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ASYMPTOTIC EXPANSION OF A CLASS OF MULTI-DIMENSIONAL INTEGRALS

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ABSTRACT. The aim of this paper is to derive the expansion of the following class of multi-dimensional integrals

$$I(\lambda) := \mathrm{fp}\int_\Omega K[x, \lambda w(\theta) r^a]\, dx,$$

with respect to the large parameter λ when Ω is a subset of $\mathbb{R}^n$, $a > 0$, w is a strictly positive and bounded function on Σ and fp means an integration in the finite part sense of Hadamard (see Section 2). This is performed for weak assumptions bearing on pseudofunction K and by extending to higher dimensional cases the tools developed in the one-dimensional context. The range of applications of the proposed results is outlined by the exhibition of several examples.

1. **Introduction.** For $n \geq 2$, we consider Ω an open subset of $\mathbb{R}^n$, O a point belonging to Ω and $d(O, \partial\Omega)$ the distance from O to its boundary $\partial\Omega$ where it is understood that $d(O, \partial\Omega) := +\infty$ if $\Omega := \mathbb{R}^n$. Moreover, $M := x = (x_1, \ldots, x_n)$ designates an arbitrary point of $\mathbb{R}^n$, dx stands for the associated Lebesgue measure and $(r, \theta_1, \ldots, \theta_{n-1})$ is a set of spherical coordinates of origin O with $r := OM$ and $\theta := (\theta_1, \ldots, \theta_{n-1}) := x/r$. It is recalled that $dx = r^{n-1}\, dr\, d\sigma$ where $d\sigma$ is the Lebesgue measure on $\Sigma := \{M \in \mathbb{R}^n, r = 1\}$ and C will designate the set of complex numbers.

If λ is a real and large parameter, f and g complex functions and ϕ is a real function, one often seeks the asymptotic expansion of the integral

$$L(\lambda) := \int_\Omega f(x) g[\lambda\phi(x)]\, dx, \tag{1.1}$$

with respect to the parameter λ. This expansion not only depends on the behavior of function g near infinity but also on the behavior both of functions f and ϕ near the critical points of Ω, *i.e.* the points of Ω satisfying $\phi(x) = 0$.

When ϕ is real analytic on Ω the problem reduces to the treatment of $\phi(x) = x_1^{l_1} \cdots x_n^{l_n}$ with $(l_1, \ldots, l_n) \in \mathbb{N}^n$ (see [3, 7]) and the question remains difficult because the critical points are not isolated (more precisely the set of critical points is $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n; \prod_{i=1}^n x_i = 0\}$). The case of $\phi(x) := x_1^{a_1} \cdots x_n^{a_n}$ with $(a_1, \ldots, a_n) \in \mathbb{R}_+^{*n}$ has been investigated in earlier papers [1, 3] for $\Omega := [0, 1]^n, g \in \mathcal{S}(\mathbb{R})$ the Schwartz space and $f(x_1, \ldots, x_n) := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \log^{l_1}[x_1] \cdots \log^{l_n}[x_n] h(x_1, \ldots, x_n)$ if $(l_1, \ldots, l_n) \in \mathbb{N}^n$, $(\alpha_1, \ldots, \alpha_n) \in C^n$ with $\mathrm{Re}(\alpha_i) \geq 0$ and $h \in \mathcal{C}^\infty(\mathbb{R}^n)$. Particular case $n = 2$ has been detailed in [8] and recently extended by [11] to weaker assumptions.

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By now, it is assumed that O is the only one critical point of ϕ in Ω. The problem turns out to be easier to tackle and one usually makes use of Mellin transform method to deal with it (see [2, 12]). Recently, [4, 5] proposed an approach based on the distributional point of view to provide the asymptotic expansion of certain multi-dimensional generalized functions. Nevertheless, the following integral

$$I(\lambda) := \mathrm{fp} \int_{\Omega} K\big(x, \lambda\phi(x)\big)\, dx, \tag{1.2}$$

where fp means an integration in the finite part sense of Hadamard and $K(x,u)$ (with u real) is a pseudofunction, do not seem to have attracted such an interest.

The aim of this paper is to derive the asymptotic expansion of $I(\lambda)$ when $\lambda \longrightarrow +\infty$, pseudofunction $K(x,u)$ offers specific kind of behaviors when $u \longrightarrow +\infty$ or $x \longrightarrow 0$ and function ϕ reduces to $\phi(x) = \phi(r,\theta) := w(\theta)r^a$ with $a > 0$ and w is a strictly positive and bounded function on Σ. This is achieved by extending to this question the results obtained for the one-dimensional case (see [10]). If Section 2 below is devoted to the introduction of appropriate notations and definitions, main theorems are established in Section 3 whereas Section 4 deals with some examples and comparisons.

2. Integration in the finite part sense of Hadamard and basic functional spaces.

This section introduces the definition of a multi-dimensional integration in the finite part sense of Hadamard for certain pseudofunctions. The reader may consult [9, 10] for detailed explanations. For $a > 0$, then $B(a) := \{M \in \mathbb{R}^n, OM := r < a\}$, $\partial B(a) := \{M \in \mathbb{R}^n, OM := r = a\}$ and also $\mathcal{B}(a) := B(a) \cup \partial B(a)$.

DEFINITION 1. A function f of $L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \{O\}, C)$ belongs to the set $\mathcal{P}(\mathbb{R}^n, C)$ if there exist positive real η_f and A_f, two complex functions $F^0 \in L^1\big(\mathcal{B}(\eta_f), C\big)$ and $F^\infty \in L^1(\mathbb{R}^n \setminus B(A_f), C)$, two families of positive integers $\big(J(i)\big)$, $\big(K(l)\big)$, two complex families (α_i), (γ_l) and two families (f^0_{ij}), (f^∞_{lk}) of elements of $L^1(\Sigma, C)$ such that

$$f(x) = f(r,\theta) = \sum_{i=0}^{I}\sum_{j=0}^{J(i)} f^0_{ij}(\theta) r^{\alpha_i} \log^j r + F^0(x), \quad \text{a.e. in } \mathcal{B}(\eta_f) \setminus \{O\}, \tag{2.1}$$
$$\mathrm{Re}(\alpha_I) < \mathrm{Re}(\alpha_{I-1}) < \cdots < \mathrm{Re}(\alpha_1) < \mathrm{Re}(\alpha_0) := -n;$$

$$f(x) = f(r,\theta) = \sum_{l=0}^{L}\sum_{k=0}^{K(l)} f^\infty_{lk}(\theta) r^{-\gamma_l} \log^k r + F^\infty(x), \quad \text{a.e. in } \mathbb{R}^n \setminus B(A_f), \tag{2.2}$$
$$\mathrm{Re}(\gamma_L) < \mathrm{Re}(\gamma_{L-1}) < \cdots < \mathrm{Re}(\gamma_1) < \mathrm{Re}(\gamma_0) := n.$$

Moreover,

$$\mathrm{fp}\int_{\mathbb{R}^n} f(x)\, dx := \mathrm{fp}\Big[\int_{B(\epsilon^{-1})\setminus B(\epsilon)} f(x)\, dx\Big] = \int_{\Sigma}\Big[\mathrm{fp}\int_0^\infty r^{n-1} f(r,\theta)\, dr\Big]\, d\sigma. \tag{2.3}$$

Hence, f presents near origin O or at infinity expansions involving a finite number of terms $a(\theta)r^\beta \log^j r$ with β and j no depending on the angular coordinates $\theta := (\theta_1, \ldots, \theta_{n-1})$. This choice authorizes the case of f possessing a conical singularity at O, *i.e.* such that (2.1) is satisfied with $f(r,\theta) = 0$ if $r < \eta_f$ and $\theta \in V \subset \Sigma$.

