J. Aust. Math. Soc. **109** (2020), 299–319
doi:10.1017/S144678871900020X

SIMPLICITY AND CHAIN CONDITIONS FOR ULTRAGRAPH LEAVITT PATH ALGEBRAS VIA PARTIAL SKEW GROUP RING THEORY

DANIEL GONÇALVES ✉ and DANILO ROYER

(Received 29 August 2018; accepted 19 March 2019; first published online 18 July 2019)

Communicated by A. Sims

Abstract

We realize Leavitt ultragraph path algebras as partial skew group rings. Using this realization we characterize artinian ultragraph path algebras and give simplicity criteria for these algebras.

2010 *Mathematics subject classification*: primary 16S35; secondary 16S99, 16P20.

Keywords and phrases: ultragraph Leavitt path algebras, partial skew group ring, simplicity, artinian ring, noetherian ring.

1. Introduction

The study of algebras associated to combinatorial objects has attracted a great deal of attention in the past years. Part of the interest in these algebras arises from the fact that many properties of the combinatorial objects translate into algebraic properties of the associated algebras and, furthermore, there are deep connections between these algebras and symbolic dynamics. As examples of algebras associated to combinatorial objects we cite graph C*-algebras, Leavitt path algebras, higher rank graph algebras, Kumjian–Pask algebras, and ultragraph C*-algebras, among others (see [1, 2] for a comprehensive list).

Notice that in the list of algebras we presented above the C*-algebraic version of the algebras was immediately followed by the algebraic analogue, except for the ultragraph case. Ultragraphs (a generalization of graphs, where the range map takes values on the power set of the vertices) were defined by Tomforde in [17] as a unifying approach to Exel–Laca and graph C*-algebras. They have proved to be a key ingredient in the study of Morita equivalence of Exel–Laca and graph C*-algebras (see [13]) and their representation theory has been studied in [6]. Very recently, ultragraph C*-algebras were connected with the symbolic dynamics of shift spaces over infinite

Partially supported by CNPq and Capes-PrInt Brazil.

alphabets (see [9, 15]) and ultragraphs were the key object behind a new proposal for the generalization of a shift of finite type to the infinite-alphabet case (see [10] for the definition and [3, 11] for further developments of the theory).

Due to the exposure above, it is natural to study the algebraic analogue of an ultragraph C*-algebra. The formalization of the definition of the algebra was given in [12], along with a study of the algebra ideals and a proof of a Cuntz–Krieger uniqueness-type theorem. Furthermore, it was shown in [12] that the class of ultragraph path algebras is strictly larger than the class of Leavitt path algebras. This raises the question of which results about Leavitt path algebras can be generalized to ultragraph path algebras and whether results from the C*-algebraic setting can be proved in the algebraic level. Our work is a first step in this direction. Building from ideas in [8], where Leavitt path algebras are realized as partial skew group rings, we realize ultragraph path algebras as partial skew group rings. This is also the algebraic version of the characterization of ultragraph C*-algebras as partial crossed products given in [10]. We highlight that the algebraic version we present is more general than the C*-algebraic version, since the latter is valid only for ultragraphs with no sinks that satisfy Condition (RFUM). Furthermore, the algebraic version we present does not rely on any topological space, rather it relies on a set. In fact, in [10] Condition (RFUM) was introduced in order to reduce the great technicalities in defining a suitable, and tractable, zero-dimensional topological space for ultragraph C*-algebras. In contrast, the definition of the set necessary in the algebraic setting falls within the grasp of a wide audience.

The theory of partial skew group rings has been in constant development recently; see, for example, [5, 7], where simplicity criteria are described, [14], where chain conditions are studied, and [4] (and the 283 references therein cited), where most of the recent developments in the theory are compiled. In our case we use partial skew ring theory to characterize artinian ultragraph path algebras and give simplicity criteria for these algebras.

Given an ultragraph $\mathcal{G}$, we realize the associated path algebra as a partial skew group ring in Section 3. For this we consider the free group on the edges of $\mathcal{G}$. In the graph case (see [8]), the free group of edges acts on a subspace of the functions in a set X, where X is the set of infinite paths in union with finite paths ending in a sink (a vertex that emits no edges). To find the correct set X in the ultragraph context is a key step in our construction. For ultragraphs, a finite path of positive length is a sequence of edges $e_1 \dots e_n$ such that $s(e_{i+1}) \in r(e_i)$. The set X is formed by the infinite paths, the pairs (α, v), where α is a finite path of positive length and v is a sink in the range of α, and the pairs (v, v), where v is a sink. After precisely defining the set X we proceed with the definition of the partial action and set up the ground to prove Theorem 3.10, which gives the isomorphism between the partial skew group ring and the ultragraph path algebra.

In light of Theorem 3.10, we use the results in [7] to characterize simplicity of ultragraph path algebras in Section 4. As is the case with Leavitt and graph C*-algebras, the criterion for simplicity we obtain coincides with the one for ultragraph

C*-algebras (the latter is given in [16]). More precisely, we show that (when R is a field) the ultragraph Leavitt path algebra is simple if, and only if, $\mathcal{G}$ satisfies Condition (L) and the unique saturated and hereditary subcollections of $\mathcal{G}^0$ are $\emptyset$ and $\mathcal{G}^0$ (this is Theorem 4.7). We remark that, using the tools developed in this section, we provide a new proof of the Cuntz–Krieger uniqueness theorem for Leavitt path algebras of ultragraphs (Corollary 4.3). We end the paper in Section 5, where we apply the results of [14] to characterize artinian ultragraph path algebras.

2. Ultragraphs and partial skew group rings

Leavitt path algebras of ultragraphs were introduced in [12]. Here we recall the main definitions and relevant results.

Definition 2.1. An *ultragraph* is a quadruple $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ consisting of two countable sets $G^0, \mathcal{G}^1$, a map $s : \mathcal{G}^1 \to G^0$, and a map $r : \mathcal{G}^1 \to P(G^0) \setminus \{\emptyset\}$, where $P(G^0)$ stands for the power set of G^0.

Definition 2.2. Let $\mathcal{G}$ be an ultragraph. Define $\mathcal{G}^0$ to be the smallest subset of $P(G^0)$ that contains $\{v\}$ for all $v \in G^0$, contains $r(e)$ for all $e \in \mathcal{G}^1$, and is closed under finite unions and nonempty finite intersections. Elements of $\mathcal{G}^0$ are called generalized vertices.

Definition 2.3. Let $\mathcal{G}$ be an ultragraph and R be a unital commutative ring. The Leavitt path algebra of $\mathcal{G}$, denoted by $L_R(\mathcal{G})$, is the universal algebra with generators $\{s_e, s_e^* : e \in \mathcal{G}^1\} \cup \{p_A : A \in \mathcal{G}^0\}$ and relations:

(1) $p_\emptyset = 0, p_A p_B = p_{A\cap B}, p_{A\cup B} = p_A + p_B - p_{A\cap B}$ for all $A, B \in \mathcal{G}^0$;
(2) $p_{s(e)} s_e = s_e p_{r(e)} = s_e$ and $p_{r(e)} s_e^* = s_e^* p_{s(e)} = s_e^*$ for each $e \in \mathcal{G}^1$;
(3) $s_e^* s_f = \delta_{e,f} p_{r(e)}$ for all $e, f \in \mathcal{G}$;
(4) $p_v = \sum_{s(e)=v} s_e s_e^*$ whenever $0 < |s^{-1}(v)| < \infty$.

Before we proceed, we quickly remind the reader of the definition of a partial action: a partial action of a group G on a set Ω is a pair $\alpha = (\{D_t\}_{t\in G}, \{\alpha_t\}_{t\in G})$, where, for each $t \in G$, D_t is a subset of Ω and $\alpha_t : D_{t^{-1}} \to D_t$ is a bijection such that $D_e = \Omega$, α_e is the identity in Ω, $\alpha_t(D_{t^{-1}} \cap D_s) = D_t \cap D_{ts}$, and $\alpha_t(\alpha_s(x)) = \alpha_{ts}(x)$ for all $x \in D_{s^{-1}} \cap D_{s^{-1}t^{-1}}$. In case Ω is an algebra or a ring, then the subsets D_t should also be ideals and the maps α_t should be isomorphisms.

Associated to a partial action of a group G in a ring A, the partial skew group ring, denoted by $A \rtimes_\alpha G$, is defined as the set of all finite formal sums $\sum_{t\in G} a_t\delta_t$, where, for all $t \in G$, $a_t \in D_t$ and δ_t is a symbol. Addition is defined componentwise and multiplication is determined by $(a_t\delta_t)(b_s\delta_s) = \alpha_t(\alpha_{-t}(a_t)b_s)\delta_{ts}$.

3. Ultragraph path algebra as a partial skew group ring

Let $\mathcal{G}$ be an ultragraph. A finite path is either an element of $\mathcal{G}^0$ or a sequence of edges $e_1 \ldots e_n$, with length $|e_1 \ldots e_n| = n$, such that $s(e_{i+1}) \in r(e_i)$ for each $i \in$

https://doi.org/10.1017/S144678871900020X Published online by Cambridge University Press

$\{0, \ldots, n-1\}$. An infinite path is a sequence $e_1e_2e_3\ldots$, with length $|e_1e_2\ldots| = \infty$, such that $s(e_{i+1}) \in r(e_i)$ for each $i \geq 0$. The set of finite paths in $\mathcal{G}$ is denoted by $\mathcal{G}^*$ and the set of infinite paths in $\mathcal{G}$ is denoted by $\mathfrak{p}^\infty$. We extend the source and range maps as follows: $r(\alpha) = r(\alpha_{|\alpha|})$, $s(\alpha) = s(\alpha_1)$ for $\alpha \in \mathcal{G}^*$ with $0 < |\alpha| < \infty$, $s(\alpha) = s(\alpha_1)$ for each $\alpha \in \mathfrak{p}^\infty$, and $r(A) = A = s(A)$ for each $A \in \mathcal{G}^0$. An element $v \in G^0$ is a sink if $s^{-1}(v) = \emptyset$ and we denote the set of sinks in G^0 by G^0_s. We say that $A \in \mathcal{G}^0$ is a sink if each vertex in A is a sink.

Define the set

$$X = \mathfrak{p}^\infty \cup \{(\alpha, v) : \alpha \in \mathcal{G}^*, |\alpha| \geq 1, v \in G^0_s \cap r(\alpha)\} \cup \{(v, v) : v \in G^0_s\}.$$

Remark 3.1. Notice that given a vertex v, the element (v, v) is an element of X if, and only if, v is a sink.

