text{mdim}(\mathcal{C}, T|_{\mathcal{C}}) \leq \mathbb{E}\text{mdim}(\mathcal{E}, T)$.

Proof. Since $\mathcal{C}$ is forward invariant, $(\mathcal{C}, T|_{\mathcal{C}})$ is a continuous bundle RDS. For every $\alpha = \{A^{(i)} : i \in I\} \in \mathfrak{C}^o_{\mathcal{C}}$, let $B^{(i)} = \mathcal{E}\backslash(\mathcal{C}\backslash A^{(i)})$. Then $\beta = \{B^{(i)} : i \in I\} \in \mathfrak{C}^o_{\mathcal{E}}$. We claim that $D(\alpha^n(\omega)) \leq D(\beta^n(\omega))$. It suffices to prove that, for every $n \geq 1$ and $\mathbb{P}$-a.a. ω, if $\gamma \succ \beta^n(\omega)$ is an open cover of $\mathcal{E}_\omega$, then there exists an open cover $\gamma' \succ \alpha^n(\omega)$ of $\mathcal{C}_\omega$ such that $\text{ord}(\gamma') \leq \text{ord}(\gamma)$. In other words, it suffices to prove that

$$\sum_{C\in\gamma'} 1_C(x) \leq \sum_{U\in\gamma} 1_U(x), \quad x \in \mathcal{C}_\omega \subset \mathcal{E}_\omega.$$

Indeed, we can take $\gamma' = \gamma \cap \mathcal{C}_\omega$. □

Example 4.1. Let $X = [0, 1]^{\mathbb{Z}}$ and let $a : \Omega \to [0, 1]$ be a random variable. Set $\mathcal{E}_\omega = \{x = (x_j) : 0 \leq x_j \leq a(\vartheta^j\omega)\}$ and let $T_\omega = \sigma$ be the left shift $(\sigma x)_j = x_{j+1}$. If ϑ is invertible and ergodic, then

$$\mathbb{P}\{\omega : a(\omega) > 0\} \leq \mathbb{E}\text{mdim}(\mathcal{E}, T) \leq 1.$$

Proof. The left hand is trivial if $a(\omega) = 0$, $\mathbb{P}$-almost surely. Assume that $\mathbb{P}\{\omega : a(\omega) > 0\} > 0$. Write $I(\omega) = \{j \in \mathbb{Z} : a(\vartheta^j\omega) > 0\}$, $I_n(\omega) = I(\omega) \cap [0, n)$. Note that ϑ is ergodic, so

$$\frac{|I_n(\omega)|}{n} = \frac{1}{n}\sum_{j=0}^{n-1} 1_{\{\omega:a(\omega)>0\}}(\vartheta^j\omega) \to \mathbb{P}\{\omega : a(\omega) > 0\}.$$

It follows that $|I(\omega)| = |\mathbb{N}|$. Define $\psi_\omega : \mathcal{E}_\omega \to X$ by

$$\psi_\omega : (x_j)_{j\in\mathbb{Z}} \mapsto \left(\frac{x_j}{a(\vartheta^j\omega)}\right)_{j\in I(\omega)}$$

and set

$$S_\omega = \begin{cases} \text{id} & \text{if } a(\omega) = 0, \\ \sigma & \text{if } a(\omega) > 0. \end{cases}$$

Then $S_\omega \circ \psi_\omega = \psi_{\vartheta\omega} \circ T_\omega$ and $S^i_\omega = \sigma^{|I_i(\omega)|}$ for $i \geq 0$. We claim that $\{\psi_\omega : \omega \in \Omega\}$ is an isomorphism from $(\mathcal{E}, T)$ to $(\Omega \times X, S)$. Note that $\mathcal{E}_\omega$ is homeomorphic to $\prod_{j\in I(\omega)}[0, a(\vartheta^j\omega)]$. It is not hard to check that $\psi_\omega : \mathcal{E}_\omega \to X$ is a homeomorphism. For any $x = (x_j) \in \mathcal{E}_\omega$, $E_j \in \mathcal{B}([0, 1])$, set $f_j(t) = x_j/t$. Then

$$\bigcap_{j\in I(\omega)} \left\{\omega : \frac{x_j}{a(\vartheta^j\omega)} \in E_j\right\} = \bigcap_{j\in I(\omega)} \{\omega : a(\vartheta^j\omega) \in (f_j)^{-1}E_j\} \in \mathcal{F},$$

so $\omega \mapsto \psi_\omega(x)$ is measurable. Then $(\omega, x) \mapsto \psi_\omega(x)$ is a Carathódory map and hence is measurable.

