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Robust Tverberg and Colourful Carathéodory Results via Random Choice

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We use the probabilistic method to obtain versions of the colourful Carathéodory theorem and Tverberg's theorem with tolerance.

In particular, we give bounds for the smallest integer $N = N(t,d,r)$ such that for any N points in $\mathbb{R}^d$, there is a partition of them into r parts for which the following condition holds: after removing any t points from the set, the convex hulls of what is left in each part intersect.

We prove a bound $N = rt + O(\sqrt{t})$ for fixed r,d which is polynomial in each parameters. Our bounds extend to colourful versions of Tverberg's theorem, as well as Reay-type variations of this theorem.

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1. Introduction

The colourful Carathéodory theorem and Tverberg's theorem are two gems of combinatorial geometry. They describe properties of sets of points in $\mathbb{R}^d$, each with a vast number of extensions and generalizations (for an introduction, see [25]). The purpose of this paper is to show how an application of the probabilistic method yields robust versions of both results; that is, the conclusions hold even if a small set of points is removed.

There are many applications of the probabilistic method in discrete geometry. Among some notable examples are the crossing number theorem [1, 35], the cutting lemma [13, 14] and the existence of epsilon-nets for families of sets with bounded VC-dimension [20]. Further examples and an introduction to the general method can be found in [2, 25].

Let us begin with the colourful Carathéodory theorem, due to Bárány [5]. If we let $\operatorname{conv}(X)$ denote the convex hull of a set $X \subset \mathbb{R}^d$, it says the following.

Theorem 1.1 (Bárány 1982). *Let $F_1, \dots, F_{d+1}$ be $d+1$ families of points in $\mathbb{R}^d$, considered as colour classes. If $0 \in \operatorname{conv}(F_i)$ for each i, there is a colourful choice $x_1 \in F_1, \dots, x_{d+1} \in F_{d+1}$ such that $0 \in \operatorname{conv}\{x_1, x_2, \dots, x_{d+1}\}$.*

Given a set $X \subset \mathbb{R}^d$ and a point p, we say X *captures* p if $p \in \text{conv}(X)$. We are interested in versions of the colourful Carathéodory theorem above where the number of colour classes is allowed to increase. There are two natural variations of this kind, which we discuss in Section 2. Our main result is Lemma 2.2, which shows the existence of a colourful choice which captures the origin even if any small subset of points is removed.

Among the numerous consequences of the colourful Carathéodory theorem, there is a strikingly short proof of Tverberg's theorem by Sarkaria [32], later simplified by Bárány and Onn [9]. For a survey regarding Sarkaria's transformation, consult [6].

Theorem 1.2 (Tverberg 1966 [37]). *Given positive integers d,r and $N = (d+1)(r-1)+1$ points in $\mathbb{R}^d$, there is a partition of them into r parts $A_1,\dots,A_r$ such that*

$$\bigcap_{j=1}^{r} \text{conv}(A_j) \neq \emptyset.$$

One of the generalizations of this result, also known as Tverberg with tolerance, consists in finding partitions where the convex hulls of the parts intersect even after any t points are removed. Stated precisely, it says the following.

Problem 1.3 (Tverberg partitions with tolerance). *Given positive integers t,d,r, find the smallest positive integer $N(t,d,r)$ such that for any set of $N(t,d,r)$ points in $\mathbb{R}^d$, there is a partition of them into r parts $A_1,A_2,\dots,A_r$ such that for any set C of at most t points*

$$\bigcap_{j=1}^{r} \text{conv}(A_j \setminus C) \neq \emptyset.$$

We refer to the parameter t as the *tolerance* of the partition. The first bound for such partitions was given by Larman [23], showing that $N(1,d,2) \leqslant 2d+3$. Larman's result is known to be optimal up to dimension four [16]. This was later improved by García-Colín to $N(t,d,2) \leqslant (t+1)(d+1)+1$ [17, 18]. She also showed that $N(t,d,r) \leqslant (t+1)(d+1)(r-1)+t+1$ for any triple (t,d,r). García-Colín conjectured a bound on N extending her result, which was proved by Soberón and Strausz [34]: $N(t,d,r) \leqslant (t+1)(d+1)(r-1)+1$.

The Soberón–Strausz bound is known not to be optimal, as shown by Mulzer and Stein for $d=1$ and in some instances for $d=2$ [28]. Recently, this was vastly improved by García-Colín, Raggi and Roldán-Pensado [19], who showed that for fixed r,d we have $N(t,d,r) = rt + o(t)$. This settles the asymptotic behaviour of $N(t,d,r)$ for large t, as the leading term matches the one for the lower bound $N(t,d,r) \geqslant rt + rd/2$, first given in [33].

However, the $o(t)$ term hides a $\text{twr}_d(O(r^2d^2))$ factor, where the tower functions $\text{twr}_i(\alpha)$ are defined by $\text{twr}_1(\alpha) = \alpha$ and $\text{twr}_{i+1}(\alpha) = 2^{\text{twr}_i(\alpha)}$. The tower function is unavoidable with the method they use, as it relies on geometric Ramsey-type results. Our main result is a new upper bound for $N(t,d,r)$. In our bound, the leading term is also rt for large t, and the bound is polynomial in r,t,d.

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Theorem 1.4. *For positive integers t,d,r, let $N(t,d,r)$ be the optimal number for Tverberg's theorem with tolerance. Then, we have*

$$N(t,d,r) = rt + \tilde{O}(r^2\sqrt{td} + r^3 d),$$

where the $\tilde{O}$ notation hides only polylogarithmic factors in t, d and r.