Definition 3.2. For an element $(\alpha, v) \in X$, we define the range and source maps by $r(\alpha, v) = v$ and $s(\alpha, v) = s(\alpha)$. In particular, for a sink v, $s(v, v) = v = r(v, v)$. We also extend the length map to the elements (α, v) by defining $|(\alpha, v)| := |\alpha|$.

Next we set up some notation necessary to define the desired partial action. Let $\mathbb{F}$ be the free group generated by $\mathcal{G}^1$ and denote by 0 the neutral element of $\mathbb{F}$. Let $W \subseteq \mathbb{F}$ be the set

$$W = \{a_1 \ldots a_n \in \mathbb{F} : a_i \in \mathcal{G}^1 \;\; \forall i \text{ and } s(a_{i+1}) \in r(a_i) \;\; \forall i \in \{1, \ldots, n-1\}\}.$$

Remark 3.3. The set W is essentially the same as the set of elements of $\mathcal{G}^*$ with positive length, with the difference that W is a subset of a group and $\mathcal{G}^*$ is a set of paths in $\mathcal{G}$.

Notation 3.4. Given an element $a \in W$ with length $|a|$, and an element $x = (\alpha, v) \in X$ with length at least $|a|$, we use the notation $x_1 \ldots x_{|a|} = a$ to mean that $\alpha_1 \ldots \alpha_{|a|} = a_1 \ldots a_{|a|}$. If $x \in X$ is such that $|x| = \infty$, that is, $x = x_1x_2x_3\ldots$, then $x_1 \ldots x_{|a|} = a$ means that $x_i = a_i$ for each $i \in \{1, \ldots, |a|\}$.

Now we define the following sets:

- for $a \in W$, let $X_a = \{x \in X : x_1 \ldots x_{|a|} = a\}$;
- for $b \in W$, let $X_{b^{-1}} = \{x \in X : s(x) \in r(b)\}$;
- for $a, b \in W$ with $r(a) \cap r(b) \neq \emptyset$, let

$$\begin{aligned} X_{ab^{-1}} = \{x \in X : |x| > |a|, \;\; x_1 \ldots x_{|a|} = a \text{ and } s(x_{|a|+1}) \in r(b) \cap r(a)\} \\ \cup \{(a, v) \in X : v \in r(a) \cap r(b)\}; \end{aligned}$$

- for the neutral element 0 of $\mathbb{F}$, let $X_0 = X$;
- for all the other elements c of $\mathbb{F}$, let $X_c = \emptyset$.

Define, for each $A \in \mathcal{G}^0$ and $b \in W$, the sets

$$X_A = \{x \in X : s(x) \in A\}$$

and

$$X_{bA} = \{x \in X_b : |x| > |b| \text{ and } s(x_{|b|+1}) \in A\} \cup \{(b, v) \in X_b : v \in A\}.$$

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Remark 3.5. Notice that for each $a, b \in W$, $X_{ab^{-1}} = X_{a(r(b))} = X_{a(r(a)\cap r(b))}$ and $X_{r(b)} = X_{b^{-1}}$. Moreover, for an element $b \in W$ and for a sink $u \in r(b)$, $X_{b\{u\}} = \{(b, u)\}$.

The following lemma follows from the definitions of the sets X_c and X_A, for $c \in \mathbb{F}$ and $A \in \mathcal{G}$, and its proof is left to the reader.

Lemma 3.6. *Let $a, b, c, d \in W$ and $A, B \in \mathcal{G}^0$. Then we have the following results.*

(1) $X_a \cap X_b = \begin{cases} X_a & \text{if } a = b\xi \text{ for some } \xi \in W \cup \{0\}, \\ \emptyset & \text{if } a_i \neq b_i \text{ for some } i, \\ X_b & \text{if } b = a\xi \text{ for some } \xi \in W. \end{cases}$

(2) $X_a \cap X_{c^{-1}} = \begin{cases} X_a & \text{if } s(a) \in r(c), \\ \emptyset & \text{otherwise.} \end{cases}$

(3) $X_a \cap X_{bc^{-1}} = \begin{cases} X_a & \text{if } a = b\xi \text{ for some } \xi \in W \text{ and } s(\xi) \in r(c), \\ X_{bc^{-1}} & \text{if } b = a\xi \text{ for some } \xi \in W \cup \{0\}, \\ \emptyset & \text{otherwise.} \end{cases}$

(4) $X_{ab^{-1}} \cap X_{cd^{-1}} = \begin{cases} X_{ab^{-1}} & \text{if } a = c\xi \text{ for some } \xi \in W \text{ and } s(\xi) \in r(d), \\ X_{cd^{-1}} & \text{if } c = a\xi \text{ for some } \xi \in W \text{ and } s(\xi) \in r(b), \\ X_{a(r(b)\cap r(d))} & \text{if } a = c, \\ \emptyset & \text{otherwise.} \end{cases}$

(5) $X_A \cap X_a = \begin{cases} X_a & \text{if } s(a) \in A, \\ \emptyset & \text{otherwise.} \end{cases}$

(6) $X_A \cap X_{ab^{-1}} = \begin{cases} X_{ab^{-1}} & \text{if } s(a) \in A, \\ \emptyset & \text{otherwise.} \end{cases}$

(7) $X_A \cap X_B = X_{A\cap B}$ *and* $X_A \cup X_B = X_{A\cup B}$.

(8) $X_{bA} \cap X_c = \begin{cases} X_{bA} & \text{if } b = c\xi \text{ for some } \xi \in W \cup \{0\}, \\ X_c & \text{if } c = b\xi \text{ for some } \xi \in W \text{ and } s(\xi) \in A, \\ \emptyset & \text{otherwise.} \end{cases}$

(9) $X_{bA} \cap X_{cd^{-1}} = \begin{cases} X_{bA} & \text{if } b = c\xi \text{ for some } \xi \in W \text{ and } s(\xi) \in r(d), \\ X_{cd^{-1}} & \text{if } c = b\xi \text{ for some } \xi \in W \text{ and } s(\xi) \in A, \\ X_{b(A\cap r(d))} & \text{if } b = c, \\ \emptyset & \text{otherwise.} \end{cases}$

(10) $X_{bA} \cap X_{cB} = \begin{cases} X_{bA} & \text{if } b = c\xi \text{ for some } \xi \in W \text{ and } s(\xi) \in B, \\ X_{cB} & \text{if } c = b\xi \text{ for some } \xi \in W \text{ and } s(\xi) \in A, \\ X_{b(A\cap B)} & \text{if } b = c, \\ \emptyset & \text{otherwise.} \end{cases}$

Our aim is to get a partial action from $\mathbb{F}$ on X. With this in mind, define the following bijective maps:

- for $a \in W$, define $\theta_a : X_{a^{-1}} \to X_a$ by

$$\theta_a(x) = \begin{cases} ax & \text{if } |x| = \infty, \\ (a\alpha, v) & \text{if } x = (\alpha, v), \\ (a, v) & \text{if } x = (v, v); \end{cases}$$

- for $a \in W$, define $\theta_a^{-1} : X_a \to X_{a^{-1}}$ as being the inverse of θ_a;
- for $a, b \in W$ with $r(a) \cap r(b) \neq \emptyset$, define $\theta_{ab^{-1}} : X_{ba^{-1}} \to X_{ab^{-1}}$ by

$$\theta_{ab^{-1}}(x) = \begin{cases} ay & \text{if } |x| = \infty \text{ and } x = by, \\ (a\alpha, v) & \text{if } x = (b\alpha, v), \\ (a, v) & \text{if } x = (b, v); \end{cases}$$

- for the neutral element $0 \in \mathbb{F}$, define $\theta_0 : X_0 \to X_0$ as the identity map;
- for all the other elements c of $\mathbb{F}$, define $\theta_c : X_{c^{-1}} \to X_c$ as the empty map.

Remark 3.7. Notice that

$$\begin{aligned} X_{bA} = \{x \in X_b : \ \theta_{b^{-1}}(x) \in X_A)\} &= \{x \in X_b : \ \theta_{b^{-1}}(x) \in X_A \cap X_{b^{-1}}\} \\ &= \theta_b(X_A \cap X_{b^{-1}}), \end{aligned}$$

that is, $X_{bA} = \theta_b(X_A \cap X_{b^{-1}})$.

It is straightforward to check that for all $c, t \in \mathbb{F}$, $\theta_c(X_{c^{-1}} \cap X_t) = X_{ct} \cap X_c$ and $\theta_c \circ \theta_t = \theta_{ct}$ in $X_{t^{-1}} \cap X_{t^{-1}c^{-1}}$ and moreover since $X_0 = X$ and $\theta_0 = Id_X$, then $(\{\theta_t\}_{t\in\mathbb{F}}, \{X_t\}_{t\in\mathbb{F}})$ is a partial action of $\mathbb{F}$ on X.

Define for each $c \in \mathbb{F}$ the set $F(X_c)$ of all the functions from X to the commutative unital ring R which vanishes out of X_c. For the neutral element $0 \in \mathbb{F}$, we denote the set $F(X_0)$ simply by $F(X)$. Notice that each $F(X_c)$ is an R-algebra, with pointwise sum and product, and, moreover, each $F(X_c)$ is an ideal of the R-algebra $F(X)$. Now, for each $c \in \mathbb{F}$, define the R-isomorphism

$$\beta_c : F(X_c^{-1}) \to F(X_c)$$

by $\beta_c(f) = f \circ \theta_{c^{-1}}$, whose inverse is the isomorphism $\beta_{c^{-1}}$. So, we get a partial action $(\{\beta_c\}_{c\in\mathbb{F}}, \{F(X_c)\}_{c\in\mathbb{F}})$ from $\mathbb{F}$ to the R-algebra $F(X)$.

To get the desired partial action, we need to restrict the partial action β to the R-subalgebra D of $F(X)$ generated by all the finite sums of all the finite products of the characteristic maps $\{1_{X_A}\}_{A\in\mathcal{G}^0}$, $\{1_{bA}\}_{b\in W, A\in\mathcal{G}^0}$, and $\{1_{X_c}\}_{c\in\mathbb{F}}$. We also define, for each $t \in \mathbb{F}$, the ideals D_t of D as being all the finite sums of finite products of the characteristic maps $\{1_{X_t}1_{X_A}\}_{A\in\mathcal{G}^0}$, $\{1_{X_t}1_{bA}\}_{b\in W, A\in\mathcal{G}^0}$, and $\{1_{X_t}1_{X_c}\}_{c\in\mathbb{F}}$.

