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THE JIANG–SU ALGEBRA AS A FRAÏSSÉ LIMIT

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Abstract. In this paper, we give a self-contained and quite elementary proof that the class of all dimension drop algebras together with their distinguished faithful traces forms a Fraïssé class with the Jiang–Su algebra as its limit. We also show that the UHF algebras can be realized as Fraïssé limits of classes of C*-algebras of matrix-valued continuous functions on [0, 1] with faithful traces.

§1. Introduction. Fraïssé theory was originally invented by Roland Fraïssé in [5], where a bijective correspondence between countable ultrahomogeneous structures and classes with certain properties of finitely generated structures was established. The classes and the corresponding ultrahomogeneous structures in question are called *Fraïssé classes* and *Fraïssé limits* of the classes, respectively.

This theory has been, among the rest, a target of generalization to the setting of metric structures. For example, a general theory was developed in [11], including connections with bounded continuous logic. In [1], Itaï Ben Yaacov concisely gave a self-contained presentation of a general theory, using a bright idea of approximate isomorphisms.

These attempts at generalization ended up successfully, and a number of metric structures are recognized as Fraïssé limits. Itaï Ben Yaacov [1] pointed out that the Urysohn universal space, the separable infinite dimensional Hilbert space, and the atomless standard probability space are examples of Fraïssé limits corresponding to suitable classes, and reconstructed the discussion in [8] where the Gurarij space had been implicitly shown to be a Fraïssé limit of the class of all finite dimensional Banach spaces. The latter result was quantized by Martino Lupini [9]: it was shown that the noncommutative Gurarij space is the Fraïssé limit of the class of all finite dimensional 1-exact operator spaces.

Among those instances are operator algebras. In [3], a more generalized version of Fraïssé theory for metric structures was presented, where the axioms of Fraïssé class were relaxed, and so the bijective correspondence established in the original theory no longer holds and the limit structures would have less homogeneity, though it is still powerful as a construction method. Using this version, the authors of the paper succeeded in realizing a family of AF algebras including the UHF algebras, the hyperfinite II_1 factor and the Jiang–Su algebra as (generalized) Fraïssé limits of

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a class of finite dimensional C*-algebras with distinguished traces, the class of finite dimensional factors and the class of dimension drop algebras with distinguished traces, respectively.

The Jiang–Su algebra was first constructed by Jiang and Su in [7] as the unique simple monotracial C*-algebra among inductive limits of prime dimension drop algebras, which is KK-equivalent to the complex numbers $\mathbb{C}$. One of the most important properties of this algebra is that it is strongly self-absorbing, because of which it plays a key role in the Elliott's classification program of separable nuclear C*-algebras via K-theoretic invariants [4]. As is pointed out in the last section of [3], the proof that the Jiang–Su algebra satisfies this property is nontrivial, and there is a reasonable prospect that Fraïssé theoretic view of this algebra will give a shortcut. However, the proof given in [3] of the fact that the Jiang–Su algebra is a Fraïssé limit was still "a bit unsatisfactory" in the authors' phrase, as it used the existence of the Jiang–Su algebra itself and relied heavily on Robert's theorem (see [3, Remark 4.8]).

In this paper, we prove that the collection of all the dimension drop algebras together with their distinguished faithful traces forms a Fraïssé class. The importance lies in that this proof is self-contained and quite elementary; in particular, it depends on neither the existence of the Jiang–Su algebra nor Robert's theorem, so that it can be considered as a solution to [3, Remark 4.8]. Also, we show that the UHF algebras are realized as a Fraïssé limit of a class of C*-algebras of matrix-valued continuous functions on the interval $[0, 1]$ together with their distinguished faithful traces. Since this class differs from the one used in [3], this result implies a different homogeneity property of the UHF algebras.

The paper consists of four sections. In the next section, we briefly introduce a version of Fraïssé theory for metric structures, which is essentially the same as the one in [3]. The third section contains the result on the UHF algebras. The argument included in this section is the basis of the fourth section, where the dimension drop algebras and the Jiang–Su algebra are dealt with.

§2. Fraïssé theory for metric structures. In this section, we present a general theory of Fraïssé limits in the context of metric structures, which is almost the same as the one in [3, Section 2]. The facts stated here are slight generalization of those of [1], and can be proved with trivial modification.

Definition 2.1. A *language* L consists of *predicate symbols* and *function symbols*. To each symbol in L is associated a natural number called its *arity*. We assume that L contains a binary predicate symbol d.

An *L-structure* is a complete metric space M together with an *interpretation* of symbols of L:

- to each n-ary predicate symbol P is assigned a continuous map $P^M\colon M^n \to \mathbb{R}$, where the distinguished binary predicate symbol d corresponds to the distance function; and
- to each n-ary function symbol f is assigned a continuous map $f^M\colon M^n \to M$.

An *embedding* of an L-structure M into another L-structure N is a map φ such that

$$f^N(\varphi(a_1), \ldots, \varphi(a_n)) = \varphi(f^M(a_1, \ldots, a_n))$$

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and

$$P^N(\varphi(a_1), \dots, \varphi(a_n)) = P^M(a_1, \dots, a_n)$$

hold for any function symbol f, any predicate symbol P and any elements $a_1, \dots, a_n$ in M.

In this paper, we focus on unital C*-algebras with distinguished traces. We assume that L consists of the binary predicate symbol d, an unary predicate symbol tr, binary function symbols $+$ and $\cdot$, an unary function symbol λ for each $\lambda \in \mathbb{C}$ which should be interpreted as multiplication by λ, an unary function symbol $*$, and 0-ary function symbols 0 and 1. Then every unital C*-algebra with trace is understood as an L-structure in the canonical manner. Note that an embedding in the sense of Definition 2.1 is an injective *trace-preserving* $*$-homomorphism in this case, which we shall call a *morphism* in the sequel.

Remark 2.2. The definition of languages and metric structures varies by paper (see [1, Remark 2.2]), and the one we adopted here is the same as [1, Definition 2.1]. Some variants such as [3, Definition 2.1] require that all the maps which appear should be bounded or uniformly continuous, in which case the language carries additional informations. A C*-algebra is seemingly not an instance of a metric structure in these cases, because it is apparently unbounded and the multiplication is not uniformly continuous. Indeed, this can be easily overcome by using the unit ball as its representative, as in [3]. Anyway, the results of Fraïssé theory in both perspectives can be easily translated into each other, so we are in the same line as [3].

Definition 2.3. A class $\mathcal{K}$ of L-structures is said to satisfy

- the *joint embedding property (JEP)* if for any $A, B \in \mathcal{K}$ there exists $C \in \mathcal{K}$ such that both A and B can be embedded in C.
- the *near amalgamation property (NAP)* if for any $A, B_1, B_2 \in \mathcal{K}$, any embeddings $\varphi_i \colon A \to B_i$, any finite subset $G \subseteq A$ and any $\varepsilon > 0$, there exist embeddings ψ_i of B_i into some $C \in \mathcal{K}$ such that $d(\psi_1 \circ \varphi_1(a), \psi_2 \circ \varphi_2(a))$ is less than ε for all $a \in G$.

An L-structure A is said to be *finitely generated* if there exists a tuple $\vec{a} = (a_1, \dots, a_n) \in A^n$ such that the smallest substructure of A containing all $a_1, \dots, a_n$ is A, for some $n \in \mathbb{N}$. (Note that we assumed L-structures to be necessarily complete.) As we focus on unital C*-algebras with distinguished traces, this definition coincides with the usual one, that is, a (unital) C*-algebra (with its distinguished trace) is finitely generated if there exists a finite subset such that its closure by addition, multiplication, scalar multiplication and $*$-operation is dense in the whole C*-algebra.

Let $\mathcal{K}$ be a class of finitely generated L-structures. For each $n \in \mathbb{N}$, we denote by $\mathcal{K}_n$ the class of all the pairs $(A, \vec{a})$, where A is a member of $\mathcal{K}$ and $\vec{a} \in A^n$ is a generator of A. If $\mathcal{K}$ satisfies JEP and NAP, then we can define a pseudometric $d^{\mathcal{K}}$ on $\mathcal{K}_n$ by

$$d^{\mathcal{K}}\big((A, \vec{a}), (B, \vec{b})\big) := \inf \max_i d(f(a_i), g(b_i)),$$

where $\vec{a} = (a_1, \dots, a_n), \vec{b} = (b_1, \dots, b_n)$ and the infimum is taken over all the embeddings f, g of A, B into some C in $\mathcal{K}$.

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DEFINITION 2.4. A class $\mathcal{K}$ of finitely generated L-structures with JEP and NAP is said to satisfy

- the *weak Polish property (WPP)* if $\mathcal{K}_n$ is separable with respect to the pseudometric $d^{\mathcal{K}}$ for all n.
- the *Cauchy continuity property (CCP)* if
  (1) for any n-ary predicate symbol P, the map $(A, (\vec{a}, \vec{b})) \mapsto P^A(\vec{a})$ from $\mathcal{K}_{n+m}$ into $\mathbb{R}$ sends Cauchy sequences to Cauchy sequences; and
  (2) for any n-ary function symbol f, the map $(A, (\vec{a}, \vec{b})) \mapsto (A, (\vec{a}, \vec{b}, f^A(\vec{a})))$ from $\mathcal{K}_{n+m}$ into $\mathcal{K}_{n+m+1}$ sends Cauchy sequences to Cauchy sequences.

REMARK 2.5. CCP implies that $d^{\mathcal{K}}((A, \vec{a}), (B, \vec{b})) = 0$ holds if and only if there is an isomorphism from A to B which sends $\vec{a}$ to $\vec{b}$ ([1, Remark 2.13(i)]). Note that if $\mathcal{K}$ is a class of finitely generated unital C*-algebras with traces and if it satisfies JEP and NAP, then it also satisfies CCP automatically, because all the relevant functions are 1-Lipschitz on the unit ball.

DEFINITION 2.6. A class $\mathcal{K}$ of finitely generated L-structures is called a *Fraïssé class* if it satisfies JEP, NAP, WPP, and CCP. A *Fraïssé limit* of a Fraïssé class $\mathcal{K}$ is a separable L-structure M which is

(1) a *$\mathcal{K}$-structure*: for any finite subset F of M and any $\varepsilon > 0$, there exists an embedding φ of a member of $\mathcal{K}$ such that the ε-neighborhood of the image of φ includes F.
(2) *$\mathcal{K}$-universal*: every member of $\mathcal{K}$ can be embedded into M.
(3) *approximately $\mathcal{K}$-homogeneous*: if A is a member of $\mathcal{K}$ and $a_1, \ldots, a_n$ are elements of A, then for any embeddings φ, ψ of A into M and any $\varepsilon > 0$, there exists an automorphism α of M with $d(\alpha \circ \varphi(a_i), \psi(a_i)) < \varepsilon$ for $i = 1, \ldots, n$.