We should stress that the bound by García-Colín, Raggi and Roldán-Pensado is quite surprising by itself. If we are given less than rt points, a trivial application of the pigeonhole principle shows that any partition of them into r parts has one with at most t points. The removal of these points shows that the tolerance of any Tverberg partition is at most $t-1$.

In other words, with a large number of points the effect of the dimension on the combinatorics behind Tverberg's theorem with tolerance fades away. Our result reinforces this counterintuitive claim by showing that, furthermore, one does not need to worry too much about the construction of the partition; a random one should suffice. For other results in discrete geometry with tolerance, see [27].

Theorem 1.4 improves all previously known bounds when t is large. If $r > d$, it can be further improved to $N(t,d,r) = rt + \tilde{O}(rd\sqrt{rt} + r^2d^2)$ (see Theorem 5.1), but both bounds remain $rt + \tilde{O}(\sqrt{t})$ for fixed r,d with the same degree for the polynomial hidden by the $\tilde{O}$ notation. The methods we use can be applied to yield versions with tolerance of several variations of Tverberg's theorem. We exhibit this for two classic variations of Tverberg's theorem. The first is the coloured Tverberg theorem with tolerance, where the set of points and the desired partitions we want to obtain are given additional combinatorial conditions. Our main result in this setting is the following theorem.

Theorem 1.5. *Let r,t,d be positive integers with $r \geqslant 3$. Suppose we are given $(1.6)t + \tilde{O}(r\sqrt{td} + r^2 d)$ families of r points each in $\mathbb{R}^d$, considered as colour classes. Then there is a partition of them into r sets $A_1,\dots,A_r$, each with exactly one point of each colour, such that even if any t colour classes are removed, the convex hulls of what is left in each A_i intersect. If $r = 2$, the same result holds with $2t + \tilde{O}(\sqrt{td} + d)$ families. The $\tilde{O}$ notation only hides polylogarithmic factors in r,t,d.*

The result above with a precise constant on the leading term is presented in Theorem 4.2.

The second variation we present is related to Reay's conjecture. Reay's conjecture is a relaxation of Tverberg's theorem. The aim is, given a set of points in $\mathbb{R}^d$, to find a partition of them into r parts where the convex hulls of any k parts intersect. We call such partitions a Reay partition. Tverberg partitions are those for which $k = r$. It is an open question whether fewer points than those for Tverberg's theorem are needed to guarantee such a partition if $k < r$. For the best bounds for this problem, see [29, 4].

In Section 5 we show bounds for the number of points that guarantee the existence for Reay partitions with tolerance. These bounds are smaller than those of Theorem 1.4. This is the first instance of a Reay-type result where the existence bounds are smaller than its Tverberg counterpart, albeit neither is known to be optimal. We prove our Tverberg-type results in Sections 3, 4 and 5.

We conclude by presenting remarks, open problems and algorithmic consequences of our results in Section 6.

2. Robust Carathéodory results

The goal of this section is to extend the colourful Carathéodory theorem if we are given N colour classes instead of $d+1$. We may try a direct approach and ask, given N colour classes, what conclusions can be reached if every colour captures the origin. Another option is, given N colour classes, to extend the contrapositive of the theorem and ask what happens if no colourful choice captures the origin.

For the latter case, a strengthening of the colourful Carathéodory theorem implies that, given $d+1$ colour classes in $\mathbb{R}^d$ such that no colourful choice captures the origin, there are *two* colours F_i, F_j whose union does not capture the origin (*i.e.* $0 \notin \text{conv}(F_i \cup F_j)$). This was proved independently in [3] and [22].

Theorem 2.1. *Let N, d be positive integers with $N \geqslant d+1$ and let $F_1, \ldots, F_N$ be sets of points in $\mathbb{R}^d$, considered as colour classes. If no colourful choice captures the origin, there are $N-d+1$ colours whose union does not capture the origin.*

Even though the result above does not follow from the colourful Carathéodory theorem, it can be proved with exactly the same arguments as in [3], as noted by Imre Bárány [7]. Later it was pointed out to the author that the Theorem 2.1 also follows from the main result of [21], which extends the colourful Carathéodory theorem to matroids. Below we include the proof following the arguments of [3].

Proof of Theorem 2.1. We may assume without loss of generality that the set of points $\bigcup_i F_i$ is in sufficiently general position; that is, there is no affine hyperplane spanned by d of the points that contains the origin. Among all colourful choices X, there must be one, X_0, which minimizes $\text{dist}(\text{conv}(X_0), 0)$. Let p be the closest point of $\text{conv}(X_0)$ to the origin. As X_0 does not capture the origin, p must be in a $(d-1)$-face of X_0. In other words, there must be at most d points of different colours $x_1, \ldots, x_d$ whose convex hull contains p.

Let H be the hyperplane orthogonal to the vector $p-0$, which passes through p. Let H^+ be the closed half-space of H that does not contain the origin. Note that $x_1, \ldots, x_d \in H^+$. The minimality of $\text{dist}(\text{conv}(X_0), 0)$ implies that the $N-d$ colours not containing $x_1, \ldots, x_d$ are contained in H^+. If one of these remaining d colours is also contained in H^+, we would have $N-d+1$ colours separated from the origin, as desired.

Let us assume that this does not happen, and look for a contradiction. Thus, we can find $u_1, \ldots, u_d \in \mathbb{R}^d \setminus H^+$ such that u_i is of the same colour as x_i. Let ℓ_1 be a ray starting from the origin in the direction of p. Let y be a point of some colour which is not represented among the x_i. We consider ℓ_2 a ray starting from the origin in the direction of $-y$. Notice that ℓ_2 is contained in $\mathbb{R}^d \setminus H^+$ since $y \in H^+$.