If $g_\theta(r) := r^{n-1}f(r,\theta)$, $G^0_\theta(r) := r^{n-1}F^0(r,\theta)$ and $G^\infty_\theta(r) := r^{n-1}F^\infty(r,\theta)$ then (2.1), for $0 < r < \eta_f$, leads to $g_\theta(r) = \sum_{i=0}^{I}\sum_{j=0}^{J(i)} f^0_{ij}(\theta) r^{\alpha_i+n-1}\log^j r + G^0_\theta(r)$ with $\mathrm{Re}(\alpha_i + n - 1) \le -1$ and $G_\theta \in L^1([0,\eta_f], C)$ and for $r > A_f$, (2.2) yields $g_\theta(r) = \sum_{l=0}^{L}\sum_{k=0}^{K(l)} f^\infty_{lk}(\theta) r^{-\gamma_l+n-1}\log^k r + G^\infty_\theta(r)$ with $\mathrm{Re}(\gamma_l + 1 - n) \le 1$ and $G^\infty_\theta(r) \in L^1(\mathbb{R}^n \setminus B(\eta_f), C)$. Thus, $\mathrm{fp}\int_0^\infty g_\theta(r)\,dr$ admits a sense for $\theta \in \Sigma_n$ (see [10]). Finally, for $O \in \Omega \subset \mathbb{R}^n$, the set $\mathcal{P}(\Omega, C)$ is the set of pseudofunctions f such that $\mathcal{F} \in \mathcal{P}(\mathbb{R}^n, C)$ if $\mathcal{F}(x) := f(x)$ for $x \in \Omega$, else $\mathcal{F}(x) := 0$ and $\mathrm{fp}\int_\Omega f(x)\,dx := \mathrm{fp}\int_{\mathbb{R}^n}\mathcal{F}(x)\,dx$.

By now, for t real and $\epsilon > 0$, the notation $A = \sum_{j,\mathrm{Re}(\alpha)\le t}^{J(i)} a_{ij}\epsilon^{\alpha_i}\log^j(\epsilon)$ means that there exist two complex families (a_{ij}) and (α_i) such that the sequence $\big(\mathrm{Re}(\alpha_i)\big)$ is strictly increasing and a family of positive integers $\big(J(i)\big)$ such that if $I := \sup\{q \in \mathbb{N}, \mathrm{Re}(\alpha_q) \le t\}$ then $A = \sum_{i=0}^{I}\sum_{j=0}^{J(i)} a_{ij}\epsilon^{\alpha_i}\log^j(\epsilon)$. Moreover, for h a complex pseudofunction of real variable such that there exist a real $s > t$, a complex function H^0_t (or H^∞_t) and a neighborhood of zero on the right (or of infinity) in which H^0_t (or H^∞_t) is bounded and $h(u) = \sum_{j,\mathrm{Re}(\alpha)\le t}^{J(i)} h^0_{ij}u^{\alpha_i}\log^j u + u^s H^0_t(u)$ (or $h(u) = \sum_{j,\mathrm{Re}(\alpha)\le t}^{J(i)} h^\infty_{ij}u^{-\alpha_i}\log^j u + u^{-s}H^\infty_t(u)$) we note $\lim_{u\to 0} h(u) = \sum_{j,\mathrm{Re}(\alpha)\le t}^{J(i)} h^0_{ij}u^{\alpha_i}\log^j u$ and set $S_0(h) := \inf\{\mathrm{Re}(\alpha_i)$, for i such that exists j with $h^0_{ij} \ne 0\}$ (or $\lim_{u\to\infty} h(u) = \sum_{j,\mathrm{Re}(\alpha)\le t}^{J(i)} h^\infty_{ij}u^{-\alpha_i}\log^j u$ and set $S_\infty(h) := \inf\{\mathrm{Re}(\alpha_i)$, for i such that exists j with $h^\infty_{ij} \ne 0\}$). If h depends on several real variables, the notation $\lim_{r\to 0} h(r,\theta) = \sum_{j,\mathrm{Re}(\alpha)\le r_1}^{J(i)} h^0_{ij}(\theta) r^{\alpha_i}\log^j r$ means that each $h^0_{ij} \in L^1(\Sigma, C)$ and that there exist $s > t$, a complex function $H^0_t(r,\theta)$ and a real $\eta > 0$ such that in $B(\eta)\setminus\{O\}$, H^0_t is bounded and $h(r,\theta) = \sum_{j,\mathrm{Re}(\alpha)\le r_1}^{J(i)} h^0_{ij}(\theta) r^{\alpha_i}\log^j r + r^s H^0_t(r,\theta)$. Moreover, $S_O(h) = \inf\{\mathrm{Re}(\alpha_i)$, for i such that h^0_{ij} is not the zero function on $\Sigma\}$. Definitions of equality $\lim_{r\to\infty} h(r,\theta) = \sum_{m,\mathrm{Re}(\gamma)\le r_2}^{M(e)} h^\infty_{em}(\theta) r^{-\gamma_e}\log^m r$ and of real $S_\infty(h)$ are also clear.

These notations authorize the definition of two useful functional spaces.

DEFINITION 2. For three real values r_1, r_2 and b such that $0 < b \le +\infty$, the sets $\mathcal{E}^{r_2}_{r_1}(]0,b[, C)$ and $\mathcal{E}^{r_2}_{r_1}(\mathbb{R}^n, C)$ are defined by

- $\mathcal{E}^{r_2}_{r_1}(]0,b[, C) := \{f, f$ is a complex pseudofunction such that $\lim_{u\to 0} f(u) = \sum_{j,\mathrm{Re}(\alpha)\le r_1}^{J(i)} f^0_{ij}u^{\alpha_i}\log^j u$; if $b < +\infty$ then $f \in L^1_{\mathrm{loc}}(]0,b], C)$ else not only $f \in L^1_{\mathrm{loc}}(]0,+\infty[, C)$ but also $\lim_{u\to\infty} f(u) = \sum_{m,\mathrm{Re}(\gamma)\le r_2}^{M(e)} f^\infty_{em}u^{-\gamma_e}\log^m u\}$,
- $\mathcal{E}^{r_2}_{r_1}(\mathbb{R}^n, C) := \{f, f \in L^1_{\mathrm{loc}}(\mathbb{R}^n\setminus\{O\}, C)$ such that notations $\lim_{r\to 0} f(r,\theta) = \sum_{j,\mathrm{Re}(\alpha)\le r_1}^{J(i)} f^0_{ij}(\theta) r^{\alpha_i}\log^j r$ and $\lim_{r\to\infty} f(r,\theta) = \sum_{m,\mathrm{Re}(\gamma)\le r_2}^{M(e)} f^\infty_{em}(\theta) r^{-\gamma_e}\log^m r$ are satisfied$\}$.

3. **The asymptotic expansion of $I(\lambda)$.** First an adequate space of pseudofunctions $K(x,u)$ is proposed.

