ROBUST STABILITY, STABILISATION AND H-INFINITY CONTROL FOR PREMIUM-RESERVE MODELS IN A MARKOVIAN REGIME SWITCHING DISCRETE-TIME FRAMEWORK

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ABSTRACT

The premium pricing process and the medium- and long-term stability of the reserve policy under conditions of uncertainty present very challenging issues in relation to the insurance world. Over the last two decades, applications of Markovian regime switching models to finance and macroeconomics have received strong attention from researchers, and particularly market practitioners. However, relatively little research has so far been carried out in relation to insurance. This paper attempts to consider how a linear Markovian regime switching system in discrete-time could be applied to model the medium- and long-term reserves and the premiums (abbreviated here as the P-R process) for an insurer. Some recently developed techniques from linear robust control theory are applied to explore the stability, stabilisation and robust H_∞-control of a P-R system, and the potential effects of abrupt structural changes in the economic fundamentals, as well as the insurer's strategy over a finite time period. Sufficient linear matrix inequality conditions are derived for solving the proposed sub-problems. Finally, a numerical example is presented to illustrate the applicability of the theoretical results.

KEYWORDS

Premium-reserve process, H-infinity control, system stability, Markovian regime switching systems, time-varying delay.

1. INTRODUCTION

1.1. Control theory in insurance

Control theory originally emerged in connection with engineering applications. For example, potential military applications during and after World War II significantly boosted the growth and popularity of control theory in engineering

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and mathematics. Subsequently, control theory has been applied in many additional fields, such as communication and networked control systems, transportation, logistics and finance.

The development of control theory was initially based on a deterministic framework, but this was soon enlarged to build on a stochastic approach. Indeed, stochastic theory is capable of offering more satisfactory explanations regarding why some "rule-of-thumb" control rules that are applied in practice have been so successful compared with the results provided by the deterministic framework. Nowadays, intensive theoretical research is carried out under the stochastic framework, although the deterministic approach has not been discarded. For example, see Fleming and Rishel (1975) and Pantelous and Papageorgiou (2013). The stochastic case includes the deterministic case, by setting the scale factor of the random source to zero.

In the field of non-life insurance, control theory is a fairly new area of research compared with the long history of actuarial mathematics. The first actuarial publications in which control theory appears were probably the famous papers by De Finetti (1957) and Borch (1967). They propose a control action for the classical risk theory problem, based on a pre-defined level of the surplus reserve. Both of these suggest a premium refund whenever the surplus exceeds a certain limiting level, see Bühlmann (1970). Under this arrangement, the premium for the tth year P_t is determined by the following equation:

$$P_{t+1} = (1+\theta)\mathbb{E}[claims] + \mathbf{1}_{(R_\tau - R_t)},$$

where $\theta > 0$ is the loading factor, $\mathbb{E}\,[claims]$ is the expected value of the claims, R_t is the reserve value at the end of the tth time period, R_τ is the pre-defined limiting level of reserve at the end of the τth time period, and

$$\mathbf{1}_{(R_\tau - R_t<0)}(R_\tau - R_t) = \begin{cases} R_\tau - R_t, \text{when } \ R_\tau - R_t < 0 \\ 0, \ \text{when } \ R_\tau - R_t > 0. \end{cases}$$

Balzer and Benjamin (1980, 1982) and Martin-Löf (1983) proposed a smooth control action for the determination of the premium of the following form:

$$P_{t+1} = (1+\theta)\mathbb{E}[claims] - \varepsilon R_{t-1}. \tag{1.1}$$

Moreover, Balzer and Benjamin (1980, 1982) discuss the effect of delay on the stability of the system over two and four years, respectively. Vandebroek and Dhaene (1990) proved that the premium Equation (1.1) is the optimal linear *feedback* controller for the premium pricing in the case that the minimisation of the probability of ruin is required, along with a smooth pattern for the development of the premiums and reserves. In order to solve this problem, they applied dynamic programming techniques.

Rantala (1986, 1988) applied elements of control theory for to simultaneously consider and optimise the premium and reserve fluctuation. He points out that a suitable control of premiums can lead to a stable and realistic development of the solvency margin. Similarly to the previous authors, Zimbidis and Haberman (2001) considered a discrete-time equation to describe the development of the reserve process for an insurance system with constant time-delay. They used an equation with same structure as the decision function in Equation (1.1) to determine the premium strategy. In their paper, the classical *Root-Locus* method (see Shinners 1964) is used to investigate the stability of the system, and an appropriate feedback factor ε is calculated using a specific premium decision function. The analysis of the stability of a premium pricing process was based on time-invariant parameters and constant delay factors, without considering any type of uncertainty.

Recently, Pantelous and Papageorgiou (2013), and Pantelous and Yang (2014, 2015) introduced time-varying delays and uncertainties in their premium pricing models under different frameworks. In those papers, the stability of a discrete-time premium pricing system with norm-bounded parameter uncertainties[1] and time-varying delay has been investigated in a deterministic and stochastic framework, respectively. They proposed H_∞ criterion to be used for the determination of the premium control rule. The majority of these papers focus on studying the properties of a given control rule, although some of them also explore feasible solutions to a specific problem employing different optimality criteria. All of the papers mentioned are based on a discrete-time approach.

1.2. Motivation: Stability, feedback control and H_∞ control

We begin with a brief analysis of the basic concepts. First, various criteria have been developed to prove the stability of dynamical systems, see Gu *et al.* (2003). The most general of these is the well-known Lyapunov method. Thus, the way to establish Lyapunov (asymptotic) stability for dynamical systems is by means of Lyapunov functions. Moreover, in feedback control (also known as closed loop control), the system output is measured and compared with the desired value, and the system continually attempts to reduce the error between the two values. The most important property of feedback control is that it always compares and adjusts the actual status in order to arrive at the target status. Therefore, the feedback approach is usually superior to the open loop approach (i.e., feedforward or open loop control, which is based only on pre-set values) for practical applications, because it is robust against unexpected disturbances and model uncertainty.[2]

Loosely speaking, the so-called H_∞ norm (see, for example, Francis and Khargonekar 1995 or Helton and Merino 1998) focuses on the worst possible case, and attempts to minimise the maximum of a (linear) loss function of the state and control variables for an arbitrary input. In other words, this rule attempts to minimise the loss in the system when the circumstances are the worst that are possible. This resembles the famous min-max decision rule in game theory, see Hansen and Sargent (2007).

In the insurance industry, the interest in time-varying parameter models has increased over recent decades. The Solvency II framework and the development of some national regulations have increased the interest in the stability and robustness of the models used to describe the behaviour of insurers. Examples of this are the studies of Pantelous and Papageorgiou (2013), and Pantelous and Yang (2014, 2015), which apply recent claim experiences and a feedback mechanism based on the surplus value to control the premium level. All of these models assume only one standard regime for the premium-reserve (P-R) system. However, in financial economics it has been indicated that statistical relationships between variables in many macroeconomic/finance models may be inconsistent. Thus, we can model, and even possibly predict, such discontinuity in many different ways, because it may contain dramatic changes in the system's behaviour. Such discontinuity is mostly associated with events such as financial or economic crises, or with significant changes in government policies. In practice, for an insurance company and its stakeholders different strategies should be implemented in different economic environments. Therefore, an "ideal" model of the P-R process should be able to take into account this significant factor. One possible technique, which is widely applied in financial economics, is given by so-called regime switching models. In these models, the studied processes are assumed to have several "regimes", with their own regime specific parameters and rules for regime switching. For a more detailed literature review on the uses of regime-switching models in insurance and finance, see Liu *et al.* (2011), Shen and Siu (2013), Freeman (2013), Fan *et al.* (2015), Siu *et al.* (2015), Chen and Delong (2015), Rambeerich and Pantelous (2016) and references therein.

1.3. What Markovian regime switching models are and why we choose them

In studies relating to quantitative finance, regime switching models attempt to capture the long-term instabilities (or structural changes) in the various variables involved in a model. Some well-motivated and popular examples are bull and bear regimes alternating in financial markets and their economic impact, the well-known phenomenon that exchange rates tend to alternate over protracted periods of depreciation and appreciation, and the fact that monetary policy can suddenly change on account of downward and upward swings in the economy.

Markovian switching represents the most widely applied and well-known class of regime switching model in both finance and macroeconomics. Many researchers apply Markov properties to describe abrupt changes in various stochastic processes. Guidolin (2011) has reviewed and summarised the research trends in the application of Markovian switching in finance for over the last 20 years. Discrete Markovian jump linear system (DMJLS) may represent a large class of regime switching systems subject to abrupt changes in structures, see Hamilton (2014). A discrete Markov chain governs the transition dynamics between the different regimes.

In the present paper, we assume that time delay and switching signals always exist, closely following the ideas of Zimbidis and Haberman (2001), Pantelous and Papageorgiou (2013) and Pantelous and Yang (2014, 2015). Thus, it is natural to apply the model of regime switching systems with time-delay to analyse the P-R pricing process. In our framework, the switching dynamics are modelled using a Markovian jump process, then a study of the stability and stabilisability is provided for the derived discrete-time Markovian jump P-R system, see Cao and Lams (1999), Xu *et al.* (2004).

The Markovian regime switching system environment employed in this paper increases the flexibility of the parameters, and hopefully allows us to model a more representative real market dynamics system. Our objective is to present a new approach to investigate and manipulate the stability of the P-R system. Furthermore, a H_∞ controller for the Markovian jump switched system is designed, which guarantees the stability of the switching system.

The remainder of this paper is organised as follows. In Section 2, the necessary notation and some key assumptions are presented. In Section 3, the new P-R system is formulated under some particular assumptions in a Markovian regime switching framework. In Sections 4 and 5, the linear matrix inequality (LMI) conditions for robust stabilisation and H_∞ controller are derived, respectively by using the concepts proposed by Boukas and Liu (2001) and Pantelous and Yang (2014), respectively. Subsequently, a numerical example is presented to demonstrate the effectiveness of the developed method in Section 6. Finally, Section 7 concludes the paper.

2. Assumptions

Here, the necessary notation and basic assumptions for our model are described. Some assumptions are almost the same as those in Zimbidis and Haberman (2001), Pantelous and Papageorgiou (2013) and Pantelous and Yang (2014, 2015), and so only a brief explanation is provided here.