The colourful facets spanned by $\{x_1, \ldots, x_d, u_1, \ldots, u_d\} = V$ are the linear image of a $(d-1)$-dimensional octahedron onto $\mathbb{R}^d$, which in turn is a continuous image of the sphere S^{d-1}. This implies that these colourful facets must intersect the topological line $\ell_1 \cup \ell_2$ in an even number

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of points. The fact that $\ell_1 \cup \ell_2$ does not intersect the facets in subfaces follows from the general position assumption.

As p is one point of intersection, let us see where the other points can be. In H^+ we can have only p, by the construction of V. In the segment $[0, p)$ we can have no point of intersection, or we would contradict the minimality of $\operatorname{dist}(\operatorname{conv}(X_0), 0)$. In ℓ_2 we can have no point of intersection, as the colourful d-tuple sustaining that point would capture the origin once y is included. This leads to the desired contradiction. □

Now assume we are given N colour classes and each captures the origin. We would like to obtain a colourful choice which does more than simply capturing the origin. For this, we use the notion of depth.

Given a finite set $X \subset \mathbb{R}^d$ and a point $p \in \mathbb{R}^d$, we define

$$\operatorname{depth}(X, p) = \min\{|H \cap X| : p \in H,\ H \text{ is a closed half-space}\}.$$

This is commonly known as Tukey depth or half-space depth [24, 36]. A direct application of the colourful Carathéodory theorem shows that given N colour classes in $\mathbb{R}^d$, each capturing the origin, there is a colourful selection X such that $\operatorname{depth}(X, 0) \geqslant \lfloor N/(d+1) \rfloor$, which is optimal. However, in many applications of the colourful Carathéodory theorem the colour classes have few points compared to the dimension. For this situation we obtain the following result.

Lemma 2.2. *Let r, N be positive integers and let $F_1, F_2, \ldots, F_N$ be N families of r points each in $\mathbb{R}^d$, considered as colour classes. If $0 \in \operatorname{conv}(F_i)$ for each i, we can make a colourful choice $x_1 \in F_1, \ldots, x_N \in F_N$ such that for the set $X = \{x_1, \ldots, x_N\}$ we have*

$$\operatorname{depth}(X, 0) \geqslant \frac{N}{r} - \sqrt{\frac{dN \ln(Nr)}{2}}.$$

For the proof of Lemma 2.2 we need the following lemma, mentioned in [15], for example.

Lemma 2.3. *Let M, d be positive integers. Given a set Y of M points in $\mathbb{R}^d$ and a point $c \in \mathbb{R}^d$, there is a family of M^d closed half-spaces containing c such that, for any subset $Z \subset Y$, if each of the half-spaces contains at least t points of Z, then we have* $\operatorname{depth}(Z, c) \geqslant t$*, for any t.*

Proof. We may assume that the affine span of $Y \cup \{c\}$ is $\mathbb{R}^d$. Otherwise, there is a hyperplane containing $Y \cup \{c\}$ and we can apply this lemma for a lower dimension. To check the $\operatorname{depth}(Z, c)$, it suffices to check half-spaces whose defining hyperplane goes through c. Given a closed half-space H^+ such that $c \in H$, consider the set $\mathcal{M} = H^+ \cap Y$. If we show that there are at most M^d possible subsets $\mathcal{M}$ we can get in this way, then the lemma would be proved.

Notice that we may move H continuously without losing c until it contains $d-1$ points $y_1, \ldots, y_d \in Y$ such that $c, y_1, \ldots, y_{d-1}$ are affinely independent and without changing the set $\mathcal{M}$ with the only possible exception of gaining a subset of $y_1, \ldots, y_{d-1}$. Therefore, there are at most $2 \cdot 2^{d-1} \cdot \binom{M}{d-1}$ possibilities for $\mathcal{M}$. Here $\binom{M}{d-1}$ comes from counting the possible $(d-1)$-tuples of Y generating H after tilting; the 2^{d-1} is the possible number of subsets of $y_1, \ldots, y_{d-1}$ that could be in $\mathcal{M}$, and the first factor 2 comes from the two possible half-spaces we can consider

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once we have H in its final position. This number is bounded above by M^d except for the case $M=3, d=2$, which can be checked by hand. □

The proof of Lemma 2.2 uses the probabilistic method, by making the colourful choice at random.

Proof of Lemma 2.2. Let $\lambda > \sqrt{dN\ln(Nr)/2}$. For each F_i we are going to choose x_i randomly and uniformly from its r points in order to form our set X.

Given a closed half-space H such that $0 \in H$, since $0 \in \operatorname{conv}(F_i)$, we have that

$$\mathbb{P}(x_i \in H) \geqslant \frac{1}{r}.$$

Thus,

$$\mathbb{E}(|X \cap H|) \geqslant \frac{N}{r}.$$

Moreover, $|X \cap H| = \sum_{i=1}^{N} \chi(x_i \in H)$. This is a sum of independent indicators each of whose probability of success is at least $1/r$. In particular, Hoeffding's inequality implies that

$$\mathbb{P}\left(|X \cap H| \leqslant \frac{N}{r} - \lambda\right) \leqslant \mathbb{P}(|X \cap H| \leqslant \mathbb{E}(|X \cap H|) - \lambda) \leqslant \exp\left(\frac{-2\lambda^2}{N}\right).$$

We can let X_H denote the random variable which is the indicator of the event $|X \cap H| \leqslant N/r - \lambda$.