DEFINITION 3. For Ω an open subset of $\mathbb{R}^n$ containing O and two real values t and t' the space $\mathcal{H}^t_{t'}(\Omega, C)$ is the set of complex pseudofunctions $K(x,u)$ such that, for λ large enough

1. The pseudofunction h such that $h(x) = h(r,\theta) := K(x,\lambda r)$ belongs to $\mathcal{P}(\Omega, C)$.
2. If the pseudofunction $\mathcal{K}(x,u)$ is defined by $\mathcal{K}(x,u) := K(x,u)$ if $x \in \Omega$, else $\mathcal{K}(x,u) := 0$ then it obeys all the following properties:

2.1. There exist a positive integer E, a complex family (γ_e) with $\mathrm{Re}(\gamma_0) < \cdots < \mathrm{Re}(\gamma_E) := t$, families of positive integers $\big(M(e)\big)$ and of complex pseudofunctions $\big(K_{em}(x)\big)$, reals $B \geq 0$ and $s > t$, a complex function $G_t(x,u)$ and a real $1 < b < +\infty$ such that for $(x,u) \in (\mathbb{R}^n \setminus \{O\}) \times [b, +\infty[$

(a)

$$\mathcal{K}(x,u) = \sum_{e=0}^{E} \sum_{m=0}^{M(e)} K_{em}(x) u^{-\gamma_e} \log^m u + u^{-s} G_t(x,u), \tag{3.1}$$

(b)

$$\left| \int_{\mathbb{R}^n \setminus B(b)} r^{-s} G_t(x, \lambda r)\, dx \right| \leq B \prec +\infty, \tag{3.2}$$

(c) $\forall e \in \{0,\dots,E\}, \forall m \in \{0,\dots,M(e)\}, K_{em} \in \mathcal{E}^{n-\mathrm{Re}(\gamma_e)}_{\mathrm{Re}(\gamma_e)-n}(\mathbb{R}^n, C)$ and also there exist a positive integer I, a complex family (α_i) such that $\mathrm{Re}(\alpha_0) < \cdots < \mathrm{Re}(\alpha_I) := t' - n$, a family of positive integers $\big(J(i)\big)$, a real $s' > t' - n$, a family (K^{ij}_{em}) of elements of $L^1(\Sigma, C)$, a complex function L_{em} bounded in a neighborhood of O in which

$$K_{em}(x) = \sum_{i=0}^{I} \sum_{j=0}^{J(i)} K^{ij}_{em}(\theta) r^{\alpha_i} \log^j r + r^{s'} L_{em}(x). \tag{3.3}$$

2.2. For almost any θ of Σ there exist a real $A_\theta \geq b \succ 0$ and $B'_\theta \geq 0$, a family of complex pseudofunctions $\big(h^{ij}(\cdot,\theta)\big)$, a complex function $H_t[(r,\theta),u]$ such that for $0 \prec r \leq W_\theta$ and $u > 0$

(a)

$$\mathcal{K}(x,u) = \mathcal{K}[(r,\theta),u] = \sum_{i=0}^{I} \sum_{j=0}^{J(i)} h^{ij}(u,\theta) r^{\alpha_i} \log^j r + r^{s'} H_{t'}[(r,\theta),u], \tag{3.4}$$

(b)

$$\left| \int_0^{A_\theta} u^{s'+n-1} H_{t'}[(u/\lambda, \theta), u]\, du \right| \leq B'_\theta \prec +\infty, \tag{3.5}$$

(c) for $i \in \{0,\dots,I\}$ and $j \in \{0,\dots,J(i)\}$, if $h^{ij}(x) := h^{ij}(r,\theta)$ where $r := OM$ and $\theta := x/r$, then $h^{ij} \in \mathcal{E}^{n+\mathrm{Re}(\alpha_i)}_{-n-\mathrm{Re}(\alpha_i)}(\mathbb{R}^n, C)$. Besides it is understood that $\lim_{u \to 0} h^{ij}(u,\theta) = \sum^{Q(p)}_{q, \mathrm{Re}(\beta) \leq -n-\mathrm{Re}(\alpha_i)} H^{ij}_{pq}(\theta) u^{\beta_p} \log^q u$ and also that there exists a complex function O_{ij} bounded in a neighborhood of infinity in which

$$h^{ij}(u,\theta) = \sum_{e=0}^{E} \sum_{m=0}^{M(e)} K^{ij}_{em}(\theta) u^{-\gamma_e} \log^m u + u^{-s} O_{ij}(u,\theta). \tag{3.6}$$

2.3. Finally, for almost any θ of Σ, the complex function $W^\theta_{t,t'}$ defined by the relation: $r^{s'+n-1} u^{-s} W^\theta_{t,t'}(r,u) := r^{n-1}\{\mathcal{K}[(r,\theta),u] - \sum_{e=0}^{E} \sum_{m=0}^{M(e)} K_{em}(r,\theta) u^{-\gamma_e} \log^m u - \sum_{i=0}^{I} \sum_{j=0}^{J(i)} \big[h^{ij}(u,\theta) - \sum_{e=0}^{E} \sum_{m=0}^{M(e)} K^{ij}_{em}(\theta) u^{-\gamma_e} \log^m u\big] r^{\alpha_i} \log^j r\}$ remains bounded in the set $]0,b] \times [A_\theta, +\infty[$.

Decomposition (3.1) provides the behavior of $\mathcal{K}(x,u)$ when u tends to infinity. Therefore, we assume that $b > 1$. If x does not belong to Ω, $K_{em}(x) = 0 = G_t(x,u)$. Note that the integrations arising in (3.2) and (3.5) are usual ones (in Lebesgue's sense). If $a := 1$ and $w = 1_\Sigma$ (the unit function on Σ), the following theorem holds.

THEOREM 1. *Assume that* $O \in \Omega \subset \mathbb{R}^n$, $a = 1$ *and* $w = 1_\Sigma$. *For t real, if* $K(x,u) \in \mathcal{H}_t^t(\Omega, C)$ *then the expansion below holds for* $I(\lambda)$, *as* λ *tends to infinity,*

$$\begin{aligned}
&\operatorname{fp}\int_\Omega K(x,\lambda r)\,dx \\
&= \sum_{m.\mathrm{Re}(\gamma)\le t}^{M(e)} \sum_{l=0}^{m} C_m^l \Big[\operatorname{fp}\int_\Omega K_{em}(x) r^{-\gamma_e} \log^{m-l}(r)\,dx\Big] \lambda^{-\gamma_e}\log^l\lambda \\
&\quad + \sum_{j.\mathrm{Re}(\alpha)\le t-n}^{J(i)} \Big\{ \sum_{l=0}^{j} C_j^l(-1)^l \Big[\operatorname{fp}\int_{\mathbb{R}^n} h^{ij}(x) r^{\alpha_i}\log^{j-l}(r)\,dx\Big] \\
&\quad - \sum_{\{p\,;\,\beta_p=-\alpha_i-n\}} \sum_{q=0}^{Q(p)} \frac{(-1)^j j!\,q!}{(1+j+q)!}\Big[\int_\Sigma H_{pq}^{ij}(\theta)\,d\sigma\Big]\log^{1+j+q-l}\lambda \\
&\quad + \sum_{\{e\,;\,\gamma_e=\alpha_i+n\}} \sum_{m=0}^{M(e)} \frac{(-1)^j j!\,m!}{(1+j+m)!}\Big[\int_\Sigma K_{em}^{ij}(\theta)\,d\sigma\Big]\log^{1+j+m-l}\lambda\Big\} \lambda^{-(\alpha_i+n)}\log^l\lambda + o(\lambda^{-t}),
\end{aligned} \tag{3.7}$$

where the sum $\sum_{j.\mathrm{Re}(\alpha)\le r}^{J(i)}$ *has been previously defined and* $C_m^l := m!/[l!\,(m-l)!]$ *for integers l and m such that* $0 \le l \le m$.