Assumption 1. *We assume that there is a binding agreement between the insurer and the insured party, indicating that all contracts will remain valid in the long term. This assumption is a strong one, but is necessary in our model. It prevents a withdrawal of the portfolio when the premium is needed to be increased because of the feedback controller effect when reserve is negative. A relaxation of this assumption will be considered in a sequel paper, as tools for game theory will be used.*

Assumption 2. *By $l_2(\Omega, \mathbb{R}^m)$, we denote the space of square-summable $\mathbb{R}^m$-valued random vectors on the probability space $(\Omega, \mathscr{F}, \mathscr{P})$, and by $l_{e_2}(\mathbb{N}; \mathbb{R}^m)$ we denote the space of m-dimensional non-anticipatory square-summable stochastic processes $f(\cdot) = (f(t))_{t\in\mathbb{N}}$ on $\mathbb{N}$ with respect to a filtration $(\mathscr{F}_t)_{t\in\mathbb{N}}$, satisfying*

$$\|f\|_{e_2}^2 = \mathbb{E}\{\sum_{t=0}^{\infty} |f(t)|^2\} = \sum_{t=0}^{\infty} \mathbb{E}\{|f(t)|^2\} < \infty.$$

$\mathbb{E}[\cdot]$ *denotes the mathematical expectation under $\mathscr{P}$, Xu* et al. *(2004). Moreover, $P_t = \sum_{i=1}^{m} P_{i,t}$ and $C_t = \sum_{i=1}^{m} C_{i,t}$ represent the sum of the premiums and the*

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incurred claim cost for a portfolio of m insurance lines at time t, respectively. Meanwhile, $\{P_{i,t}\}$ *and* $\{C_{i,t}\}$ *are adapted to the filtration* $\{\mathscr{F}_t\}$.

Assumption 3. *The relationship between the administration expenses, the relative operation costs, the desired profit margin and the corresponding premium can be expressed by the equation*

$$\textbf{\textit{Operation Costs}} + \textbf{\textit{Profit Margin}} = (1-e)\,P_t.$$

A typical feature of the operational costs is that they can be predicted. For simplification, we assume that the sum of operation costs and the desired profit is a constant percentage of the premium. Let us also recall that dividends have already been included in the concept of the desired profits. We will not consider dividends in great detail here.

Assumption 4. *Let* $\{\sigma_t;\, t \geq 0\}$ *be a discrete-time Markov chain with a finite state space* $\mathcal{S} = \{1, 2 \cdots N\}$. *Denote the state transition matrix by* $P = [p_{ij}]_{i,j\in\mathcal{S}}$, *i.e., the transition probabilities of* $\{\sigma_t,\, t \geq 0\}$ *are given by*

$$Pr[\sigma_{t+1} = j|\sigma_t = i] = p_{ij} \quad for \quad i, j \in \mathcal{S},$$

with $p_{ij} \geq 0$ *for* $i, j \in S$, *and* $\sum_{j=1}^{N} p_{ij} = 1$, *for* $i \in \mathcal{S}$. *The transition probability is time-independent. Thus, the resulting Markov chain is time-homogeneous.*

Assumption 5. *The positive integer* τ_i *represents the time delay when the system operates in the regime* i. *Then, we denote*

$$\tau_{\max} = \max\{\tau_i,\, i \in \mathcal{S}\},$$

$$\tau_{\min} = \min\{\tau_i,\, i \in \mathcal{S}\}.$$

We consider a mode-dependent delay, τ_{σ_t}, *that is upper and lower bounded, i.e.* $\tau_{\min} \leq \tau_{\sigma_t} \leq \tau_{\max}$ *with* $\tau_{\min}, \tau_{\max} \in \mathbb{N}$. *Therefore, considering a specific time-delay interval, at the end of each year* $[t, t+1)$, *we have the exact information up to the end of the year* $t - \tau_{\sigma_t}$. *The value for* τ_i *can be estimated using past experience and statistical data. Moreover, the national and international regulatory policy might be also applied to define the upper bound of this interval, see Moon* et al. *(2001), Gu* et al. *(2003), Xu* et al. *(2004).*

Assumption 6. *When the model is applied in studying general financial strength conditions, it is useful at first to define a basic case (nominal system,) in which certain specified values are fixed for the parameters of the model. Then, a sensitivity analysis can be carried out. By varying the size of the portfolio, its composition, and other basic parameters, it is possible to study how the business reacts to various external and internal impulses.*

Assumption 7. *The portfolio of m individual insurance lines (or products or policies) can be either* independent *or* dependent*. The different lines are dependent when there are interactions between the different reserve accounts.*

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Assumption 8. *The state of the insurer is described by one variable only, namely the reserve or risk capital. Similarly, the controller for premiums is the only control variable.*

3. Model formulation

Throughout the paper, matrices are assumed to have compatible dimensions. The superscript T denotes matrix transposition. $diag\{\cdots\}$ indicates a block-diagonal matrix. For a symmetric matrix, $P > 0$ (< 0) means that P is positive (negative) definite. I represents the identity matrix, and 0 denotes the zero matrix. $\mathbb{R}^m$ denotes m dimensional Euclidean space.

3.1. The reserve process

Let $\underline{R}_t = (R_{1,t}\, R_{2,t} \cdots R_{m,t})^T$ be the vector expression of the reserves, where $R_{i,t}$ is the reserve of the ith insurance line at time t. The reserve, $\underline{R}_t$, evolves according to

$$\underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t + e\underline{P}_{t+1} - \underline{C}_{t+1}. \tag{3.1}$$

J_{σ_t} represents the investment return matrix at time t for the risk-free asset. It is also possible to include risky assets, but we leave this for a future study. Practically speaking, it is true that such short-term insurance lines (relative to non-life-insurance policies) are invested predominately in standard bank accounts or/and in short-term *secure* bonds (with a duration of less than 6 months). Switching the signal σ_t is a piecewise constant function of time, which takes the value i in the finite set $\mathcal{S} = [1 \quad 2 \cdots N]$. The Markov chain states represent different system regimes. We assume that the switching signal σ_t is governed by a Markovian jump process (see Assumption 4). The premiums are assumed to be the *earned premiums* and claims are *incurred claims*. The investment income consists of cash yield and changes in the value of assets. All of the variables in the basic equation (except for e) are stochastic. From the Equations (3.4) and (3.1), we obtain that

$$\begin{aligned}\underline{R}_{t+1} &= [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t + e\{\hat{\underline{C}}_{t+1} - [E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - [Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t\} - \underline{C}_{t+1}\\ &= [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - e[Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t + \underline{w}_{t+1}.\end{aligned}$$

The parameters J_i, E_i and Z_i are real constant base matrices. $\Delta J_{i,t}$, $\Delta E_{i,t}$, and $\Delta Z_{i,t}$ are the respective parameter uncertainties.[3] For the purpose of the modelling process, J_i and E_i respectively could represent a risk-free interest rate and a constant-base return to the policyholders, respectively. Then, Z_i is an amplifier of the control input. Finally, $\Delta J_{i,t}$, $\Delta E_{i,t}$ and $\Delta Z_{i,t}$ are unknown matrices representing the time-varying parameter uncertainties, and they are assumed to be of the form:

$$[\Delta J_{i,t} \quad -e\Delta E_{i,t} \quad -e\Delta Z_{i,t}] = M_i F_t [N_{1i} \quad N_{2i} \quad N_{3i}], \tag{3.2}$$

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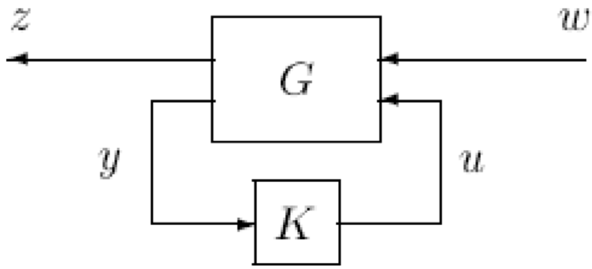

FIGURE 1: Control system.

where M_i, N_{1i}, N_{2i} and N_{3i} are known real constant matrices and $F_t : \mathbb{N} \to \mathbb{R}^{s \times j}$ is an unknown time-varying matrix function satisfying

$$F_t^T F_t \leq I, \quad \forall t \in \mathbb{N}, \tag{3.3}$$

$\Delta J_{i,t}$, $\Delta E_{i,t}$ and $\Delta Z_{i,t}$ are said to be admissible[4] if they satisfy both (3.2) and (3.3). Thus, we have the following discrete time Markovian jump linear P-R system:

$$\underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - e[Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t + \underline{w}_{t+1},$$

with known initial values

$$\underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0]. \tag{Θ}$$

The system has N system regimes. We denote the system Θ without the controller element $\underline{U}_t$ and disturbance $\underline{w}_{t+1}$ by Θ_1. The system Θ without the disturbance $\underline{w}_{t+1}$ is denoted by Θ_2. The system Θ without the controller element $\underline{U}_t$ is denoted by Θ_3. The observation (see the next remark) is denoted by $\underline{z}_t$, where $\underline{z}_t = C\underline{R}_t$ is the control output.

Remark 1. Many H_∞ control problems can be illustrated by the Figure 1, where z is called the controlled output or observation and w is an outside disturbance. Obviously, u is the controller and G is the system/plant. In some systems, it is not possible to directly detect (observe) the status of the state variable y accurately, and we may design and use some observation tools (for instance, we use a thermometer to gauge the temperature in a heating system). In this situation, we rely on the observer z instead of the state variable y to analyse the system process. Intuitively speaking, the H_∞ control minimises the maximum impact of w on the observer z (please note that this is not y). In our case, the P-R process is studied in a robust H_∞ control framework. The full system Θ should be

$$\underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - e[Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t + \underline{w}_{t+1},$$

$$\underline{z}_t = [C_{\sigma_t} + \Delta C_{\sigma_t,t}]\underline{R}_t - e[C'_{\sigma_t} + \Delta C'_{\sigma_t,t}]\underline{R}_{t-\tau_t} - e[C''_{\sigma_t} + \Delta C''_{\sigma_t,t}]\underline{U}_t + C'''\underline{w}_{t+1},$$

$$\underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0].$$

However, for simplicity we let $\Delta C_{\sigma_t,t}$, $[C'_{\sigma_t} + \Delta C'_{\sigma_t,t}]$, $[C''_{\sigma_t} + \Delta C''_{\sigma_t,t}]$, C''' be equal to 0, and $C_{\sigma_t} = C$, so that the control output becomes $\underline{z}_t = C\underline{R}_t$ and C is the identity matrix in our numerical example. Practically, this means that we always assume that the observation from the system is exactly the accumulated reserve value itself, which does not require any further modification. In other words, the current state of the accumulated reserve accounts $\underline{R}_t$ can be accurately and directly gauged, although the current value is not the true value, owing to the time-delay factor. Certainly, we can give our P-R system in additional practical meaning by creating a more complicated structured observer, $\underline{z}_t$. This will be considered in a followup paper.