The definition here is more relaxed than that of [1] and close to [3, Definition 2.6]: our Fraïssé class is incomplete and lacks the hereditary property (see [1, Definitions 2.5(ii) and 2.12]). Consequently, we cannot establish a bijective correspondence between Fraïssé classes and separable structures with homogeneity, which is a part of the main result of Fraïssé theory. The following theorem summarizes what remains in our framework.

THEOREM 2.7. *Every Fraïssé class $\mathcal{K}$ admits a unique limit. Moreover, for any L-structure A_0 in $\mathcal{K}$, there exists a sequence of embeddings $A_0 \xrightarrow{\varphi_0} A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} \cdots$ such that A_i belongs to $\mathcal{K}$ for all i and its inductive limit L-structure coincides with the Fraïssé limit of $\mathcal{K}$.*

§3. UHF algebras.

A *supernatural number* is a formal product

$$\nu = \prod_{p\colon \text{prime}} p^{n_p},$$

where n_p is either a non-negative integer or ∞ for each p such that $\sum_p n_p = \infty$. In [6, Theorem 1.12], it was proved that one can associate to each UHF algebra a supernatural number as a complete invariant. Now, given a supernatural number ν, we denote by $\mathbb{N}(\nu)$ the set of all natural numbers which formally divide ν, and by $\mathcal{K}(\nu)$ the class of all the pairs $\langle C[0,1] \otimes \mathbb{M}_n, \tau \rangle$, where n is in $\mathbb{N}(\nu)$ and τ is a faithful

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trace on the C*-algebra $C[0,1] \otimes \mathbb{M}_n$. Our goal in this section is to show that $\mathcal{K}(v)$ is a Fraïssé class the limit of which is the UHF algebra with v as its associated supernatural number.

First, note that $C[0,1] \otimes \mathbb{M}_n$ is canonically isomorphic to $C([0,1], \mathbb{M}_n)$, the C*-algebra of all continuous $\mathbb{M}_n$-valued functions on the interval $[0,1]$. In the sequel, we shall denote this C*-algebra by $\mathcal{A}_n$ for simplicity. Next, let τ be a probability Radon measure on $[0,1]$, which is identified with a state on $C[0,1]$ by integration. Then $\tau \otimes \mathrm{tr}$ is clearly a trace on $\mathcal{A}_n$, where tr is the unique normalized trace on $\mathbb{M}_n$. It is easy to see that every trace on $\mathcal{A}_n$ is of this form, so a probability Radon measure on $[0,1]$ may also be identified with a trace on $\mathcal{A}_n$. In the sequel, we simply write τ instead of $\tau \otimes \mathrm{tr}$ and use the same adjectives for measures and traces in common. For example, a measure is said to be faithful if its corresponding trace is faithful. Also, all the measures are assumed to be probability Radon measures so that they always correspond to traces.

A measure is said to be *diffuse* or *atomless* if any measurable set of nonzero measure can be partitioned into two measurable sets of nonzero measure. The following is often used in the sequel without further mention.

LEMMA 3.1. *Let σ, τ be faithful measures. If σ is diffuse, then there exists a unique nondecreasing continuous function β from* $[0,1]$ *onto* $[0,1]$ *with $\beta_*(\sigma) = \tau$. Moreover, τ is diffuse if and only if β is a homeomorphism.*

PROOF. We first assume that the measure σ is equal to the Lebesgue measure λ and set $\alpha(t) := \tau([0,t))$. Note that α is a strictly increasing lower semicontinuous function from $[0,1]$ into $[0,1]$. Let β be the unique nondecreasing function extending α^{-1}. Then

$$\beta_*(\lambda)([0,t)) = \lambda\big(\beta^{-1}\big[[0,t)\big]\big) = \lambda\big([0,\alpha(t))\big) = \alpha(t) = \tau([0,t)),$$

so $\beta_*(\lambda)$ is equal to τ. Also, if τ is diffuse, then α is continuous, whence $\beta = \alpha^{-1}$ is a homeomorphism.

For the general case, let β_σ, β_τ be the nondecreasing continuous functions such that $(\beta_\sigma)_*(\lambda) = \sigma$ and $(\beta_\tau)_*(\lambda) = \tau$. Then β_σ is a homeomorphism and $\beta := \beta_\tau \circ \beta_\sigma^{-1}$ satisfies $\beta_*(\sigma) = \tau$, which completes the proof. ⊣

The next propositions are immediate corollaries of the preceding lemma. Recall that a morphism between elements of $\mathcal{K}(v)$ is an injective unital trace-preserving $*$-homomorphism.

PROPOSITION 3.2. *Let τ be a faithful trace on $\mathcal{A}_n$. Then for any faithful diffuse trace σ on $\mathcal{A}_n$, there is a morphism $\varphi\colon \langle \mathcal{A}_n, \tau\rangle \to \langle \mathcal{A}_n, \sigma\rangle$.*

PROOF. Let $\beta\colon [0,1] \to [0,1]$ be the nondecreasing continuous function satisfying $\beta_*(\sigma) = \tau$. Then $\varphi := \beta^*$ is the desired morphism. ⊣

PROPOSITION 3.3. *The class $\mathcal{K}(v)$ satisfies JEP.*

PROOF. Suppose n_1, n_2 are in $\mathbb{N}(v)$ and put $n := \mathrm{lcm}(n_1, n_2)$. Then n is also in $\mathbb{N}(v)$. Also, $\langle \mathcal{A}_{n_i}, \lambda\rangle$ is clearly embeddable into $\langle \mathcal{A}_n, \lambda\rangle$ by amplification. This fact together with Proposition 3.2 implies that $\mathcal{K}(v)$ satisfies JEP. ⊣

Next, we shall show that the class $\mathcal{K}(v)$ satisfies NAP. For this, we begin with proving that all the morphisms between members of $\mathcal{K}(v)$ are approximately diagonalizable.

DEFINITION 3.4. A morphism $\varphi\colon \langle \mathcal{A}_n, \tau\rangle \to \langle \mathcal{A}_m, \sigma\rangle$ is said to be *diagonalizable* if there are a unitary $u \in \mathcal{A}_m$ and continuous maps $\xi_1, \dots, \xi_k\colon [0,1] \to [0,1]$ such that

$$\varphi(f) = u \begin{pmatrix} f\circ\xi_1 & & 0 \\ & \ddots & \\ 0 & & f\circ\xi_k \end{pmatrix} u^* \tag{1}$$

for all $f \in \mathcal{A}_n$.

In general, for a unitary v in a C*-algebra $\mathcal{A}$, the associated inner automorphism $a \mapsto vav^*$ is denoted by $\mathrm{Ad}(v)$. Also, if $a_1, \dots, a_k$ are elements of $\mathcal{A}$, then $\mathrm{diag}[a_1, \dots, a_k]$ denotes the diagonal element of $\mathbb{M}_k(\mathcal{A})$ such that the (i,i)-entry is equal to a_i. Using these notations, we can rewrite equation (1) as

$$\varphi(f) = \mathrm{Ad}(u)\big(\mathrm{diag}[f\circ\xi_1, \dots, f\circ\xi_k]\big). \tag{1'}$$

In this paper, we shall call this equation a *diagonal expression* of φ, and u and $\xi_1, \dots, \xi_k$ its *associated unitary and maps*. Note that the union of the images of the maps associated to a diagonal expression is equal to $[0,1]$, as morphisms are necessarily faithful. Also, compositions of diagonalizable morphisms are again diagonalizable.

PROPOSITION 3.5. *Let $\varphi\colon \langle \mathcal{A}_n, \tau\rangle \to \langle \mathcal{A}_m, \sigma\rangle$ be a morphism. Then for any finite subset $G \subseteq \mathcal{A}_n$ and any $\varepsilon > 0$, there exists a diagonalizable morphism $\psi\colon \langle \mathcal{A}_n, \tau\rangle \to \langle \mathcal{A}_m, \sigma\rangle$ with $\|\varphi(g) - \psi(g)\| < \varepsilon$ for all $g \in G$. Moreover, we can take ψ so that the maps $\xi_1, \dots, \xi_k$ associated to a diagonal expression of ψ satisfy $\xi_1 \le \cdots \le \xi_k$.*

For the proof, we shall introduce a notation. A *multiset* is a generalization of the concept of a set such that each element is allowed to occur multiple times. In order to distinguish multisets from usual sets, we use double braces. For example, $\{\{a, a, b\}\}$ denotes the multiset which consists of two a's and one b.

PROOF OF PROPOSITION 3.5. For $t \in [0,1]$, let $\mathrm{ev}_t\colon \mathcal{A}_m \to \mathbb{M}_m$ be the evaluation $*$-homomorphism. Then $\mathrm{ev}_t \circ \varphi$ is a unital $*$-homomorphism from $\mathcal{A}_n$ to the finite dimensional C*-algebra $\mathbb{M}_m$, so there exist a unitary $v_t \in \mathbb{M}_m$ and real numbers $s_1^t, \dots, s_k^t \in [0,1]$ such that the equation

$$\mathrm{ev}_t \circ \varphi(f) = \mathrm{Ad}(v_t)\big(\mathrm{diag}\big[f(s_1^t), \dots, f(s_k^t)\big]\big)$$

holds for all $f \in \mathcal{A}_n$. Note that $\{\{s_1^t, \dots, s_k^t\}\}$ coincides with the spectrum of $\mathrm{ev}_t \circ \varphi(\mathrm{id}_{[0,1]} \otimes 1_{\mathbb{M}_n})$ as multisets. By continuity, if t_1 and t_2 are close to each other, then so are the spectra of $\mathrm{ev}_{t_1} \circ \varphi(\mathrm{id}_{[0,1]} \otimes 1_{\mathbb{M}_n})$ and $\mathrm{ev}_{t_2} \circ \varphi(\mathrm{id}_{[0,1]} \otimes 1_{\mathbb{M}_n})$ with respect to the Hausdorff distance. Therefore, if we define

$$\xi_1(t) := \max\{\{s_1^t, \dots, s_k^t\}\};$$
$$\xi_i(t) := \max\Big[\{\{s_1^t, \dots, s_k^t\}\} \setminus \{\{\xi_1(t), \dots, \xi_{i-1}(t)\}\}\Big],$$

then obviously $\xi_1, \dots, \xi_k$ are continuous functions from $[0,1]$ into $[0,1]$ satisfying $\xi_1 \le \cdots \le \xi_k$ and $\{\{\xi_1(t), \dots, \xi_k(t)\}\} = \{\{s_1^t, \dots, s_k^t\}\}$. By multiplying an appropriate permutation unitary from the right if necessary, we may assume in the rest of the proof that the unitary v_t satisfies

$$\mathrm{ev}_t \circ \varphi(f) = \mathrm{Ad}(v_t)\Big(\mathrm{diag}\big[f(\xi_1(t)), \dots, f(\xi_k(t))\big]\Big)$$

for each t.