PROOF. For large λ, property 1 of Definition 3 ensures the existence of $I(\lambda)$ and if $J_\theta(\lambda) := \operatorname{fp}\int_0^b r^{n-1}\mathcal{K}[(r,\theta),\lambda r]\,dr$ it is actually possible to write

$$I(\lambda) =:= \operatorname{fp}\int_{\mathbb{R}^n}\mathcal{K}(x,\lambda r)\,dx = \operatorname{fp}\int_{\mathbb{R}^n\setminus B(b)}\mathcal{K}(x,\lambda r)\,dx + \int_{\Sigma_n} J_\theta(\lambda)\,d\sigma. \tag{3.8}$$

For λ large enough, use of decomposition (3.1) immediately leads to

$$\operatorname{fp}\int_{\mathbb{R}^n\setminus B(b)}\mathcal{K}(x,\lambda r)\,dx = \sum_{e=0}^{E}\sum_{m=0}^{M(e)}\Big[\operatorname{fp}\int_{\mathbb{R}^n\setminus B(b)} K_{em}(x) r^{-\gamma_e}\log^m(\lambda r)\,dx\Big]\lambda^{-\gamma_e} + R(\lambda), \tag{3.9}$$

where $\operatorname{fp}\int_{\mathbb{R}^n\setminus B(b)} K_{em}(x) r^{-\gamma_e}\log^m(\lambda r)\,dx$ indeed admits a sense because $K_{em} \in \mathcal{E}_{\mathrm{Re}(\gamma_e)-n}^{n-\mathrm{Re}(\gamma_e)}(\mathbb{R}^n, C)$ and also according to the assumption $s > t$ and inequality (3.2), $R(\lambda) := \lambda^{-s}\int_{\mathbb{R}^n\setminus B(b)} r^{-s} G_t(x,\lambda r)\,dx = o(\lambda^{-t})$. Since $K(x,u) \in\in \mathcal{H}_t^t(\Omega, C)$ the reader may check that for almost any $\theta \in \Sigma$ then $F_\theta(r,u) := r^{n-1}\mathcal{K}[(r,\theta),u]$ belongs to $\mathcal{F}_{t-1}^t(]0,b[,C)$ a space introduced in [10]. Thus, Theorem 1 derived in [10] yields

$$\begin{aligned}
J_\theta(\lambda) &= \sum_{m.\mathrm{Re}(\gamma)\le t}^{M(e)} \Big[\operatorname{fp}\int_0^b K_{em}(r,\theta) r^{-\gamma_e+n-1}\log^m(\lambda r)\,dr\Big]\lambda^{-\gamma_e} \\
&\quad + \sum_{j.\mathrm{Re}(\alpha)\le t-n}^{J(i)} \sum_{l=0}^{j} C_j^l(-1)^l\Big[\operatorname{fp}\int_0^\infty h^{ij}(v,\theta) v^{\alpha_i+n-1}\log^{j-l}(v)\,dv \\
&\quad - \sum_{\{p\,;\,\beta_p=-\alpha_i-n\}}\sum_{q=0}^{Q(p)} \frac{H_{pq}^{ij}(\theta)}{1+j+q-l}\log^{1+j+q-l}\lambda \\
&\quad + \sum_{\{e\,;\,\gamma_e=\alpha_i+n\}}\sum_{m=0}^{M(e)} \frac{K_{em}^{ij}(\theta)}{1+j+m-l}\log^{1+j+m-l}\lambda\Big]\lambda^{-(\alpha_i+n)}\log^l\lambda + o(\lambda^{-t}).
\end{aligned} \tag{3.10}$$

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Finally, integration over Σ of (3.10) together with relation $\sum_{l=0}^{j} C_j^l(-1)^l/(1+j+q-l) = (-1)^j j!\, q!/(1+j+q)!$ ensure result (3.7).

In order to deal with the general case another functional space is proposed.

DEFINITION 4. For Ω an open subset of $\mathbb{R}^n$ containing O, $a \in \mathbb{R}_+^*$, w a strictly positive and bounded function on Σ and a real number τ, a complex pseudofunction $K(x,u)$ belongs to $\mathcal{A}_\tau(\Omega, w, a, C)$ if and only if it satisfies the following properties for λ large enough:

1. The pseudofunction h such that $h(x) = h(r,\theta) := K\big(x, \lambda w(\theta) r^a\big)$ belongs to $\mathcal{P}(\Omega, C)$.
2. The pseudofunction $\mathcal{K}(x,u)$ obeys properties 2.1 and 2.2 of Definition 3 with $t := \tau$ and $t' := a\tau$ except inequalities (3.2) and (3.5) which are replaced by

$$\left|\int_{\mathbb{R}^n\backslash B(d)} w(\theta)^{-s} r^{-as} G_t[x, \lambda w(\theta) r^a]\, dx\right| \leq B < +\infty, \tag{3.11}$$

$$\left|\int_0^{A_\theta} u^{s'+n-1} H_{t'}\left[\left(\frac{u}{\lambda}, \theta\right), u^a\right] du\right| \leq B'_\theta < +\infty, \tag{3.12}$$

where $d := b^{1/a} > 1$ (because $a > 0$ and $b > 1$ in Definition 3) and the assumptions $K_{em} \in \mathcal{E}^{n-\mathrm{Re}(\gamma_e)}_{\mathrm{Re}(\gamma_e)-n}(\mathbb{R}^n, C)$ and $h^{ij} \in \mathcal{E}^{n+\mathrm{Re}(\alpha_i)}_{-n-\mathrm{Re}(\alpha_i)}(\mathbb{R}^n, C)$ are respectively replaced by $K_{em} \in \mathcal{E}^{n-a\,\mathrm{Re}(\gamma_e)}_{a\tau}(\mathbb{R}^n, C)$ and $h^{ij} \in \mathcal{E}^{\tau}_{-(n+\mathrm{Re}(\alpha_i))/a}(\mathbb{R}^n, C)$.

This definition leads to the next result.

THEOREM 2. *For t real, if $K(x,u) \in \mathcal{A}_t(\Omega, w, a, C)$ the integral $I(\lambda)$ presents the following and asymptotic expansion with respect to the large parameter λ*

$$\begin{aligned}
I(\lambda) &:= \mathrm{fp}\int_\Omega K[x, \lambda w(\theta) r^a]\, dx \\
&= \sum_{m,\mathrm{Re}(\gamma)\leq t}^{M(e)} \sum_{l=0}^{m} C_m^l \left[\mathrm{fp}\int_\Omega w(\theta)^{-\gamma_e} K_{em}(x) r^{-a\gamma_e} \log^{m-l}[w(\theta) r^a]\, dx\right] \lambda^{-\gamma_e} \log^l \lambda \\
&\quad + \sum_{j,\mathrm{Re}(\alpha)\leq at-n}^{J(i)} \left\{ \sum_{\{e\,;\,a\gamma_e=\alpha_i+n\}} \sum_{m=0}^{M(e)} \sum_{l_2=0}^{1+j+m} \frac{(-1)^j j!\, m!\, C_{1+j+m}^{l_2}}{a^{1+j}(1+j+m)!} \mathcal{H}_{em}^{ij}(l_2) \log^{l_2}\lambda \right. \\
&\quad + \sum_{l=0}^{j} \frac{C_j^l(-1)^l}{a^j}\left[\mathrm{fp}\int_{\mathbb{R}^n} w(\theta)^{-\frac{\alpha_i+n}{a}} h^{ij}(r^a,\theta) r^{\alpha_i} \log^{j-l}\big(w(\theta)^{-1} r^a\big)\, dx\right] \log^l \lambda \\
&\quad \left. - \sum_{\{p\,;\,a\beta_p=-\alpha_i-n\}} \sum_{q=0}^{Q(p)} \sum_{l_1=0}^{1+j+q} \frac{(-1)^j j!\, q!\, C_{1+j+q}^{l_1}}{a^{1+j}(1+j+q)!} \mathcal{H}_{pq}^{ij}(l_1) \log^{l_1}\lambda \right\} \lambda^{-(\alpha_i+n)/a} + o(\lambda^{-t}),
\end{aligned} \tag{3.13}$$