3.2. The premium rating rule

The rating of premiums usually depends on the available claims experience, general and specific market conditions, the strategy and restriction of the company, and so on. Therefore, the mathematical modelling of this very complicated process is not an easy task. Moreover, it is difficult to determine an "ideal" mathematical formula to accurately cover all aspects of the premium setting, thus it is necessary to attempt to approximate rules for the anticipated behaviour of the insurer. Zimbidis and Haberman (2001) and Pantelous and Papageorgiou (2013) propose a premium rating formula that embeds the feedback mechanism, as follows:

$$\underline{P}_{t+1} = \underline{\hat{C}}_{t+1} - [E + \Delta E_t]\underline{R}_{t-\tau_t},$$

where $\underline{\hat{C}}$ is the *claim estimator*, which is described in more details in the next section. Here, $\underline{P}_t = (P_{1,t} P_{2,t} \cdots P_{m,t})^T$ for $t \in \mathbb{N}$ is the vector representing the *premium* paid in the insurance lines $1, \ldots, m$ in one time interval. E is a known real positive matrix, which adjusts the premiums based on the level of the reserve with time lag τ_t, and ΔE_t is a parameter uncertainty, which varies over time. Note that $E + \Delta E_t$ should normally lie in the interval [0, 1]. τ_t denotes the time delay (see Assumption 5). Moreover, in Pantelous and Yang (2014), an additional controller $\underline{U}_t$ is introduced in the premium $\underline{P}_t$ to stabilise the reserve process. The process for determining $\underline{U}_t$ will be explained later. Thus, in Pantelous and Yang (2014, 2015) the premium process is formulated as follows:

$$\underline{P}_{t+1} = \underline{\hat{C}}_{t+1} - [E + \Delta E_t]\underline{R}_{t-\tau_t} - [Z + \Delta Z_t]\underline{U}_t.$$

Now, we assume that the above equation can be applied in different regimes with regime-specific parameters. Hence, the model is developed as a Markovian jump linear system, and the premium process is formulated as

$$\underline{P}_{t+1} = \underline{\hat{C}}_{t+1} - [E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - [Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t. \tag{3.4}$$

The above equation means that the premium $\underline{P}_t$ at time $t+1$ is $\underline{\hat{C}}_{t+1}$, plus a correction that depends linearly on the past reserve $\underline{R}_{t-\tau_t}$ and the current reserve $\underline{R}_t$ values through $\underline{U}_t$. This dependence can be controlled by varying the values

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of the involved parameters. A time delay on information is also considered. $\underline{U}_t \in \mathbb{R}^m$ is the control input that has been added to the original system. However, for simplicity the state feedback controller is considered to depend on the latest value of R, $\underline{U}_t = K_{1i}\underline{R}_t$, where the matrix K_{1i} should be determined by solving an appropriate LMI (convex optimisation) problem.

In this model, the insurer can control its financial position. A suitable control of premiums can result in a stable and realistic evolution of the reserve, as well as the solvency margin.

3.3. Claims estimator

The claims have been incurred by the end of the accounting year. Because a substantial portion of the incurred claims are usually unknown when the balance sheet is compiled, their total value has to be estimated. This estimate is given for the claims incurred, and is subject to a considerable degree of error. Meanwhile, the total of the claims in one year would not be cleared until many years in the future, and even in one decade for some insurance lines or cases.

The premium P_{t+1} for the $(t+1)$th year is calculated by the *claim estimator* $\hat{C}_{t+1}$. As in Zimbidis and Haberman (2001), $\hat{C}_{t+1}$ is determined by the inflation-weighted average of the most recent available claim experience of the f years $[C_{t-\tau_t-f}, C_{t-\tau_t-f+1}, \ldots, C_{t-\tau_t}]$, and a feedback mechanism using the past reserve value of $R_{t-\tau}$.

$$\hat{C}_{t+1} = \frac{1}{Me}[(1+j)^{f+\tau_t} C_{t-\tau_t-f} + (1+j)^{f+\tau_t-1} C_{t-\tau_t-f+1} + \cdots + (1+j)^{\tau_t} C_{t-\tau_t},$$

$$M = \sum_{k=0}^{f} (1+j)^{f+\tau_t-k},$$

where j is the inflation rate. An inaccurate claims estimation is misleading in many ways and can have fatal consequences. For instance, an underestimation of the claims incurred can result in unprofitable premium level. An underestimation of the claims also lead to a higher probability of insolvency, which can delay corrective action by the management. In this paper, $\underline{w}_{t+1}$ a disturbance to system that is caused by an error between estimated claim value and actual incurred value.

$$\underline{w}_{t+1} = e\underline{\hat{C}}_{t+1} - \underline{C}_{t+1} \in l_{e_2}(\mathbb{N}; \mathbb{R}^m),$$

where e has been explained in Assumption 3, $\underline{C}_t = (C_{1,t} C_{2,t} \cdots C_{m,t})^T$ for $t \in \mathbb{N}$ is the vector representing the incurred claims, which is assumed to follow a stochastic process.

Remark 2. Under the linear control theory framework, the financial position is governed by a linear equation, where the reserve at time t depends linearly on the previous state, the previous control action, and the disturbance $\underline{w}_{t+1}$.

Both the premium and reserve processes have a linear relationship with the original claims process. The claims process is a driving force in the system, and the control equation determines how the total "energy" of the claims process is channelled via the system to the premium and the reserve, respectively. In real world applications, this may be a part of an insurance portfolio, a line or a company.

4. Robust stability

In this section, the robust stability is considered, Xu *et al.* (2004). Before we proceed further, we recapitulate the following lemma which is required later.

Lemma 1 (Xie *et al.*, 1992). *Given appropriately dimensioned matrices Σ_1, Σ_2, Σ_3, with $\Sigma_1^T = \Sigma_1$. Then*

$$\Sigma_1 + \Sigma_3 F_t \Sigma_2 + \Sigma_2^T F_t^T \Sigma_3^T < 0,$$

holds for all F_t, satisfying $F_t^T F_t \leq I$, if and only if for some $\epsilon > 0$, it holds that

$$\Sigma_1 + \epsilon \Sigma_2^T \Sigma_2 + \epsilon^{-1} \Sigma_3 \Sigma_3^T < 0.$$

Lemma 2 (Schur complement). *Let the matrix X be given by*

$$X = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix}. \tag{4.1}$$

Then, X is negative definite if and only if C and $A - BC^{-1}B^T$ are both negative definite.

$$X < 0 \Longleftrightarrow C < 0,\ A - B^T C^{-1} B < 0.$$

Definition 1. *The uncertain stochastic discrete time-delay system Θ_1 is said to be robustly stochastically stable if there exists a scalar $c > 0$ such that for all admissible uncertainties it holds that*

$$\mathbb{E}\left[\sum_{t=0}^{\infty} |\underline{R}_t|^2 | \underline{R}_0, \sigma_0\right] \leq c \sup_{-\tau_{\max} \leq t \leq 0} \mathbb{E}[|\underline{\varphi}_t|]^2, \tag{4.2}$$

when $\underline{w}_{t+1} = 0$, where $\underline{R}_t$ denotes the reserve at time t under the initial conditions.

Remark 3. This definition means that the total value of the reserve process in the system is bounded by a finite number, i.e., for any "admissible" input the reaction of R is also bounded in the expected value sense.

4.1. Stability of the system Θ_1

In this subsection, we consider the uncertain discrete time system Θ with state feedback controller $\underline{U}_t = 0$ and disturbance $\underline{w}_{t+1} = 0$. This means that the value of the actual incurred claims is exactly the same as the estimation.

$$\underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t},$$
$$\underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0]. \qquad (\Theta_1)$$

Theorem 1. *For given scalars $\tau_{\max} > \tau_{\min} \geq 0$, the system Θ_1 is robustly asymptotically stable, if there exist matrices $X_i > 0$, $L > 0$, and $\epsilon_i > 0$, $\forall i \in \mathcal{S}$, such that the following LMI condition holds:*

$$\begin{bmatrix} -X_i & 0 & X_i J_i^T H_i & X_i N_{1i}^T & X_i \\ 0 & -L & -eLE_i^T H_i & LN_{2i}^T & 0 \\ H_i^T J_i X_i & -eH_i^T E_i L & \Lambda_i & 0 & 0 \\ N_{1i} X_i & N_{2i} L & 0 & -\epsilon_i I & 0 \\ X_i & 0 & 0 & 0 & -\frac{1}{\varrho} L \end{bmatrix} < 0, \qquad (4.3)$$

where

$$\mathcal{X} = diag\{X_1, \ldots, X_i\}, \quad \Lambda_i = -\mathcal{X} + \epsilon_i H_i^T M_i M_i^T H_i,$$
$$H_i = (\sqrt{p_{i1}} I \cdots \sqrt{p_{iN}} I), \quad \frac{1}{\varrho} = 1 + (1 - p_{\min})(\tau_{\max} - \tau_{\min}),$$

and $p_{\min} = \min\{p_{ii}, i \in \mathcal{S}\}$ for $i \in \mathcal{S}$.