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Next, fix $t_0 \in [0,1]$. We claim that there exists $\delta(t_0) > 0$ with the following property: if $|t - t_0| < \delta(t_0)$, then there exists a unitary $w_{t_0} \in \mathbb{M}_m$ with $\|v_t - w_{t_0}\| < \varepsilon$ such that the equation

$$\mathrm{ev}_{t_0} \circ \varphi(f) = \mathrm{Ad}(w_{t_0})\Big(\mathrm{diag}\big[f(\xi_1(t_0)), \dots, f(\xi_k(t_0))\big]\Big)$$

holds for all $f \in \mathcal{A}_n$. To see this, let $\{s_1, \dots, s_l\}$ be the set of *distinct* eigenvalues of $\mathrm{ev}_{t_0} \circ \varphi(\mathrm{id}_{[0,1]} \otimes 1_{\mathbb{M}_n})$ and take mutually orthogonal non-negative continuous functions $f_1, \dots, f_l$ such that f_i is constantly equal to 1 on some neighborhood of s_i for each i. Note that if $\{e_{p,q}\}$ is the system of standard matrix units of $\mathbb{M}_n$, then $\{\mathrm{ev}_{t_0} \circ \varphi(f_i \otimes e_{p,q})\}_{i,p,q}$ forms a system of matrix units which spans $\mathrm{Im}(\mathrm{ev}_{t_0} \circ \varphi)$, and if t is sufficiently close to t_0, then $\{\mathrm{ev}_t \circ \varphi(f_i \otimes e_{p,q})\}_{i,p,q}$ is a system of matrix units in $\mathrm{Im}(\mathrm{ev}_t \circ \varphi)$ which is close to $\{\mathrm{ev}_{t_0} \circ \varphi(f_i \otimes e_{p,q})\}_{i,p,q}$. Hence, as in the proof of [2, Lemma III.3.2], we can find a unitary w with $\|w - 1\| < \varepsilon$ such that

$$w\big(\mathrm{ev}_{t_0} \circ \varphi(f_i \otimes e_{p,q})\big)w^* = \mathrm{ev}_t \circ \varphi(f_i \otimes e_{p,q}),$$

and $w_{t_0} := v_t w$ has the desired property.

Now take $\delta_0 > 0$ sufficiently small so that the inequalities

$$\|g(\xi_i(s)) - g(\xi_i(t))\| < \varepsilon, \qquad \|\mathrm{ev}_s \circ \varphi(g) - \mathrm{ev}_t \circ \varphi(g)\| < \varepsilon$$

hold for all $g \in G$ whenever $|s - t| < \varepsilon$, and consider an open covering

$$\mathcal{U} := \big\{ U_\delta(t) \;\big|\; t \in [0,1] \;\&\; \delta < \min\{\delta(t), \delta_0\} \big\}$$

of $[0,1]$, where $U_\delta(t)$ denotes the open ball of radius δ and center t. Since $[0,1]$ is compact, there exists a finite subcovering, say $\{I_1, \dots, I_r\}$. We denote the center of I_j by c_j, and without loss of generality, we may assume $c_1 < \cdots < c_r$ and $I_j \cap I_{j+1} \neq \varnothing$ for all j. Take small $\eta > 0$ and $b_j \in I_j \cap I_{j+1} \cap (c_j + \eta, c_{j+1} - \eta)$ for each j, and find a unitary $u \in \mathcal{A}_m$ such that

- $u(b_j)$ is equal to v_{b_j} for all j;
- the image of u on $[c_j + \eta, c_{j+1} - \eta]$ is included in the ε-ball of center $u(b_j)$; and
- the image of u on $[c_j - \eta, c_j + \eta]$ is included in the path-connected subset

  $$\Big\{ w \;\Big|\; \mathrm{ev}_{c_j} \circ \varphi(f) = \mathrm{Ad}(w)\Big(\mathrm{diag}\big[f(\xi_1(c_j)), \dots, f(\xi_k(c_j))\big]\Big) \Big\}$$

  of unitaries,

which is possible by the claim we proved in the previous paragraph.

We shall set

$$\psi(f) := \mathrm{Ad}(u)\big(\mathrm{diag}[f \circ \xi_1, \dots, f \circ \xi_k]\big)$$

and show that this ψ has the desired property. First, it is clear from the definition of ξ_i that ψ is trace-preserving. Now, let g be a member of G. Without loss of generality, we may assume that the norm of g is less than 1. Then, for $t \in [c_j + \eta, c_{j+1} - \eta]$ we have

$$\begin{aligned}
\mathrm{ev}_t \circ \psi(g) &= \mathrm{Ad}(u(t))\Big(\mathrm{diag}\big[g(\xi_1(t)), \dots, g(\xi_k(t))\big]\Big)\\
&\sim_{3\varepsilon} \mathrm{Ad}(u(b_j))\Big(\mathrm{diag}\big[g(\xi_1(b_j)), \dots, g(\xi_k(b_j))\big]\Big) = \mathrm{ev}_{b_j} \circ \varphi(g)\\
&\sim_{\varepsilon} \mathrm{ev}_t \circ \varphi(g).
\end{aligned}$$

On the other hand, if $t \in [c_j - \eta, c_j + \eta]$, then

$$\begin{aligned} \mathrm{ev}_t \circ \psi(g) &= \mathrm{Ad}(u(t))\Big(\mathrm{diag}\big[g(\xi_1(t)), \dots, g(\xi_k(t))\big]\Big) \\ &\sim_\varepsilon \mathrm{Ad}(u(t))\Big(\mathrm{diag}\big[g(\xi_1(c_j)), \dots, g(\xi_k(c_j))\big]\Big) = \mathrm{ev}_{c_j} \circ \varphi(g) \\ &\sim_\varepsilon \mathrm{ev}_t \circ \varphi(g). \end{aligned}$$

Consequently, it follows that $\|\varphi(g) - \psi(g)\| < 4\varepsilon$ for all $g \in G$, which completes the proof. ⊣

Proposition 3.6. *Let τ, σ be faithful diffuse measures on $[0, 1]$. Then for any $n \in \mathbb{N}(\nu)$ and any $\varepsilon > 0$, there exist $m \in \mathbb{N}(\nu)$ and a diagonalizable morphism $\varphi\colon \langle \mathcal{A}_n, \tau\rangle \to \langle \mathcal{A}_m, \sigma\rangle$ such that the images of the maps associated to a diagonal expression of φ have diameters less than ε.*

The following proof is presented with the intention of helping the reader to understand the proof of Proposition 4.8. For a more straightforward proof, see Remark 3.7.

Proof of Proposition 3.6. Since τ is diffuse, there exists $\delta > 0$ such that $\tau([t_1, t_2]) < \delta$ implies $|t_2 - t_1| < 1/6$. Take $k \in \mathbb{N}$ so that $m_1 := nk$ is in $\mathbb{N}(\nu)$ and $1/k$ is smaller than δ. Then there are $t_0 \in (1/2, 2/3)$ and $r \in \mathbb{N}$ with $\tau([0, t_0]) = r/k$. We set $\tau_1 := \frac{k}{r}\tau|_{[0,t_0]}$ and $\tau_2 := \frac{k}{k-r}\tau_{[t_0,1]}$, so $\tau = \frac{r}{k}\tau_1 + \frac{k-r}{k}\tau_2$. By Lemma 3.1, one can find increasing maps $\eta_1\colon [0, 1] \to [0, t_0]$ and $\eta_2\colon [0, 1] \to [t_0, 1]$ such that τ_i is equal to $(\eta_i)_*(\sigma)$ for $i = 1, 2$. We set

$$\xi^1_j := \begin{cases} \eta_1 & \text{if } j = 1, \dots, r, \\ \eta_2 & \text{if } j = r + 1, \dots, k, \end{cases}$$

and define $\varphi_1\colon \mathcal{A}_n \to \mathcal{A}_{m_1}$ by

$$\varphi_1(f) = \mathrm{diag}[f \circ \xi^1_1, \dots, f \circ \xi^1_k].$$

Then it can be easily verified that φ_1 is a morphism from $\langle \mathcal{A}_n, \tau\rangle$ to $\langle \mathcal{A}_{m_1}, \sigma\rangle$, and that the images of the maps $\xi^1_1, \dots, \xi^1_k$ are either $[0, t_0]$ or $[t_0, 1]$, so their diameters are less than $2/3$.

Now take $d \in \mathbb{N}$ large enough so that $(2/3)^d$ is less than ε, and repeat the procedure above for d times to obtain a sequence

$$\langle \mathcal{A}_n, \tau\rangle \xrightarrow{\varphi_1} \langle \mathcal{A}_{m_1}, \sigma\rangle \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{d-1}} \langle \mathcal{A}_{m_d}, \sigma\rangle.$$

Then $\varphi := \varphi_{d-1} \circ \cdots \circ \varphi_1$ has the desired property. ⊣

Remark 3.7. Here is a shorter and more natural proof of Proposition 3.6. Take a natural number $k \in \mathbb{N}$ so that there exists a partition $0 = t_0 < \cdots < t_k = 1$ of $[0, 1]$ with $|t_i - t_{i-1}| < \varepsilon$ and $\tau([t_{i-1}, t_i]) = 1/k$ for all i, and that the number $m := nk$ is in $\mathbb{N}(\nu)$. By Lemma 3.1, there exists a function ξ_i from $[0, 1]$ onto $[t_{i-1}, t_i]$ such that $\frac{1}{k}(\xi_i)_*(\sigma) = \tau|_{[t_{i-1},t_i]}$. Then the map $\varphi\colon \mathcal{A}_n \to \mathcal{A}_m$ defined by $\varphi(f) = \mathrm{diag}[f \circ \xi_1, \dots, f \circ \xi_k]$ has the desired property.

Proposition 3.8. *The class $\mathcal{K}(\nu)$ satisfies NAP.*

Proof. Let φ_1 and φ_2 be morphisms from $\langle \mathcal{A}_{n_0}, \tau_0\rangle$ into $\langle \mathcal{A}_{m'}, \sigma'\rangle$ and $\langle \mathcal{A}_{m''}, \sigma''\rangle$, respectively, and G be a finite subset of $\mathcal{A}_{n_0}$. Our goal is to show that given $\varepsilon > 0$, we can find morphisms ψ_1 and ψ_2 from $\langle \mathcal{A}_{m'}, \sigma'\rangle$ and $\langle \mathcal{A}_{m''}, \sigma''\rangle$ into some

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$\langle \mathcal{A}_{n_2}, \tau_2 \rangle \in \mathcal{K}(v)$ with $\|\psi_1 \circ \varphi_1(g) - \psi_2 \circ \varphi_2(g)\| < \varepsilon$ for all $g \in G$. To see this, by Propositions 3.3, 3.2, and 3.5, we may assume from the outset that $m' = m'' =: n_1$ and $\sigma' = \sigma'' =: \tau_1$, that τ_1 is diffuse, and that both φ_1 and φ_2 are diagonalizable.