Remark 3.8. From now on we will use the notation 1_A, 1_{bA}, and 1_t instead of 1_{X_A}, $1_{X_{bA}}$, and 1_{X_t} for $A \in \mathcal{G}^0$, $b \in W$, and $t \in \mathbb{F}$. It follows directly from Lemma 3.6 that

$$D = \operatorname{span}\{1_A, 1_c, 1_{bA} : A \in \mathcal{G}^0, c \in \mathbb{F} \setminus \{0\}, b \in W\}$$

and that, for each $t \in \mathbb{F}$,

$$D_t = \operatorname{span}\{1_t 1_A, 1_t 1_c, 1_t 1_{bA} : A \in \mathcal{G}^0, c \in \mathbb{F}, b \in W\},$$

where ‘span’ means linear span.

Our aim is to restrict the partial action β to the ideals $\{D_t\}_{t\in\mathbb{F}}$ of D. The next proposition tells us that $\beta_t(D_{t^{-1}}) = D_t$ for each $t \in \mathbb{F}$.

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Proposition 3.9. (1) *For all* $t, c \in \mathbb{F}$, $\beta_c(1_{c^{-1}}1_t) = 1_c 1_{ct}$.

(2) *For* $b \in W$ *and* $A \in \mathcal{G}^0$, $\beta_b(1_b^{-1}1_A) = 1_b 1_{bA}$.

(3) *For* $t = ab^{-1}$, *with* $b \in W$ *and* $a \in W \cup \{0\}$, *and* $A \in \mathcal{G}^0$,

$$\beta_t(1_{t^{-1}}1_A) = \begin{cases} 1_t & \text{if } s(b) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

(4) *For* $b, c \in W$ *and* $A \in \mathcal{G}^0$,

$$\beta_c(1_{c^{-1}}1_{bA}) = \begin{cases} 1_c 1_{cbA} & \text{if } s(b) \in r(c), \\ 0 & \text{otherwise.} \end{cases}$$

(5) *For* $b, c, d \in W$ *and* $A \in \mathcal{G}^0$,

$$\beta_{dc^{-1}}(1_{cd^{-1}}1_{bA}) = \begin{cases} 1_{dc^{-1}}1_{d\xi A} = 1_{d\xi A} & \text{if } b = c\xi \text{ for some } \xi \in W \text{ and } s(\xi) \in r(d), \\ 1_{dc^{-1}} & \text{if } c = b\xi \text{ for some } \xi \in \text{ and } s(\xi) \in A, \\ 1_{dc^{-1}}1_{dA} = 1_{d(r(c)\cap A)} & \text{if } b = c, \\ 0 & \text{otherwise.} \end{cases}$$

(6) *For* $a, b, c \in W$ *and* $A \in \mathcal{G}^0$,

$$\beta_{c^{-1}}(1_c 1_{bA}) = \begin{cases} 1_{c^{-1}}1_{\xi A} = 1_{\xi A} & \text{if } b = c\xi \text{ for some } \xi \in W \cup \{0\}, \\ 1_{c^{-1}} & \text{if } c = b\xi \text{ for some } \xi \in W \text{ and } s(\xi) \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. To simplify this proof we use the boolean notation, that is, we use the notation $[P] = 1$ if P is true and $[P] = 0$ otherwise.

The first item follows from the fact that $\theta_c(X_{c^{-1}} \cap X_t) = X_c \cap X_{ct}$, since $\beta_c(1_{c^{-1}}1_t)(x) = [\theta_{c^{-1}}(x) \in X_{c^{-1}} \cap X_t] = [x \in \theta_c(X_{c^{-1}} \cap X_t)] = [x \in (X_c \cap X_{ct})] = 1_c(x)1_{tc}(x)$.

To see that the second item holds, note that $\beta_b(1_{b^{-1}}1_A)(x) = [\theta_{b^{-1}}(x) \in (X_{b^{-1}} \cap X_A)] = [x \in \theta_b(X_{b^{-1}} \cap X_A)] = [x \in X_{bA}] = 1_{bA}(x)$, where the second to last equality follows from Remark 3.7.

The third item follows from item (6) of Lemma 3.6.

To see that item (4) holds, note that, for $x \in X_c$,

$$\begin{aligned} \beta_c(1_{bA}1_{c^{-1}})(x) &= [\theta_{c^{-1}}(x) \in X_{bA} \cap X_{c^{-1}}] = [x \in \theta_c(X_{bA} \cap X_{c^{-1}})] \\ &= [x \in X_{cbA} \cap X_c] = 1_c(x)1_{cbA}(x) \end{aligned}$$

and, for $x \notin X_c$, $\beta_c(1_{bA}1_{c^{-1}})(x) = 0 = 1_c(x)1_{cbA}(x)$.

Item (5) follows from item (9) of Lemma 3.6 and the last item follows from item (8) of the same lemma. □

By the previous proposition we get that, for each $t \in \mathbb{F}$, $\beta_t(D_{t^{-1}}) \subseteq D_t$ and, consequently, $\beta_t(D_{t^{-1}}) = D_t$ for each $t \in \mathbb{F}$. So, we may consider the restriction of the partial action β to the subsets $\{D_t\}_{t\in\mathbb{F}}$ of D. We denote this restriction also by β and so we get a partial action $(\{\beta_t\}_{t\in\mathbb{F}}, \{D_t\}_{t\in\mathbb{F}})$ of $\mathbb{F}$ in D. Now we are ready to prove the following theorem.

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THEOREM 3.10. *Let $\mathcal{G}$ be an ultragraph, let R be a unital commutative ring, and let $L_R(\mathcal{G})$ be the Leavitt path algebra of $\mathcal{G}$. Then there exists an R-isomorphism $\phi : L_R(\mathcal{G}) \to D \rtimes_\beta \mathbb{F}$ such that $\phi(p_A) = 1_A\delta_0$, $\phi(s_e^*) = 1_{e^{-1}}\delta_{e^{-1}}$, and $\phi(s_e) = 1_e\delta_e$ for each $A \in \mathcal{G}^0$ and $e \in \mathcal{G}^1$.*

PROOF. First we show that the sets $\{1_A\delta_0\}_{A\in\mathcal{G}^0}$ and $\{1_e\delta_e, 1_{e^{-1}}\delta_{e^{-1}}\}_{e\in\mathcal{G}^1}$ satisfy the relations which define the algebra $L_R(\mathcal{G})$.

The first relation of Definition 2.3 follows from item (7) of Lemma 3.6.

To verify the second relation, let $e \in \mathcal{G}^1$ and note that $1_{s(e)}\delta_0 1_e\delta_e = 1_{s(e)}1_e\delta_e = 1_e\delta_e$ and $1_e\delta_e 1_{r(e)}\delta_0 = \beta_e(\beta_{e^{-1}}(1_e)1_{r(e)})\delta_e = \beta_e(1_{e^{-1}}1_{r(e)})\delta_e = 1_e 1_{er(e)}\delta_e = 1_e\delta_e$, where the second to last equality follows from item (2) of Proposition 3.9. Moreover, $1_{r(e)}\delta_0 1_{e^{-1}}\delta_{e^{-1}} = 1_{e^{-1}}\delta_{e^{-1}}$ and $1_{e^{-1}}\delta_{e^{-1}}1_{s(e)}\delta_0 = \beta_{e^{-1}}(\beta_e(1_{e^{-1}})1_{s(e)})\delta_{e^{-1}} = \beta_{e^{-1}}(1_e 1_{s(e)})\delta_{e^{-1}} = \beta_{e^{-1}}(1_e)\delta_{e^{-1}} = 1_{e^{-1}}\delta_{e^{-1}}$.

Next we verify the third relation of Definition 2.3. Let $e, f \in \mathcal{G}^1$. Then

$$1_{e^{-1}}\delta_{e^{-1}}1_f\delta_f = \beta_{e^{-1}}(1_e 1_f)\delta_{e^{-1}f}.$$

If $e \neq f$ then $1_e 1_f = 0$ and if $e = f$ then $\beta_{e^{-1}}(1_e 1_f)\delta_{e^{-1}f} = \beta_{e^{-1}}(1_e)\delta_0 = 1_{e^{-1}}\delta_0 = 1_{r(e)}\delta_0$.

To verify the last relation of Definition 2.3, note first that $1_e\delta_e 1_{e^{-1}}\delta_{e^{-1}} = 1_e\delta_0$ for each edge e. Now let v be a vertex such that $0 < |s^{-1}(v)| < \infty$. Then $X_v = \bigcup_{e\in s^{-1}(v)} X_e$, from where $1_v = \sum_{e\in s^{-1}(v)} 1_e$, and so

$$\sum_{e\in s^{-1}(v)} 1_e\delta_e 1_{e^{-1}}\delta_{e^{-1}} = \sum_{e\in s^{-1}(v)} 1_e\delta_0 = 1_v\delta_0.$$

So, by the universality of $L_R(\mathcal{G})$, there exists an R-homomorphism $\phi : L_R(\mathcal{G}) \to D \rtimes_\beta \mathbb{F}$ such that $\phi(p_A) = 1_A\delta_0$, $\phi(s_e) = 1_e\delta_e$, and $\phi(s_e^*) = 1_{e^{-1}}\delta_{e^{-1}}$ for each $A \in \mathcal{G}^0$ and each edge e.

Now we prove that ϕ is surjective. For each $a = a_1 \dots a_{|a|} \in W$ and $d = d_1 \dots d_{|d|} \in W$, we use the notation $\phi(s_a)$, $\phi(s_d^*)$, and $\phi(s_a s_d^*)$ to denote the elements $\phi(s_{a_1}) \dots \phi(s_{a_{|a|}})$, $\phi(s_{d_{|d|}}^*) \dots \phi(s_{d_1}^*)$, and $\phi(s_{a_1}) \dots \phi(s_{a_{|a|}})\phi(s_{d_{|d|}}^*) \dots \phi(s_{d_1}^*)$, respectively.

Claim 1. For each $a, d \in W$, $\phi(s_a)\phi(s_a^*) = 1_a\delta_0$, $\phi(s_d^*)\phi(s_d) = 1_{d^{-1}}\delta_0$, and $\phi(s_a s_d^*)$ $\phi(s_d s_a^*) = 1_{ad^{-1}}\delta_0$.

The equalities $\phi(s_a)\phi(s_a^*) = 1_a\delta_0$ and $\phi(s_d^*)\phi(s_d) = 1_{d^{-1}}\delta_0$ follow by induction on the lengths of a and d and from the first item of Proposition 3.9. To prove the other equality, write $a = eg$, where $|e| = 1$ and $|g| = |a| - 1$, and suppose by inductive arguments that $\phi(s_g)\phi(s_d^*)\phi(s_d)\phi(s_g)^* = 1_{gd^{-1}}\delta_0$. Then

$$\begin{aligned}\phi(s_a s_d^*)\phi(s_d s_a^*) &= \phi(s_e)\phi(s_g)\phi(s_d^*)\phi(s_d)\phi(s_g^*)\phi(s_e) = \phi(s_e)1_{gd^{-1}}\phi(s_e)\\ &= 1_e\delta_e 1_{gd^{-1}}1_{e^{-1}}\delta_{e^{-1}} = \alpha_e(1_{e^{-1}}1_{gd^{-1}})\delta_0 = 1_e 1_{egd^{-1}}\delta_0 = 1_{ad^{-1}}\delta_0,\end{aligned}$$

where the second to last equality follows from the first item of Proposition 3.9. So, Claim 1 is proved.