Pick a sequence $\mathcal{A}_k$ of finite open covers of X such that $\operatorname{diam} \mathcal{A}_k \to 0$. Since $S_\omega^i = \sigma^{|I_i(\omega)|}$ for $i \geq 0$, and $\bigvee_{i=0}^{n-1} (S_\omega^i)^{-1}\mathcal{A}_k \succ \bigvee_{i=0}^{|I_n(\omega)|} (\sigma^i)^{-1}\mathcal{A}_k$,

$$
\begin{aligned}
\mathbb{E}\mathrm{mdim}(\Omega \times X, S) &= \lim_{k\to\infty} \lim_{n\to\infty} \frac{1}{n} D(\mathcal{A}_k^n(\omega)) \\
&= \lim_{k\to\infty} \lim_{n\to\infty} \frac{|I_n(\omega)|}{n} \frac{1}{|I_n(\omega)|} D\bigg(\bigvee_{i=0}^{n-1} (S_\omega^i)^{-1}\mathcal{A}_k\bigg) \\
&\geq \mathbb{P}\{\omega : a(\omega) > 0\}\, \mathrm{mdim}(X, \sigma).
\end{aligned}
$$

By Lindenstrauss and Weiss [**35**, Proposition 3.3], we know that $\mathrm{mdim}(X, \sigma) = 1$. By Proposition 4.3, $\mathbb{E}\mathrm{mdim}(\mathcal{E}, T) \geq \mathbb{P}\{\omega : a(\omega) > 0\}$. On the other hand, since $(\mathcal{E}, T)$ can be embedded into $(\Omega \times X, \sigma)$, it follows from Proposition 4.5 that $\mathbb{E}\mathrm{mdim}(\mathcal{E}, T) \leq \mathbb{E}\mathrm{mdim}(\Omega \times X, \sigma) = 1$. □

For each $n \geq 1$ and a positive $\epsilon > 0$, define a family of metrics d_n^ω on $\mathcal{E}_\omega$ by the formula

$$
d_n^\omega(x, y) = \max_{0\leq k<n} d(T_\omega^k x, T_\omega^k y), \quad x, y \in \mathcal{E}_\omega.
$$

Fix ω and let $\alpha = \{A_i : i = 1, 2, \ldots, l\}$ be a finite open cover of $\mathcal{E}_\omega$. Define the mesh of α with respect to the metric d_n^ω by

$$
\operatorname{diam}(\alpha, d_n^\omega) = \max_{1\leq i\leq l} \operatorname{diam}(A_i, d_n^\omega),
$$

where

$$
\operatorname{diam}(A_i, d_n^\omega) = \sup\{d_n^\omega(x, y) : x, y \in A_i\}.
$$

Let $\mathrm{cov}(\omega, \epsilon, n)$ be the minimal cardinality of finite open covers of $\mathcal{E}_\omega$ by sets of d_n^ω-diameter less than ϵ, that is,

$$
\mathrm{cov}(\omega, \epsilon, n) = \inf\{|\alpha| : \alpha \in \mathrm{C}_{\mathcal{E}_\omega}^o, \operatorname{diam}(\alpha, d_n^\omega) < \epsilon\}.
$$