Proof. Let the matrices $P_i = X_i^{-1}$ and $Q = L^{-1}$. We can construct the Lyapunov functional candidate as follows:

$$V_{\sigma_t}(\underline{R}_t) = V^1_{\sigma_t}(\underline{R}_t) + V^2_{\sigma_t}(\underline{R}_t) + V^3(\underline{R}_t), \qquad (4.4)$$

where

$$V^1_{\sigma_t}(\underline{R}_t) \triangleq \underline{R}_t^T P_{\sigma_t} \underline{R}_t, \qquad (4.5)$$

$$V^2_{\sigma_t}(\underline{R}_t) \triangleq \sum_{l=t-\tau_{\sigma_t}}^{t-1} \underline{R}_l^T Q \underline{R}_l, \qquad (4.6)$$

$$V^3(\underline{R}_t) \triangleq \sum_{k=-\tau_{\max}+1}^{-\tau_{\min}+1} \sum_{l=t+k-1}^{t-1} \underline{R}_l^T \tilde{Q} \underline{R}_l, \qquad (4.7)$$

and $\tilde{Q} = (1 - p_{\min})Q$. We define $\Delta V_{\sigma_t}(\underline{R}_t) = E[V_{\sigma_{t+1}}(\underline{R}_{t+1})|\underline{R}_t] - V_{\sigma_t}(\underline{R}_t)$. Then, based on the results in Boukas and Liu (2001), Xu *et al.* (2004), and Theorem 1 in Pantelous and Yang (2014), the following equality holds

$$
\begin{aligned}
\mathbb{E}[V^1(\underline{R}_{t+1})|\underline{R}_t, \sigma_t = i] - V^1_{\sigma_t}(\underline{R}_t) = {} & \underline{R}_t^T[(J_i + \Delta J_{i,t})^T G_i (J_i + \Delta J_{i,t}) - P_i]\underline{R}_t \\
& + 2\underline{R}_t^T[J_i + \Delta J_{i,t} - e(E_i + \Delta E_{i,t})]^T \\
& \quad \times G_i[-e(E_i + \Delta E_{i,t})]\underline{R}_t \\
& + \underline{R}_{t-\tau_i}^T[-e(E_i + \Delta E_{i,t})]^T \\
& \quad \times G_i[-e(E_i + \Delta E_{i,t})]\underline{R}_{t-\tau_i},
\end{aligned} \tag{4.8}
$$

where $\mathcal{P} = diag\{P_1, \ldots, P_i\}$ and $G_i = H_i \mathcal{P} H_i^T$. Meanwhile,

$$
\begin{aligned}
\mathbb{E}[V^2_{\sigma_t}(\underline{R}_{t+1})|\underline{R}_t, \sigma_t = i] - V^2_{\sigma_t}(\underline{R}_t) = {} & p_{ii}\left[\sum_{l=t-\tau_i+1}^{t} - \sum_{l=t-\tau_i}^{t-1}\right]\underline{R}_l^T Q \underline{R}_l \\
& + \sum_{i \neq j} p_{ij}\left[\sum_{l=t-\tau_j+1}^{t} - \sum_{l=t-\tau_i}^{t-1}\right]\underline{R}_l^T Q \underline{R}_l \\
= {} & p_{ii}\left[\underline{R}_t^T Q \underline{R}_t - \underline{R}_{t-\tau_i}^T Q \underline{R}_{t-\tau_i}\right] \\
& + \sum_{i \neq j} p_{ij}\left[\sum_{l=t-\tau_j+1}^{t} - \sum_{l=t-\tau_i+1}^{t-1}\right]\underline{R}_l^T Q \underline{R}_l \\
& - \sum_{j \neq i} p_{ij}\underline{R}_{t-\tau_i}^T Q \underline{R}_{t-\tau_i} \\
= {} & \underline{R}_t^T Q \underline{R}_t - \underline{R}_{t-\tau_i}^T Q \underline{R}_{t-\tau_i} \\
& + \sum_{i \neq j} p_{ij}\left[\sum_{l=t-\tau_j+1}^{t-1} - \sum_{l=t-\tau_i+1}^{t-1}\right]\underline{R}_l^T Q \underline{R}_l.
\end{aligned}
$$

Note that

$$
\sum_{l=t-\tau_j+1}^{t-1}\underline{R}_l^T Q \underline{R}_l = \sum_{l=t-\tau_{\min}+1}^{t-1}\underline{R}_l^T Q \underline{R}_l + \sum_{l=t-\tau_j+1}^{t-\tau_{\min}}\underline{R}_l^T Q \underline{R}_l.
$$

Therefore,

$$\mathbb{E}[V^2_{\sigma_t}(\underline{R}_{t+1})|\underline{R}_t, \sigma_t = i] - V^2_{\sigma_t}(\underline{R}_t) = \underline{R}_t^T Q \underline{R}_t - \underline{R}_{t-\tau_i}^T Q \underline{R}_{t-\tau_i} + \sum_{i \neq j} p_{ij} \left[\sum_{l=t-\tau_{\min}+1}^{t-1} + \sum_{l=t-\tau_j+1}^{t-\tau_{\min}} - \sum_{l=t-\tau_i+1}^{t-1} \right] \underline{R}_l^T Q \underline{R}_l.$$

Because

$$\sum_{l=t-\tau_{\min}+1}^{t-1} \underline{R}_l^T Q \underline{R}_l \leq \sum_{l=t-\tau_i+1}^{t-1} \underline{R}_l^T Q \underline{R}_l$$

and

$$\sum_{i \neq j} p_{ij} = 1 - p_{ii} \leq 1 - p_{\min},$$

$$\begin{aligned} \mathbb{E}[V^2_{\sigma_t}(\underline{R}_{t+1})|\underline{R}_t, \sigma_t = i] - V^2_{\sigma_t}(\underline{R}_t) &\leq \underline{R}_t^T Q \underline{R}_t - \underline{R}_{t-\tau_i}^T Q \underline{R}_{t-\tau_i} + \sum_{i \neq j} p_{ij} \sum_{l=t-\tau_j+1}^{t-\tau_{\min}} \underline{R}_l^T Q \underline{R}_l \\ &\leq \underline{R}_t^T Q \underline{R}_t - \underline{R}_{t-\tau_i}^T Q \underline{R}_{t-\tau_i} + (1 - p_{\min}) \sum_{l=t-\tau_{\max}+1}^{t-\tau_{\min}} \underline{R}_l^T Q \underline{R}_l. \end{aligned} \quad (4.9)$$

Furthermore,

$$\mathbb{E}[V^3(\underline{R}_{t+1})|\underline{R}_t, \sigma_t = i] - V^3(\underline{R}_t) = (\tau_{\max} - \tau_{\min}) \underline{R}_t^T \tilde{Q} \underline{R}_t - \sum_{l=t-\tau_{\max}}^{t-\tau_{\min}} \underline{R}_l^T \tilde{Q} \underline{R}_l. \quad (4.10)$$

From Equations (4.8)–(4.10), we can show that

$$\begin{aligned} \mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})|\underline{R}_t, \sigma_t = i] - V_i(\underline{R}_t) &\leq \underline{R}_t^T[(J_i + \Delta J_{i,t})^T G_i (J_i + \Delta J_{i,t}) - P_i]\underline{R}_t \\ &+ 2\underline{R}_t^T [J_i + \Delta J_{i,t} - e(E_i + \Delta E_{i,t})]^T G_i [-e(E_i + \Delta E_{i,t})]\underline{R}_t \\ &+ \underline{R}_{t-\tau_i}^T [-e(E_i + \Delta E_{i,t})]^T G_i [-e(E_i + \Delta E_{i,t})]\underline{R}_{t-\tau_i} \\ &+ \underline{R}_t^T Q \underline{R}_t - \underline{R}_{t-\tau_i}^T Q \underline{R}_{t-\tau_i} + (1 - p_{\min})(\tau_{\max} - \tau_{\min}) \underline{R}_t^T Q \underline{R}_t \end{aligned} \quad (4.11)$$

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Equation (4.11) is equivalent to

$$\mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})|\underline{R}_t] - V_{\sigma_t}(\underline{R}_t) \leq \xi^T(t)\Psi_{\sigma_t}\xi(t), \tag{4.12}$$

where

$$\xi(t) = [\underline{R}_t^T \quad \underline{R}_{t-\tau_s}^T]^T,$$

$$\forall i \in \mathcal{S}, \Psi_i = \begin{bmatrix} A_{1i} & A_{2i} \\ A_{3i} & A_{4i} \end{bmatrix},$$

$$A_{1i} = (J_i + \Delta J_{i,t})^T G_i (J_i + \Delta J_{i,t}) - P_i + \varrho Q,$$

$$A_{2i} = [J_i + \Delta J_{i,t} - e(E_i + \Delta E_{i,t})]^T G_i[-e(E_i + \Delta E_{i,t})],$$

$$A_{3i} = [-e(E_i + \Delta E_{i,t})]^T G_i[J_i + \Delta J_{i,t} - e(E_i + \Delta E_{i,t})],$$

$$A_{4i} = [-e(E_i + \Delta E_{i,t})]^T G_i[-e(E_i + \Delta E_{i,t})] - Q.$$

By the Schur complement and that fact that $G_i = H_i \mathcal{P} H_i^T$, we can derive a matrix Ω_i from Ψ_i. Therefore, it holds that

$$\Omega_i = \Sigma_1 + \Sigma_3 F_t \Sigma_2 + \Sigma_2^T F_t^T \Sigma_3^T, \tag{4.13}$$

where

$$\Sigma_1 = \begin{bmatrix} -P_i + \varrho Q & 0 & J_i^T H_i \\ 0 & -Q & -eE_i^T H_i \\ H_i J_i & -eH_i E_i & -\mathcal{P}^{-1} \end{bmatrix} < 0, \tag{4.14}$$

$$\Sigma_2 = [0 \quad 0 \quad M_i^T H_i]^T,$$

$$\Sigma_3 = [N_{1i} \quad N_{2i} \quad 0].$$

Similar to the method given in Pantelous and Yang (2014, and references therein), Equation (4.2) leads to the following inequality, by the Schur complement presented in Lemma 2:

$$\begin{bmatrix} -X_i + \varrho X_i L^{-1} X_i & 0 & X_i J_i^T H_i & X_i N_{1i}^T \\ 0 & -L & -eLE_i^T H_i & LN_{2i}^T \\ H_i^T J_i X_i & -eH_i^T E_i L & \Lambda_i & 0 \\ N_{1i} X_i & N_{2i} L & 0 & -\epsilon_i I \end{bmatrix} < 0. \tag{4.15}$$