where coefficients $\mathcal{H}_{pq}^{ij}(k)$ and $\mathcal{H}_{em}^{ij}(k)$ are for positive integer k defined by

$$\mathcal{H}_{pq}^{ij}(k) := \int_\Sigma w(\theta)^{-\frac{\alpha_i+n}{a}} H_{pq}^{ij}(\theta) \log^{1+j+q-k}[w(\theta)]\, d\sigma,$$
$$\mathcal{H}_{em}^{ij}(k) := \int_\Sigma w(\theta)^{-\frac{\alpha_i+n}{a}} K_{em}^{ij}(\theta) \log^{1+j+m-k}[w(\theta)]\, d\sigma.$$

PROOF. For $d := b^{1/a}$ it is indeed possible to write $I(\lambda) = I_1(\lambda) + I_2(\lambda)$ with

$$I_1(\lambda) = \mathrm{fp} \int_{\mathbb{R}^n \setminus B(d)} \mathcal{K}[x, \lambda w(\theta) r^a]\, dx, \quad I_2(\lambda) = \mathrm{fp} \int_{B(d)} \mathcal{K}[x, \lambda w(\theta) r^a]\, dx. \tag{3.14}$$

For $\lambda > w_i^{-1}$ if $w_i := \inf\{w(\theta), \theta \in \Sigma\}$, the assumption $r \geq d := b^{1/a}$ ensures that $u := \lambda w(\theta) r^a \geq b$. Thus, decomposition (3.1) is valid and when combined with inequality (3.11) and assumption $s > t$ it yields for $I_1(\lambda)$

$$I_1(\lambda) = \sum_{e=0}^{E} \sum_{m=0}^{M(e)} \Big[\mathrm{fp} \int_{\mathbb{R}^n \setminus B(d)} w(\theta)^{-\gamma_e} K_{em}(\theta) r^{-a\gamma_e} \log^m[\lambda w(\theta) r^a]\, dx \Big] \lambda^{-\gamma_e} + o(\lambda^{-t}).$$

Observe that each integral in above equality exists because $K_{em} \in \mathcal{E}_{at-n}^{n-a\,\mathrm{Re}(\gamma_e)}(\mathbb{R}^n, C)$ and w is strictly positive and bounded on Σ. If $\beta = \beta(\theta) := [\lambda w(\theta)]^{1/a}$ and $\mathcal{K}^a(x,u) := \mathcal{K}(x, u^a)$, one gets

$$I_2(\lambda) = \int_\Sigma \Big[\mathrm{fp} \int_0^d r^{n-1} \mathcal{K}\big(x, [\beta(\theta) r]^a\big)\, dr \Big]\, d\sigma = \int_\Sigma J_\theta^a(\beta)\, d\sigma \tag{3.15}$$

where

$$J_\theta^a(\beta) := \mathrm{fp} \int_0^d r^{n-1} \mathcal{K}^a(x, \beta r)\, dr \quad \text{with } x = (r, \theta). \tag{3.16}$$

If λ is large enough, $\beta(\theta) := [\lambda w(\theta)]^{1/a}$ is large too and the asymptotic expansion of complex $J_\theta^a(\beta)$ with respect to $\beta(\theta)$ is required up to order $o[\beta(\theta)^{-at}]$. Such an expansion is obtained by introducing the function $F_\theta^a(r,u) := r^{n-1}\mathcal{K}^a(r,u) = r^{n-1}\mathcal{K}(r,u^a)$. The reader may easily check that $F_\theta^a \in \mathcal{F}_{at-1}^{at}(]0,d[, C)$. Accordingly,

$$\begin{aligned} J_\theta^a(\beta) = & \sum_{m.\mathrm{Re}(\gamma) \leq t}^{M(e)} \Big[\mathrm{fp} \int_0^d a^m K_{em}(r,\theta) r^{-a\gamma_e + n - 1} \log^m(\beta r)\, dr \Big] \beta^{-a\gamma_e} \\ & + \sum_{j.\mathrm{Re}(\alpha) \leq at-n}^{J(i)} \sum_{l=0}^{j} C_j^l (-1)^l \Big[\mathrm{fp} \int_0^\infty h^{ij}(v^a, \theta) v^{\alpha_i + n - 1} \log^{j-l}(v)\, dv \\ & - \sum_{\{p\,;\,a\beta_p = -\alpha_i - n\}} \sum_{q=0}^{Q(p)} \frac{a^q H_{pq}^{ij}(\theta)}{1 + j + q - l} \log^{1+j+q-l} \beta \\ & + \sum_{\{e\,;\,a\gamma_e = \alpha_i + n\}} \sum_{m=0}^{M(e)} \frac{a^m K_{em}^{ij}(\theta)}{1 + j + m - l} \log^{1+j+m-l} \beta \Big] \beta^{-(\alpha_i + n)} \log^l \beta + o(\beta^{-at}). \end{aligned} \tag{3.17}$$

The expansion of $I_2(\lambda)$ is deduced by using the link $\beta^a = \lambda w(\theta)$ and by integrating expansion (3.17) over Σ. After some algebra, one gets formula (3.13).

Of great interest for the applications is also the special case $K(x,u) = f(x)H(u)$, *i.e.* the case of separated variables x and u. For adequate assumptions both bearing on pseudofunctions f and H, Theorem 2 provides a useful result.

THEOREM 3. *Consider Ω a domain of $\mathbb{R}^n$ containing O, $a \in \mathbb{R}_+^*$ and w a strictly positive and bounded function on Σ. For complex pseudofunctions f and H, assume that it is possible to introduce real values $S_0(f)$ ($S_\infty(f)$ if Ω is not bounded), $S_0(H)$ and also $S_\infty(H)$. If t, v and T are real numbers $T :=$ $\mathrm{Max}\big[-t, -\big(S_0(f) + n\big)/a\big]$ and if Ω is bounded*