Let $\mathcal{H}$ be the family of at most $(Nr)^d$ half-spaces from Lemma 2.3 with $Y = \cup_i F_i$ and $c = 0$. We have that

$$\mathbb{E}\left(\sum_{H \in \mathcal{H}} X_H\right) = \sum_{H \in \mathcal{H}} \mathbb{E}(X_H) \leqslant \sum_{H \in \mathcal{H}} \exp\left(\frac{-2\lambda^2}{N}\right) \leqslant \exp(d\ln(Nr)) \exp\left(\frac{-2\lambda^2}{N}\right) < 1,$$

where the last inequality follows from the choice of λ. Thus, there must be an instance of X where $\sum_{H \in \mathcal{H}} X_H = 0$. By the choice of $\mathcal{H}$, we have that $\operatorname{depth}(X, 0) \geqslant N/r - \lambda$, as desired. □

One curious aspect of the colourful Carathéodory theorem is the case $r = 2$. In this instance, the colour classes are simply the endpoints of $d+1$ segments each containing the origin. To show the existence of a colourful choice as the theorem indicates, it suffices to use a non-trivial linear dependence of the $d+1$ directions of the segments. The signs of the coefficients indicate which endpoint should be taken for each segment. Likewise, Lemma 2.2 has a similar version when $r = 2$.

Corollary 2.4. *Let N, d be positive integers and*

$$t = \left\lceil \frac{N}{2} - \sqrt{\frac{dN\ln(2N)}{2}} \right\rceil - 1.$$

Given a set S of N vectors in $\mathbb{R}^d$, there is an assignment Γ of signs $(+)$ or $(-)$ to the element of S such that, if any t vectors are removed from S, the remaining vectors have a non-trivial linear dependence where the signs of all non-zero coefficients agree with Γ.

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3. Robust Tverberg results

Let us restate Theorem 1.4 in the form which we aim to prove. In order to show that the version below implies the one in the Introduction, it suffices to write $N = rt + p$, and notice that with $p = \tilde{O}(r^2\sqrt{td} + dr^3)$, the tolerance provided by the result below is at least t.

Theorem 3.1. *Let r, N, d be positive integers and*

$$t = \left\lceil \frac{N}{r} - \sqrt{\frac{(d+1)(r-1)N\ln(Nr)}{2}} \right\rceil - 1.$$

Suppose we are given a set X of N points in $\mathbb{R}^d$. Then there is a partition of X into r parts $A_1, A_2, \ldots, A_r$ such that for any set C of at most t points, we have

$$\bigcap_{j=1}^{r} \operatorname{conv}(A_j \setminus C) \neq \emptyset.$$

Since Tverberg's theorem can be deduced using the colourful Carathéodory theorem, it should come as no surprise that variations of the latter often translate to variations of both. The result above is the consequence of applying Lemma 2.2.

Proof. Let S be a set of N points in $\mathbb{R}^d$, $S = \{a_1, \ldots, a_N\}$. Let $u_1, u_2, \ldots, u_r$ be the vertices of a regular simplex in $\mathbb{R}^{r-1}$ centred at the origin. Notice that any linear combination $\beta_1 u_1 + \cdots + \beta_r u_r$ gives the zero vector if and only if $\beta_1 = \cdots = \beta_r$.

We construct the points $b_i = (a_i, 1) \in \mathbb{R}^{d+1}$, and for each i consider the family

$$F_i = \{b_i \otimes u_j : 1 \leqslant j \leqslant r\} \subset \mathbb{R}^{(d+1)(r-1)},$$

where $\otimes$ denotes the standard tensor product. Notice that the barycentre of each F_i is the origin in $\mathbb{R}^{(d+1)(r-1)}$. Thus, we may apply Lemma 2.2 and obtain a colourful choice $x_i = b_i \otimes u_{j_i}$ for each i such that for the set $X = \{x_1, \ldots, x_N\}$ we have $\operatorname{depth}(X, 0) \geqslant t+1$. If we remove any set C of at most t points, we still have $\operatorname{depth}(X \setminus C, 0) \geqslant 1$. In other words, $0 \in \operatorname{conv}(X \setminus C)$.

Consider the sets $I_j = \{i : j_i = j\}$ and $A_j = \{u_i : i \in I_j\}$. The sets $A_1, A_2, \ldots, A_r$ form the partition of S induced by X. Given any set $C \subset \{1, \ldots, N\} = [N]$ of at most t indices, we know that there are coefficients $\{\alpha_i : i \in [N] \setminus C\}$ of a convex combination such that

$$\sum_{i \in [N] \setminus C} \alpha_i b_i \otimes u_{j_i} = 0.$$

If we factor each u_j, we get

$$\left(\sum_{i \in I_1 \setminus C} \alpha_i b_i\right) \otimes u_1 + \left(\sum_{i \in I_2 \setminus C} \alpha_i b_i\right) \otimes u_2 + \cdots + \left(\sum_{i \in I_r \setminus C} \alpha_i b_i\right) \otimes u_r = 0.$$

The choice of $u_1, \ldots, u_r$ implies that

$$\sum_{i \in I_1 \setminus C} \alpha_i b_i = \cdots = \sum_{i \in I_r \setminus C} \alpha_i b_i.$$

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Using the fact that the last coordinate of each b_i is equal to 1, by a simple scaling we can assume that $\sum_{i \in I_j \setminus C} \alpha_i = 1$ for each j. If we look at the first d coordinates, we have

$$\sum_{i \in I_1 \setminus C} \alpha_i a_i = \cdots = \sum_{i \in I_r \setminus C} \alpha_i a_i,$$

where each expression is a convex combination. If we define $C' = \{u_i : i \in C\}$, the convex combinations above translate to

$$\bigcap_{j=1}^{r} \operatorname{conv}(A_j \setminus C') \neq \emptyset,$$

as desired. □

4. Coloured Tverberg with tolerance

One of the most important open problems around Tverberg's theorem is a long-standing conjecture by Bárány and Larman, also known as the coloured Tverberg theorem [8].