Let $\zeta_1^i, \ldots, \zeta_l^i$ be the maps associated to a diagonal expression of φ_i. Take $\delta > 0$ so that $|s - t| < \delta$ implies $|g(s) - g(t)| < \varepsilon$ for any $g \in G$, and apply Proposition 3.6 to obtain a morphism ρ from $\langle \mathcal{A}_{n_1}, \tau_1 \rangle$ into some $\langle \mathcal{A}_{n_2}, \tau_2 \rangle$ such that the images of the maps associated to a diagonal expression of $\rho \circ \tilde{\varphi}_i$ have diameters less than $\delta/3$ for each i. Then apply Proposition 3.3 to find a diagonalizable morphism Φ_i such that the inequaily $\|\rho \circ \varphi_i(g) - \Phi_i(g)\| < \varepsilon$ holds for $g \in G$, and that the maps $\xi_1^i, \ldots, \xi_k^i$ associated to a diagonal expression of Φ_i satisfies $\xi_1^i \leq \cdots \leq \xi_k^i$. Recalling the proof of Proposition 3.5, one can easily check that the diameters of the images of ξ_j^i is still less than $\delta/3$.

We claim that the inequality $\|\xi_j^1 - \xi_j^2\| < \delta$ holds for all j. To see this, suppose on the contrary that $\xi_j^1(t) \geq \xi_j^2(t) + \delta$ at some point $t \in [0, 1]$, and set $c := \max \xi_j^2$, $d := \min \xi_{j+1}^1$. (If j is equal to k, then set $d := 1$ instead.) Then it follows that

- the image of ξ_l^2 is included in $[0, c]$ if $1 \leq l \leq j$; and
- if the image of ξ_l^1 intersects with $[0, d)$, then l is less than or equal to j.

Since d is larger than c by at least $\delta/3$, and since Φ_i is trace-preserving, we have

$$j = \sum_{l=1}^{j} \tau_2\big((\xi_l^2)^{-1}[0, c]\big) \leq n_2 \tau_0\big([0, c]\big) < n_2 \tau_0\big([0, d)\big) \leq \sum_{l=1}^{j} \tau_2\big((\xi_l^1)^{-1}[0, 1]\big) = j,$$

which is a contradiction. Therefore, $\|\xi_j^1 - \xi_j^2\|$ must be smaller than δ, as desired.

Now, let u_i be a unitary such that the equality

$$\Phi_i(f) = \mathrm{Ad}(u_i)\big(\mathrm{diag}[f \circ \xi_1^i, \ldots, f \circ \xi_k^i]\big)$$

holds for all $f \in \mathcal{A}_{n_0}$, and put $\psi_1 := \rho$ and $\psi_2 := \mathrm{Ad}(u_1 u_2^*) \circ \rho$. Then for $g \in G$, we have

$$\begin{aligned}
\psi_2 \circ \varphi_2(g) &= \mathrm{Ad}(u_1 u_2^*) \circ \rho \circ \varphi_2(g) \\
&\sim_\varepsilon \mathrm{Ad}(u_1 u_2^*) \circ \Phi_2(g) \\
&= \mathrm{Ad}(u_1)\big(\mathrm{diag}[g \circ \xi_1^2, \ldots, g \circ \xi_k^2]\big) \\
&\sim_\varepsilon \mathrm{Ad}(u_1)\big(\mathrm{diag}[g \circ \xi_1^1, \ldots, g \circ \xi_k^1]\big) \\
&= \Phi_1(g) \\
&\sim_\varepsilon \psi_1 \circ \varphi_1(g),
\end{aligned}$$

which completes the proof. ⊣

THEOREM 3.9. *The class $\mathcal{K}(v)$ is a Fraïssé class.*

PROOF. We have already shown that $\mathcal{K}(v)$ satisfies JEP and NAP in Propositions 3.3 and 3.8. Also, it can be easily verified form the proof of Proposition 3.2 that $\mathcal{K}(v)$ satisfies WPP. Since $\mathcal{K}(v)$ automatically satisfies CCP, as is noted in Remark 2.5, it follows that $\mathcal{K}(v)$ is a Fraïssé class. ⊣

We close this section by showing that the Fraïssé limit of $\mathcal{K}(v)$ is the unique UHF algebra $\mathbb{M}_v$ corresponding to the supernatural number v. The following lemma will be needed for the proof.

LEMMA 3.10. *Let $\langle \mathcal{A}_n, \tau \rangle$ be a member of $\mathscr{K}(v)$. Then for any finite subset $F \subseteq \mathcal{A}_n$ and any $\varepsilon > 0$, there exist a morphism from $\langle \mathcal{A}_n, \tau \rangle$ into some $\langle \mathcal{A}_m, \sigma \rangle \in \mathscr{K}(v)$ and a finite dimensional C*-subalgebra $\mathcal{B} \subseteq \mathcal{A}_m$ such that the image $\varphi[F]$ is included in the ε-neighborhood of $\mathcal{B}$.*

PROOF. We may assume $F = \{\mathrm{id}_{[0,1]} \otimes 1_{\mathbb{M}_n}\} \cup \{1_{C[0,1]} \otimes e_{i,j} \mid i, j = 1, \ldots, n\}$ where $\{e_{i,j}\}$ is the system of standard matrix units of $\mathbb{M}_n$, because this set generates $\mathcal{A}_n$. Also, we may assume that τ is diffuse by Proposition 3.2. Now, let φ be as in Proposition 3.6. Then

$$\begin{aligned}\varphi(\mathrm{id}_{[0,1]} \otimes 1_{\mathbb{M}_n}) &= \mathrm{diag}[\xi_1, \ldots, \xi_k] \\ &\sim_\varepsilon 1_{C[0,1]} \otimes \mathrm{diag}[\xi_1(0), \ldots, \xi_k(0)],\end{aligned}$$

so $\varphi[F]$ is included in the ε-neighborhood of the finite dimensional C*-subalgebra $1_{C[0,1]} \otimes \mathbb{M}_m$, as desired. ⊣

We use the following theorem of George A. Elliott. Recall that for a unital AF-algebra $\mathcal{A}$, its *scaled dimension group* is defined as its ordered K_0-group $\langle K_0(\mathcal{A}), K_0(\mathcal{A})^+ \rangle$ together with the canonical ordered unit $[1_\mathcal{A}]_0$. (For the precise definition of these objects and a proof of the theorem, see [10, Definitions 3.1.4 and 5.1.4 and Theorem 7.3.4]).

THEOREM 3.11 (Elliott's classification theorem (for unital AF-algebras)). *Two AF-algebras $\mathcal{A}$ and $\mathcal{B}$ are isomorphic if and only if their scaled dimension groups $\langle K_0(\mathcal{A}), K_0(\mathcal{A})^+, [1_\mathcal{A}]_0 \rangle$ and $\langle K_0(\mathcal{B}), K_0(\mathcal{B})^+, [1_\mathcal{B}]_0 \rangle$ are isomorphic.*

THEOREM 3.12. *The Fraïssé limit of $\mathscr{K}(v)$ is $\langle \mathbb{M}_v, \mathrm{tr} \rangle$, where* tr *is the unique trace on $\mathbb{M}_v$.*

PROOF. Let $\langle \mathcal{A}, \theta \rangle$ be the Fraïssé limit of $\mathscr{K}$. By $\mathscr{K}$-universality and Theorem 2.7, it is clear that $\mathcal{A}$ and $\mathbb{M}_v$ have the same K-theory. Therefore, in view of Theorem 3.11, it suffices to show that $\mathcal{A}$ is an AF algebra. For this, let F be a finite subset of $\mathcal{A}$. Then given $\varepsilon > 0$, we can find a morphism φ of some $\langle \mathcal{A}_n, \tau \rangle \in \mathscr{K}(v)$ into $\langle \mathcal{A}, \theta \rangle$ and a finite subset $F' \subseteq \mathcal{A}_n$ such that F is included in the ε-neighborhood of $\varphi[F']$. On the other hand, by Lemma 3.10, there is a morphism ψ from $\langle \mathcal{A}_n, \tau \rangle$ into some $\langle \mathcal{A}_m, \sigma \rangle \in \mathscr{K}(v)$ such that $\psi[F']$ is included in the ε-neighborhood of a finite dimensional C*-subalgebra of $\mathcal{A}_m$. Since $\langle \mathcal{A}, \theta \rangle$ is $\mathscr{K}$-universal and approximately $\mathscr{K}$-homogeneous, there is a morphism $\iota \colon \langle \mathcal{A}_m, \sigma \rangle \to \langle \mathcal{A}, \theta \rangle$ such that $d(\varphi(f), \iota \circ \psi(f))$ is less than ε for all $f \in F'$. It follows that F is included in the 3ε-neighborhood of a finite dimensional C*-subalgebra of $\mathcal{A}$, so by [2, Theorem III.3.4], $\mathcal{A}$ is an AF algebra, which completes the proof. ⊣

§4. The Jiang–Su algebra. Let p, q be natural numbers. We shall begin with the well-known observation that if $\{e_{ij}\}_{i,j}$ and $\{f_{kl}\}_{k,l}$ are the systems of standard matrix units of $\mathbb{M}_p$ and $\mathbb{M}_q$, respectively, then $\{e_{ij} \otimes f_{kl}\}_{(i,k),(j,l)}$ is a system of matrix units which spans $\mathbb{M}_p \otimes \mathbb{M}_q$, so $\mathbb{M}_p \otimes \mathbb{M}_q$ is canonically identified with $\mathbb{M}_{pq}$. Now, the *dimension drop algebra* $\mathcal{Z}_{p,q}$ is defined by

$$\mathcal{Z}_{p,q} := \{f \in \mathcal{A}_{pq} \mid f(0) \in \mathbb{M}_p \otimes 1_{\mathbb{M}_q} \ \&\ f(1) \in 1_{\mathbb{M}_p} \otimes \mathbb{M}_q\},$$

where we took over the notation $\mathcal{A}_n = C([0,1], \mathbb{M}_n)$ from Section 3. It is said to be *prime* if p and q are coprime. We denote by $\mathscr{K}$ the class of all pairs $\langle \mathcal{Z}_{p,q}, \tau \rangle$, where $\mathcal{Z}_{p,q}$ is a prime dimension drop algebra and τ is a faithful trace on it.