One familiar with the classic topological entropy theory in the deterministic case will have no difficulty in extending the notion of separated sets to the random case. A set $F \subset \mathcal{E}_\omega$ is said to be (ω, ϵ, n)-separated if $x, y \in F$, $x \neq y$ implies that $d_n^\omega(x, y) \geq \epsilon$. We define

$$
\mathrm{sep}(\omega, \epsilon, n) = \sup\{|F| : F \subset \mathcal{E}_\omega \text{ is an } (\omega, \epsilon, n)\text{-separated set}\}
$$

as the maximum cardinality of (ω, ϵ, n)-separated sets of $\mathcal{E}_\omega$. As in the deterministic case [**7**, Lemma 2.5.1],

$$
\mathrm{cov}(\omega, 2\epsilon, n) \leq \mathrm{sep}(\omega, \epsilon, n) \leq \mathrm{cov}(\omega, \epsilon, n). \tag{3}
$$

Set

$$
S(\omega, \epsilon) = \lim_{n\to\infty} \frac{1}{n} \log \mathrm{cov}(\omega, \epsilon, n).
$$

Notice that $\log \mathrm{cov}(\omega, \epsilon, n)$ is a subadditive function of n and the limit above can be replaced by an infimum. Obviously, S is monotone non-decreasing as non-random

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$\epsilon \to 0$, and, for the purpose of measuring how fast it increases, we define the metric mean dimension of T on the fiber $\mathcal{E}_\omega$ as

$$\mathrm{mdim_M}(T, \omega) = \liminf_{\epsilon \to 0} \frac{S(\omega, \epsilon)}{|\log \epsilon|}.$$

It follows from (3) and the fact that $\lim_{\epsilon \to 0} (\log \epsilon / \log 2\epsilon) = 1$ that

$$\begin{aligned}\mathrm{mdim_M}(T, \omega) &= \liminf_{\epsilon \to 0} \frac{1}{|\log \epsilon|} \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{sep}(\omega, \epsilon, n) \\ &= \liminf_{\epsilon \to 0} \frac{1}{|\log \epsilon|} \liminf_{n \to \infty} \frac{1}{n} \log \mathrm{sep}(\omega, \epsilon, n).\end{aligned}$$

Since $\mathrm{sep}(\omega, \epsilon, n)$ is measurable (even for random $\epsilon(\omega)$, [**27**, Lemma 1.2]), it follows that $\mathrm{mdim_M}(T, \omega)$ is measurable as well. We have the following definition.

Definition 4.2. The metric mean topological dimension of bundle RDS T, denoted by $\mathbb{E}\mathrm{mdim_M}(T)$, is defined by

$$\mathbb{E}\mathrm{mdim_M}(T) = \int \mathrm{mdim_M}(T, \omega)\, d\mathbb{P}(\omega).$$

Set

$$S'(\omega, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log \mathrm{sep}(\omega, \epsilon, n)$$

and compare

$$\mathbb{E}\mathrm{mdim_M}(T) = \int \liminf_{\epsilon \to 0} \frac{S'(\omega, \epsilon)}{|\log \epsilon|}\, d\mathbb{P}(\omega)$$

with [**27**, Proposition 1.6]

$$h_{\mathrm{top}}(T) = \int \lim_{\epsilon \to 0} S'(\omega, \epsilon)\, d\mathbb{P}(\omega).$$

We can immediately state the relationship between the metric mean dimension and the topological entropy of a system.

PROPOSITION 4.6. *If* $\mathbb{E}\mathrm{mdim_M}(T) \neq 0$, *then* $h_{top}(T) = \infty$.

Proof. If $\mathbb{E}\mathrm{mdim_M}(T) > 0$, then there exists a set $A \subset \Omega$ with positive measure such that $\mathrm{mdim_M}(T, \omega) > 0$ on A. Let

$$A_k = \left\{ \omega \in \Omega : \liminf_{\epsilon \to 0} \frac{S'(\omega, \epsilon)}{|\log \epsilon|} \geq \frac{1}{k} \right\}$$

and we know that $\bigcup_{k=1}^{\infty} A_k = A$, so there exists an integer m such that $\mathbb{P}(A_m) > 0$ and, for any $\omega \in A_m$,

$$\lim_{\epsilon \to 0} S'(\omega, \epsilon) = \infty.$$

Hence

$$h_{\mathrm{top}}(T) = \int \lim_{\epsilon \to 0} S'(\omega, \epsilon)\, d\mathbb{P}(\omega) \geq \int_{A_m} \lim_{\epsilon \to 0} S'(\omega, \epsilon)\, d\mathbb{P}(\omega) = \infty. \qquad \square$$

COROLLARY 4.1. *If* T *has finite topological entropy, then* $\mathbb{E}\mathrm{mdim_M}(T) = 0$.