Because $X_i = P_i^{-1}$, and $L = Q^{-1}$, we can pre- and post-multiply both sides of Equation (4.15) by $diag\{P_i, Q, I, I\}$ to obtain

$$\begin{bmatrix} -P_i + \varrho Q & 0 & J_i^T H_i & N_{1i}^T \\ 0 & -Q & -eE_i^T H_i & N_{2i}^T \\ H_i^T J_i & -eH_i^T E_i & -\mathcal{P}^{-1} + \epsilon_i H_i^T M_i M_i^T H_i & 0 \\ N_{1i} & N_{2i} & 0 & -\epsilon_i I \end{bmatrix} < 0. \tag{4.16}$$

Therefore, if LMI condition (4.3) is satisfied, we can show that

$$\Sigma_1 + \epsilon_i \Sigma_2 \Sigma_2^T + \epsilon_i^{-1} \Sigma_3^T \Sigma_3 < 0.$$

$$\begin{bmatrix} -P_i + \varrho Q & 0 & J_i^T H_i \\ 0 & -Q & -e E_i^T H_i \\ H_i J_i & -e H_i E_i & -\mathcal{P}^{-1} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \epsilon_i H_i^T M_i M_i^T H_i \end{bmatrix}$$
$$+ \begin{bmatrix} N_{1i}^T \\ N_{2i}^T \\ 0 \end{bmatrix} \epsilon_i^{-1} \begin{bmatrix} N_{1i}^T & N_{2i}^T & 0 \end{bmatrix} < 0. \tag{4.17}$$

According to Lemma 1, which is a result given in Xie *et al.*(1992), this indicates that:

$$\Omega_i = \Sigma_1 + \Sigma_3 F_t \Sigma_2 + \Sigma_2^T F_t^T \Sigma_3^T < 0.$$

This means that the LMI condition (4.3) guarantees that $\Omega_i < 0$. In particular, it follows that

$$\Omega_i < \begin{bmatrix} -\delta I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{4.18}$$

$$\Psi_i < \begin{bmatrix} -\delta I & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.19}$$

where δ is a positive scalar. Because $\tau_{\min} \le \tau_i \le \tau_{\max}$ and $\tau_{\max} - \tau_{\min} \geqslant 1$, we get that

$$V_{\sigma_t}(\underline{R}_t) \le \underline{R}_t^T P \underline{R}_t + \sum_{l=t-\tau_{\max}}^{t-1} \underline{R}_l^T Q \underline{R}_l + \sum_{k=-\tau_{\max}+1}^{-\tau_{\min}+1} \sum_{l=t-\tau_{\max}}^{t-1} \underline{R}_l^T Q \underline{R}_l.$$

Then, we obtain that $\lambda_{\max}(P)|\underline{R}_t|^2 \ge \underline{R}_t^T P \underline{R}_t$ and $\lambda_{\max}(Q)|\underline{R}_t|^2 \ge \underline{R}_t^T Q \underline{R}_t$, where $\lambda_{\max}(\quad)$ denotes the maximum eigenvalue of the respective matrix. Thus, closely following Pantelous and Papageorgiou (2013), and Pantelous and Yang (2014), we can derive that

$$V_{\sigma_t}(\underline{R}_t) \le \lambda |\underline{R}_t|^2 + \lambda(\tau_{\max} - \tau_{\min} + 1) \sum_{l=t-\tau_{\max}}^{t-1} |\underline{R}_l|^2, \tag{4.20}$$

where $\lambda = \max[\lambda_{\max}(P), \quad \lambda_{\max}(Q), \quad \lambda_{\max}(L)]$. Hence, from Equations (4.12) and (4.19) it is easy to deduce that

$$\mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})|\underline{R}_t] - V_{\sigma_t}(\underline{R}_t) < -\delta |\underline{R}_t|^2. \tag{4.21}$$

Now, summing up both sides of Equation (4.21) over time t, we have that

$$\mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})|\underline{R}_t] - V_{\sigma_0}(\underline{R}_0) < -\delta \sum_{s=0}^{t} |\underline{R}_s|^2. \tag{4.22}$$

Then, after taking the expectation on both sides of the above equation, it follows that

$$\mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})] - \mathbb{E}[V_{\sigma_0}(\underline{R}_0)] < -\delta\mathbb{E}[\sum_{s=0}^{t} |\underline{R}_s|^2]. \tag{4.23}$$

Thus,

$$\mathbb{E}[\sum_{s=0}^{t} |\underline{R}_s|^2] \leq \frac{1}{\delta}\mathbb{E}[V_{\sigma_0}(R_0)]. \tag{4.24}$$

By applying Equation (4.20) at time $t = 0$ and rearranging, we have that

$$\begin{aligned} V_{\sigma_0}(\underline{R}_0) &\leq \lambda|\underline{R}_0|^2 + \lambda(\tau_{\max} - \tau_{\min} + 1) \sum_{l=-\tau_{\max}}^{-1} |\underline{R}_l|^2 \\ &\leq \lambda(\tau_{\max} - \tau_{\min} + 1) \sum_{l=-\tau_{\max}}^{0} |\underline{R}_l|^2. \end{aligned}$$

Therefore, after using a mathematical transformation, the expectation becomes

$$\mathbb{E}[V_{\sigma_0}(\underline{R}_0)] \leq \lambda(\tau_{\max} - \tau_{\min} + 1)(\tau_{\max} + 1) \sup_{-\tau_{\max} \leq t \leq 0} \mathbb{E}[|\underline{\varphi}_t|]^2. \tag{4.25}$$

Then, by following the calculations Equations (4.24) and (4.25), we get that

$$\mathbb{E}[\sum_{s=0}^{t}|\underline{R}_s|^2] \leq c \sup_{-\tau_{\max} \leq t \leq 0} \mathbb{E}[|\underline{\varphi}_t|]^2, \tag{4.26}$$

where $c = \frac{1}{\delta}\lambda[(\tau_{\max} - \tau_{\min} + 1)(\tau_{\max} + 1)] > 0$. The above calculations indicate that the positive scalar c has a relationship with the upper and lower bounds on time delay, which extends the result given in Theorem 1 in Boukas and Liu (2001). From Equation (4.26), we have that

$$\lim_{t\to\infty} \mathbb{E}[\sum_{s=0}^{t}|\underline{R}_s|^2] \leq c \sup_{-\tau_{\max} \leq t \leq 0} \mathbb{E}[|\underline{\varphi}_t|]^2.$$

This shows that the system Θ_1 is robustly stochastically stable when the LMI condition (4.3) is satisfied. ■

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4.2. Stabilisation of the system Θ_2

The system Θ with state feedback controller $\underline{U}_t \neq 0$ and disturbance $\underline{w}_{t+1} = 0$, which we denote as Θ_2, is given by

$$\underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} - e[Z_{\sigma_t} + \Delta Z_{\sigma_t,t}]\underline{U}_t,$$
$$\underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0], \qquad (\Theta_2)$$

and

$$\underline{U}_t = K_{1i}\underline{R}_t.$$

Theorem 2. *Consider the uncertain regime switching system Θ_2. This system is robustly stochastically stabilisable if there exist matrices $L > 0$, $X_i > 0$, $Y_i > 0$, and $\epsilon_i > 0$, $\forall i \in \mathcal{S}$, such that the following LMI condition holds:*

$$\begin{bmatrix} -X_i & 0 & X_i J_i^T H_i - eY_i Z_i^T H_i & X_i N_{1i}^T + Y_i N_{3i}^T & X_i \\ 0 & -L & -eLE_i^T H_i & LN_{2i}^T & 0 \\ H_i^T J_i X_i - eH_i^T Z_i Y_i & -eH_i^T E_i L & \Lambda_i & 0 & 0 \\ N_{1i}X_i + N_{3i}Y_i & N_{2i}L & 0 & -\epsilon_i I & 0 \\ X_i & 0 & 0 & 0 & -\frac{1}{\varrho}L \end{bmatrix} < 0. \qquad (4.27)$$

In this case, an appropriate robust stabilising state feedback controller can be chosen as $\underline{U}_t = Y_i X_i^{-1}\underline{R}_t$.

Proof. From Theorem 1, it follows that the LMI (4.27) guarantees that the following system Equation (4.28) is robust stochastically stable. (The parameters J_{σ_t} and ΔJ_{σ_t} are replaced by $J_{\sigma_t} + Z_{\sigma_t}K_{\sigma_t}$ and $\Delta J_{\sigma_t} + \Delta Z_{\sigma_t}K_{\sigma_t}$.)

$$\underline{R}_{t+1} = [J_{\sigma_t} + Z_{\sigma_t}K_{1\sigma_t} + \Delta J_{\sigma_t} + \Delta Z_{\sigma_t}K_{1\sigma_t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t}]\underline{R}_{t-\tau_t},$$
$$\underline{R}_t = \varphi_t \text{ for } t \in [-\tau_{\max}, 0]. \qquad (4.28)$$

Therefore, we have that Θ_2 is robust stochastically stable, because Θ_2 and Equation (4.28) describe the same system. The proof is completed. ■

Remark 4. The above theorem provides a sufficient condition for the solvability of the robust stabilisation problem for an uncertain regime switching system Θ_2. A desired state feedback controller can be obtained by solving the LMI in Equation (4.27).

5. Robust H_∞ stability and H_∞ controller synthesis

5.1. Robust H_∞ stability

In this subsection, H_∞ stability is considered. Intuitively, H_∞ stability means that the magnitude of the movement in the output owing to the system

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disturbance is bounded by γ. In our application, it means that the worst impact of disturbance in a claim process on the reserve level is bounded when the system is robustly stochastically stable, see Cao and Lams (1999), Xu *et al.* (2004).

Definition 2. *The uncertain stochastic discrete time-delay system Θ is said to be robustly stochastically stable with disturbance attenuation level γ if it is robustly stable and the Equation (5.1) is satisfied:*

$$||\underline{z}_t|\underline{R}_0, \sigma_0||_{e_2} \leq \gamma||\underline{w}_t||_{e_2}, \tag{5.1}$$

for all non-zero $\underline{w}_t \in l_{e_2}([0,\infty);\mathbb{R}^m)$ and where $\underline{w}_t$ is $\mathscr{F}_{t-1}$ measurable for all $t \in \mathbb{N}$, $\gamma > 0$ is a given scalar, and $\underline{z}_t = C\underline{R}_t$ is the control output.