$t \geq -S_0(H)$, else $t \geq \mathrm{Max}\big[-S_0(H), \big(n - S_\infty(f)\big)/a\big]$, $v := \mathrm{Max}[n - at, n - aS_\infty(H)]$ then for $f \in \mathcal{E}^v_{at-n}(\Omega, C)$ and $H \in \mathcal{E}^t_T(]0, +\infty[, C)$ with the following behavior

$$H(u) = \sum_{q.\mathrm{Re}(\beta)\leq T}^{Q(p)} H^0_{pq} u^{\beta_p} \log^q u + u^{T'} H^0_T(u), \quad T' > T \tag{3.18}$$

$$H(u) = \sum_{e.\mathrm{Re}(\gamma)\leq t}^{M(e)} H^\infty_{em} u^{-\gamma_e} \log^m u + u^{-s} H^\infty_T(u), \quad s > t \tag{3.19}$$

$$f(x) = \sum_{j.\mathrm{Re}(\alpha)\leq at-n}^{J(i)} f^0_{ij}(\theta) r^{\alpha_i} \log^j r + r^{s'} F^0(x), \quad s' > at - n \tag{3.20}$$

if Ω is not bounded

$$f(x) = \sum_{k.\mathrm{Re}(\delta)\leq v}^{K(l)} f^\infty_{lk}(\theta) r^{-\delta_l} \log^k r + r^{-v'} F^\infty(x), \quad v' > v \tag{3.21}$$

and also $h_\lambda(x) = h_\lambda(r,\theta) := f(x)H(\lambda w(\theta) r^a) \in L^1_{\mathrm{loc}}(\Omega \setminus \{O\}, C)$, the expansion below is valid for large λ

$$\begin{aligned}
&L(\lambda) \\
&:= \mathrm{fp} \int_\Omega f(x) H[\lambda w(\theta) r^a]\, dx \\
&= \sum_{m.\mathrm{Re}(\gamma)\leq t}^{M(e)} \sum_{l=0}^{m} C^l_m H^\infty_{em} \Big[\mathrm{fp} \int_\Omega f(x) w(\theta)^{-\gamma_e} r^{-a\gamma_e} \log^{m-l}[w(\theta) r^a]\, dx\Big] \lambda^{-\gamma_e} \log^l \lambda \\
&+ \sum_{j.\mathrm{Re}(\alpha)\leq at-n}^{J(i)} \Bigg\{ \sum_{\{e\,;\,a\gamma_e = \alpha_i + n\}} \sum_{m=0}^{M(e)} \sum_{l_2=0}^{1+j+m} \frac{(-1)^j j!\, m!\, C^{l_2}_{1+j+m}}{a^{1+j}(1+j+m)!} H^\infty_{em} F^m_{ij}(l_2) \log^{l_2} \lambda \\
&+ \sum_{l=0}^{j} \frac{C^l_j (-1)^l}{a^j} \Big[\mathrm{fp} \int_{\mathbb{R}^n} f^0_{ij}(\theta) w(\theta)^{-\frac{\alpha_i+n}{a}} H(r^a) r^{\alpha_i} \log^{j-l}\big(w(\theta)^{-1} r^a\big)\, dx\Big] \log^l \lambda \\
&- \sum_{\{p\,;\,a\beta_p = -\alpha_i - n\}} \sum_{q=0}^{Q(p)} \sum_{l_1=0}^{1+j+q} \frac{(-1)^j j!\, q!\, C^{l_1}_{1+j+q}}{a^{1+j}(1+j+q)!} H^0_{pq} F^q_{ij}(l_1) \log^{l_1} \lambda \Bigg\} \lambda^{-(\alpha_i+n)/a} + o(\lambda^{-t}),
\end{aligned} \tag{3.22}$$

with, for positive integers i, j, q, k: $F^q_{ij}(k) := \int_\Sigma f^0_{ij}(\theta) w(\theta)^{-\frac{\alpha_i+n}{a}} \log^{1+j+q-k}[w(\theta)]\, d\sigma$.

Observe that for Ω not bounded, the behavior (3.21) of function f near infinity and at a large enough order is not involved in this result despite it is requested for the derivation. Since the derivation makes use of Theorem 2 and looks like the one given in [10] for Theorem 3, it is let to the reader.

4. Applications. In this section, several examples are exhibited in order to highlight the range of applications of the proposed theorems. By the way, comparisons with results obtained in [4] are also given.

PROPOSITION 1. *Consider Φ a complex pseudofunction for which it is possible to introduce $S_0(\Phi)$ and $S_\infty(\Phi)$. If $\gamma \in C$, $a \in \mathbb{R}^*_+$, $l \in \mathbb{N}$ and w is a strictly positive* *and bounded function on Σ, then for real t such that $t \geq n - [S_\infty(\Phi) + \mathrm{Re}(\gamma)]/a$, $\Phi \in \mathcal{E}^{n-at+\mathrm{Re}(\gamma)}_{at-n-\mathrm{Re}(\gamma)}(\mathbb{R}^n, C)$ and λ large*

$$J(\lambda) = \mathrm{fp}\int_{\mathbb{R}^n} \Phi(x) r^{\gamma} \log^l(r) \mathrm{e}^{-\lambda w(\theta) r^a}\, dx$$

$$\begin{aligned}(4.1)\quad = \sum_{j.\mathrm{Re}(\alpha)\leq at-n-\mathrm{Re}(\gamma)}^{J(i)} \Bigg\{ &\sum_{k=0}^{j+l}\sum_{h=0}^{j+l-k} \frac{C^k_{j+l} C^h_{j+l-k}(-1)^{j+l-h}}{a^{j+l+1}} J^l_{ij}(k+h) N_h\Big(\frac{\alpha_i+\gamma+n}{a}\Big)\log^k\lambda \\ &- \sum_{\{p\,;\,ap=-\alpha_i-\gamma-n\}} \sum_{l_1=0}^{1+j+l} \frac{(-1)^{j+l+p} C^{l_1}_{1+j+l}}{a^{1+j+l}(1+j+l)p!} J^1_{ij}(l_1)\log^{l_1}\lambda \Bigg\} \lambda^{-(\alpha_i+\gamma+n)/a} \\ &+ o(\lambda^{-t}),\end{aligned}$$

with the notation $\lim_{r\to 0}\Phi(r,\theta) = \sum^{J(i)}_{j.\mathrm{Re}(\alpha)\leq at-n-\mathrm{Re}(\gamma)} A_{ij}(\theta) r^{\alpha_i}\log^j r$ *and for positive integers k, m and $\beta \in C$*

$$N_k(\beta) := \mathrm{fp}\int_0^\infty v^{\beta-1}\log^k(v)\mathrm{e}^{-v}\, dv, \tag{4.2}$$

$$J^m_{ij}(k) := \int_\Sigma A_{ij}(\theta)\log^{m+j-k}[w(\theta)]\, w(\theta)^{-(\alpha_i+\gamma+n)/a}\, d\sigma. \tag{4.3}$$

Since $\Phi \in \mathcal{E}^{n-at+\mathrm{Re}(\gamma)}_{at-n-\mathrm{Re}(\gamma)}(\mathbb{R}^n, C)$, $\Phi \in L^1_{\mathrm{loc}}(\mathbb{R}^n\setminus\{O\}, C)$ and consequently $h_\lambda(x) := \Phi(x) r^\gamma \log^l(r)\mathrm{e}^{-\lambda w(\theta) r^a} \in L^1_{\mathrm{loc}}(\mathbb{R}^n\setminus\{O\}, C)$. Formula (4.1) is thereafter obtained by applying Theorem 3 with $f(r,\theta) := r^\gamma \log^l(r)\Phi(x)$ and $H(u) := \mathrm{e}^{-u}$.