Claim 2. For each $b \in W$ and $A \in \mathcal{G}^0$, $\phi(s_b)\phi(p_A)\phi(s_b^*) = 1_{bA}\delta_0$.

https://doi.org/10.1017/S144678871900020X Published online by Cambridge University Press

For $|b| = 1$, note that $\phi(s_b)\phi(p_A)\phi(s_b^*) = \beta_b(1_{b^{-1}}1_A)\delta_0 = 1_b 1_{bA}\delta_0 = 1_{b_A}\delta_0$, where the second to last equality follows from item (2) of Proposition 3.9. Now, for $|b| > 1$, write $b = ed$ with $|e| = 1$ and $|d| = |b| - 1$. By inductive arguments,

$$\begin{aligned}\phi(s_b)\phi(p_A)\phi(s_b^*) &= \phi(s_e)\phi(s_d)\phi(p_A)\phi(s_d^*)\phi(s_e^*)\\ &= \phi(s_e)1_{dA}\delta_0\phi(s_e^*) = \beta_e(1_{e^{-1}}1_{dA})\delta_0 = 1_e 1_{edA}\delta_0 = 1_{bA}\delta_0,\end{aligned}$$

where the second to last equality follows by similar arguments to the ones used in the proof of item (2) of Proposition 3.9. So, Claim 2 is proved.

By Remark 3.8, to prove that ϕ is surjective, it is enough to prove that

$$\{1_A\delta_0, 1_c\delta_0, 1_{bA}\delta_0 : A \in \mathcal{G}^0, c \in \mathbb{F} \setminus \{0\}, b \in W\} \subseteq \mathrm{Im}(\phi)$$

and, for each $t \in \mathbb{F}$,

$$\{1_t 1_A\delta_t, 1_t 1_c\delta_t, 1_t 1_{bA}\delta_t : A \in \mathcal{G}^0, c \in \mathbb{F}, b \in W\} \subseteq \mathrm{Im}(\phi).$$

Claim 3. $\{1_A\delta_0, 1_c\delta_0, 1_{bA}\delta_0 : A \in \mathcal{G}^0, c \in \mathbb{F} \setminus \{0\}, b \in W\} \subseteq \mathrm{Im}(\phi)$.

Recall that for each $A \in \mathcal{G}^0$, $\phi(p_A) = 1_A\delta_0$. Moreover, for $c \in \mathbb{F} \setminus \{0\}$ with $c = ad^{-1}$, where $a, d \in W \cup \{0\}$, we get by Claim 1 that $1_c\delta_0 \in \mathrm{Im}(\phi)$ (for all the other $c \in \mathbb{F} \setminus \{0\}$ we also have $1_c\delta_0 \in \mathrm{Im}(\phi)$, since $1_c = 0$). To finish, notice that, by Claim 2, we get that $1_{bA}\delta_0 \in \mathrm{Im}(\phi)$ for each $b \in W$ and $A \in \mathcal{G}^0$. So, Claim 3 is proved.

Claim 4. For each $t \in \mathbb{F} \setminus \{0\}$,

$$\{1_t 1_A\delta_t, 1_t 1_c\delta_t, 1_t 1_{bA}\delta_t : A \in \mathcal{G}^0, c \in \mathbb{F} \setminus \{0\}, b \in W\} \subseteq \mathrm{Im}(\phi).$$

First, for $e \in W$, with $|e| = 1$, recall that $1_e\delta_e = \phi(e)$. Now let $c \in W$ with $|c| > 1$, write $c = ed$ with $|e| = 1$, and suppose (by inductive arguments on $|c|$) that $\phi(d) = 1_d\delta_d$. Then

$$\phi(s_c) = \phi(s_e)\phi(s_d) = 1_e\delta_e 1_d\delta_d = \beta_e(1_{e^{-1}}1_d)\delta_{ed} = 1_{e1_{ed}}\delta_{ed} = 1_c\delta_c,$$

where the second to last equality follows from item (1) of Proposition 3.9. Analogously, we get that $\phi(s_d^*) = 1_{d^{-1}}\delta_{d^{-1}}$ for each $d \in W$. Now, for $c, d \in W$,

$$\phi(s_c)\phi(s_d^*) = 1_c\delta_c 1_{d^{-1}}\delta_{d^1} = \beta_c(1_{c^{-1}}1_{d^{-1}})\delta_{cd^{-1}} = 1_c 1_{cd^{-1}}\delta_{cd^{-1}} = 1_{cd^{-1}}\delta_{cd^{-1}},$$

where, again, the second to last equality follows from item (1) of Proposition 3.9. So, we get $1_t\delta_t \in \mathrm{Im}(\phi)$ for each $t \in \mathbb{F} \setminus \{0\}$.

Now, for $t, c \in \mathbb{F} \setminus \{0\}$, $b \in W$, and $A \in \mathcal{G}^0$, note that $1_t 1_{bA}\delta_t = 1_{bA}\delta_0 1_t\delta_t \in \mathrm{Im}(\phi)$ and similarly one shows that $1_t 1_A\delta_t, 1_t 1_c\delta_t \in \mathrm{Im}(\phi)$. So, we get that ϕ is surjective.

It remains to show that ϕ is injective. To prove this, we will use the graded uniqueness theorem; see [12, Theorem 3.2]. For each integer number n, define

$$F_n = \mathrm{span}\{f_{ab^{-1}}\delta_{ab^{-1}} : f_{ab^{-1}} \in D_{ab^{-1}},\ a, b \in W \cup \{0\} \text{ and } |a| - |b| = n\}.$$

Note that $D \rtimes_\beta \mathbb{F}$ is $\mathbb{Z}$-graded by the gradation $\{F_n\}_{n\in\mathbb{Z}}$. Moreover, $L_R(\mathcal{G})$ is a $\mathbb{Z}$-graded ring with the grading

$$L_R(\mathcal{G})_n = \mathrm{span}\{s_a p_A s_b^* : a, b \in \mathcal{G}^*, A \in \mathcal{G}^0 \text{ and } |a| - |b| = n\}$$

introduced in [12]. It is easy to see that ϕ is a graded ring homomorphism. Since $X_A \neq \emptyset$, then $\phi(\tau p_A) = \tau 1_A \neq 0$ for each $A \in \mathcal{G}^0$ and $\tau \in R \setminus \{0\}$. It follows from [12, Theorem 3.2] that ϕ is injective and hence an isomorphism. □

https://doi.org/10.1017/S144678871900020X Published online by Cambridge University Press

4. Simplicity and maximal commutativity

In this section we use the realization of ultragraph Leavitt path algebras as partial skew group rings to describe simplicity criteria for these algebras. Recall that from [7, Theorem 2.3], the algebra $D \rtimes_\beta \mathbb{F}$ is simple if, and only if, D is $\mathbb{F}$-simple and $D\delta_0$ is maximal commutative in $D \rtimes_\beta \mathbb{F}$. Aiming at the simplicity criteria given for ultragraph C*-algebras in [16], we will characterize maximal commutativity in terms of Condition (L) and $\mathbb{F}$ simplicity in terms of hereditary and saturated subcollections $\mathcal{G}^0$.

Recall that a cycle in an ultragraph $\mathcal{G}$ is a path $\alpha = e_1 \dots e_{|\alpha|}$ with $|\alpha| \geq 1$ and $s(\alpha) \in r(\alpha)$, and an exit for α is an edge e with $s(e) = s(e_i)$ for some $i \in \{1, \dots, |\alpha|\}$ and $e \neq e_i$. The ultragraph $\mathcal{G}$ satisfies Condition (L) if each cycle $\alpha = e_1 \dots e_{|\alpha|}$ has an exit or if $r(e_i)$ contains a sink for some i.

Before we state our next result, we recall the notion of maximal commutativity: the centralizer of a nonempty subset S of a ring R, which we denote by $C_R(S)$, is the set of all elements of R that commute with each element of S. If $C_R(S) = S$ holds, then S is said to be a *maximal commutative subring* of R.

Theorem 4.1. *Let $\mathcal{G}$ be an ultragraph. Then $D\delta_0$ is maximal commutative in $D \rtimes_\beta \mathbb{F}$ if, and only if, $\mathcal{G}$ satisfies condition (L).*

Proof. First suppose that $\mathcal{G}$ satisfies condition (L). Suppose, by contradiction, that there exists $x = \sum a_t\delta_t$, with some $t \neq 0$, that commutes with $a\delta_0$ for all $a \in D$. Then there exists $t \in \mathbb{F} \setminus \{0\}$, and $a_t \in D_t$ with $a_t \neq 0$, such that $a_t\delta_t a_0\delta_0 = a_0\delta_0 a_t\delta_t$ for each $a_0 \in D$. From the last equality,

$$\beta_t(\beta_{t^{-1}}(a_t)a_0) = a_t a_0 \tag{1}$$

for each $a_0 \in D_0$. Since $a_t \neq 0$, then $t = a$, $t = b^{-1}$, or $t = ab^{-1}$ with $a, b \in W$.

Notice that, since

$$D_t = \operatorname{span}\{1_t 1_c, 1_t 1_{bA}, 1_t 1_A : c \in \mathbb{F}, b \in W, A \in \mathcal{G}^0\},$$

then, for each $\xi \in X_t$ with $|\xi| = \infty$, there exists an $m \in \mathbb{N}$ such that, if $\eta \in X_t$ and $\eta_1\eta_2 \dots \eta_m = \xi_1\xi_2 \dots \xi_m$, then $a_t(\eta) = a_t(\xi)$.

We now divide the proof into three cases.

Case 1. Suppose that $t \in W$.

If we take $a_0 = 1_{t^{-1}}$ in Equation (1), we get that $a_t = a_t 1_{t^{-1}}$. Hence, the support of a_t is contained in $X_t \cap X_{t^{-1}}$ and therefore t is a closed path. If we take $a_0 = 1_t 1_{t^{-1}}$, then, from Equation (1), we have that $\beta_t(\beta_{t^{-1}}(a_t)1_t) = a_t 1_t 1_{t^{-1}} = a_t$ and, from Remark 3.8 and Proposition 3.9, we get $\beta_t(\beta_{t^{-1}}(a_t)1_t) = a_t 1_{tt}$. Therefore, $a_t 1_{tt} = a_t$. With the same arguments, if we take $a_0 = 1_{t^2}$, we get $a_t 1_{t^3} = a_t$ and inductively we get $a_t 1_{t^n} = a_t$ for each $n \in \mathbb{N}$.