Here, we consider the P-R system Θ_3, which takes the impact of an outside disturbance $\underline{w}_{t+1}$ into account, and does not include a controller. Then, the P-R process reduces to

$$\underline{R}_{t+1} = [J_{\sigma_t} + \Delta J_{\sigma_t,t}]\underline{R}_t - e[E_{\sigma_t} + \Delta E_{\sigma_t,t}]\underline{R}_{t-\tau_t} + \underline{w}_{t+1},$$

$$\underline{R}_t = \underline{\varphi}_t \text{ for } t \in [-\tau_{\max}, 0]. \tag{Θ_3}$$

Theorem 3. *For given scalars $\tau_{\max} > \tau_{\min} \geq 0$, the system Θ_3 is robustly stochastically stable with disturbance attenuation level $\gamma > 0$, if there exist matrices $L > 0$, $X_i > 0$, and $\varepsilon_i > 0$, such that the following LMI condition holds $\forall i \in \mathcal{S}$:*

$$\begin{bmatrix} -X_i & 0 & 0 & X_iC^T & X_iJ_i^TH_i & X_iN_{1i}^T & X_i \\ 0 & -L & 0 & 0 & -eLE_i^TH_i & LN_{2i}^T & 0 \\ 0 & 0 & -\gamma^2 I & 0 & H_i & 0 & 0 \\ CX_i & 0 & 0 & -I & 0 & 0 & 0 \\ H_i^TJ_iX_i & -eH_i^TE_iL & H_i^T & 0 & \Lambda_i & 0 & 0 \\ N_{1i}X_i & N_{2i}L & 0 & 0 & 0 & -\epsilon_i I & 0 \\ X_i & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\varrho}L \end{bmatrix} < 0. \tag{5.2}$$

Proof. Again, denote

$$V_{\sigma_t}(\underline{R}_t) = V^1_{\sigma_t}(\underline{R}_t) + V^2_{\sigma_t}(\underline{R}_t) + V^3(\underline{R}_t), \tag{5.3}$$

where $V^1_{\sigma_t}(\underline{R}_t)$, $V^2_{\sigma_t}(\underline{R}_t)$ and $V^3(\underline{R}_t)$ are defined in Equations (4.5)–(4.7). Following the same procedure as in Theorem 1, we can obtain formulas similar to Equations (4.3) and (4.10). From Equation (5.2), it is easy to deduce the

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following matrix:

$$\begin{bmatrix} -X_i & 0 & X_i J_i^T H_i & X_i N_{1i}^T & X_i \\ 0 & -L & -eLE_i^T H_i & LN_{2i}^T & 0 \\ H_i^T J_i X_i & -eH_i^T E_i L & \Lambda_i & 0 & 0 \\ N_{1i} X_i & N_{2i} L & 0 & -\epsilon_i I & 0 \\ X_i & 0 & 0 & 0 & -\frac{1}{\varrho} L \end{bmatrix} < 0. \tag{5.4}$$

Therefore, Θ_3 is robustly stable In the next step, our aim is to show that $||\underline{z}_t||_{e_2} \leq \gamma||\underline{w}_t||_{e_2}$ holds for all non-zero $\underline{w}_t \in l_{e_2}[0, \infty)$ and $\gamma > 0$. To prove this, we need to define

$$T_{H_\infty} = \mathbb{E}\left\{\sum_{t=0}^{N}(\underline{z}_t^T \underline{z}_t - \gamma^2 \underline{w}_t^T \underline{w}_t)|\underline{R}_0, \sigma_t\right\}. \tag{5.5}$$

With a zero initial condition, we know that $V_{\sigma_0}(\underline{R}_0) = 0$. On the other hand, we have shown in Theorem 1 that $\mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})] - \mathbb{E}[V_{\sigma_t}(\underline{R}_t)] \leq 0$. Therefore, for any time $\mathbb{T}$ we have that $\mathbb{E}\left(\sum_{t=0}^{\mathbb{T}} \mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})|\underline{R}_t] - V_{\sigma_t}(\underline{R}_t)\right) \leq 0$ and $V_{\mathbb{T}}(\underline{R}_{\mathbb{T}}) \geq 0$, which after taking $\mathbb{T} \to \infty$ will give us

$$\mathbb{E}\left(\sum_{t=0}^{\infty} \mathbb{E}[V_{\sigma_{t+1}}(\underline{R}_{t+1})|\underline{R}_t] - V_{\sigma_t}(\underline{R}_t)\right) \leq 0.$$

By using this relation and the definition of T_{H_∞}, we obtain that

$$\begin{aligned} T_{H_\infty} &= \mathbb{E}\left(\sum_{t=0}^{\infty}[\mathbb{E}[V_{t+1}(\underline{R}_{t+1})|\underline{R}_t] - V_t(\underline{R}_t) + \underline{z}_t^T \underline{z}_t - \gamma^2 \underline{w}_t^T \underline{w}_t]\right) \\ &\quad -\mathbb{E}\left(\sum_{t=0}^{\infty}[\mathbb{E}[V_{t+1}(\underline{R}_{t+1})|\underline{R}_t] - V_t(\underline{R}_t)]\right) \\ &= \mathbb{E}\left(\sum_{t=0}^{\infty}[\xi^T(t)\Psi_{\sigma_t}\xi(t) + \underline{z}_t^T \underline{z}_t - \gamma^2 \underline{w}_t^T \underline{w}_t]\right) \\ &\quad - V_{\mathbb{T}+1}(\underline{R}_{\mathbb{T}+1}) + V_{\sigma_0}(\underline{R}_0), \mathbb{T} \to \infty \\ &\leq \mathbb{E}\left(\sum_{t=0}^{\infty}[\xi^T(t)\Psi_{\sigma_t}\xi(t) + \underline{z}_t^T \underline{z}_t - \gamma^2 \underline{w}_t^T \underline{w}_t]\right) \\ &= \mathbb{E}\sum_{t=0}^{\infty} \eta^T(t)\tilde{\Psi}_{\sigma_t}\eta(t), \end{aligned} \tag{5.6}$$

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where $\eta(t) = [\underline{R}_t^T \quad \underline{R}_{t-\tau_{\sigma_t}}^T \quad \underline{w}_{t+1}^T]^T$,

$$\forall i \in \mathcal{S}, \quad \tilde{\Psi}_i = \begin{bmatrix} A_{1i} + C^T C & A_{2i} & 0 \\ A_{3i} & A_{4i} & 0 \\ 0 & 0 & -\gamma^2 I \end{bmatrix}.$$

A_{1i}, A_{2i}, A_{3i}, A_{4i} are defined in proof of Theorem 1. With the Schur complement, the inequalities conditions in Theorem 3 guarantee that for each $i \in \mathcal{S}$, it holds that $\tilde{\Psi}_i < 0$, and therefore we get that $T_{H_\infty} < 0$, under zero initial conditions. Then, the system is robustly stochastically stable with an H_∞ norm bounded by γ.

■

5.2. H_∞ controller of the system Θ

Here, we consider the uncertain discrete time system Θ with state feedback controller $\underline{U}_t \neq 0$ and disturbance $\underline{w}_{t+1} \neq 0$. This means that the value of the actual incurred claims is not the same as the estimation. We use the following LMI condition to determine a feasible state H_∞ controller to control this process.

Theorem 4. *Consider the uncertain regime switching system Θ. This system is robustly stochastically stabilisable with disturbance attenuation level $\gamma > 0$ if there exist matrices $X_i > 0$, $Y_i > 0$, $L > 0$, and $\epsilon_i > 0$, such that the following LMI condition holds:*

$$\begin{bmatrix} -X_i & 0 & 0 & X_i C^T & X_i J_i^T H_i - e Y_i Z_i^T H_i & X_i N_{1i}^T + Y_i N_{3i}^T & X_i \\ 0 & -L & 0 & 0 & -e L E_i^T H_i & L N_{2i}^T & 0 \\ 0 & 0 & -\gamma^2 I & 0 & H_i & 0 & 0 \\ C X_i & 0 & 0 & -I & 0 & 0 & 0 \\ H_i^T J_i X_i - e H_i^T Z_i Y_i & -e H_i^T E_i L & H_i^T & 0 & \Lambda_i & 0 & 0 \\ N_{1i} X_i + N_{3i} X_i & N_{2i} L & 0 & 0 & 0 & -\epsilon_i I & 0 \\ X_i & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\varrho} L \end{bmatrix} < 0. \tag{5.7}$$

In this case, an appropriate robust stabilising state feedback controller can be

$$\underline{U_t} = K_{1i} \underline{R_t}, \quad K_{1i} = Y_i X_i^{-1}.$$

Proof. The proof of Theorem 4 is similar to that of Theorem 2, and so it is omitted. ■

5.3. Special case: One-dimensional insurance line

So far, the state variable in the model is considered as a multidimensional vector, which means that it can be applied in an insurance company with multiple

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lines. In order to obtain a better understanding and wider applicability of the main result of this paper, let us assume here that the system Θ contains only one insurance line. Therefore, the parameters and state variables are scalar:

$$R_{t+1} = [j_{\sigma_t} + \Delta j_{\sigma_t,t}]R_t - e[\varepsilon_{\sigma_t} + \Delta\varepsilon_{\sigma_t,t}]R_{t-\tau_t} - e[z_{\sigma_t} + \Delta z_{\sigma_t,t}]U_t + w_{t+1},$$
$$R_t = \varphi_t \text{ for } t \in [-\tau_{\max}, 0]. \tag{5.8}$$

Proposition 1. *Consider the above scalar system, This system is robustly stochastically stabilisable with disturbance attenuation level γ if there exist scalars $x_i > 0$, $y_i > 0$, $l > 0$, and $p_i > 0$, such that following condition holds*

$$\begin{bmatrix} -x_i & 0 & 0 & x_i c & x_i j_i h_i - e y_i z_i h_i & x_i n_{1i} + y_i n_{3i} & x_i \\ 0 & -l & 0 & 0 & -ele_i h_i & l n_{2i} & 0 \\ 0 & 0 & -\gamma^2 I & 0 & h_i & 0 & 0 \\ c x_i & 0 & 0 & -1 & 0 & 0 & 0 \\ h_i j_i x_i - e h_i z_i y_i & -e h_i e_i L & h_i & 0 & \Lambda_i & 0 & 0 \\ n_{1i} x_i + n_{3i} x_i & n_{2i} L & 0 & 0 & 0 & -p_i & 0 \\ x_i & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\varrho} L \end{bmatrix} < 0. \tag{5.9}$$

In this case, an appropriate robust stabilising state feedback controller can be

$$U_t = K_{1i} R_t, \ \ where \ K_{1i} = y_i x_i^{-1}.$$

6. Numerical application

In this section, a numerical application involving an insurance company is formulated to illustrate the applicability of the theoretical results. We assume that the company runs three different insurance lines, which are mutually correlated. Then, we use the result from Theorem 4 to determine the H_∞ controller such that the total reserve process is stabilised with a particular disturbance attenuation level γ. Let us recall that when the model is applied by a particular insurer, the basic parameters, the parameter uncertainty and the disturbance distribution have to be estimated based on real data and realistic assumptions. Here, we assume that the Markovian switching state space is $\mathcal{S} = [1, 2]$, which indicates that there are two different system regimes for the system Θ. In the following paragraphs, the necessary parameters are described in detail.