Conjecture 4.1. *Let r, d be positive integers. Suppose $F_1, \dots, F_{d+1}$ are families of r points each in $\mathbb{R}^d$, considered as colour classes. Then there is a partition of them into r sets $A_1, A_2, \dots, A_r$ such that each part has exactly one point of each colour and*

$$\bigcap_{j=1}^{r} \operatorname{conv}(A_j) \neq \emptyset.$$

This conjecture has only been verified for $d = 2$ by Bárány and Larman [8], whose paper also included a proof for the case $r = 2$ by Lovász, and when $r + 1$ is prime [11, 12]. One relaxation which gives a positive result is if we are given $(r-1)d+1$ colour classes instead of $d+1$ [33], where we can impose further conditions on the coefficients of the intersecting convex combinations. Another is if we allow each colour class to have $2r-1$ points instead of r [10], although in this case the sets A_i do not form a partition.

Given a family of sets F_j of r points, each considered colour classes, we will say that a partition of their union into r sets $A_1, \dots, A_r$ is a colourful partition if each A_i has exactly one point of each F_j. We show a version of Tverberg's theorem with tolerance which holds for coloured classes.

Theorem 4.2. *Let r, t, d be positive integers. There is a constant c_r that depends only on r such that the following hold. Given $c_r t + \tilde{O}(r\sqrt{td} + rd)$ colour classes of r points each in $\mathbb{R}^d$, there is a colourful partition of them into r sets $A_1, \dots, A_r$ such that, even if we remove any t colour classes, the convex hulls of what is left in each A_i still intersect. The $\tilde{O}(\cdot)$ notation only hides polylogarithmic factors in r, t, d. Moreover, $c_r \to e/(e-1) \sim 1.581\dots$ as $r \to \infty$.*

It should be noted that the adaptation of Sarkaria's methods described in [33] in combination with Theorem 1.4 gives a coloured version similar to the one above. This would have much stronger conditions on the coefficients of the convex combinations that give the point of intersection, but would require $rt + o(t)$ colour classes instead of $c_r t + o(t)$. The value c_r also satisfies the bounds $c_2 \leqslant 2$ and $c_r \leqslant 1.6$ for $r \geqslant 3$. To prove Theorem 4.2, we need the following lemma.

Lemma 4.3. *Let r be a positive integer and consider $[r] = \{1, 2, \dots, r\}$. Suppose that we are given r forbidden values $v_1, v_2, \dots, v_r$ not necessarily distinct in $[r]$. The probability that a random permutation σ satisfies $\sigma(i) \neq v_i$ for all i is maximized when all the v_i are different.*

Proof. Let S_r be the set of all permutations $\sigma : [r] \to [r]$. For each permutation $\sigma \in S_r$, let $n(\sigma)$ be the number of indices i such that $\sigma(i) = v_i$. If the forbidden values are not all different, we can assume without loss of generality that $v_1 = v_2$. Thus, there is at least one number $\tau \in [r]$ which is not forbidden. Consider a new list of forbidden values $(\tau, v_2, \dots, v_r)$, and let $m(\sigma)$ be the number of indices i such that $\sigma(i)$ is the ith element of the new list. Let $F : S_r \to S_r$ be the bijection such that $F(\sigma) = \sigma$ if $\sigma(1) \neq v_1$ and $\sigma(1) \neq \tau$, and F switches the values of $\sigma^{-1}(v_1)$ and $\sigma^{-1}(\tau)$ otherwise. If $n(\sigma) = 0$, then $m(F(\sigma)) = 0$, but there may be permutations with $n(\sigma) \neq 0$ and $m(F(\sigma)) = 0$. Thus, if we repeat this process until every pair of forbidden values is different, the number of permutations σ with $n(\sigma) = 0$ only increases, as desired. □

It is known that the probability $p(r)$ that a random permutation of $[r]$ has at least one fixed point tends to $1 - 1/e$ as $r \to \infty$. Thus, the lemma above implies that, given at least one forbidden value for each element in $[r]$, the probability that a random permutation of $[r]$ hits at least one of them is at least $p(r)$.

We say that a family of r sets $F_1, \dots, F_r$ in $\mathbb{R}^d$ is a coloured r-block if

- each F_i has r points,
- each F_i captures the origin, and
- all points in the r-block are coloured with one of r possible colours in such a way that each F_i has exactly one point of each colour.

Given a coloured r-block, we say that a subset of its points is a colourful choice if it has exactly one point of each colour and exactly one point of each F_i. Given a family of coloured r-blocks, we say a subset of their union is a colourful choice if its restriction to each r-block is also a colourful choice.

Corollary 4.4. *Let r, d be positive integers. Suppose that we are given a coloured r-block in $\mathbb{R}^d$ and a closed half-space H containing the origin. If we pick a random colourful choice, the probability that we have at least one point in H is at least $p(r)$.*

Proof. Let $F_1, \dots, F_r$ be the sets in the block. It suffices to notice that each colourful choice corresponds to a permutation $\sigma : [r] \to [r]$ so that $\sigma(i) = j$ if and only if the point chosen from F_i is of colour j. Then, if we forbid all points in H (at least one forbidden point from each F_i), we have reduced the problem to the lemma above. □

Theorem 4.5. *Let r, N, d be positive integers. Suppose we are given N coloured r-blocks in $\mathbb{R}^d$, all using the same r colours. Then there is a colourful choice M such that each half-space* *containing the origin contains points of at least t different r-blocks, as long as*

$$t \leqslant p(r)N - \sqrt{\frac{dN\ln(Nr^2)}{2}}.$$

Proof. Let $\lambda > \sqrt{dN\ln(Nr^2)/2}$ and let H be a closed half-space containing the origin. For each coloured r-block B we choose independently at random a coloured choice X_B; their union is a random coloured choice X for the whole family. Let $x_B = \chi(X_B \cap H \neq \emptyset)$. Then

- $\mathbb{E}(x_B) \geqslant p(r)$,
- $\sum_B x_B$ is the number of coloured r-blocks whose colourful choice has at least one point in H, so $\mathbb{E}(\sum_B x_B) \geqslant p(r)N$,
- for $B \neq B'$, x_B and $x_{B'}$ are independent.