Let $\xi \in X_t$ be such that $a_t(\xi) \neq 0$. Then $a_t(\xi)1_{t^n}(\xi) \neq 0$ for each $n \in \mathbb{N}$ and so $|\xi| = \infty$. Let $m \in \mathbb{N}$ be such that if $\eta \in X_t$ and $\eta_1 \dots \eta_m = \xi_1 \dots \xi_m$, then $a_t(\eta) = a_t(\xi)$.

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Since $\mathcal{G}$ satisfies condition (L), the closed path $t = t_1 \ldots t_{|t|}$ either has an exit or some $r(t_i)$ contains a sink.

Suppose first that t has an exit, that is, there exists an edge e such that $s(e) \in r(t_i)$ for some i and $e \neq t_{i+1}$. Let $k \in \mathbb{N}$ be such that $k|t| \geq m$ and let η be such that $\eta = t^k t_1 t_2 \ldots t_i e y$ (for some y). Then we get that $0 \neq a_t(\xi) = a_t(\eta) = (a_t 1_{t^{k+1}})(\eta) = 0$, which is a contradiction.

Now suppose that $r(t_i)$ contains a sink v for some i. Then, again, let $k \in \mathbb{N}$ be such that $k|t| \geq m$ and let $\eta = (t^k t_1 t_2 \ldots t_i, v)$, which is an element of X_t. Then we have that $0 \neq a_t(\xi) = (a_t 1_{t^{k+1}})(\xi) = (a_t 1_{t^{k+1}})(\eta) = 0$, which is also a contradiction.

So, we conclude that $t \notin W$.

Case 2. $t = d^{-1}$ *with* $d \in W$.

From Equation (1), we get that $\beta_{d^{-1}}(\beta_d(a_{d^{-1}})a_0) = a_{d^{-1}} a_0$ and so $\beta_d(a_{d^{-1}})a_0 = \beta_d(a_{d^{-1}} a_0)$. Let $c_d = \beta_d(a_{d^{-1}})$. Then $\beta_{d^{-1}}(c_d) = a_{d^{-1}}$ and so we get the equality

$$\beta_d(\beta_{d^{-1}}(c_d)a_0) = c_d a_0$$

for each $a_0 \in D_0$. Now, by Case 1, we get a contradiction and hence it is not possible that $t = d^{-1}$ with $d \in W$.

Case 3. $t = cd^{-1}$ *with* $c, d \in W$.

As in Case 1, we get that $a_t = a_t 1_{t^n}$ for each $n \in \mathbb{N}$. Hence, since $a_t \neq 0$, we have that $X_{t^n} \neq \emptyset$ for each n. Therefore, either $c = db$ or $d = cb$ with $b \in W$.

If $c = db$, then $t^n = db^n d^{-1}$ and so b is a closed path. Let $\xi \in X_t$ with $|\xi| = \infty$ and $a_t(\xi) \neq 0$. Proceeding from this point as in Case 1, we get a contradiction.

If $d = cb$ for some $b \in W$, then we also get a similar contradiction, by considering the equality $\beta_{t^{-1}}(\beta_t(u_{t^{-1}})a_0) = u_{t^{-1}} a_0$ obtained from Equation (1), where $u_{t^{-1}} = \beta_{t^{-1}}(a_t)$.

So, we proved that if $\mathcal{G}$ satisfies condition (L), then D is maximal commutative in $D \rtimes_\beta \mathbb{F}$. Next we prove the converse.

Suppose that $\mathcal{G}$ does not satisfy condition (L). Then there exist a closed path $t = t_1 \ldots t_{|t|}$ in $\mathcal{G}$ such that t has no exit and $r(t_i)$ contains exactly one vertex for each t_i. We show that $1_t\delta_t$ commutes with $D_0\delta_0$. By Remark 3.8, it is enough to show that $1_t\delta_t$ commutes with $1_c\delta_0$ for each $c \in \mathbb{F} \setminus \{0\}$, and with $1_A\delta_0$ and $1_{bA}\delta_0$ for each $A \in \mathcal{G}^0$ and $b \in W$.

Let $A \in \mathcal{G}^0$. If $r(t) = s(t) \in A$, then $1_A\delta_0 1_t\delta_t = 1_t\delta_t = \beta_t(1_{t^{-1}})\delta_t = \beta_t(1_{t^{-1}} 1_A)\delta_t = \beta_t(\beta_{t^{-1}}(1_t)1_A)\delta_t = 1_t\delta_t 1_A\delta_0$ and, if $s(t) = r(t) \notin A$, then $1_A\delta_0 1_t\delta_t = 0 = 1_t\delta_t 1_A\delta_0$.

Now let $A \in \mathcal{G}^0$ and $b \in W$. Note that $1_t\delta_t 1_{bA}\delta_0 = \beta_t(1_{t^{-1}} 1_{bA})\delta_t$ and $1_{bA}\delta_0 1_t\delta_t = 1_{bA} 1_t\delta_t$. If $s(b) \notin r(t)$, then $1_{t^{-1}} 1_{bA} = 0 = 1_t 1_{bA}$ and we are done. Suppose that $s(b) \in r(t)$. Then, by Proposition 3.9, $\beta_t(1_{t^{-1}} 1_{bA}) = 1_t 1_{tbA}$. So, it remains to show that $1_t 1_{tbA} = 1_t 1_{bA}$. Notice that $X_t = \{\xi\}$, where ξ is the infinite path $\xi = tt\ldots$. Then to verify the desired equality it is enough to show that $\xi \in X_{tbA}$ if, and only if, $\xi \in X_{bA}$. Suppose that $\xi \in X_{tbA}$. Then $\xi = tby$, where y is a path such that $s(y) \in A$. Therefore, there exists an $n \in \mathbb{N}$ such that $b = t^n t_1 \ldots t_i$ for some i and note that $s(y) = r(t_i)$. Hence,

$$\xi = tby = tt^n t_1 \ldots t_i y = t^n t_1 \ldots t_i t_{i+1} \ldots t_{|t|} t_1 \ldots t_i y = b t_{i+1} \ldots t_{|t|} t_1 \ldots t_i y.$$

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Now note that $bt_{i+1}\dots t_{|t|}t_1\dots t_i y\in X_{bA}$, since $s(t_{i+1})=r(t_i)=s(y)\in A$. Similarly, one shows that if $\xi\in X_{bA}$, then $\xi\in X_{tbA}$. So, $1_t1_{tbA}=1_t1_{bA}$.

Finally, we show that $1_t\delta_t1_c\delta_0=1_c\delta_01_t\delta_t$ for each $c\in\mathbb{F}\setminus\{0\}$. To prove this, it is sufficient to show that $\beta_t(1_{t^{-1}}1_c)=1_t1_c$ for each $c\in\mathbb{F}\setminus\{0\}$. By Proposition 3.9, we have that $\beta_t(1_{t^{-1}}1_c)=1_t1_{tc}$ and hence we have to show that $1_t1_{tc}=1_t1_c$. Notice that to prove this last equality it is enough to show that $\xi=tt\dots$ is an element of X_{tc} if, and only if, $\xi\in X_c$. This follows by arguments similar to the previous case, splitting the proof into cases depending on whether $c=a$, $c=b$, or $c=ab^{-1}$ with $a,b\in W$. □

The next proposition will be useful in the characterization of $\mathbb{F}$-simplicity of D.

Proposition 4.2. *Let $x_0\delta_0$ be a nonzero element of $D\delta_0$ and let I be the ideal generated by $x_0\delta_0$ in $D\rtimes_\beta\mathbb{F}$. Then there exist a vertex $v\in G^0$ and a nonzero element $h\in R$ such that $(h1_v)\delta_0\in I$.*

Proof. First note that by Remark 3.8,

$$x_0=\sum_{i=1}^m\alpha_i1_{a_ib_i^{-1}}+\sum_{j=1}^n\beta_j1_{e_jA_j}+\sum_{k=1}^p\gamma_k1_{B_k}$$

with $a_i,b_i,e_j\in W$ and $a_ib_i^{-1}\neq 0$, $A_j,B_k\in\mathcal{G}^0$, and $\alpha_i,\beta_j,\gamma_k\in R$. Let $A=\{s(a_i):1\le i\le m\}\cup\{s(e_j):1\le j\le n\}\bigcup_{k=1}^p B_k$, which is an element of $\mathcal{G}^0$, and note that $1_Ax_0=x_0$. Let $\xi\in X$ be such that $x_0(\xi)\neq 0$ and let $v=s(\xi)$. Then $v\in A$ and so $1_v(\xi)x_0(\xi)=1_A(\xi)x_0(\xi)=x_0(\xi)\neq 0$. Therefore, $1_vx_0\neq 0$.

If v is a sink, then $1_vx_0=\sum_{k=1}^p\gamma_k1_v1_{B_k}=\sum_{k\in\{1\dots p\}:v\in B_k}\gamma_k1_v=h1_v$. So, $(h1_v)\delta_0\in I$.

Now suppose that v is not a sink. Let $M=\max\{|a_i|,|e_j|:1\le i\le m,1\le j\le n\}$. Note that since v is not a sink, then we can write

$$X_v=\dot{\bigcup_{c\in J}}X_{c\{u\}}\dot{\bigcup_{d\in L}}X_d,$$

where J is the set of all elements c of W such that $s(c)=v$, $|c|<M$, and such that there is a sink $u\in r(c)$, and L is the set of all the elements d such that $s(d)=v$ and $|d|=M+1$.

Since $1_vx_0\neq 0$, then $1_{c\{u\}}x_0\neq 0$ for some $c\in J$ and some sink $u\in r(c)$, or $1_dx_0\neq 0$ for some $d\in L$.