For $\beta \in C$ consider $N_0(\beta) = \mathrm{fp}\int_0^\infty v^{\beta-1}\mathrm{e}^{-v}\, dv$. Clearly, if $\mathrm{Re}(\beta) > 0$ then $N_0(\beta) = \Gamma(\beta)$ where Γ is the usual Gamma function. By using as many integrations by parts as necessary two case arise for $\mathrm{Re}(\beta) \leq 0$: if $\beta = -p$ with $p \in \mathbb{N}$ then $N_0(-p) = (-1)^p\Psi(p+1)/p!$ where $\Psi(p+1) := \sum_{l=1}^p l^{-1} - C_e$ is the usual digamma function, else $N_0(\beta) = \Gamma(\beta+m)/[\beta(\beta+1)\cdots(\beta+m-1)]$ where m is any positive integer such that $\mathrm{Re}(\beta) + m > 0$.

EXAMPLE 1. Suppose that $\gamma = 0$, $l = 0$ and $J(i) = 0$ with $A_{ij}(\theta) = A_i(\theta)$. In such circumstances and under the assumptions of Proposition 1, formula (4.1) rewrites

$$\begin{aligned}(4.4)\quad M(\lambda) &= \mathrm{fp}\int_{\mathbb{R}^n}\Phi(x)\mathrm{e}^{-\lambda w(\theta) r^a}\, dx \\ &= \frac{1}{a}\sum_{j.\mathrm{Re}(\alpha)\leq at-n}\Bigg\{\Big[\int_\Sigma A_i(\theta) w(\theta)^{-\frac{\alpha_i+n}{a}}\, d\sigma\Big] N_0\Big(\frac{\alpha_i+n}{a}\Big) \\ &\quad - \sum_{\{p\,;\,ap=-\alpha_i-n\}} \frac{(-1)^p}{p!}\Big[\int_\Sigma A_i(\theta) w(\theta)^{-\frac{\alpha_i+n}{a}}\log[w(\theta)]\, d\sigma \\ &\quad + \log[\lambda]\int_\Sigma A_i(\theta) w(\theta)^{-\frac{\alpha_i+n}{a}}\, d\sigma\Big]\Bigg\}\lambda^{-(\alpha_i+n)/a} + o(\lambda^{-t}).\end{aligned}$$

If $\forall i$, $\mathrm{Re}(\alpha_i + n) > 0$, the second contribution on the right-hand side of (4.4) vanishes. In such a case, $N_0(\frac{\alpha_i+n}{a}) = \Gamma(\frac{\alpha_i+n}{a})$. This agrees with the proposed Example 9 given in [5] if one sets $A_i(\theta) := 0$ outside $V \subset \Sigma$. Nevertheless, if there exist positive integers p and i such that $-p \leq t$ and $\alpha_i = -ap - n$ this second contribution introduces a logarithmic term $(\log[\lambda]\lambda^p)$ in the asymptotic expansion.

EXAMPLE 2. By now, it is assumed that $\Phi(x) = \Phi(r,\theta) := A(\theta)\phi(x)$ with $A \in L^1(\Sigma, C)$ and $\phi \in \mathcal{D}_N(\mathbb{R}^n, C)$ the set of complex functions ϕ which admit on a compact neighborhood of origin O continuous derivatives up to order positive integer N. Thanks to the usual Taylor expansion, there exists for each $\phi \in \mathcal{D}_N(\mathbb{R}^n, C)$ a neighborhood of O where the function R_N^ϕ below is bounded and

$$\phi(x) = \phi(r,\theta) = \sum_{|\alpha|=0}^{N-1} \frac{A_\alpha(\theta)}{\alpha!} D^\alpha\phi(O) r^{|\alpha|} + r^N R_N^\phi(x), \tag{4.5}$$

where for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we set: $\alpha! := \alpha_1! \cdots \alpha_n!, |\alpha| := \alpha_1 + \cdots + \alpha_n$ and if $x = (r,\theta) = (x_1, \ldots, x_n)$ then $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} := r^{|\alpha|} A_\alpha(\theta), D^\alpha\phi := \partial^{|\alpha|}\phi / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$. Note also that (4.5) holds for $N = 0$ by setting $\sum_{|\alpha|=0}^q := 0$ for $q < 0$ and by choosing $R_0^\phi = \phi \in \mathcal{D}_0(\mathbb{R}^n, C)$. For $t \geq n - [S_\infty(\phi) + \mathrm{Re}(\gamma)]/a$ and $\phi \in \mathcal{D}_{N_t}(\mathbb{R}^n, C) \cap \mathcal{E}_{at-n-\mathrm{Re}(\gamma)}^{n-at+\mathrm{Re}(\gamma)}(\mathbb{R}^n, C)$ if $N_t := [[1 + at - n - \mathrm{Re}(\gamma)]]$ where $[[b]]$ denotes the integer part of real b, Proposition 1 yields

$$\begin{aligned} K(\lambda) &= \mathrm{fp} \int_{\mathbb{R}^n} A(\theta)\phi(x) r^\gamma \log^l(r) \mathrm{e}^{-\lambda w(\theta) r^a}\, dx \\ &= \sum_{|\alpha|=0}^{[[at-n-\mathrm{Re}(\gamma)]]} \frac{D^\alpha\phi(O)}{\alpha!} \Big\{ \sum_{k=0}^{l} \sum_{h=0}^{l-k} \frac{(-1)^{l-h} l!\, J_\alpha(l-k-h)}{k!\, h!\, (l-k-h)!\, a^{l+1}} N_h\Big(\frac{|\alpha| + \gamma + n}{a}\Big) \log^k \lambda \\ &\quad - \sum_{\{p\,;\, ap = -|\alpha| - \gamma - n\}} \sum_{l_1=0}^{l+1} \frac{(-1)^{l+p} C_{l+1}^{l_1} J_\alpha(1-l_1)}{a(l+1)p!} \log^{l_1}\lambda \Big\} \lambda^{-(|\alpha|+\gamma+n)/a} + o(\lambda^{-t}), \end{aligned} \tag{4.6}$$

with $J_\alpha(k) := \int_\Sigma A(\theta) A_\alpha(\theta) \log^k[w(\theta)] w(\theta)^{-\frac{|\alpha|+\gamma+n}{a}}\, d\sigma$ and $\sum_{|\alpha|=0}^p := 0$ for $p < 0$.

The case $w = A = 1_\Sigma, a = 2$ and $l = 0$ is considered in [5] (Examples 1 and 4). Under these assumptions $J_\alpha(k) = 0$ for $k \geq 1$ and $J_\alpha(0) = \int_\Sigma A_\alpha(\theta)\, d\sigma$. If $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $J_\alpha(0) = 0$ for any α_l odd and if $\alpha = 2\beta = (2\beta_1, \ldots, 2\beta_n) \in \mathbb{N}^n$ it is well known that $J_{2\beta}(0) = 2\Gamma(\beta_1 + 1/2) \cdots \Gamma(\beta_n + 1/2)/\Gamma(|\beta| + n/2)$ with $\Gamma(j + 1/2) = \pi^{1/2}(2j)!/2^{2j}j!$. This leads to $J_\alpha(0)/\alpha! = J_{2\beta}(0)/(2\beta)! = 2\pi^{n/2}/[2^{2|\beta|}\beta!\,\Gamma(|\beta| + n/2)]$. For $\gamma \in C$, suppose that real t is such that $\big[[2t - n - \mathrm{Re}(\gamma)]\big] \in \mathbb{N}$ (else expansion (4.6) rewrites $K(\lambda) = o\big(\lambda^{-t}\big)$) and introduce for $k \in \mathrm{N}, E(k)$ such that $E(2m) = E(2m+1) := m$. If γ is not equal to $-n - 2m$ with $m \in \mathbb{N}$ it is impossible to find $p \in \mathbb{N}$ such that $2p = -2|\beta| - \gamma - n$. Consequently the second term on the right-hand side of (4.6) is zero and

$$\begin{aligned} &\mathrm{fp} \int_{\mathbb{R}^n} \phi(x) r^\gamma \mathrm{e}^{-\lambda r^2}\, dx \\ &= \sum_{k=0}^{E[[2t-n-\mathrm{Re}(\gamma)]]} \sum_{|\beta|=k} \frac{D^{2\beta}\phi(O)\pi^{n/2} N_0\big(k + (\gamma+n)/2\big)}{2^{2k}\beta!\,\Gamma(k + n/2)} \lambda^{-(2k+\gamma+n)/2} + o(\lambda^{-t}). \end{aligned} \tag{4.7}$$