We conclude this section by pointing out that the mean topological dimension is not larger than the metric mean dimension for a continuous bundle RDS as well as in the deterministic case.

By Remark 4.4, it suffices to prove that, for any $\mathcal{A} = \{A_i\}_{i=1}^{l} \in \mathrm{C}_X^o$ and $\mathbb{P}$-a.a. ω,

$$\lim_{n\to\infty} \frac{1}{n} D(\mathcal{A}^n(\omega)) \le \mathrm{mdim}_{\mathrm{M}}(T, \omega).$$

By Lemma 4.1, without loss of generality, we can refine $\mathcal{A}$ to be of the form

$$\mathcal{A} = \bigvee_{i=1}^{r} \{U_i, V_i\},$$

where, for every $1 \le i \le r$, $\{U_i, V_i\}$ is a two element open cover of X.

Define $\phi_i^{\omega} : \mathcal{E}_\omega \to [0, 1]$ by

$$\phi_i^{\omega}(x) = \frac{d(x, X\backslash V_i(\omega))}{d(x, X\backslash V_i(\omega)) + d(x, X\backslash U_i(\omega))}$$

and let $\Phi^{\omega}(x) = (\phi_1^{\omega}(x), \phi_2^{\omega}(x), \ldots, \phi_r^{\omega}(x))$. Notice that ϕ_i^{ω} is Lipschitz. Let L_ω be a bound on the Lipschitz constants of all the $(\phi_i^{\omega})_{i=1}^{r}$. For any integer $b \ge 1$, define $F_b^{\omega} : \mathcal{E}_\omega \to [0, 1]^{rb}$ by

$$F_b^{\omega}(x) = (\Phi^{\omega}(x), \Phi^{\vartheta\omega}(T_\omega x), \ldots, \Phi^{\vartheta^{b-1}\omega}(T_\omega^{b-1} x)).$$

From Lemma 4.4, we see that $F_b^{\omega} \succ \mathcal{A}^b(\omega)$, and

$$\|F_b^{\omega}(x) - F_b^{\omega}(y)\|_\infty \le L_\omega d_b^{\omega}(x, y).$$

As usual, if $S \subset \{1, 2, \ldots, rb\}$, denote by $F_b^{\omega}(x)_S \in [0, 1]^{|S|}$ the projection of $F_b^{\omega}(x)$ to the coordinates in the index set S. We need the following two fiberwise lemmas, which correspond to [**35**, Lemmas 4.3 and 4.4].

LEMMA 4.7. *Let* $\epsilon > 0$, $D(\omega) = \mathrm{mdim}_{\mathrm{M}}(T, \omega)$. *If b is larger than some $N(\omega, \epsilon)$, there is a $\xi(\omega) \in (0, 1)^{rb}$ such that, for any $|S| \ge (D(\omega) + \epsilon)b$,*

$$\xi(\omega)_S \notin F_b^{\omega}(\mathcal{E}_\omega)_S.$$

LEMMA 4.8. *If $\pi : F_b^{\omega}(\mathcal{E}_\omega) \to [0, 1]^{rb}$ satisfies for both $a = 0$ and 1, and, for all $\xi \in [0, 1]^{rb}$,*

$$\{1 \le k \le rb : \xi_k = a\} \subset \{1 \le k \le rb : \pi(\xi)_k = a\},$$

then $\pi \circ F_b^{\omega}$ is compatible with $\mathcal{A}^b(\omega)$.

The main idea from Lindenstrauss and Weiss still works for the random transformations between different fibers [**35**, Theorem 4.2], the major change being the substitution of T_ω for T, and we omit the proof.