Suppose that $1_{c\{u\}}x_0\neq 0$. Note that for each $i\in\{1,\dots,m\}$, $j\in\{1,\dots,n\}$, and $k\in\{1,\dots,p\}$, we have that $1_{c\{u\}}1_{a_ib_i^{-1}}=0$ or $1_{c\{u\}}1_{a_ib_i^{-1}}=1_{c\{u\}}$, $1_{c\{u\}}1_{e_jA_j}=0$ or $1_{c\{u\}}1_{e_jA_j}=1_{c\{u\}}$, and $1_{c\{u\}}1_{B_k}=0$ or $1_{c\{u\}}1_{B_k}=1_{c\{u\}}$. Then

$$0\neq 1_{c\{u\}}x_0=\Bigg(\sum_{i:1_{c\{u\}}1_{a_ib_i^{-1}}\neq 0}\alpha_i+\sum_{j:1_{c\{u\}}1_{e_jA_j}\neq 0}\beta_j+\sum_{k:1_{c\{u\}}1_{B_k}\neq 0}\gamma_k\Bigg)1_{c\{u\}}=h1_{c\{u\}}.$$

Therefore, $(h1_{c\{u\}})\delta_0\in I$. Since I is an ideal, then $1_{c^{-1}}\delta_{c^{-1}}h1_{c\{u\}}\delta_01_c\delta_c=h\beta_{c^{-1}}(1_c1_{c\{u\}})\delta_0=(h1_u)\delta_0$ belongs to I.

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Now assume that $1_d x_0 \neq 0$ for some $d \in L$. Since $|d| > |a_i|$, then $1_d 1_{a_i b_i^{-1}} = 1_d$ or $1_d 1_{a_i b_i^{-1}} = 0$ for each $i \in \{1, \ldots, m\}$, and similarly $1_d 1_{e_j} A_j = 1_d$ or $1_d 1_{e_j} A_j = 0$ for each $j \in \{1, \ldots, n\}$. Moreover, $1_d 1_{B_k} = 1_d$ if $s(d) \in B_k$ and $1_d 1_{B_k} = 0$ if $s(b) \notin B_k$ for each $k \in \{1, \ldots, p\}$. Then

$$0 \neq 1_d x_0 = \Big(\sum_{i : 1_d 1_{a_i b_i^{-1}} \neq 0} \alpha_i + \sum_{j : 1_d 1_{e_j A_j} \neq 0} \beta_j + \sum_{k : 1_d 1_{B_k} \neq 0} \gamma_k \Big) 1_d = h 1_d$$

and so $(h 1_d)\delta_0 \in I$. Hence, $(h 1_{r(d)})\delta_0 = 1_{d^{-1}}\delta_{d^{-1}} h 1_d \delta_0 1_d \delta_d$ belongs to I. Then, for each vertex $w \in r(d)$, we get that $(h 1_w)\delta_0 = 1_w \delta_0 (h 1_{r(d)})\delta_0$ belongs to I. □

As a consequence of the above proposition, we can provide a new proof of the Cuntz–Krieger uniqueness theorem for Leavitt path algebras of ultragraphs.

Corollary 4.3. *Let $\mathcal{G}$ be an ultragraph that satisfies Condition (L), let R be a unital commutative ring, and let $\pi : L_R(\mathcal{G}) \to S$ be a homomorphism such that $\pi(r p_A) \neq 0$ for each $A \in \mathcal{G}^0$ and nonzero $r \in R$. Then π is injective.*

Proof. Let $I = \ker(\pi)$ and suppose that $I \neq 0$. Since $\mathcal{G}$ satisfies Condition (L), then, by Theorem 4.1, $D\delta_0$ is maximal commutative. Therefore, by [7, Theorem 2.1], $I \cap D\delta_0 \neq 0$. Let $0 \neq x_0\delta_0 \in I \cap D_0\delta_0$. By Proposition 4.2, there exist a nonzero $h \in R$ and a vertex v such that $(h 1_v)\delta_0 \in I$, which is a contradiction. Therefore, $\ker(\pi) = 0$. □

As in the C* setting, the characterization of simplicity of ultragraph Leavitt path algebras relies on the notion of hereditary and saturated collections. For the reader's convenience, we recall these below.

Definition 4.4. Let $\mathcal{G}$ be an ultragraph. A subcollection $H \subseteq \mathcal{G}^0$ is called hereditary if:

(1) $s(e) \in H$ implies that $r(e) \in H$ for each $e \in \mathcal{G}^1$;
(2) $A \cup B \in H$ for all $A, B \in H$;
(3) $A \in H, B \in \mathcal{G}^0$ and $B \subseteq A$ imply that $B \in H$.

Moreover, H is called saturated if for any $v \in G^0$ with $0 < |s^{-1}(v)| < \infty$,

$$\{r(e) : e \in \mathcal{G}^1 \text{ and } s(e) = v\} \subseteq H \text{ implies that } v \in H.$$

The next lemma is key in the characterization of $\mathbb{F}$-simplicity in terms of existence of hereditary and saturated subcollections of $\mathcal{G}^0$.

Lemma 4.5. *Let R be a unital commutative domain and let I be an $\mathbb{F}$-invariant ideal of D_0. Then the collection*

$$H = \{A \in \mathcal{G}^0 : h 1_A \in I \textit{ for some nonzero } h \in R\}$$

is hereditary and saturated.

https://doi.org/10.1017/S144678871900020X Published online by Cambridge University Press

Proof. First we show that H is hereditary. Let $e \in \mathcal{G}^1$ be such that $s(e) \in H$ and let $h \in R$ be a nonzero element such that $h1_{s(e)} \in I$. Then $h1_e = h1_e1_{s(e)} \in I \cap D_e$ and, since I is $\mathbb{F}$-invariant, we have that $h1_{r(e)} = h\beta_{e^{-1}}(1_e) \in I$ and so $r(e) \in H$. Let $A, B \in H$ and let h, k be nonzero elements in R such that $h1_A \in I$ and $k1_B \in I$. Then $hk \neq 0$, since R is a domain. Moreover, $hk1_{A\cup B} = hk1_A + hk1_B - hk1_A1_B \in I$, since I is an ideal. Finally, let $A \in H$, and $B \in \mathcal{G}^0$ with $B \subseteq A$. Take a nonzero element $h \in R$ such that $h1_A \in I$. Note that $h1_B = h1_B1_A \in I$. Hence, $B \in H$ and H is hereditary.

Now we show that H is saturated. Let $v \in G^0$ be such that $0 < |s^{-1}(v)| < \infty$. Suppose that for each $e \in s^{-1}(v)$, $r(e) \in H$. Then for each $e \in s^{-1}(v)$ there is a nonzero $h_e \in R$ such that $h_e1_{r(e)} \in I$. Since I is $\mathbb{F}$-invariant, then $h_e1_e = h_e\beta_e(1_{e^{-1}}) = \beta_e(h_e1_{r(e)}) \in I$. Define $h = \prod_{e\in s^{-1}(v)} h_e$, which is nonzero since R is a domain. Then $h1_e \in I$ for each $e \in s^{-1}(v)$ and so $h1_v = \sum_{e\in s^{-1}(v)} h1_e \in I$, from where we get that $v \in H$ and H is saturated. □

We can now describe the relation between $\mathbb{F}$-simplicity of D and hereditary and saturated subcollections of $\mathcal{G}^0$.

Theorem 4.6. *Let R be a field. Then the algebra D is $\mathbb{F}$-simple if, and only if, the only hereditary and saturated subcollections of $\mathcal{G}^0$ are $\emptyset$ and $\mathcal{G}^0$.*

Proof. Suppose first that the only saturated and hereditary subcollections of $\mathcal{G}^0$ are $\emptyset$ and $\mathcal{G}^0$. Let $I \subseteq D$ be a nonzero $\mathbb{F}$-invariant ideal. We show that $I = D$. Let J be the set of all finite sums $\sum a_t\delta_t$ with $a_t \in D_t \cap I$. Notice that J is nonzero and is an ideal of $D \rtimes_\beta \mathbb{F}$, since I is F-invariant. Then, by Proposition 4.2, there exist a $v \in G^0$ and a nonzero $h \in R$ such that $h1_v\delta_0 \in J$. Since $J \cap D_0\delta_0 = I\delta_0$, then $h1_v \in I$. Let $H = \{A \in \mathcal{G}^0 : h1_A \in I \text{ for some nonzero } h \in R\}$. By Lemma 4.5, H is hereditary and saturated (and $H \neq \emptyset$, since $v \in H$) and hence $H = \mathcal{G}^0$. Then, for each $A \in \mathcal{G}^0$, there exists a nonzero element $h \in R$ such that $h1_A \in I$. Since R is a field, we have that $1_A \in I$ and it follows that $I = D_0$.

Now suppose that D_0 is $\mathbb{F}$-simple. Let $H \subseteq \mathcal{G}^0$ be nonempty, hereditary, and saturated. We need to show that $H = \mathcal{G}^0$.

Let I be the ideal in $D \rtimes_\beta \mathbb{F}$ generated by the set $\{1_A\delta_0 : A \in H\}$, that is, I is the linear span of all the elements of the form $a_r\delta_r1_A\delta_0a_s\delta_s$ with $r, s \in \mathbb{F}$, $a_r \in D_r$, and $a_s \in D_s$. Let $J = \{a : a\delta_0 \in D\delta_0 \cap I\}$, which is a nonzero ideal of D. Moreover, J is $\mathbb{F}$ invariant, since if $a_t \in J \cap D_t$, then $a_t\delta_0 \in I$ and $\beta_{t^{-1}}(a_t)\delta_0 = 1_{t^{-1}}\delta_{t^{-1}}a_t\delta_01_t\delta_t \in I$. Since D is $\mathbb{F}$-simple, then $J = D$.

Our next step is to show that $\{u\} \in H$ for each vertex $u \in G^0$.

Let $u \in G^0$. Then we can write

$$1_u\delta_0 = \sum_t x_t\delta_t1_{A_t}\delta_0y_{t^{-1}}\delta_{t^{-1}} = \sum_t \beta_t(\beta_{t^{-1}}(x_t)1_{A_t}y_{t^{-1}})\delta_0$$

with $A_t \in H$. Multiplying the above equation by $1_u\delta_0$,

$$1_u = \sum_{t\in T} 1_u\beta_t(\beta_{t^{-1}}(x_t)1_{A_t}y_{t^{-1}}), \tag{2}$$

https://doi.org/10.1017/S144678871900020X Published online by Cambridge University Press

where $T = \{t : 1_u\beta_t(\beta_{t^{-1}}(x_t)1_{A_t}y_{t^{-1}}) \neq 0\}$. In particular, for each $t \in T$, we have that $1_u 1_t \neq 0$ and $1_{A_t}1_{t^{-1}} \neq 0$.