Thus, after applying Hoeffding's inequality we get

$$\mathbb{P}\left(\sum_B x_B \leqslant p(r)N - \lambda\right) \leqslant \exp\left(\frac{-2\lambda^2}{N}\right).$$

Notice that among all the coloured r-blocks we have Nr^2 points. Thus, to find the depth of 0 from a subset it is sufficient to check a family $\mathcal{H}$ of at most $(Nr^2)^d$ half-spaces. If we call a half-space H *bad* if less than $p(r)N - \lambda$ of the r-blocks have a point in it, the probability that there is at least one bad half-space is at most

$$\sum_{H\in\mathcal{H}} \exp\left(\frac{-2\lambda^2}{N}\right) = (Nr^2)^d \exp\left(\frac{-2\lambda^2}{N}\right) = \exp\left(d\ln(Nr^2) - \frac{2\lambda^2}{N}\right) < 1.$$

Thus, there is at least one colourful choice with no bad hyperplanes. □

Now we are ready to prove Theorem 4.2.

Proof of Theorem 4.2. We will show that given N colour classes of r points each in $\mathbb{R}^d$, we can find a colourful partition of them into r sets $A_1, \ldots, A_r$ so that, even if we remove any r colour classes, the convex hulls of what is left in each A_i still intersect as long as

$$t \leqslant p(r)N - \sqrt{\frac{(d+1)(r-1)N\ln(Nr^2)}{2}} - 1.$$

This implies the result in the theorem with $c_r = 1/(p(r))$.

Let $n = (r-1)(d+1)$. If we apply the Sarkaria transformation to the original set of points, notice that each colour class of r points is represented by r^2 points in $\mathbb{R}^n$. Moreover, we can introduce r colours in $\mathbb{R}^n$ and colour points of the form $(x,1)\otimes u_i$ with colour i. This turns the r^2 points in $\mathbb{R}^n$ into a coloured r-block. Notice also that for a colour class in $\mathbb{R}^d$, assigning each of its points to a different A_i corresponds to a colourful choice in its corresponding coloured r-block in $\mathbb{R}^n$. Thus, we can use Theorem 4.5 to finish the proof. □

5. Reay partitions with tolerance

Let $R^* = R^*(d,r,k)$ be the smallest integer such that among any R^* points in $\mathbb{R}^d$, there is a partition of them into r parts such that the convex hulls of any k parts intersect. We call such partitions *Reay*

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partitions. Tverberg's theorem asserts that $R^*(d,r,r) = (d+1)(r-1)+1$. However, there are no cases known for which $R^*(d,r,k) < R^*(d,r,r)$. In 1979 [30] Reay conjectured that $R^*(d,r,k) = (d+1)(r-1)+1$ for any $r \geqslant k \geqslant 2$. Reay's conjecture remains open. There have been several advances improving lower bounds for R^* [4, 29].

It is natural to extend Reay partitions to the setting with tolerance. We define the integer $R = R(t,d,r,k)$ as the smallest R such that among any R points in $\mathbb{R}^d$, there is a partition $\mathcal{P}$ of them into r parts with the following property. For any set C of at most t points and any k-tuple $\mathcal{K}$ of parts of $\mathcal{P}$, even if the points of C are removed, the convex hulls of what is left in each part of $\mathcal{K}$ intersect.

Theorem 5.1. *For any positive integers t,d,r,k with $r \geqslant k$, we have that*

$$R(t,d,r,k) = rt + \tilde{O}(r\sqrt{dkrt} + r^2dk).$$

Moreover, an application of Helly's theorem shows that $N(t,d,r) = R(t,d,r,d+1)$, so the bounds above improve Theorem 1.4 if r is large.

Proof. Suppose we are given M points in $\mathbb{R}^d$. We can colour them randomly and independently with one of r colours. Let us bound the probability that a given k-tuple of colours can be separated if we remove t points. In other words, we want to bound from above the probability that there is a set of at most t points such that after removing them, the convex hulls of what is left in each part of the k-tuple do not intersect.

We apply Sarkaria's trick, but use only k vectors $u_1,\dots,u_k$ in $\mathbb{R}^{k-1}$ instead of r vectors in $\mathbb{R}^{r-1}$. Every point $a_i \in \mathbb{R}^d$ is represented by the set F_i made by the k points of the form $(a_i,1)\otimes u_j$ for $1 \leqslant j \leqslant k$. If our chosen k-tuples consist of the first k colours, then assigning a colour to a_i corresponds to possibly choosing an element of F_i as follows:

- if a_i is coloured with colour j and $j \leqslant k$, then we choose $(a_i,1)\otimes u_j$,
- if a_i is coloured with some colour $j \geqslant k+1$, we do not make a choice from F_i.

This turns our partition in $\mathbb{R}^d$ into a colourful choice X in $\mathbb{R}^{(k-1)(d+1)}$, where some classes F_i may not have an element selected.