THEOREM 4.1. *For any continuous bundle RDS T,* $\mathbb{E}\mathrm{mdim}(T) \le \mathbb{E}\mathrm{mdim}_{\mathrm{M}}(T)$.

COROLLARY 4.2. *If T has finite topological entropy, then* $\mathbb{E}\mathrm{mdim}(T) = 0$.

5. *Small sets and the small boundary property*

Denote by $\mathcal{P}_{\mathbb{P}}(\Omega \times X)$ the space of probability measures on $\Omega \times X$ whose marginal on Ω is $\mathbb{P}$, and set $\mathcal{P}_{\mathbb{P}}(\mathcal{E}) = \{\mu \in \mathcal{P}_{\mathbb{P}}(\Omega \times X) : \mu(\mathcal{E}) = 1\}$. Let $\mathcal{I}_{\mathbb{P}}(\mathcal{E})$ be the space of all Θ-invariant measures in $\mathcal{P}_{\mathbb{P}}(\mathcal{E})$. Any $\mu \in \mathcal{I}_{\mathbb{P}}(\mathcal{E})$ on $\mathcal{E}$ disintegrates as $d\mu(\omega, x) = d\mu_\omega(x)\, d\mathbb{P}(\omega)$ [**15**, §10.2], where the μ_ω are regular conditional probabilities with respect to the sub σ-algebra $\mathcal{F}_{\mathcal{E}}$ formed by all sets $(A \times X) \cap \mathcal{E}$ with $A \in \mathcal{F}$.

Denote by $\mathbb{L}^1(\Omega, C(X))$ the space of functions $f(\omega, x)$ that are measurable in ω and continuous in x and satisfy

$$\|f\| = \int \sup_{x\in X} |f(\omega, x)|\, d\mathbb{P} < \infty.$$

We say that a sequence μ_n converges to μ in the weak convergence topology if $\int f\, d\mu \to \int f\, d\mu_n$ for each $f \in \mathbb{L}^1(\Omega, C(X))$.

Definition 5.1. Let T be a bundle RDS and let E a measurable subset of $\mathcal{E}$. We define the ω-orbit capacity of the random set E to be

$$\text{ocap}(E, \omega) = \lim_{n\to\infty} \frac{1}{n} \sup_{x\in\mathcal{E}_\omega} \sum_{i=0}^{n-1} 1_{E(\vartheta^i\omega)}(T^i_\omega x).$$

The set E is called T-small (or simply small) if $\text{ocap}(E, \omega) = 0$ for $\mathbb{P}$-a.a. ω.

Let

$$b_n(\omega) = \sup_{x\in\mathcal{E}_\omega} \sum_{i=0}^{n-1} 1_{E(\vartheta^i\omega)}(T^i_\omega x).$$

We remark that the limit above exists for $\mathbb{P}$-a.a. ω by Lemma 4.6, since $b_n(\omega) \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ and $b_{n+m}(\omega) \le b_n(\omega) + b_m(\vartheta^n\omega)$. Then, for any measurable set $E \subset \mathcal{E}$, $\omega \mapsto \text{ocap}(E, \omega)$ is measurable and satisfies

$$\int \text{ocap}(E, \omega)\, d\mathbb{P}(\omega) = \lim_{n\to\infty} \frac{1}{n} \int b_n(\omega)\, d\mathbb{P}(\omega).$$

Remark 5.1. One could define the orbit capacity for a random set $E \subset \Omega \times X$, but, in fact, the orbit capacity depends only on $E \cap \mathcal{E}$.

PROPOSITION 5.1. *For every closed random set $E \subset \mathcal{E}$,*

$$\int \text{ocap}(E, \omega)\, d\mathbb{P}(\omega) = \sup\{\mu(E) : \mu \in \mathcal{I}_{\mathbb{P}}(\mathcal{E})\}.$$