If $u \in r(b)$ for some $b \in W$ with $\{s(b)\} \in H$, then $\{u\} \in H$, since H is hereditary. If $0 < |s^{-1}(u)| < \infty$ and $r(e) \in H$ for each $e \in s^{-1}(u)$, then $\{u\} \in H$, since H is saturated. So, we are left with the cases when there is no path b with $\{s(b)\} \in H$ and $u \in r(b)$ and $s^{-1}(u) = \emptyset$, $|s^{-1}(u)| = \infty$, or $0 < |s^{-1}(u)| < \infty$ but $r(e) \notin H$ for some $e \in s^{-1}(u)$. Since there is no path $b \in W$ such that $\{s(b)\} \in H$ and $u \in r(b)$, then, for each $b \in W$,

$$1_u\beta_{b^{-1}}(\beta_b(x_{b^{-1}})1_A y_b) = 0$$

(notice that if $b \in W$ is such that $u \in r(b)$, then, since H is hereditary, $s(b) \notin A$ and hence $1_A 1_{s(b)} y_b = 0$). So, each nonzero element $t \in T$ is of the form $t = ab^{-1}$ with $a \in W$ and $b \in W \cup \{0\}$.

Case 1. $s^{-1}(u) = \emptyset$ *and there is no path* b *with* $s(b) \in H$ *and* $u \in r(b)$.

For each $t = ab^{-1} \in \mathbb{F}$ with $a \in W$ and $b \in W \cup \{0\}$, we have that $1_u 1_t = 0$, since u is a sink, and so $t = ab^{-1} \notin T$. So, $T = \{0\}$ and then $1_u = 1_u x_0 1_{A_0} y_0$ with $A_0 \in H$. Therefore, $u \in A_0$ and so $\{u\} \in H$.

Case 2. $|s^{-1}(u)| = \infty$ *and there is no path* b *with* $\{s(b)\} \in H$ *and* $u \in r(b)$.

Suppose that $0 \notin T$. Then each $t \in T$ is of the form $t = ab^{-1}$ with $a \in W$ and $b \in W \cup \{0\}$. Since $|s^{-1}(u)| = \infty$, then there exists $\xi \in X$ such that $s(\xi) \neq s(a)$ for each $ab^{-1} \in T$. So, we get that $1 = 1_u(\xi) = \sum_{t \in T} 1_u\beta_t(\beta_{t^{-1}}(x_t)1_{A_t}y_{t^{-1}})(\xi) = 0$, which is a contradiction. Hence, $0 \in T$ and so $1_u x_0 1_{A_0} y_0 \neq 0$. Therefore, $\{u\} \subseteq A_0 \in H$ and, since H is hereditary, we have that $\{u\} \in H$.

Note that it follows from Cases 1 and 2 and by the fact that H is hereditary, that if u is a vertex such that $|s^{-1}(u)| = 0$ or $|s^{-1}(u)| = \infty$, then $\{u\} \in H$.

Case 3. $0 < |s^{-1}(u)| < \infty$, *there is an edge* $e \in s^{-1}(u)$ *with* $r(e) \notin H$, *and there is no path* b *with* $\{s(b)\} \in H$ *and* $u \in r(b)$.

Let us first prove the following claim.

Claim. *If* e *is an edge such that* $r(e) \notin H$, *then there is a vertex* $v \in r(e)$ *such that* $\{v\} \notin H$.

Let $w = s(e)$. Notice that $\{w\} \notin H$, since H is hereditary. Also note that there is no path d with $s(d) \in \{H\}$ and $w \in r(d)$. Therefore, since $J = D$, proceeding as we did for u,

$$1_w = \sum_{t \in S} 1_w\beta_t(\beta_{t^{-1}}(x'_t)1_{A_t}y'_{t^{-1}}),$$

where $A_t \in H$ for each $t \in S$, each $1_w\beta_t(\beta_{t^{-1}}(x'_t)1_{A_t}y'_{t^{-1}})$ is nonzero, $0 \notin S$ because $\{w\} \notin H$, and each t is of the form $t = ab^{-1}$ with $a \in W$ and $b \in W \cup \{0\}$.

For each $t = ab^{-1} \in T$, let $c_t = 1_w\beta_t(\beta_{t^{-1}}(x'_t)1_{A_t}y'_{t^{-1}})$, so that

$$1_w = \sum_{t \in S} c_t. \tag{3}$$

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Since $1_w 1_t \neq 0$, then $w = s(a)$ and, since $1_{A_t} 1_{t^{-1}} \neq 0$, we have that $\{s(b)\} \subseteq A_t \in H$. Since H is hereditary, then $\{s(b)\} \in H$ and therefore $r(b) \in H$ and also $r(b) \cap r(a) \in H$. For $t = a \in W$, we get $A_t \cap r(t) \in H$.

By multiplying Equation (3) on the left-hand side by $1_{e^{-1}}\delta_{e^{-1}}$ and by $1_e\delta_e$ on the right-hand side,

$$1_{r(e)} = \sum_S \beta_{e^{-1}}(1_e c_t). \tag{4}$$

Notice that for $t = a_1 \ldots a_{|a|} b^{-1} \in S$ with $a_1 \neq e$, $\beta_{e^{-1}}(1_e c_t) = 0$. Let $M = \max\{|a| : ab^{-1} \in S \text{ and } a_1 = e\}$ and let $S_i = \{ab^{-1} \in S : |a| = i \text{ and } a_1 = e\}$ for $1 \leq i \leq M$. In particular, note that each element of S_1 is of the form $t = eb^{-1}$ with $b \in W \cup \{0\}$.

If $e \notin S_1$, define

$$A_1 = \bigcup_{ab^{-1} \in S_1} r(e) \cap r(b)$$

and, if $e \in S_1$, define

$$A_1 = \Big(\bigcup_{ab^{-1} \in S_1, b \neq 0} r(e) \cap r(b) \Big) \cup (r(e) \cap A_e).$$

Notice that $A_1 \subseteq r(e)$ and that $A_1 \in H$, since $r(e) \cap r(b) \in H$ for each $eb^{-1} \in S_1$ and $r(e) \cap A_e \in H$.

From Equation (4), we get $1_{r(e)} = \sum_{i=1}^M \sum_{t \in S_i} \beta_{e^{-1}}(1_e c_t)$.

Now we show that $M > 1$. Seeking a contradiction, suppose that $M = 1$. Then

$$1_{r(e)} = \sum_{eb^{-1} \in S_1} \beta_{e^{-1}}(1_e c_{ab^{-1}}).$$

Since $A_1 \subseteq r(e)$, $A_1 \in H$, and $r(e) \notin H$, we have that A_1 is a proper subset of $r(e)$. So, there is a vertex v such that $v \in r(e) \setminus A_1$. Let $\xi \in X$ be such that $s(\xi) = v$ (notice that by the paragraph just above the statement of Case 3, v is not a sink). Then, for each $t = eb^{-1} \in S_1$,

$$1_{eb^{-1}}(e\xi) = 1_{e(r(e) \cap r(b))}(e\xi) = 0,$$

since $r(e) \cap r(b) \subseteq A_1$. Therefore,

$$1 = 1_{r(e)}(\xi) = \sum_{eb^{-1} \in S_1} \beta_{e^{-1}}(1_e c_{eb^{-1}})(\xi) = \sum_{eb^{-1} \in S_1} 1_{eb^{-1}} c_{eb^{-1}}(e\xi) = 0,$$

which is a contradiction. Therefore, $M > 1$.

Recall now that for each $ab^{-1} \in S_2 \cup \cdots \cup S_M$, the element a is of the form $a = a_1 a_2 \ldots a_{|a|} = e a_2 \ldots a_{|a|}$. We want to show that $\{s(a_2)\} \notin H$ for some $ab^{-1} \in S_2 \cup S_M$. Again seeking a contradiction, suppose that $\{s(a_2)\} \in H$ for each $ab^{-1} \in S_2 \cup \cdots \cup S_M$. Let A_2 be the set of all those vertices (the vertices $s(a_2)$). Notice that $A_2 \in H$ (since we are supposing that each $\{s(a_2)\} \in H$ and H is hereditary) and that $A_2 \subseteq r(e)$ (since $s(a_2) \in r(a_1) = r(e)$). So, we get that $A_1 \cup A_2 \subseteq r(e)$ and, since $A_1 \cup A_2 \in H$ and $r(e) \notin H$, there exist a vertex $v_0 \in r(e) \setminus (A_1 \cup A_2)$. Let $\xi \in X$with $s(\xi) = v_0$.

For each $eb^{-1} \in S_1$, we get $1_{eb^{-1}}(e\xi) = 0$, since $s(\xi) \notin A_1$, and, for each $ea_2 \dots a_{|a|}b^{-1} \in S_2 \cup \dots \cup S_M$, we get $1_{ea_2 \dots a_{|a|}b^{-1}}(e\xi) = 0$, since $s(\xi) \neq s(a_2)$ (because $s(\xi) \notin A_2$). Therefore,

$$1 = 1_{r(e)}(\xi) = \sum_{i=1}^{M} \sum_{ab^{-1} \in S_i} \beta_{e^{-1}}(1_e c_{ab^{-1}})(\xi) = 0,$$

which is a contradiction.

So, there is an element $ab^{-1} \in S_2 \cup \dots \cup S_M$ (where $a = ea_2 \dots a_{|a|}$) with $\{s(a_2)\} \notin H$. Since $s(a_2) \in r(e)$, we proved the claim.

Now we prove Case 3.

First write 1_u as in Equation (2), that is,

$$1_u = \sum_{t \in T} 1_u \beta_t(\beta_{t^{-1}}(x_t) 1_{A_t} y_{t^{-1}}),$$

where $1_u \beta_t(\beta_{t^{-1}}(x_t) 1_{A_t} y_{t^{-1}}) \neq 0$.

To show that $\{u\} \in H$, it is enough to show that $0 \in T$, because in this case $0 \neq 1_u 1_{A_0}$, which implies that $u \in A_0$ and, since $A_0 \in H$, then $\{u\} \in H$.

Suppose, by contradiction, that $0 \notin T$. Then each $t \in T$ is of the form $t = ab^{-1}$ with $a \in W$ and $b \in W \cup \{0\}$. Recall that for each $t = ab^{-1}$, $r(a) \cap r(b) \in H$ and, for $t = a$, $r(a) \cap A_a \in H$.

Let $M = \max\{|a| : ab^{-1} \in T, a \in W, b \in W \cup \{0\}\}$.