The tolerance with which the convex hulls of our k-tuple intersect is equal to $\operatorname{depth}(X,0)-1$. With a computation similar to those for the proof of Theorem 1.1, we get that for any $\lambda > 0$

$$\mathbb{P}\left(\operatorname{depth}(X,0) < \frac{M}{r} - \lambda\right) \leqslant \exp\left((d+1)(k-1)\ln(Mr) - \frac{2\lambda^2}{M}\right).$$

So the probability that there is a k-tuple with tolerance smaller than $M/r - \lambda - 1$ is at most

$$\binom{r}{k}\exp\left((d+1)(k-1)\ln(Mr) - \frac{2\lambda^2}{M}\right).$$

Thus, by choosing

$$\lambda > \sqrt{\frac{1}{2}\left[(d+1)(k-1)\ln(Mr) + \ln\binom{r}{k}\right]},$$

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we know there is an instance where this does not happen. If we check how large M must be to guarantee that the given tolerance is at least t, we get the asymptotic bound of the theorem. One should note that $\ln \binom{r}{k} \leqslant k\ln(r/k) = \tilde{O}(k)$, which reduces the number of terms we get in the final expression. □

We should stress that the lower bound mentioned in the Introduction, $N(t,d,r) \geqslant rt + rd/2$, extends to Reay's setting. This is because the construction has the property that for every partition into r parts, there is a set of t points such that their removal makes one of the parts separable from the rest of the set by a hyperplane. In other words,

$$N(t,d,r) = R(t,d,r,r) \geqslant R(t,d,r,k) \geqslant R(t,d,r,2) \geqslant rt + \frac{rd}{2}.$$

6. Remarks

It would be interesting to see if in Theorem 4.2 the constant $c_r \sim 1.582\ldots$ could be replaced by 1, namely, determining if the following result holds.

Conjecture 6.1. *Let r,d be fixed positive integers. There is an integer $M = M(t,d,r) = t(1+o(1))$ such that, given any M families of r points each in $\mathbb{R}^d$, there is a colourful partition $A_1,\ldots,A_r$ of them such that for any family C of at most t colours*

$$\bigcap_{j=1}^{r} \operatorname{conv}(A_j \setminus C) \neq \emptyset.$$

This may seem counterintuitive at first sight, as we are removing almost all the points. One of the novel methods to obtain colourful Tverberg results is the 'constraints' method by Blagojević, Frick and Ziegler [10]. It is tempting to use Theorem 1.4 with the constraints method to tackle the conjecture above. However, since the tolerance given by Theorem 1.4 is at most a $(1/r)$-fraction of the total number of points, Conjecture 6.1 seems out of reach.

Theorem 1.4 does not improve the previous bounds for $N(t,d,r)$ for low values of t. It is still possible that Larman's result is optimal.

Problem 6.2 (Larman [23]). *Let d be a positive integer. Determine if there is a set S of $2d+2$ points in $\mathbb{R}^d$ with the property that, for any partition A,B of S, there is a point $x \in S$ such that*

$$\operatorname{conv}(A \setminus \{x\}) \cap \operatorname{conv}(B \setminus \{x\}) = \emptyset.$$

Finally, the topological versions of Tverberg's theorem with tolerance remain open, even in the cases where r is a prime number or a prime power.

Problem 6.3. *Let r,t,d be positive integers. Given an integer n, let Δ^n denote the n-dimensional simplex, with $n+1$ vertices. Find the smallest integer $N^* = N^*(t,d,r)$ such that the following*

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holds. Given any continuous map $f : \Delta^{N^*} \to \mathbb{R}^d$, there is a partition of the vertices of Δ^{N^*} into r sets $A_1, \ldots, A_r$ such that, for any set C of at most t vertices,

$$\bigcap_{j=1}^{r} f([A_j \setminus C]) \neq \emptyset,$$

where $[X]$ denotes the face spanned by X, for any set of vertices X.

So far, not even the Soberón–Strausz bound $N^* \leqslant (t+1)(r-1)(d+1)$ is known to hold. The only bound at the moment is $N^* \leqslant (t+1)(r-1)(d+1)+t$, when r is a prime power. This follows from taking $t+1$ topological Tverberg partitions, as removing t vertices leaves one of the partitions unaffected.

If one is interested in non-deterministic algorithms that yield Tverberg theorems with tolerance, the proof of our results can be extended to the following.

Theorem 6.4. *Let N,t,d,r be positive integers and let $\varepsilon > 0$ be a real number. Given N points in $\mathbb{R}^d$, a random partition of them into r parts is a Tverberg partition with tolerance t with probability at least $1-\varepsilon$, as long as*

$$t+1 \leqslant \frac{N}{r} - \sqrt{\frac{1}{2}\left[(d+1)(r-1)N\ln(Nr) + N\ln\left(\frac{1}{\varepsilon}\right)\right]}.$$

The advantage of the result above is that generating the partition is trivial, taking time N. The problem of finding Tverberg partitions, in both its deterministic and non-deterministic versions, is interesting. See, for instance, [15, 26, 28, 31].

References

[1] Ajtai, M., Chvátal, V., Newborn, M. M. and Szemerédi, E. (1982) Crossing-free subgraphs. In *Theory and Practice of Combinatorics*, Vol. 60 of North-Holland Mathematics Studies, North-Holland, pp. 9–12.

[2] Alon, N. and Spencer, J. H. (2008) *The Probabilistic Method*, third edition, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley.

[3] Arocha, J. L., Bárány, I., Bracho, J., Fabila, R. and Montejano, L. (2009) Very colorful theorems. *Discrete Comput. Geom.* **42** 142–154.