Proof. Let $\mu \in \mathcal{I}_{\mathbb{P}}(\mathcal{E})$. For any $\epsilon > 0$ and large enough n,

$$\begin{aligned}
\mu(E) &= \int 1_E(\omega, x)\, d\mu(\omega, x)\\
&= \frac{1}{n} \int \sum_{i=0}^{n-1} 1_E(\Theta^i(\omega, x))\, d\mu(\omega, x)\\
&= \frac{1}{n} \iint \sum_{i=0}^{n-1} 1_E(\Theta^i(\omega, x))\, d\mu_\omega(x)\, d\mathbb{P}(\omega)\\
&= \int \left(\frac{1}{n} \int \sum_{i=0}^{n-1} 1_{E(\vartheta^i\omega)}(T^i_\omega x)\, d\mu_\omega(x)\right) d\mathbb{P}(\omega)\\
&\le \int \text{ocap}(E, \omega)\, d\mathbb{P}(\omega) + \epsilon.
\end{aligned}$$

Since ϵ is arbitrary, $\mu(E) \le \int \text{ocap}(E, \omega)\, d\mathbb{P}$.

Conversely, for any $\epsilon > 0$, consider

$$G_n = \left\{(\omega, x) \in \mathcal{E} : \frac{1}{n}\sum_{i=0}^{n-1} 1_{E(\vartheta^i\omega)}(T^i_\omega x) > \text{ocap}(E, \omega) - \epsilon\right\}.$$

Since

$$\sum_{i=0}^{n-1} 1_{E(\vartheta^i\omega)}(T^i_\omega x) = \sum_{i=0}^{n-1} 1_E(\Theta^i(\omega, x))$$

and $\text{ocap}(E, \omega)$ are both $\mathcal{F} \otimes \mathcal{B}_X$ measurable, G_n is $\mathcal{F} \otimes \mathcal{B}_X$ measurable. By the definition of $\text{ocap}(E, \omega)$,

$$\bigcup_{n=1}^{\infty} \bigcap_{m \geq n} \pi_\Omega G_m = \Omega.$$

Set $\Omega_n = \bigcap_{m\geq n} \pi_\Omega G_m$. Applying Lemma 2.1 to $G_n \cap (\Omega_n \times X)$, there exists an $\mathcal{F} \cap \Omega_n, \mathcal{B}_X$ measurable function $\gamma_n : \Omega_n \to X$ such that

$$\frac{1}{n}\sum_{i=0}^{n-1} 1_{E(\vartheta^i\omega)}(T^i_\omega \gamma_n(\omega)) > \text{ocap}(E, \omega) - \epsilon, \quad \omega \in \Omega_n.$$

For $\omega \notin \Omega_n$, applying Lemma 2.1 again, but to $\mathcal{E}$, we get a measurable function $e(\omega) \in \mathcal{E}_\omega$. Now we can define a sequence of probability measures $\nu^{(n)}$ on $\mathcal{E}$ via their measurable disintegrations

$$\nu^{(n)}_\omega = \delta_{\zeta_n(\omega)} \quad \text{where } \zeta_n(\omega) = \begin{cases} \gamma_n(\omega) & \text{if } \omega \in \Omega_n, \\ e(\omega) & \text{if } \omega \notin \Omega_n, \end{cases}$$

so that $d\nu^{(n)}(\omega, x) = d\nu^{(n)}_\omega(x)\, d\mathbb{P}(\omega)$. It is not difficult to see that $\nu^{(n)}$ can be obtained from setting

$$\nu^{(n)}(A \times U) = \int_A \nu^{(n)}_\omega(U)\, d\mathbb{P}(\omega), \quad A \in \mathcal{F}, U \in \mathcal{B}_X$$

and then extending to a probability measure on the product sigma algebra $\mathcal{F} \otimes \mathcal{B}_X$ with $\nu^{(n)}(\mathcal{E}) = 1$. Set

$$\mu^{(n)} = \frac{1}{n}\sum_{i=0}^{n-1} \nu^{(n)} \circ \Theta^{-i}.$$

Then any limit point of $\mu^{(n)}$ in the topology of weak convergence is in $\mathcal{I}_\mathbb{P}(\mathcal{E})$. In fact, suppose that $\mu^{(n)} \to \mu \in \mathcal{P}_\mathbb{P}(\mathcal{E})$ and, for any $f \in \mathbb{L}^1(\Omega, C(X))$,

$$\begin{aligned}\left|\int f \circ \Theta\, d\mu - \int f\, d\mu\right| &= \lim_{n\to\infty}\left|\int f \circ \Theta\, d\mu^{(n)} - \int f \circ\, d\mu^{(n)}\right| \\ &= \lim_{n\to\infty}\frac{1}{n}\left|\int (f \circ \Theta^n - f)\, d\nu^{(n)}\right| \\ &\leq \lim_{n\to\infty}\frac{2}{n}\|f\| = 0.\end{aligned}$$