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In [7], Jiang and Su constructed the Jiang–Su algebra as an inductive limit of prime dimension drop algebras, and proved that it is the unique monotracial simple C*-algebra among such inductive limits. Our goal here is to show that the Jiang–Su algebra together with its unique trace is the Fraïssé limit of the class $\mathscr{K}$. The direction of the proof is the same as that of Section 3, but we need some additional observations because of the pinching condition.

We shall say a $*$-homomorphism φ from $\mathcal{Z}_{p,q}$ into $\mathcal{Z}_{p',q'}$ is diagonalizable if it is of the form

$$\varphi(f) = \mathrm{Ad}(u)\big(\mathrm{diag}[f \circ \xi_1, \dots, f \circ \xi_k]\big)$$

for some unitary u in $\mathcal{A}_{p'q'}$ and some continuous maps $\xi_1, \dots, \xi_k$ from $[0,1]$ into $[0,1]$. Lemma 4.2 below helps us construct diagonalizable morphisms between members of $\mathscr{K}$.

Notation 4.1. *Let $\xi = (\xi_1, \dots, \xi_k)$ be a tuple of functions from* $[0,1]$ *to* $[0,1]$. *For $s = 0, 1$, we set $F_s(\xi) = \{\xi_1(s), \dots, \xi_k(s)\}$. Also, for $t \in F_s(\xi)$, we denote by $n_s^t(\xi)$ the number of i with $\xi_i(s) = t$. If the family ξ under consideration is apparent from context, then $F_s(\xi)$ and $n_s^t(\xi)$ are simply written as F_s and n_s^t, respectively.*

Lemma 4.2. *Let $\varphi\colon \mathcal{Z}_{p,q} \to \mathcal{A}_{p'q'}$ be a $*$-homomorphism of the form*

$$\varphi(f) = \mathrm{diag}[f \circ \xi_1, \dots, f \circ \xi_k],$$

where $\xi_1, \dots, \xi_k$ are continuous functions from $[0,1]$ *into* $[0,1]$, *and $n_s^t = n_s^t(\xi)$ be as in Notation* 4.1. *Then the following are equivalent.*

(i) *There exists a unitary $u \in \mathcal{A}_{p'q'}$ such that the image of* $\mathrm{Ad}(u) \circ \varphi$ *is included in $\mathcal{Z}_{p',q'}$.*

(ii) *The congruence equations*

$$\begin{aligned} qn_0^0 \equiv pn_0^1 \equiv 0 \pmod{q'}, \qquad & qn_1^0 \equiv pn_1^1 \equiv 0 \pmod{p'}, \\ n_0^t \equiv 0 \pmod{q'}, \qquad n_1^t \equiv 0 & \pmod{p'} \qquad (t \neq 0, 1) \end{aligned} \tag{2}$$

hold.

Moreover, if $\mathcal{Z}_{p,q}$ is prime, then there exists a unitary $v \in \mathcal{A}_{p'q'}$ with the following property: for any $\psi\colon \mathcal{Z}_{p,q} \to \mathcal{A}_{p'q'}$ of the form

$$\psi(f) = \mathrm{diag}[f \circ \zeta_1, \dots, f \circ \zeta_k],$$

where $\zeta_1 \leq \dots \leq \zeta_k$ are continuous functions from $[0,1]$ *into* $[0,1]$, *if the numbers $n_s^t(\zeta)$ satisfies equations* (2), *then the image of* $\mathrm{Ad}(v) \circ \psi$ *is included in $\mathcal{Z}_{p,q}$.*

Proof. First, we shall prove (i) $\Rightarrow$ (ii). Let $F_s = F_s(\xi)$ be as in Notation 4.1. For $t \in F_0$, take $f^t \in \mathcal{Z}_{p,q}$ such that $f^t(t)$ is a minimal projection in $\mathrm{ev}_t[\mathcal{Z}_{p,q}]$ and $f^t(s)$ vanishes if $s \in F_0 \setminus \{t\}$. If $t \neq 0, 1$, then $\mathrm{ev}_0 \circ \varphi(f^t)$ is a projection of rank n_0^t in $\mathbb{M}_{p'} \otimes 1_{\mathbb{M}_{q'}}$. Since the rank of any projection in $\mathbb{M}_{p'} \otimes 1_{\mathbb{M}_{q'}}$ is necessarily a multiple of q', it follows that $n_0^t \equiv 0 \pmod{q'}$. On the other hand, $f^0(0)$ and $f^1(1)$ are minimal projections in $\mathbb{M}_p \otimes 1_{\mathbb{M}_q}$ and $1_{\mathbb{M}_p} \otimes \mathbb{M}_q$, so that their ranks are q and p, respectively. Therefore, $\mathrm{ev}_0 \circ \varphi(f^0)$ and $\mathrm{ev}_0 \circ \varphi(f^1)$ are projections of ranks qn_0^0 and pn_0^1, which implies $qn_0^0 \equiv pn_0^1 \equiv 0 \pmod{q'}$. The other congruence equations in equations (2) follow by similar arguments.

Next, in order to see (ii) $\Rightarrow$ (i), suppose equations (2) hold. If $\xi_i(0) = t$, then by definition of n_s^t, there are distinct suffixes $i = i_1, \dots, i_{n_0^t}$ such that $\xi_{i_1}(0) = \dots = \xi_{i_{n_0^t}}(0) = t$, so that the matrices $f(\xi_{i_1}(0)), \dots, f(\xi_{i_{n_0^t}}(0))$ are equal for each $f \in \mathcal{Z}_{p,q}$.

On one hand, for $t \neq 0, 1$, the number n_0^t is a multiple of q' by assumption; on the other hand, if $t = 0$ or $t = 1$, then $f(\xi_i(0))$ is included in $\mathbb{M}_p \otimes 1_{\mathbb{M}_q}$ or $1_{\mathbb{M}_p} \otimes \mathbb{M}_q$, respectively, so the congruence equation $qn_0^0 \equiv pn_0^1 \equiv 0 \pmod{q'}$ implies the existence of a permutation unitary w such that $\mathrm{diag}[f(\xi_{i_1}(0)), \dots, f(\xi_{i_{n_0^t}}(0))]$ is equal to $\mathrm{Ad}(w)(a_f \otimes 1_{\mathbb{M}_{q'}})$ for some matrix a_f. Consequently, there is a permutation unitary $u_0 \in \mathbb{M}_{p'q'}$ such that the image of $\mathrm{Ad}(u_0) \circ \mathrm{ev}_0 \circ \varphi$ is included in $\mathbb{M}_{p'} \otimes 1_{\mathbb{M}_{q'}}$. Similarly, we can find a unitary $u_1 \in \mathbb{M}_{p'q'}$ such that the image of $\mathrm{Ad}(u_1) \circ \mathrm{ev}_1 \circ \varphi$ is included in $1_{\mathbb{M}_{q'}} \otimes \mathbb{M}_{p'}$. Since the unitary group of $\mathbb{M}_{p'q'}$ is path-connected, there is a unitary $u \in \mathcal{A}_{p'q'}$ with $u(0) = u_0$ and $u(1) = u_1$, so that the image of $\mathrm{Ad}(u) \circ \varphi$ is included in $\mathcal{Z}_{p',q'}$, as desired.

Finally, suppose that $\mathcal{Z}_{p,q}$ is prime. In order to prove the existence of the unitary v in the latter claim, we only have to show the congruence equations

$$n_0^0(\xi) \equiv n_0^0(\zeta) \pmod{q'}, \qquad n_0^1(\xi) \equiv n_0^1(\zeta) \pmod{q'},$$
$$n_1^0(\xi) \equiv n_1^0(\zeta) \pmod{p'}, \qquad n_1^1(\xi) \equiv n_1^1(\zeta) \pmod{p'},$$

recalling the construction of the unitary u in the preceding paragraph and taking the assumption $\zeta_1 \leq \cdots \leq \zeta_k$ into account. To see these congruence equations, note that by what we proved in the preceding paragraphs we have

$$qn_0^0(\xi) \equiv pn_0^1(\xi) \equiv 0 \pmod{q'}, \qquad qn_0^0(\zeta) \equiv pn_0^1(\zeta) \equiv 0 \pmod{q'},$$

and

$$n_0^0(\xi) + n_0^1(\xi) \equiv k \equiv n_0^0(\zeta) + n_0^1(\zeta) \pmod{q'},$$

since $n_0^t \equiv 0 \pmod{q'}$ for $t \neq 0, 1$. Consequently, it follows that

$$p(n_0^0(\xi) - n_0^0(\zeta)) \equiv p(n_0^1(\zeta) - n_0^1(\xi)) \equiv 0 \pmod{q'},$$
$$q(n_0^1(\xi) - n_0^1(\zeta)) \equiv q(n_0^0(\zeta) - n_0^0(\xi)) \equiv 0 \pmod{q'},$$

and so

$$n_0^0(\xi) \equiv n_0^0(\zeta) \pmod{q'}, \qquad n_0^1(\xi) \equiv n_0^1(\zeta) \pmod{q'},$$

since p and q are coprime. The other equivalences follow similarly, which completes the proof. ⊣

We note that every trace on a dimension drop algebra bijectively corresponds to a probability Radon measure on $[0, 1]$, as in the case of $\mathcal{A}_n$. The following proposition is an immediate corollary of Lemma 3.1. The proof is the same as Proposition 3.2, so we omit it.