[4] Asada, M., Chen, R., Frick, F., Huang, F., Polevy, M., Stoner, D., Tsang, L. H. and Wellner, Z. (2016) On Reay's relaxed Tverberg conjecture and generalizations of Conway's thrackle conjecture. arXiv:1608.04279

[5] Bárány, I. (1982) A generalization of Carathéodory's theorem. *Discrete Math.* **40** 141–152.

[6] Bárány, I. (2015) Tensors, colours, octahedra. In *Geometry, Structure and Randomness in Combinatorics* (J. Matoušek *et al.*, eds), Edizione della Normale, pp. 1–17.

[7] Bárány, I. Personal communication.

[8] Bárány, I. and Larman, D. G. (1992) A colored version of Tverberg's theorem. *J. London Math. Soc.* **s2-45** 314–320.

[9] Bárány, I. and Onn, S. (1997) Colourful linear programming and its relatives. *Math. Oper. Res.* **22** 550–567.

[10] Blagojević, P. V. M., Frick, F. and Ziegler, G. M. (2014) Tverberg plus constraints. *Bull. London Math. Soc.* **46** 953–967.

https://doi.org/10.1017/S0963548317000591 Published online by Cambridge University Press

[11] Blagojević, P. V. M., Matschke, B. and Ziegler, G. M. (2011) Optimal bounds for a colorful Tverberg–Vrećica type problem. *Adv. Math.* **226** 5198–5215.
[12] Blagojević, P. V. M., Matschke, B. and Ziegler, G. M. (2015) Optimal bounds for the colored Tverberg problem. *J. Eur. Math. Soc.* **17** 739–754.
[13] Chazelle, B. and Friedman, J. (1990) A deterministic view of random sampling and its use in geometry. *Combinatorica* **10** 229–249.
[14] Clarkson, K. L. (1987) New applications of random sampling in computational geometry. *Discrete Comput. Geom.* **2** 195–222.
[15] Clarkson, K. L., Eppstein, D., Miller, G. L., Sturtivant, C. and Teng, S.-H. (1996) Approximating center points with iterative Radon points. *Internat. J. Comput. Geom. Appl.* **6** 357–377.
[16] Forge, D., Las Vergnas, M. and Schuchert, P. (2001) 10 points in dimension 4 not projectively equivalent to the vertices of a convex polytope. *Europ. J. Combin.* **22** 705–708.
[17] García-Colín, N. (2007) Applying Tverberg type theorems to geometric problems. PhD thesis, University College London.
[18] García-Colín, N. and Larman, D. (2015) Projective equivalences of k-neighbourly polytopes. *Graphs Combin.* **31** 1403–1422.
[19] García-Colín, N., Raggi, M. and Roldán-Pensado, E. (2017) A note on the tolerant Tverberg theorem. *Discrete Comput. Geom.* **58**, no. 3, 746–754.
[20] Haussler, D. and Welzl, E. (1987) ε-nets and simplex range queries. *Discrete Comput. Geom.* **2** 127–151.
[21] Holmsen, A. F. (2016) The intersection of a matroid and an oriented matroid. *Adv. Math.* **290** 1–14.
[22] Holmsen, A. F., Pach, J. and Tverberg, H. (2008) Points surrounding the origin. *Combinatorica* **28** 633–644.
[23] Larman, D. G. (1972) On sets projectively equivalent to the vertices of a convex polytope. *Bull. London Math. Soc.* **4** 6–12.
[24] Liu, R. Y., Serfling, R. J. and Souvaine, D. L. (2006) *Data Depth: Robust Multivariate Analysis, Computational Geometry, and Applications*, Vol. 72 of DIMAC Series in Discrete Mathematics and Theoretical Computer Science, AMS.
[25] Matoušek, J. (2002) *Lectures on Discrete Geometry*, Vol. 212 of Graduate Texts in Mathematics, Springer.
[26] Miller, G. L. and Sheehy, D. R. (2009) Approximate center points with proofs. In *SCG '09: Twenty-Fifth Annual Symposium on Computational Geometry*, ACM, pp. 153–158.
[27] Montejano, L. and Oliveros, D. (2011) Tolerance in Helly-type theorems. *Discrete Comput. Geom.* **45** 348–357.
[28] Mulzer, W. and Stein, Y. (2013) Algorithms for tolerated Tverberg partitions. In *ISAAC 2013: International Symposium on Algorithms and Computation*, Springer, pp. 295–305.
[29] Perles, M. A. and Sigron, M. (2016) Some variations on Tverberg's theorem. *Israel J. Math.* **216** 957–972.
[30] Reay, J. R. (1979) Several generalizations of Tverberg's theorem. *Israel J. Math.* **34** 238–244.
[31] Rolnick, D. and Soberón, P. (2016) Algorithms for Tverberg's theorem via centerpoint theorems. arXiv:1601.03083v2
[32] Sarkaria, K. S. (1992) Tverberg's theorem via number fields. *Israel J. Math.* **79** 317–320.
[33] Soberón, P. (2015) Equal coefficients and tolerance in coloured Tverberg partitions. *Combinatorica* **35** 235–252.
[34] Soberón, P. and Strausz, R. (2012) A generalisation of Tverberg's theorem. *Discrete Comput. Geom.* **47** 455–460.
[35] Székely, L. A. (1997) Crossing numbers and hard Erdős problems in discrete geometry. *Combin. Probab. Comput.* **6** 353–358.
[36] Tukey, J. W. (1975) Mathematics and the picturing of data. In *Proceedings of the International Congress of Mathematicians*, Vol. 2, Canadian Mathematical Congress, pp. 523–531.
[37] Tverberg, H. (1966) A generalization of Radon's theorem. *J. London Math. Soc.* **41** 123–128.