Then $\mu \in \mathcal{I}_\mathbb{P}(\mathcal{E})$. Since, for any $0 \leq i < n$,

$$\begin{aligned}\nu^{(n)}(\Theta^{-i}E) &= \iint 1_E(\Theta^i(\omega, x))\, d\nu^{(n)}_\omega(x)\, d\mathbb{P}(\omega) \\ &= \int_\Omega 1_{E(\vartheta^i\omega)}(T^i_\omega \zeta_n(\omega))\, d\mathbb{P}(\omega),\end{aligned}$$

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we see that when n is large enough,

$$\begin{aligned}\mu^{(n)}(E) &= \frac{1}{n}\sum_{i=0}^{n-1}\nu^{(n)}(\Theta^{-i}E)\\ &= \int_\Omega \frac{1}{n}\sum_{i=0}^{n-1} 1_{E(\vartheta^i\omega)}(T_\omega^i\zeta_n(\omega))\,d\mathbb{P}(\omega)\\ &\geq \int_{\Omega_n} \frac{1}{n}\sum_{i=0}^{n-1} 1_{E(\vartheta^i\omega)}(T_\omega^i\gamma_n(\omega))\,d\mathbb{P}(\omega)\\ &> \int_{\Omega_n} \operatorname{ocap}(E,\omega)\,d\mathbb{P}(\omega) - \epsilon\mathbb{P}(\Omega_n)\\ &\geq \int_\Omega \operatorname{ocap}(E,\omega)\,d\mathbb{P}(\omega) - \epsilon - \int_{\Omega\setminus\Omega_n} \operatorname{ocap}(E,\omega)\,d\mathbb{P}(\omega).\end{aligned}$$

Recall that $\mu^{(n)} \to \mu$ weakly implies that $\mu(E) \geq \limsup_{n\to\infty}\mu^{(n)}(E)$ for each closed random set E [**2**, Lemma 1.6.6].

$$\int \operatorname{ocap}(E,\omega)\,d\mathbb{P}(\omega) = \sup\{\mu(E) : \mu \in \mathcal{I}_{\mathbb{P}}(\mathcal{E})\}. \qquad \square$$

COROLLARY 5.1. *For a closed random set $E \subset \mathcal{E}$, E is T-small if and only if E is a μ-null set for all $\mu \in \mathcal{I}_{\mathbb{P}}(\mathcal{E})$.*

Definition 5.2. A continuous bundle RDS T has the small boundary property if, for every random point $\xi \subset \mathcal{E}$ and every open random set $U \subset \Omega \times X$ such that $\xi(\omega) \in U(\omega) \cap \mathcal{E}_\omega$, there is a open random set $V \subset \Omega \times X$ such that

$$\xi(\omega) \in V(\omega) \cap \mathcal{E}_\omega \subset U(\omega) \cap \mathcal{E}_\omega$$

and $\partial(V \cap \mathcal{E}) = \{(\omega, x) : x \in \partial(V(\omega) \cap \mathcal{E}_\omega)\}$ is T-small, where ∂ denotes the boundary operator.

Remark 5.2. It is equivalent if we release the condition $\xi \subset \mathcal{E}$ and neglect all the intersections with $\mathcal{E}_\omega$ in the definition.

Recall that the isomorphism between two continuous bundle RDSs is a family of homeomorphisms between the corresponding fibers from each bundle. It sends every random point and random neighborhood from one bundle to the other. So it is trivial to see that isomorphisms preserve the SBP.