Proposition 4.3. *Let τ be a faithful trace on $\mathcal{Z}_{p,q}$. Then for any faithful diffuse trace σ on $\mathcal{Z}_{p,q}$, there is a morphism $\varphi\colon \langle \mathcal{Z}_{p,q}, \tau \rangle \to \langle \mathcal{Z}_{p,q}, \sigma \rangle$.*

Proposition 4.4. *Let p and q be coprime natural numbers. Then there exists $M(p,q) \in \mathbb{N}$ such that if p' and q' are coprime natural numbers larger than $M(p,q)$ and if pq divides $p'q'$, then for any faithful diffuse measures τ, σ on* $[0, 1]$, *we can find a morphism φ from $\langle \mathcal{Z}_{p,q}, \tau \rangle$ into $\langle \mathcal{Z}_{p',q'}, \sigma \rangle$.*

Proof. Let c, d be divisors of p, q, respectively. Then, since c and d are coprime, there exists $N(c,d) \in \mathbb{N}$ such that for any $n \in \mathbb{N}$, one can find $l, m \in \mathbb{N}$ with $(lc + md) < N(c,d)$ and $lc + md \equiv n \pmod{cd}$. We set

$$M(p,q) := \max_{c|p, d|q} \frac{pq}{cd} N(c,d).$$

Now suppose that p', q' are coprime natural numbers larger than $M(p,q)$ and that pq divides $p'q'$. Set $r := p'/(g_{p,p'}g_{q,p'})$ and $s := q'/(g_{p,q'}g_{q,q'})$, where $g_{n,m}$ denotes the greatest common divisor of n and m. Note that since p' and q' are coprime and pq divides $p'q'$, the equations $p = g_{p,p'}g_{p,q'}$ and $q = g_{q,p'}g_{q,q'}$ hold. Since $r > M(p,q)/(g_{p,p'}g_{q,p'}) \geq N(g_{p,q'}, g_{q,q'})$ and similarly $s > N(g_{p,p'}, g_{q,q'})$, we can find $l_r, m_r, l_s, m_s \in \mathbb{N}$ such that both $r - l_r g_{p,q'} - m_r g_{q,q'}$ and $s - l_s g_{p,p'} - m_s g_{q,p'}$ are positive and can be divided by $g_{p,q'}g_{q,q'}$ and $g_{p,p'}g_{q,p'}$, respectively. We shall put

$$a_0 := l_r g_{p,q'} s, \qquad b_0 := rs - a_0,$$
$$a_1 := l_s g_{p,p'} r, \qquad b_1 := rs - a_1.$$

Suppose $a_0 > a_1$ and set $c := a_0 - a_1$. We cut $[0,1]$ into three intervals $I_1 = [0, t_1]$, $I_2 = [t_1, t_2]$ and $I_3 = [t_2, 1]$ so that

$$\tau(I_1) = \frac{a_1 + 1/3}{rs}, \qquad \tau(I_2) = \frac{c - 2/3}{rs}, \qquad \tau(I_3) = \frac{b_0 + 1/3}{rs}.$$

Let τ_i be the normalization of $\tau|_{I_i}$. An argument similar to the proof of Lemma 3.1 enables us to find continuous functions η_1, η_2, η_3 such that

- η_1 is a surjection from $[0,1]$ onto I_1 with $\eta_1(0) = \eta_1(1) = 0$ and $(\eta_1)_*(\sigma) = \tau_1$;
- η_2 is the increasing surjection from $[0,1]$ onto $[0,1]$ with $(\eta_2)_*(\sigma) = \frac{1}{3c}(\tau_1 + \tau_3) + (1 - \frac{2}{3c})\tau_2$; and
- η_3 is a surjection from $[0,1]$ onto I_3 with $\eta_3(0) = \eta_3(1) = 1$ and $(\eta_3)_*(\sigma) = \tau_3$.

Then put

$$\xi_i := \begin{cases} \eta_1 & \text{if } i = 1, \ldots, a_1, \\ \eta_2 & \text{if } i = a_1 + 1, \ldots, a_0, \\ \eta_3 & \text{if } i = a_0 + 1, \ldots, rs, \end{cases}$$

and consider the $*$-homomorphism $\varphi\colon \mathcal{Z}_{p,q} \to \mathcal{A}_{p'q'}$ defined by

$$\varphi(f) = \operatorname{diag}[f \circ \xi_1, \ldots, f \circ \xi_{rs}].$$

It is not difficult to see from the definition of η_i that φ is trace-preserving. We shall check that this φ satisfies condition (ii) in Lemma 4.2. Indeed, the functions η_1, η_2, η_3 are defined so that the equations

$$n_0^0 = a_0, \qquad n_0^1 = b_0, \qquad n_1^0 = a_1, \qquad n_1^1 = b_1$$

hold, where $n_s^t = n_s^t(\xi)$ is as in Notation 4.1. Now, it follows that

$$qn_0^0 = qa_0 = ql_r g_{p,q'} s = g_{q,p'} l_r q' \equiv 0 \pmod{q'},$$

and

$$pn_0^1 = p(rs - a_0) = p(r - l_r g_{p,q'} - m_r g_{q,q'})s + pm_r g_{q,q'} s \equiv 0 \pmod{q'}.$$

The other congruences in equations (2) can be similarly verified, so there exists a unitary $u \in \mathcal{A}_{p'q'}$ such that $\operatorname{Ad}(u) \circ \varphi$ is a morphism from $\langle \mathcal{Z}_{p,q}, \tau \rangle$ into $\langle \mathcal{Z}_{p',q'}, \sigma \rangle$, which completes the proof of the case $a_0 > a_1$. The cases $a_0 < a_1$ and $a_0 = a_1$ can be shown similarly. ⊣

Corollary 4.5. *The class $\mathcal{K}$ satisfies JEP.*

Proof. Let $\langle \mathcal{Z}_{p_1,q_1}, \tau_1 \rangle$, $\langle \mathcal{Z}_{p_2,q_2}, \tau_2 \rangle$ be members of $\mathcal{K}$. Find coprime $p_3, q_3 \in \mathbb{N}$ such that both p_3 and q_3 are larger than $\max\{M(p_1,q_1), M(p_2,q_2)\}$ and p_3q_3 is divided by $\operatorname{lcm}(p_1q_1, p_2q_2)$. Then by Propositions 4.3 and 4.4, there is a morphism φ_i from $\langle \mathcal{Z}_{p_i,q_i}, \tau_i \rangle$ into $\langle \mathcal{Z}_{p_3,q_3}, \tau_3 \rangle$, where τ_3 is a diffuse faithful trace. ⊣

One might surmise that every morphism of $\mathscr{K}$ can be approximated by a diagonalizable one. However, as the following example shows, this claim is false.

EXAMPLE 4.6. We shall construct an injective $*$-homomorphism from $\mathcal{Z}_{2,3}$ to $\mathcal{Z}_{7,8}$. Define $c_2\colon \mathbb{M}_2 \otimes 1_{\mathbb{M}_3} \to \mathbb{M}_2$ and $c_3\colon 1_{\mathbb{M}_2} \otimes \mathbb{M}_3 \to \mathbb{M}_3$ by $c_2(a \otimes 1) = a$ and $c_3(1 \otimes b) = b$, respectively, and consider the map ψ from $\mathcal{Z}_{2,3}$ to $\mathcal{A}_{56}$ defined by

$$\mathrm{ev}_t \circ \psi(f) = \mathrm{diag}\big[c_2(f(0)), \underbrace{f(0), \dots, f(0)}_{5}, \underbrace{f(t), \dots, f(t)}_{4}\big].$$

Then $\mathrm{ev}_0 \circ \psi(f)$ is equal to $c_2(f(0)) \otimes 1_{\mathbb{M}_{28}}$, while $\mathrm{ev}_1 \circ \psi(f)$ is unitarily similar to $1_{\mathbb{M}_8} \otimes \mathrm{diag}\big[c_2(f(0)), c_2(f(0)), c_3(f(1))\big]$, so one can find a unitary $u \in \mathcal{A}_{56}$ such that the image of the $*$-homomorphism $\mathrm{Ad}(u) \circ \psi$ is included in $\mathcal{Z}_{7,8}$. This map is clearly not diagonalizable, because in general there exists a diagonalizable $*$-homomorphism from $\mathcal{Z}_{p,q}$ to $\mathcal{Z}_{p',q'}$ only if pq divides $p'q'$, but $56 = 7 \times 8$ is not a multiple of $6 = 2 \times 3$.

As we pointed out in the preceding example, in order for there to be a diagonalizable $*$-homomorphism from $\mathcal{Z}_{p,q}$ to $\mathcal{Z}_{p',q'}$, it is necessary that pq divides $p'q'$. We shall show in the next proposition that if p and q are coprime, then this condition is indeed sufficient for every $*$-homomorphism from $\mathcal{Z}_{p,q}$ to $\mathcal{Z}_{p',q'}$ to be approximately diagonalizable.

PROPOSITION 4.7. *Let $\varphi\colon \langle \mathcal{Z}_{p,q}, \tau\rangle \to \langle \mathcal{Z}_{p',q'}, \sigma\rangle$ be a morphism, and suppose that p and q are coprime and that pq divides $p'q'$. Then for any finite subset $G \subseteq \mathcal{Z}_{p,q}$ and any $\varepsilon > 0$, there exists a diagonalizable morphism $\psi\colon \langle \mathcal{Z}_{p,q}, \tau\rangle \to \langle \mathcal{Z}_{p',q'}, \sigma\rangle$ with $\|\varphi(g) - \psi(g)\| < \varepsilon$ for all $g \in G$. Moreover, we can take ψ so that the maps $\xi_1, \dots, \xi_k$ associated to a diagonal expression of ψ satisfy $\xi_1 \leq \dots \leq \xi_k$.*

PROOF. Let $c_p\colon \mathbb{M}_p \otimes 1_{\mathbb{M}_q} \to \mathbb{M}_p$ and $c_q\colon 1_{\mathbb{M}_p} \otimes \mathbb{M}_q \to \mathbb{M}_q$ be the maps defined by $c_p(a \otimes 1) = a$ and $c_q(1 \otimes b) = b$, respectively. As in the proof of Proposition 3.5, one can find a unitary $v_t \in \mathbb{M}_{p'q'}$ and real numbers $s_1^t, \dots, s_k^t \in [0,1]$ for each $t \in [0,1]$ such that the equation

$$\mathrm{ev}_t \circ \varphi(f) = \mathrm{Ad}(v_t)\Big(\mathrm{diag}\big[\overbrace{c_p(f(0)), \dots, c_p(f(0))}^{a}, f(s_1^t), \dots, f(s_k^t), \underbrace{c_q(f(1)), \dots, c_q(f(1))}_{b}\big]\Big)$$

holds for all $f \in \mathcal{Z}_{p,q}$. We may assume without loss of generality that the inequalities $0 \leq a < q$ and $0 \leq b < p$ hold, because

$$\mathrm{diag}\big[\underbrace{c_p(f(0)), \dots, c_p(f(0))}_{q}\big] = f(0)$$

and

$$\mathrm{diag}\big[\underbrace{c_q(f(1)), \dots, c_q(f(1))}_{p}\big] = f(1).$$

We claim that a and b are equal to 0. To see this, note that $pa + qb$ is a multiple of pq, because pq divides $p'q'$. Since p and q are coprime, it follows that $a \equiv 0 \pmod{q}$ and $b \equiv 0 \pmod{p}$, so $a = b = 0$, as desired. Consequently, we can find continuous maps $\xi_1, \dots, \xi_k$ from $[0,1]$ to $[0,1]$ with $\xi_1 \leq \dots \leq \xi_k$ such that

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$$\mathrm{ev}_t \circ \varphi(f) = \mathrm{Ad}(v_t)\Big(\mathrm{diag}\big[f(\xi_1(t)), \dots, f(\xi_k(t))\big]\Big)$$

for all $f \in \mathcal{Z}_{p,q}$, as in the proof of Proposition 3.5.