If there exists $m \in \mathbb{N}$ such that $\gamma = -n - 2m$ for $0 \leq |\beta| \leq E\big(\big[[2t - n - \mathrm{Re}(\gamma)]\big]$ the integer $p := m - |\beta|$ provides a logarithmic term and

$$\mathrm{fp} \int_{\mathbb{R}^n} \phi(x) r^{-(n+2m)} \mathrm{e}^{-\lambda r^2}\, dx = o(\lambda^{-t}) + \sum_{k=0}^{E[[2t-n-\mathrm{Re}(\gamma)]]} \sum_{|\beta|=k} \frac{D^{2\beta}\phi(O)\pi^{n/2}\lambda^{m-k}}{2^{2k}\beta!\,\Gamma(k + n/2)}$$

$$\left\{N_0\Big(k+(\gamma+n)/2\Big)-\left[\sum_{p=0}^{m-k}\frac{(-1)^p}{p!}\right]\log\lambda\right\}. \tag{4.8}$$

Observe that both results (4.7) and (4.8) disagree with those proposed in [5]. In fact, when dealing with integration in the finite part sense a change of scale may generate corrective terms. For instance, if $\lambda>0$ and $\delta\in C$ (see [10], Lemma 2),

$$\mathrm{fp}\int_0^\infty r^\delta \mathrm{e}^{-\lambda r^2}\,dr=\lambda^{-(\delta+1)/2}\,\mathrm{fp}\int_0^\infty t^\delta \mathrm{e}^{-t^2}\,dt \quad \text{if } \delta\neq -1-2p, p\in\mathbb{N}, \tag{4.9}$$

$$\mathrm{fp}\int_0^\infty r^{-1-2p}\mathrm{e}^{-\lambda r^2}\,dr=\lambda^p\left\{\mathrm{fp}\int_0^\infty t^{-1-2p}\mathrm{e}^{-t^2}\,dt-\frac{(-1)^p}{p!}\log\lambda\right\}. \tag{4.10}$$

Consequently, if $\phi_\beta(x):=x^{2\beta}=A_{2\beta}(\theta)r^{2|\beta|}$ with $\beta=(\beta_1,\ldots,\beta_n)\in\mathbb{N}^n$, one obtains

$$\begin{aligned}I(\lambda)&=\mathrm{fp}\int_{\mathbb{R}^n}\phi_\beta(x)r^\gamma\mathrm{e}^{-\lambda r^2}\,dx\\&=\left[\int_\Sigma A_{2\beta}(\theta)\,d\sigma\right]\left[\mathrm{fp}\int_0^\infty r^{2|\beta|+\gamma+n-1}\mathrm{e}^{-\lambda r^2}\,dr\right]=\frac{1}{2}\left[\int_\Sigma A_{2\beta}(\theta)\,d\sigma\right]\\&\qquad\left\{\mathrm{fp}\int_0^\infty v^{\frac{2|\beta|+\gamma+n}{2}-1}\mathrm{e}^{-v}\,dv-\sum_{\{p\,;\,2p=-\gamma-n-2|\beta|\}}\frac{(-1)^p}{p!}\log\lambda\right\}\lambda^{-\frac{2|\beta|+\gamma+n}{2}}\end{aligned}$$

because $2\,\mathrm{fp}\int_0^\infty t^\delta\mathrm{e}^{-t^2}\,dt=\mathrm{fp}\int_0^\infty v^{\frac{\delta+1}{2}-1}\mathrm{e}^{-v}\,dv$. Obviously, above equality agrees with (4.7) and (4.8). Observe that $\phi_\beta\in\mathcal{P}(\mathbb{R}^n):=\{$smooth and complex functions ϕ such that $\lim_{r\to+\infty}\mathrm{e}^{-br}D^\alpha\phi(x)=0$, $\forall b>0$ and $\alpha\in\mathbb{N}^n\}$ and that formula (3.10) and (3.13) proposed by paper [5] don't apply for ϕ_β. In fact, it seems that the "moment expansion" proposed in this latter paper does not hold if $\gamma=-n-2m$. More precisely, for $\phi\in\mathcal{P}(\mathbb{R}^n)$ relations (4.9) and (4.10) show that in such a case

$$\langle\mathrm{fp}[(\sqrt{\lambda}r)^{-(n+2m)}\mathrm{e}^{-(\sqrt{\lambda}r)^2}],\phi(x)\rangle\neq\langle\mathrm{fp}[r^{-(n+2m)}\mathrm{e}^{-r^2}],\phi(x/\sqrt{\lambda})\rangle, \tag{4.11}$$

and that extra terms occur at this stage.

PROPOSITION 2. *Consider $c\in\mathbb{R}_+$ and A a positive and bounded function on Σ. For Ω a bounded subset of $\mathbb{R}^n$ containing origin O, an integer $N\geq n$ and a function $f\in L^1_{\mathrm{loc}}(\Omega\cup\partial\Omega,C)\cap\mathcal{D}_{N-n+1}(\Omega,C)$ (see Example 2 for the definition of $\mathcal{D}_q(\Omega,C)$) then*

$$\begin{aligned}I(\lambda)&=\int_\Omega\frac{f(x)\,dx}{1+A(\theta)r^c+\lambda r}\\&=\sum_{e=0}^{N-1}\left[\mathrm{fp}\int_\Omega(-1)^e[1+A(\theta)r^c]^e f(x)r^{-(e+1)}\,dx\right]\lambda^{-(e+1)}\\&\quad+\sum_{(i,j)\in A_N}\left[\int_\Sigma\sum_{|\alpha|=i}\frac{D^\alpha f(O)}{\alpha!}A_\alpha(\theta)A_j(\theta)\,d\sigma\right]\left\{\mathrm{fp}\int_0^\infty\frac{(-1)^j r^{i+cj+n-1}}{(1+r)^{j+1}}\,dr\right.\\&\quad-\frac{H(i+cj+n)(-1)^{(1-c)j-i-n}\Big((1-c)j-i-n\Big)!}{j!\,(-i-cj-n)!}\\&\quad\left.+\Delta(j,i+cj+n-1)(-1)^{i+cj+n-1}C^j_{i+cj+n-1}\right\}\lambda^{-(i+cj+n)}+o(\lambda^{-N}),\end{aligned} \tag{4.12}$$

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where $A_N := \{(i,j) \in \mathbb{N}^2, a_i^j := i + cj \leq N - n\}$ *and for* $k \in \mathbb{N}$, $a \in \mathbb{R}$ *functions* Δ *and* H *obey* $\Delta(k, a) = 1 = H(-a)$ *if* $a \in \mathbb{N}$ *and* $k \leq a$; *else* $\Delta(k, a) = 0 = H(-a)$.

The reader may easily deduce (4.12) by carefully applying Theorem 1.

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