By hypothesis, there is an edge $e_0 \in s^{-1}(u)$ such that $r(e_0) \notin H$. By the previous claim, there is a vertex $v_1 \in r(e_0)$ such that $\{v_1\} \notin H$. It follows from the paragraph immediately after Case 2 that $0 < |s^{-1}(v_1)| < \infty$. Since H is saturated, there is an edge $e_1 \in s^{-1}(v_1)$ such that $\{r(e_1)\} \notin H$. By applying the previous argument repeatedly, we get a path $e_0 \dots e_M$ such that $s(e_i) = v_i$ and $\{v_i\} \notin H$ for each $i \in \{1, \dots, M\}$. Let $\xi \in X$ be such that $s(\xi) \in r(e_M)$. Then $e_0 e_1 \dots e_M \xi \in X$ and, for each $t = ab^{-1} \in T$,

$$1_{ab^{-1}}(e_0 e_1 \dots e_M \xi) = 1_{a(r(a) \cap r(b))}(e_0 e_1 \dots e_M \xi) = 0,$$

since $s(e_{|a|}) \notin H$ and $r(a) \cap r(b) \in H$. The same holds for $t = a \in T$. So, $1_t(e_0 \dots e_M \xi) = 0$ for each $t \in T$. Finally,

$$\begin{aligned} 1 = 1_u(e_0 \dots e_M \xi) &= \sum_{t \in T} 1_u \beta_t(\beta_{t^{-1}}(x_t) 1_{A_t} y_{t^{-1}})(e_0 \dots e_M \xi) \\ &= \sum_{t \in T} 1_u \beta_t(\beta_{t^{-1}}(x_t) 1_{A_t} y_{t^{-1}})(e_0 \dots e_M \xi) 1_t(e_0 \dots e_M \xi) = 0, \end{aligned}$$

which is a contradiction. Therefore, $0 \in T$ and Case 3 is proved.

So, we get that $\{u\} \in H$ for each $u \in \mathcal{G}^0$.

To end the proof, notice that, by [17, Lemma 2.12], any $A \in \mathcal{G}^0$ can be written as

$$\bigcap_{e \in X_1} r(e) \cup \dots \cup \bigcap_{e \in X_n} r(e) \cup F,$$

where $X_1, \dots, X_n$ are finite subsets of $\mathcal{G}^1$ and F is a finite subset of G^0. Since H is hereditary and $\{s(e)\} \in H$, we have that $r(e) \in H$ for each $e \in \mathcal{G}^1$. The result now follows from the fact that H is hereditary. $\square$

https://doi.org/10.1017/S144678871900020X Published online by Cambridge University Press

We can now prove the simplicity criteria for the Leavitt path algebra of an ultragraph $\mathcal{G}$, $L_R(\mathcal{G})$, via partial skew group ring theory.

THEOREM 4.7. *Let $\mathcal{G}$ be an ultragraph and R be a field. Then $L_R(\mathcal{G})$ is simple if, and only if, $\mathcal{G}$ satisfies condition (L) and the unique saturated and hereditary subcollections of $\mathcal{G}^0$ are $\emptyset$ and $\mathcal{G}^0$.*

PROOF. By Theorem 3.10, $L_R(\mathcal{G})$ and $D \rtimes_\beta \mathbb{F}$ are isomorphic algebras. By [7, Theorem 2.3], the algebra $D \rtimes_\beta \mathbb{F}$ is simple if, and only if, D is $\mathbb{F}$-simple and $D\delta_0$ is maximal commutative in $D \rtimes_\beta \mathbb{F}$. The result now follows from Theorems 4.1 and 4.6. □

In [16, Theorem 3.11], Tomforde gave a complete combinatorial description of ultragraphs such that the associated ultragraph C*-algebra is simple. Since this description is obtained based only on the description of simplicity via hereditary and saturated collections, the theorem above implies that we have the same description for $L_R(\mathcal{G})$. For the reader's convenience, we state the theorem below, but for this we need to recall a few definitions.

For an ultragraph $\mathcal{G}$ and $v, w \in G^0$, the notation $w \geq v$ means that there is a path α with $s(\alpha) = w$ and $v \in r(\alpha)$. Also, $G^0 \geq \{v\}$ means that $w \geq v$ for each $w \in G^0$. The ultragraph $\mathcal{G}$ is said to be cofinal if for each infinite path $\alpha = e_1 e_2 \ldots$, and each vertex $v \in G^0$, there is an $i \in \mathbb{N}$ such that $v \geq s(e_i)$. Moreover, for $v \in G^0$ and $A \subseteq G^0$, we write $v \to A$ to mean that there are paths $\alpha_1, \ldots, \alpha_n$ such that $s(\alpha_i) = v$ for all $1 \leq i \leq n$ and $A \subseteq \bigcup_{i=1}^n r(\alpha_i)$.

THEOREM 4.8. *Let $\mathcal{G}$ be an ultragraph and R be a field. Then $L_R(\mathcal{G})$ is simple if and only if:*

(1) *$\mathcal{G}$ satisfies condition (L);*
(2) *$\mathcal{G}$ is cofinal;*
(3) *$G^0 \geq \{v\}$ for every singular vertex $v \in G^0$;*
(4) *if $e \in \mathcal{G}^1$ is an edge for which the set $r(e)$ is infinite, then for every $w \in G^0$ there exists a set $A_w \subseteq r(e)$ for which $r(e) \setminus A_w$ is finite and $v \to A_w$.*

PROOF. The proof of this theorem relies only on the fact that the only hereditary and saturated subcollections of $\mathcal{G}^0$ are $\emptyset$ and $\mathcal{G}^0$. So, the proof given in [16, Theorem 3.11] applies. □

5. Chain conditions

In [14], chain conditions were described for partial skew groupoid rings. As an application, a new proof of the criterion for a Leavitt path algebra to be artinian is given. Namely, a Leavitt path algebra associated to a graph E is artinian if and only if E is finite and acyclic (a graph (ultragraph) is called acyclic if there are no cycles in the graph (ultragraph)). Building from the ideas in [14], we show that this same criterion is true for ultragraph Leavitt path algebras. In our proof we will use that any

https://doi.org/10.1017/S144678871900020X Published online by Cambridge University Press

ultragraph Leavitt path algebra of a finite acyclic ultragraph is isomorphic to a Leavitt path algebra of a finite acyclic graph, a result we state precisely below.

Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be a finite ultragraph. Enumerate G^0, say

$$G^0 = \{v_1, \dots, v_n\}.$$

Define a map $c : \mathcal{G}^1 \to \{0, 1\}^n$ by $c(e) = (y_i)$, where

$$y_i = \begin{cases} 1 & \text{if } v_i \in r(e), \\ 0 & \text{if } v_i \notin r(e). \end{cases}$$

Consider the graph $\mathcal{F} = (G^0, \mathcal{F}^1, r, s)$, where the set of edges $\mathcal{F}^1$ consists of all edges defined as follows: for each edge $e \in \mathcal{G}^1$ and $i \in \{1, \dots, n\}$ such that $c(e)_i = 1$, let f_{e_i} be the edge such that $s(f_{e_i}) = s(e)$ and $r(f_{e_i}) = v_i$. We can now state the following proposition, a proof of which is left to the reader.

Proposition 5.1. *Let $\mathcal{G}$ be a finite ultragraph, that is, suppose that G^0 and $\mathcal{G}^1$ are finite, and let $\mathcal{F}$ be the associated graph as defined above. Then $L_R(\mathcal{G})$ is isomorphic to $L_R(\mathcal{F})$. Furthermore, if $\mathcal{G}$ is acyclic, then $\mathcal{F}$ is acyclic.*

We end the paper with the characterization of artinian ultragraph Leavitt path algebras. Recall that a ring is left (right) artinian if it satisfies the descending chain condition on left (right) ideals, and artinian if it is both left and right artinian.

Theorem 5.2. *Let R be a field and let $\mathcal{G}$ be an ultragraph. Consider $L_R(\mathcal{G})$, the ultragraph Leavitt path algebra of $\mathcal{G}$. Then the following five assertions are equivalent:*

(1) *$\mathcal{G}$ is finite and acyclic;*
(2) *$L_R(\mathcal{G})$ is left artinian;*
(3) *$L_R(\mathcal{G})$ is right artinian;*
(4) *$L_R(\mathcal{G})$ is artinian;*
(5) *$L_R(\mathcal{G})$ is unital and semisimple.*

Proof. All we need to prove is that (2) $\Rightarrow$ (1). The other implications follow from Proposition 5.1 and [14, Theorem 5.2].

(2) $\Rightarrow$ (1) The proof of this implication will follow closely the proof of [14, Theorem 5.2] for Leavitt path algebras. We include it here for completeness.

Suppose that $L_K(E) \cong D \rtimes_\beta \mathbb{F}$ is left artinian. By [14, Theorem 1.3], we get that $D_g = \{0\}$ for all but finitely many $g \in \mathbb{F}$, and D is left artinian.

Assume that there exists an infinite path $p = e_1e_2e_3\dots$ in $\mathcal{G}$. Then the ideals D_{e_1}, $D_{e_1e_2}$, $D_{e_1e_2e_3}, \dots$ are all nonzero, which is a contradiction. Therefore, there is no infinite path in $\mathcal{G}$ and hence $\mathcal{G}$ must be acyclic.

Next we prove that $\mathcal{G}$ is finite. Notice that if $G^0 = \{v_1, v_2, v_3, \ldots\}$ is infinite, then

$$\bigoplus_{v\in E^0\setminus\{v_1\}} L_R(\mathcal{G})v \supseteq \bigoplus_{v\in E^0\setminus\{v_1,v_2\}} L_R(\mathcal{G})v \supseteq \bigoplus_{v\in E^0\setminus\{v_1,v_2,v_3\}} L_R(\mathcal{G})v \supseteq \cdots$$

is a descending chain of left ideals of $L_R(\mathcal{G})$ that never stabilizes (since every pair of vertices in G^0 are orthogonal idempotents). Hence, $L_R(\mathcal{G})$ is not left artinian, which is a contradiction. Therefore, G^0 is finite.

We finish the proof of showing that $\mathcal{G}^1$ is finite. Since G^0 is finite, it is enough to prove that G^0 contains no infinite emitter. Seeking a contradiction, suppose that there is a vertex $v \in G^0$ which is an infinite emitter. Since G^0 is finite, there must exist some $u \in G^0$ such that the set $I = \{e \in E^1 \mid s(e) = v \text{ and } u \in r(e)\}$ is infinite. If u is a sink, then $(u, u) \in X_{e^{-1}}$ for all $e \in I$ and hence $D_{e^{-1}}$ is nonzero for infinitely many $e \in I$, which is a contradiction. Suppose that u is not a sink. Then there exists a path $\eta \in X$ such that $s(\eta) = u$. Hence, $X_{e^{-1}}$ contains η for each $e \in I$. Therefore, $D_{e^{-1}}$ is nonzero for infinitely many $e \in I$, which is a contradiction. □

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DANIEL GONÇALVES, Department of Mathematics,
Federal University of Santa Catarina,
88040900, Brazil
e-mail: daemig@gmail.com

DANILO ROYER, Department of Mathematics,
Federal University of Santa Catarina,
88040900, Brazil
e-mail: daniloroyer@gmail.com