- First, the value of the reserve accounts at $t = 0$ is given by the following matrix:

$$\underline{R}_0 = \begin{bmatrix} R_0(1) \\ R_0(2) \\ R_0(3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

i.e., at time $t = 0$, we assume that the reserve account for each insurance line is £0 pounds.

- For the time delay, we assume that the mode-dependent delays are $\tau(1) = 3$ for Regime 1 and $\tau(2) = 1$ for Regime 2. Therefore, for $\tau_1 = 1$ and $\tau_2 = 3$:

$$\underline{R}_{-3} = \begin{bmatrix} R_{-3}(1) \\ R_{-3}(2) \\ R_{-3}(3) \end{bmatrix} = \underline{R}_{-2} = \begin{bmatrix} R_{-2}(1) \\ R_{-2}(2) \\ R_{-2}(3) \end{bmatrix} = \underline{R}_{-1} = \begin{bmatrix} R_{-1}(1) \\ R_{-1}(2) \\ R_{-1}(3) \end{bmatrix} = \begin{bmatrix} \$270,000 \\ \$340,000 \\ \$160,000 \end{bmatrix}.$$

- In our model, it is assumed that the insurer can invest the reserve in risk-free investments (T-bills) in order to generate additional income. Because dependencies between the three insurance lines exist, we have to use weights in the parameter matrix. We assume that the corresponding rates of income are given by the following matrix:
  For Regime 1

$$J_1 = \begin{bmatrix} 1.021 * w_{11} & 1.021 * w_{12} & 1.021 * w_{13} \\ 1.021 * w_{21} & 1.021 * w_{22} & 1.021 * w_{23} \\ 1.021 * w_{31} & 1.021 * w_{32} & 1.021 * w_{33} \end{bmatrix}.$$

  For Regime 2

$$J_2 = \begin{bmatrix} 1.039 * w_{11} & 1.039 * w_{12} & 1.039 * w_{13} \\ 1.039 * w_{21} & 1.039 * w_{22} & 1.039 * w_{23} \\ 1.039 * w_{31} & 1.039 * w_{32} & 1.039 * w_{33} \end{bmatrix}.$$

- The weight ratios w_{nm} which demonstrate the solvency relation between each line have the following values:

$$\begin{aligned} w_{11} &= 0.86, w_{12} = 0.07, \quad \textit{and} \quad w_{13} = 0.07, \\ w_{21} &= 0.10, w_{22} = 0.87, \quad \textit{and} \quad w_{23} = 0.03, \\ w_{31} &= 0.08, w_{32} = 0.09, \quad \textit{and} \quad w_{33} = 0.83. \end{aligned}$$

- The parameter E comes from the mechanism proposed by Balzer and Benjamin (1980, 1986). The value of E could be the constant base return rate of policy holder rather than the issuer.
  In the examples, we assume that the value of the parameter matrix E is as follows:

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For Regime 1

$$E_1 = \begin{bmatrix} 0.13 * w_{11} & 0.13 * w_{12} & 0.13 * w_{13} \\ 0.13 * w_{21} & 0.13 * w_{22} & 0.13 * w_{23} \\ 0.13 * w_{31} & 0.13 * w_{32} & 0.13 * w_{33} \end{bmatrix}.$$

For Regime 2

$$E_2 = \begin{bmatrix} 0.18 * w_{11} & 0.18 * w_{12} & 0.18 * w_{13} \\ 0.18 * w_{21} & 0.18 * w_{22} & 0.18 * w_{23} \\ 0.18 * w_{31} & 0.18 * w_{32} & 0.18 * w_{33} \end{bmatrix}.$$

- For the parameter e, we let $e = 0.8$, which means that $1 - 0.8 = 0.2$ (or 20%) of the premium revenue is used to cover the administration and operating costs and give the company a reasonable profit margin.
- $\gamma = 3.7$. This is the value of the maximum impact level of the disturbance to the reserves.
- The time-varying unknown parameter uncertainties $\Delta J_{i,t}$, $\Delta E_{i,t}$, and $\Delta Z_{i,t}$, $i \in [1, 2]$ are defined by

$$[\Delta J_{i,t} \quad -e\Delta E_{i,t} \quad -e\Delta Z_{i,t}] = M_i F_t [N_{1i} \quad N_{2i} \quad N_{3i}],$$

where

$$M_1 = \begin{bmatrix} 0.002 & 0 & 0 \\ 0 & 0.003 & 0 \\ 0 & 0 & 0.002 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0.005 & 0 & 0 \\ 0 & 0.005 & 0 \\ 0 & 0 & 0.004 \end{bmatrix},$$

$$N_{11} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}, N_{21} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}, N_{31} = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}.$$

$$N_{12} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 1 \\ 1 & 3 & 1 \end{bmatrix}, N_{22} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}, N_{32} = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 3 & 2 \end{bmatrix}.$$

- We assume that the insurer will alter the operating regime under the influence of some key economic and market factors that are not constant. In this application, it is assumed that the insurer can switch between two regimes. Thus, two different transition probabilities are required. Type 2 switching transits

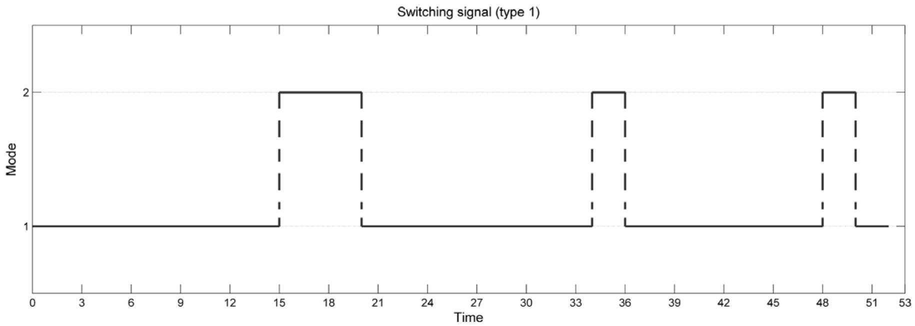

FIGURE 2: Switching signal: Type 1.

more frequently than Type 1.
Transition probability (Type 1 switching)

$$\Pi_1 = \begin{bmatrix} 0.9 & 0.1 \\ 0.5 & 0.5 \end{bmatrix}.$$

Transition probability (Type 2 switching)

$$\Pi_2 = \begin{bmatrix} 0.7 & 0.3 \\ 0.5 & 0.5 \end{bmatrix}.$$

Here, the performance of system under different Markovian switching signals is presented. The simulation results are provided for the time-period of $t = 52$ weeks.

By applying the result of Theorem 4 , the H_∞-controller is derived, and we obtain the feedback controller for each regime separately under **Type 1 switching signal** (see Figure 2) as below:

If the system is in Regime 1:

$$K_{11} = \begin{bmatrix} 0.9491 & 0.0197 & -0.0047 \\ 0.0867 & 1.1114 & 0.0364 \\ 0.0029 & -0.0794 & 1.0063 \end{bmatrix}.$$

If the system is in Regime 2:

$$K_{12} = \begin{bmatrix} 0.9381 & 0.0025 & -0.0189 \\ 0.0792 & 1.1370 & 0.0211 \\ 0.0021 & -0.1045 & 1.0176 \end{bmatrix}.$$

It is clear that under the Type 1 switching signal (Figure 2), few changes are proposed between the two modes (regimes). Generally speaking, this can be considered as a reasonably stable case.

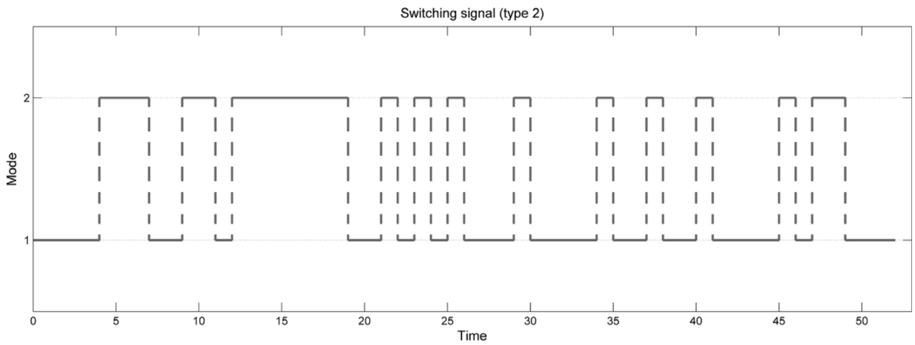

FIGURE 3: Switching signal: Type 2.

Now, when the model is under the **Type 2 switching signal** (see Figure 3), the controller for each regime is follows.

If the system is in Mode 1:

$$K_{11} = \begin{bmatrix} 0.9479 & 0.0216 & -0.0066 \\ 0.0909 & 1.1135 & 0.0382 \\ 0.0012 & -0.0817 & 1.0075 \end{bmatrix}.$$

If the system is in Mode 2:

$$K_{12} = \begin{bmatrix} 0.9365 & 0.0013 & -0.0199 \\ 0.0788 & 1.1373 & 0.0204 \\ 0.0021 & -0.1055 & 1.0173 \end{bmatrix}.$$

On the contrary under the Type 2 switching signal (see Figure 3), changes between the two modes (regimes) occur frequently. Thus, this can be seen as a reasonably volatile case.

In Figures 4 and 5, the movement of the charged premium is presented for the three lines under the Type 1 and 2 signals, respectively. From those figures, we can clearly observe that the controlled premium for each dependent line fluctuates around £150,000 (although no drift is observed for any of the available lines for either signal). Moreover, it should be mentioned that the premium for each dependent line remains positive for the whole duration of each simulation.