It can be easily seen that the rest of the proof of Proposition 3.5 works even if $\mathcal{A}_n$ and $\mathcal{A}_m$ are replaced by $\mathcal{Z}_{p,q}$ and $\mathcal{A}_{p'q'}$, respectively, so one can easily obtain a morphism ψ from $\langle \mathcal{Z}_{p,q}, \tau\rangle$ into $\langle \mathcal{A}_{p'q'}, \sigma\rangle$ with $\|\varphi(g) - \psi(g)\| < \varepsilon$ for all $g \in G$. Moreover, a careful reading and a trivial modification of the third paragraph of the proof of Proposition 3.5 enable us to take the unitary u so that $\mathrm{ev}_0 \circ \varphi(f) = \mathrm{ev}_0 \circ \psi(f)$ and $\mathrm{ev}_1 \circ \varphi(f) = \mathrm{ev}_1 \circ \psi(f)$ hold for all $f \in \mathcal{Z}_{p,q}$. Therefore, we can take ψ so that its image is included in $\mathcal{Z}_{p',q'}$, which completes the proof. ⊣

Proposition 4.8. *Let τ, σ be faithful diffuse measures on* $[0,1]$. *Then for any coprime $p, q \in \mathbb{N}$, there exist coprime $p', q' \in \mathbb{N}$ and a diagonalizable morphism $\varphi\colon \langle \mathcal{Z}_{p,q}, \tau\rangle \to \langle \mathcal{Z}_{p',q'}, \sigma\rangle$ such that the images of the maps associated to a diagonal expression of φ have diameters less than ε.*

Proof. The proof is similar to that of Proposition 3.6, but this time, instead of dividing $[0,1]$ into two intervals $[0, t_0]$ and $[t_0, 1]$, we divide $[0,1]$ into three intervals $[0, t_0]$, $[t_0, t_1]$, and $[t_1, 1]$ the diameters of which are close to $1/3$, and use Lemma 4.2.

Take integers $l > 2q$, $m > 2p$ so that $p_1 := lp$ and $q_1 := mq$ are coprime, and set $k := lm$. Then let r, s be natural numbers such that

$$r \equiv k \pmod{q_1}, \qquad s \equiv k \pmod{p_1}, \qquad r + s < k.$$

We can always find such r and s because $k = lm > \max\{2p_1, 2q_1\}$. Since τ is diffuse, there are $t_0 < t_1$ in $(0,1)$ such that $\tau([0, t_0]) = r/k$ and $\tau([t_1, 1]) = s/k$. Here, we may assume that the diameters of $[0, t_1]$ and $[t_2, 1]$ are arbitrarily close to $1/3$, because l and m can be arbitrarily large and so q_1/k and p_1/k can be arbitrarily small.

Now, put $I_1 := [0, t_0]$, $I_2 := [t_0, t_1]$, and $I_3 := [t_1, 1]$, and let τ_i be the normalization of $\tau|_{I_i}$. By Lemma 3.1, we can find continuous functions η_i from $[0,1]$ onto I_i such that

- η_1 and η_3 are increasing;
- η_2 is decreasing; and
- $(\eta_i)_*(\sigma) = \tau_i$.

We shall set

$$\xi_j := \begin{cases} \eta_1 & \text{if } j = 1, \dots, r, \\ \eta_2 & \text{if } j = r+1, \dots, k-s, \\ \eta_3 & \text{if } j = k-s+1, \dots, k, \end{cases}$$

and check condition (ii) in Lemma 4.2. Let $n_s^t = n_s^t(\xi)$ be as in Notation 4.1. Then clearly $n_0^1 = n_1^0 = 0$. Also,

$$q n_0^0 = qr \equiv qk = q_1 l \equiv 0 \pmod{q_1},$$

and

$$p n_1^1 = ps \equiv pk = p_1 m \equiv 0 \pmod{p_1},$$

so equations (2) hold, as desired. Consequently, there is a diagonalizable morphism $\varphi_1\colon \langle \mathcal{Z}_{p,q}, \tau\rangle \to \langle \mathcal{Z}_{p',q'}, \sigma\rangle$ such that the ranges of the maps associated to φ_1 have their diameters arbitrarily close to $1/3$. The rest of the proof is now the same as Proposition 3.6, so we are done. ⊣

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LEMMA 4.9. *Let φ, ψ be $*$-homomorphisms from $\mathcal{Z}_{p,q}$ into $\mathcal{Z}_{p',q'}$ of the forms*

$$\varphi(f) = \operatorname{Ad}(u)\big(\operatorname{diag}[f \circ \xi_1, \dots, f \circ \xi_k]\big),$$
$$\psi(f) = \operatorname{Ad}(v)\big(\operatorname{diag}[f \circ \xi_1, \dots, f \circ \xi_k]\big).$$

Then, for any finite subset $G \subseteq \mathcal{Z}_{p,q}$ and any $\varepsilon > 0$, there exists a unitary $w \in \mathcal{A}_{p'q'}$ such that the inner automorphism $\operatorname{Ad}(w)$ of $\mathcal{A}_{p'q'}$ preserves $\mathcal{Z}_{p',q'}$ and $\|\operatorname{Ad}(w) \circ \varphi(g) - \psi(g)\| < \varepsilon$ holds for all $g \in G$.

PROOF. Let ρ be the $*$-homomorphism from $\mathcal{Z}_{p,q}$ into $\mathcal{A}_{p'q'}$ defined by

$$\rho(f) = \operatorname{diag}[f \circ \xi_1, \dots, f \circ \xi_k],$$

and put

$$\mathcal{B}_s := \operatorname{ev}_s \circ \rho[\mathcal{Z}_{p,q}],$$
$$\mathcal{C}_s^{\varphi} := \operatorname{ev}_s \circ \operatorname{Ad}(u^*)[\mathcal{Z}_{p',q'}], \quad \text{and} \quad \mathcal{C}_s^{\psi} := \operatorname{ev}_s \circ \operatorname{Ad}(v^*)[\mathcal{Z}_{p',q'}]$$

for $s = 0, 1$. Then, $\mathcal{C}_s^{\varphi}$ and $\mathcal{C}_s^{\psi}$ are simple subalgebras of $\mathbb{M}_{p'q'}$ which are isomorphic to each other, and $\mathcal{B}_s$ is included in both of them. It is not difficult to find a unitary w'_s in the commutant $(\mathcal{B}_s)'$ of $\mathcal{B}_s$ which induces the isomorphism of $\mathcal{C}_s^{\varphi}$ onto $\mathcal{C}_s^{\psi}$.

Now, take $\delta > 0$ so that $|t_1 - t_2| < \delta$ implies $\|g(\xi_i(t_1)) - g(\xi_i(t_2))\| < \varepsilon/2$, and let w' be a unitary in $\mathcal{A}_{p'q'}$ such that

- $w'(0) = w'_0$ and $w'(1) = w'_1$;
- $w'(t) = 1$ for $t \in [\delta, 1-\delta]$; and
- the images of $w'|_{[0,\delta]}$ and $w'|_{[1-\delta,1]}$ is included in $(\mathcal{B}_0)'$ and $(\mathcal{B}_1)'$, respectively.

Then, clearly the inner automorphism induced by $w := vw'u^*$ preserves $\mathcal{Z}_{p',q'}$. Also, for $g \in G$ and $t \in [0, \delta]$,

$$\begin{aligned}
\operatorname{ev}_t \circ \operatorname{Ad}(w) \circ \varphi(g) &= \operatorname{Ad}\big(v(t)w'(t)\big)\Big(\operatorname{diag}\big[g(\xi_1(t)), \dots, g(\xi_k(t))\big]\Big) \\
&\sim_{\varepsilon/2} \operatorname{Ad}\big(v(t)w'(t)\big)\Big(\operatorname{diag}\big[g(\xi_1(0)), \dots, g(\xi_k(0))\big]\Big) \\
&= \operatorname{Ad}(v(t))\Big(\operatorname{diag}\big[g(\xi_1(0)), \dots, g(\xi_k(0))\big]\Big) \\
&\sim_{\varepsilon/2} \operatorname{Ad}(v(t))\Big(\operatorname{diag}\big[g(\xi_1(t)), \dots, g(\xi_k(t))\big]\Big) \\
&= \operatorname{ev}_t \circ \psi(g).
\end{aligned}$$

Similarly, it follows that $\operatorname{ev}_t \circ \operatorname{Ad}(w) \circ \varphi(g) \sim_{\varepsilon} \operatorname{ev}_t \circ \psi(g)$ if t is in $[1-\delta, 1]$, and it is obvious that $\operatorname{ev}_t \circ \operatorname{Ad}(w) \circ \varphi(g) = \operatorname{ev}_t \circ \psi(g)$ if t is in $[\delta, 1-\delta]$. Consequently, $\|\operatorname{Ad}(w) \circ \varphi(g) - \psi(g)\|$ is less than ε for all $g \in G$, which completes the proof. ⊣

PROPOSITION 4.10. *The class $\mathcal{K}$ satisfies NAP.*

PROOF. Let φ_1 and φ_2 be morphisms from $\langle \mathcal{Z}_{p_0,q_0}, \tau_0 \rangle$ to $\langle \mathcal{Z}_{p',q'}, \tau' \rangle$ and $\langle \mathcal{Z}_{p'',q''}, \tau'' \rangle$, respectively, G be a finite subset included in the unit ball of $\mathcal{Z}_{p,q}$, and ε be a positive real number. Our goal is to find morphisms ψ_1 and ψ_2 from $\langle \mathcal{Z}_{p',q'}, \tau' \rangle$ and $\langle \mathcal{Z}_{p'',q''}, \tau'' \rangle$ into some $\langle \mathcal{Z}_{p_2,q_2}, \tau_2 \rangle$ such that the inequality $\|\psi_1 \circ \varphi_1(g) - \psi_2 \circ \varphi_2(g)\| < \varepsilon$ holds for all $g \in G$. To see this, by Corollary 4.5 and Propositions 4.4 and 4.7, we may assume from the outset that $\langle \mathcal{Z}_{p',q'}, \tau' \rangle = \langle \mathcal{Z}_{p'',q''}, \tau'' \rangle =: \langle \mathcal{Z}_{p_1,q_1}, \tau_1 \rangle$, that p_0q_0 divides p_1q_1, and that the morphisms φ_1 and φ_2 are diagonalizable. Also, we may assume that the trace τ_1 is diffuse, by Proposition 4.3.