COROLLARY 5.2. *If Θ is uniquely ergodic, then T has the SBP.*

Proof. Suppose that $\mathcal{I}_{\mathbb{P}}(\mathcal{E}) = \{\mu\}$. For any random point $\xi \subset \mathcal{E}$ with open random neighborhood U, let

$$E_r = \{(\omega, y) : d(\xi(\omega), y) = r\}.$$

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Then the set

$$R = \{r > 0 : E_r \text{ is not } T\text{-small}\} = \{r > 0 : E_r \text{ has positive } \mu \text{ measure}\}$$

is at most countable. By Lemma III.39 in [**8**], $\omega \mapsto \inf\{d(\xi(\omega), y) : y \in X \backslash U(\omega)\}$ is measurable. So we can apply Lemma 2.1 to the set

$$G = \{(\omega, r) : 0 < r < d(\xi(\omega), X \backslash U(\omega)), r \notin R\}$$

and select a measurable $r : \Omega \to \mathbb{R}$ such that $(\omega, r(\omega)) \in G$. Let $V(\omega) = \{y : d(\xi(\omega), y) < r(\omega)\}$. Then $\xi(\omega) \in V(\omega) \cap \mathcal{E}_\omega \subset U(\omega) \cap \mathcal{E}_\omega$ and $V \cap \mathcal{E}$ has small boundary. So T has the SBP. □

As a consequence of the definition of the SBP, the following result relates the orbit capacity to a partition of unity subordinate to an open cover $\mathcal{A}$ of X and allows us to define an $\mathcal{A}^b(\omega)$-compatible map in the next theorem.

PROPOSITION 5.2. *If T has the SBP, then, for every $\mathcal{A} \in \mathrm{C}_X^o$, every $\epsilon > 0$, $\mathbb{P}$-a.a. ω_0 and sufficiently large b, there is a subordinate partition of unity $\phi_j^\omega : \mathcal{E}_\omega \to [0, 1]$, $(j = 1, 2, \ldots, |\mathcal{A}|)$ with respect to $\mathcal{A}(\omega) = \mathcal{A} \cap \mathcal{E}_\omega$, $\omega \in \Omega$, such that*

$$\frac{1}{b} \sum_{i=0}^{b-1} 1_{A(\vartheta^i \omega_0)}(T_{\omega_0}^i x) < \epsilon, \quad \textit{where } A(\omega) = \bigcup_{j=1}^{|\mathcal{A}|} (\phi_j^\omega)^{-1}(0, 1). \tag{4}$$

Proof. The proof of this result is much the same as that of its deterministic analogue [**35**, Proposition 5.3]. □

Now take $\mathcal{A} \in \mathrm{C}_X^o$ and $\epsilon > 0$. Suppose $|\mathcal{A}| = k$. Construct for $\mathbb{P}$-a.a. ω_0, according to the above proposition, an $\mathcal{A}(\omega)$ subordinate partition of unity (ϕ_j^ω) which obeys (4).

Define $\Phi^\omega : \mathcal{E}_\omega \to \mathbb{R}^k$ by

$$x \mapsto (\phi_1^\omega(x), \phi_2^\omega(x), \ldots, \phi_k^\omega(x)).$$

Define the map $F_b^\omega : \mathcal{E}_\omega \to \mathbb{R}^{kb}$ by

$$F_b^\omega(x) = (\Phi^\omega(x), \Phi^{\vartheta\omega}(T_\omega x), \ldots, \Phi^{\vartheta^{b-1}\omega}(T_\omega^{b-1} x)).$$

By proving that $F_b^{\omega_0}(\mathcal{E}_{\omega_0})$ is a subset of a finite number of ϵkb dimensional affine subspaces of $\mathbb{R}^{kb}$ [**35**, Theorem 5.4], one can deduce that $D(\mathcal{A}^b(\omega_0)) < \epsilon kb$. So we obtain the following theorem.

THEOREM 5.1. *If T has the SBP, then* $\mathbb{E}\mathrm{mdim}(T) = 0$.

COROLLARY 5.3. *If Θ is uniquely ergodic, then* $\mathbb{E}\mathrm{mdim}(T) = 0$.

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