It is also clear from Figures 4 and 5, that the state feedback controller $\underline{U}_t$ helps to reduce the impact of the disturbance, and leads to the stabilisation of system quickly. Thus, in Figures 6 and 7 the movement of the charged reserve is presented for the three lines under the Type 1 and 2 signals, respectively. Finally, it is interesting to observe Figure 8, where the total reserve is presented and a comparison is provided for both types of signals. Obviously, the reason that the reserve is not converging exactly to 0 is related to the fact that new random disturbances affect the system, see also Pantelous and Yang (2014). As expected,

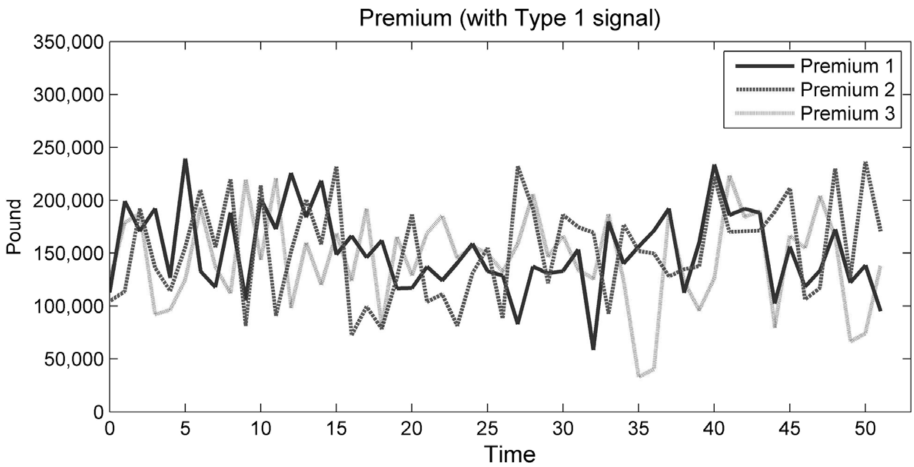

FIGURE 4: The evolution of the three premiums under the Type 1 signal.

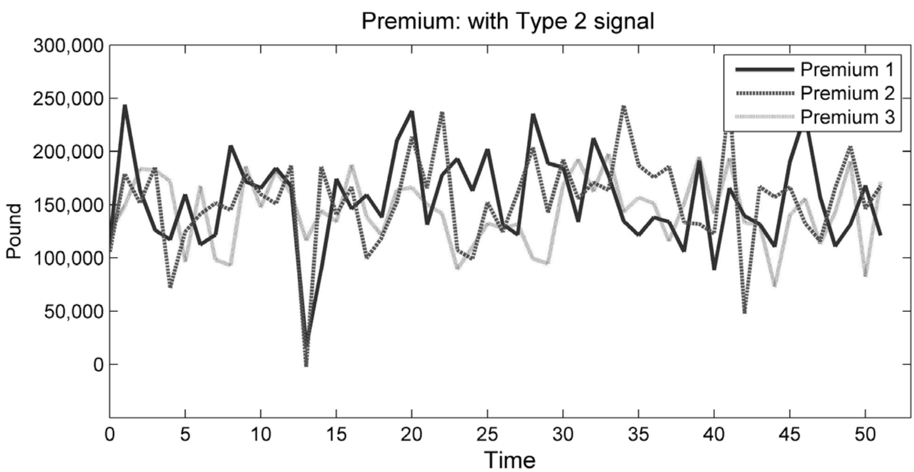

FIGURE 5: The evolution of the three premiums under the Type 2 signal.

the Type 2 signal gives a higher fluctuation in comparison with the Type 1 signal.

To summarise, by using the robust H_∞ tool to generate the state feedback controller $\underline{U}_t$, in this application, we manipulate the stability of the system even though the system disturbance $\underline{w}_t \neq 0$.

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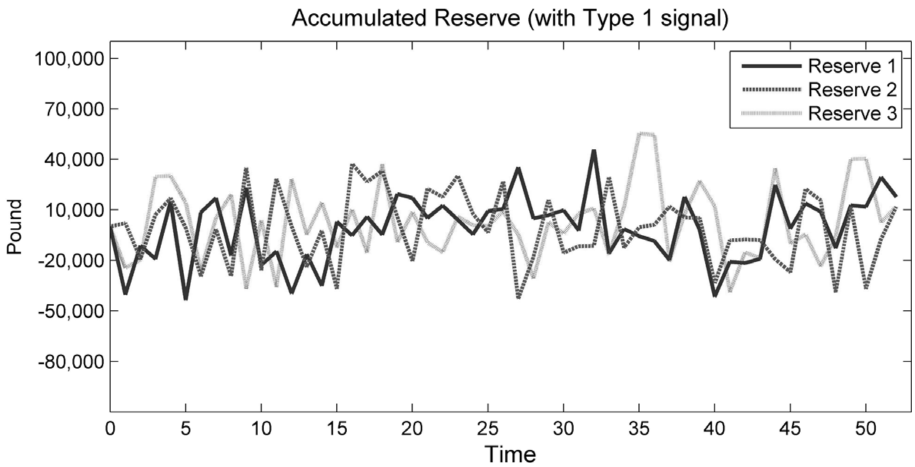

FIGURE 6: The evolution of the accumulated reserves under the Type 1 signal.

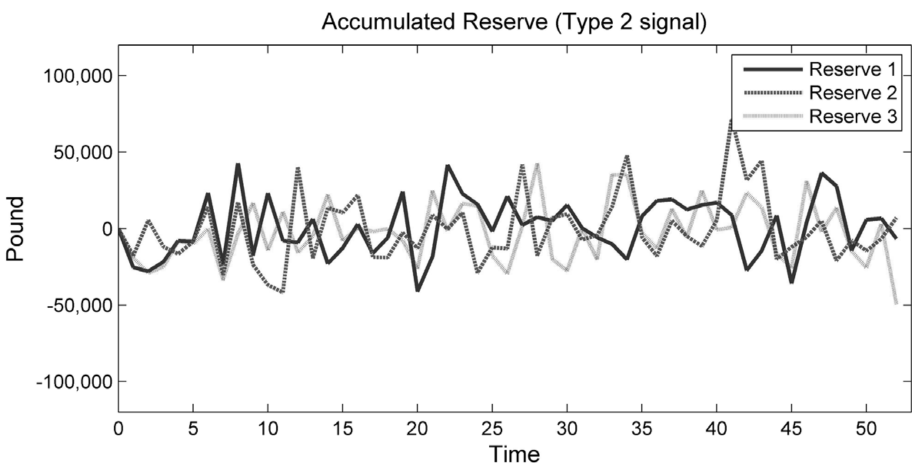

FIGURE 7: The evolution of the accumulated reserves under the Type 2 signal.

7. CONCLUSIONS

In this paper, a Markovian regime switching P-R model for different insurance lines has been proposed, in order to describe abrupt changes in structures. This regime switching model considers a negative feedback mechanism for the reserves, invests the surplus in short-term risk-free (T-bills) assets, and also assumes time-varying and bounded delays for the reserves in a stochastic, discrete-time framework. The parameter uncertainties for the coefficients involved in the model are also norm-bounded. Thus, the new model significantly extends the

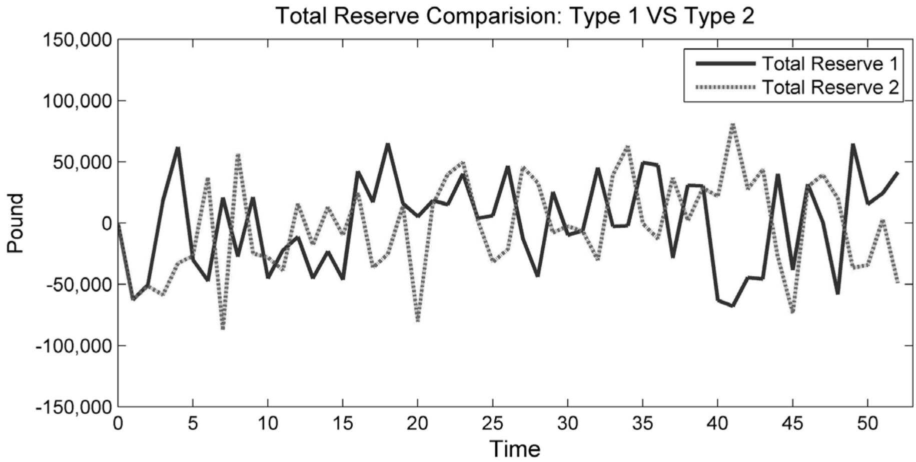

FIGURE 8: The comparison of the total reserve: Type 1 vs. Type 2 switching.

models proposed by Zimbidis and Haberman (2001), Pantelous and Papageorgiou (2013) and Pantelous and Yang (2014, 2015).

In addition, a control parameter has been introduced into the system Θ, and some new ideas have been presented to generate an effective state feedback controller for the P-R system. The LMI conditions for the robust stabilisation and a feasible H_∞ controller have been derived through a series of lemmas and theorems. Thus, for the first time according to the authors' knowledge, a linear robust control theory for Markovian regime switching systems has been implemented in the P-R model. With the H_∞ controller, the premium can be adjusted to reasonable levels for different modes (regimes). Both robust stochastic stability and a pre-specified disturbance attenuation level can be guaranteed for all admissible uncertainties. The corresponding results have been illustrated by presenting a numerical example.

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NOTES

1. In further details, for a parameter ΔA with matrix entries, $\|\Delta A\| \leq \alpha$ (α is a scalar). As we will see in the next section, ΔA describes the uncertain part of the parameter A, see Francis and Khargonekar 1995.

2. Obviously, the robustness of any system depends on the norm used and its corresponding topology. As we will see later, the norm used and its corresponding topology are specified when discussing robustness.

3. In the modelling process, we assume that the exact values of J_i, E_i and Z_i are unknown and they cannot be controlled in the process.

4. In reality, most systems can be affected by different types of uncertainty and/or disturbances, in which case a control decision should be made that accounts for these uncertainties. In our approach, one seeks a solution satisfying all admissible uncertainty realisations (worst-case approach). Different other types of structured dependence on the uncertainty can be considered in a followup paper.

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