Take $\delta > 0$ so that $|s - t| < \delta$ implies $\|g(s) - g(t)\| < \varepsilon$. By Propositions 4.8 and 4.7, we can find a diagonalizable morphism ρ from $\langle \mathcal{Z}_{p_1,q_1}, \tau_1 \rangle$ into some $\langle \mathcal{Z}_{p_2,q_2}, \tau_2\rangle \in \mathcal{K}$ and diagonalizable morphisms Φ_1, Φ_2 from $\langle \mathcal{Z}_{p_0,q_0}, \tau_0\rangle$ into $\langle \mathcal{Z}_{p_2,q_2}, \tau_2\rangle$ with the following properties, as in the proof of Proposition 3.8:

- the inequality $\|\rho \circ \varphi_i(g) - \Phi_i(g)\| < \varepsilon$ holds for all $g \in G$; and
- there is a diagonal expression
$$\Phi_i(f) = \mathrm{Ad}(u_i)\big(\mathrm{diag}[f \circ \xi_1^i, \dots, f \circ \xi_k^i]\big),$$
such that $\xi_1^i \le \cdots \le \xi_k^i$ for each i, and $\|\xi_j^1 - \xi_j^2\| < \delta$ for all j.

By Lemma 4.2, there exists a unitary $v \in \mathcal{A}_{p_2q_2}$ such that for each i the image of $\Psi_i := \mathrm{Ad}(vu_i^*) \circ \Phi_i$ is included in $\mathcal{Z}_{p_2,q_2}$. Then by Lemma 4.9, there exists a unitary $w_i \in \mathcal{A}_{p_2q_2}$ such that the inner automorphism $\mathrm{Ad}(w_i)$ preserves $\mathcal{Z}_{p_2,q_2}$, and that $\|\mathrm{Ad}(w_i) \circ \Phi_i(g) - \Psi_i(g)\| < \varepsilon$ holds for all $g \in G$. We put $\psi_1 := \rho$ and $\psi_2 := \mathrm{Ad}(w_1^* w_2) \circ \rho$. Then for $g \in G$, we have

$$\begin{aligned}
\psi_2 \circ \varphi_2(g) &= \mathrm{Ad}(w_1^* w_2) \circ \rho \circ \varphi_2(g) \\
&\sim_\varepsilon \mathrm{Ad}(w_1^* w_2) \circ \Phi_2(g) \\
&\sim_\varepsilon \mathrm{Ad}(w_1^*) \circ \Psi_2(g) \\
&= \mathrm{Ad}(w_1^* v)\big(\mathrm{diag}[g \circ \xi_1^2, \dots, g \circ \xi_k^2]\big) \\
&\sim_\varepsilon \mathrm{Ad}(w_1^* v)\big(\mathrm{diag}[g \circ \xi_1^1, \dots, g \circ \xi_k^1]\big) \\
&= \mathrm{Ad}(w_1^*) \circ \Psi_1(g) \\
&\sim_\varepsilon \Phi_1(g) \\
&\sim_\varepsilon \psi_1 \circ \varphi_1(g),
\end{aligned}$$

which completes the proof. ⊣

The following theorem can be shown in almost the same way as Theorem 3.9. We omit details.

THEOREM 4.11. *The class $\mathcal{K}$ is a Fraïssé class.*

We close this section by showing that the Fraïssé limit of $\mathcal{K}$ is simple and monotracial. Since every inductive limit of a sequence of dimension drop algebras is isomorphic to the Jiang–Su algebra if it is simple and monotracial ([7, Theorem 6.2]), and since the Fraïssé limit of a Fraïssé class is obtained as the inductive limit of a sequence of members of the class by Theorem 2.7, this fact implies that the Fraïssé limit of $\mathcal{K}$ is indeed isomorphic to the Jiang–Su algebra.

LEMMA 4.12. *For a measure τ on* $[0, 1]$, *let $E(\tau)$ be the set of all morphisms from $\langle \mathcal{Z}_{1,1}, \tau\rangle$ into some $\langle \mathcal{Z}_{p,q}, \tau'\rangle \in \mathcal{K}$. If τ is diffuse and faithful, and if σ is a measure with $E(\sigma) \supseteq E(\tau)$, then $\sigma = \tau$.*

PROOF. Suppose $\sigma \neq \tau$. Then there exists $s \in (0, 1)$ with $\sigma([0, s]) \neq \tau([0, s])$. If $\sigma([0, s]) > \tau([0, s])$, then since τ is diffuse there exists $t > s$ such that $\sigma([0, s]) = \tau([0, t])$. Now apply Proposition 4.8 to find a diagonalizable morphism $\varphi\colon \langle \mathcal{Z}_{1,1}, \tau\rangle \to \langle \mathcal{Z}_{p,q}, \tau\rangle$ such that the images of the maps $\xi_1, \dots, \xi_k$ associated to a diagonal expression of φ have diameters less than $(t - s)/3$, and set

$$S = \{\xi_i \mid \mathrm{Im}\, \xi_i \cap [0, s] \neq \varnothing\}, \qquad T = \{\xi_i \mid \mathrm{Im}\, \xi_i \subseteq [0, t]\}.$$

Then clearly $S \subsetneq T$, and since $\varphi \in E(\tau) \subseteq E(\sigma)$, it follows that

$$\sigma([0, s]) \le \frac{\#S}{pq} < \frac{\#T}{pq} \le \tau([0, t]) = \sigma([0, s]),$$

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which is a contradiction. The inequality $\sigma([0,s]) < \tau([0,s])$ implies a similar contradiction, so $\sigma = \tau$. ⊣

PROPOSITION 4.13. *Let $\langle \mathcal{Z}, \tau\rangle$ be the Fraïssé limit of the class $\mathscr{K}$. Then τ is the unique trace on $\mathcal{Z}$.*

PROOF. Let $\langle \mathcal{Z}_{p_1,q_1}, \tau_1\rangle$ and $\langle \mathcal{Z}_{p_2,q_2}, \tau_2\rangle$ be members of $\mathscr{K}$. We first claim that if ι and φ are morphisms from $\langle \mathcal{Z}_{p_1,q_1}, \tau_1\rangle$ to $\langle \mathcal{Z}, \tau\rangle$ and $\langle Z_{p_2,q_2}, \tau_2\rangle$, respectively, then there exists a net $\{\psi_\lambda\}_\lambda$ of morphisms from $\langle \mathcal{Z}_{p_2,q_2}, \tau_2\rangle$ to $\langle \mathcal{Z}, \tau\rangle$ such that the net $\{\psi_\lambda \circ \varphi\}_\lambda$ converges to ι with respect to the point-norm topology. Indeed, by $\mathscr{K}$-universality, there exists a morphism ψ from $\langle \mathcal{Z}_{p_2,q_2}, \tau_2\rangle$ to $\langle \mathcal{Z}, \tau\rangle$. Since $\langle \mathcal{Z}, \tau\rangle$ is approximately $\mathscr{K}$-homogeneous, for any finite $G \subseteq \mathcal{Z}_{p,q}$ and any $\varepsilon > 0$ there exists an automorphism $\alpha_{G,\varepsilon}$ of $\langle \mathcal{Z}, \tau\rangle$ with $\|\alpha_{G,\varepsilon} \circ \psi \circ \varphi(g) - \iota(g)\| < \varepsilon$ for all $g \in G$. Therefore, the net $\{\alpha_{G,\varepsilon} \circ \psi\}_{G,\varepsilon}$ has the desired property.

Now, let σ be a faithful trace on $\mathcal{Z}$ and suppose $\sigma \neq \tau$. Since $\langle \mathcal{Z}, \tau\rangle$ is the inductive limit of a sequence of members of $\mathscr{K}$, and since $\mathcal{Z}_{1,1} = C[0,1]$ is canonically isomorphic to the center of any dimension drop algebra, there exists an embedding $\iota\colon \mathcal{Z}_{1,1} \to \mathcal{Z}$ such that $\iota^*(\tau) \neq \iota^*(\sigma)$. By what we proved in the preceding paragraph and Proposition 4.3, we may assume without loss of generality that $\iota^*(\tau)$ is diffuse. Now assume $\varphi\colon \langle \mathcal{Z}_{1,1}, \iota^*(\tau)\rangle \to \langle \mathcal{Z}_{p,q}, \rho\rangle$ is in $E(\iota^*(\tau))$, and find a net $\{\psi_\lambda\}_\lambda$ of morphisms from $\langle \mathcal{Z}_{p,q}, \rho\rangle$ to $\langle \mathcal{Z}, \tau\rangle$ such that $\{\psi_\lambda \circ \varphi\}$ converges to ι with respect to the point-norm topology. Set $\rho'_\lambda := (\psi_\lambda)^*(\sigma)$ and let ρ' be a limit point of $\{\rho'_\lambda\}_\lambda$ in the set $T(\mathcal{Z}_{p,q})$ of all traces on $\mathcal{Z}_{p,q}$. Then it is clear that $\varphi\colon \langle \mathcal{Z}_{1,1}, \iota^*(\sigma)\rangle \to \langle \mathcal{Z}_{p,q}, \rho'\rangle$ is trace-preserving, so φ is in $E(\iota^*(\sigma))$. This contradicts with Lemma 4.12, whence σ is equal to τ. ⊣

PROPOSITION 4.14. *The Fraïssé limit $\langle \mathcal{Z}, \tau\rangle$ is simple.*

PROOF. Let $\mathcal{I} \subseteq \mathcal{Z}$ be a nontrivial ideal. For each embedding $\iota\colon \mathcal{Z}_{1,1} \to \mathcal{Z}$, let Σ_ι be the closed subset of $[0,1]$ which corresponds to the ideal $\mathcal{I}_\iota := \mathcal{Z}_{1,1} \cap \iota^{-1}[\mathcal{I}]$. For each $\varepsilon > 0$, choose a function $f_\iota^\varepsilon \in \mathcal{I}_\iota$ such that the inequality $|f_\iota^\varepsilon(t)| \geq \varepsilon$ holds if $\operatorname{dist}(t, \Sigma_\iota) \geq \varepsilon$. Now, by Proposition 4.8, there exists a diagonalizable morphism φ from $\langle \mathcal{Z}_{1,1}, \lambda\rangle$ into some $\langle \mathcal{Z}_{p,q}, \lambda\rangle$ such that the images of the maps $\xi_1, \dots, \xi_k$ associated to a diagonal expression of φ have diameters less than ε, and as in the proof of Proposition 4.13, we can find an embedding $\eta\colon \mathcal{Z}_{1,1} \to \mathcal{Z}$ which factors through φ and satisfies $\|\iota(f_\iota^\varepsilon) - \eta(f_\iota^\varepsilon)\| < \varepsilon$. Then Σ_η is included in the ε-neighborhood of Σ_ι, since $\operatorname{dist}(f_\iota^\varepsilon, \mathcal{I}_\eta) = \operatorname{dist}(\eta(f_\iota^\varepsilon), \mathcal{I}) < \varepsilon$ (see the proof of [2, Lemma III.4.1], for example). On the other hand, clearly $\Sigma_\eta \cap \operatorname{Im} \xi_i$ is nonempty for all i, so Σ_ι intersects every 4ε-ball. Since this is true for any $\varepsilon > 0$ and any ι, it follows that Σ_ι is equal to $[0,1]$ for any ι, so $\mathcal{I} = 0$. ⊣

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