ANZIAM J. **62** (2020), 209–234
doi:10.1017/S1446181120000164

MEAN–VARIANCE EQUILIBRIUM ASSET-LIABILITY MANAGEMENT STRATEGY WITH COINTEGRATED ASSETS

MEI CHOI CHIU [1]

(Received 11 September, 2019; accepted 17 June, 2020; first published online 6 November, 2020)

Abstract

This paper investigates asset-liability management problems in a continuous-time economy. When the financial market consists of cointegrated risky assets, institutional investors attempt to make profit from the cointegration feature on the one hand, while on the other hand they need to maintain a stable surplus level, that is, the company's wealth less its liability. Challenges occur when the liability is random and cannot be fully financed or hedged through the financial market. For mean–variance investors, an additional concern is the rational time-consistency issue, which ensures that a decision made in the future will not be restricted by the current surplus level. By putting all these factors together, this paper derives a closed-form feedback equilibrium control for time-consistent mean–variance asset-liability management problems with cointegrated risky assets. The solution is built upon the Hamilton–Jacobi–Bellman framework addressing time inconsistency.

2020 *Mathematics subject classification*: primary 91G10; secondary 91B70.

Keywords and phrases: asset-liability management, cointegration, mean–variance, time inconsistency.

1. Introduction

The 2003 Nobel Memorial Prize was awarded to Granger and Engle for their discovery of cointegration [19] and generalised autoregressive conditional heteroscedasticity through time series models. Specifically, the concept of cointegration [18] asserts that a pair of nonstationary time series can have a stationary linear combination or a lower degree of integration than the original series. In other words, this is an alternative description of comovements beyond the conventional correlation coefficient. Cointegration has been applied to a variety of economic models, including the relationships between capital and output, real wages and labour productivity, nominal exchange rates and relative prices, consumption and disposable income, long- and short-term interest

[1]Department of Mathematics and Information Technology, The Education University of Hong Kong, 10 Lo Ping Road, Tai Po, New Territories, Hong Kong; e-mail: mcchiu@eduhk.hk.

rates, money velocity and interest rates, price of shares and dividends, production and sales, and many other financial variables.

In finance, cointegration has been an important concept for enhancing asset pricing and portfolio management. Specifically, Alexander [1] and Alexander et al. [2] pioneered the use of cointegration in portfolio management, while Duan and Pliska [17] recognised the application in commodity option pricing. Empirical evidence for the applications of cointegration included, but was not limited to, exchange rates [3], stocks [6], global financial indices [25] and commodities [15].

Using the Markowitz [23] mean–variance (MV) criteria and the continuous-time cointegration model [24], Chiu and Wong [10] established the first MV pairs-trading rule for cointegrated risky assets. Their work has been useful in insurers' investment problems [11], asset-liability management (ALM) [12], robust decision rule [14] and the mortality hedging [28]. While the framework of Chiu and Wong and the subsequent extensions are based on a combination of work by Li and Ng [22], Zhou and Li [30], and Chiu and Li [9], the resulting strategies violate the dynamic programming principles (DPP) and hence cause the issue of time inconsistency.Time inconsistency in dynamic MV portfolio problems has recently attracted a great deal of attention. In particular, Cui et al. [16] showed that time inconsistency could result in a suboptimal solution. They demonstrated that the classical time-inconsistent MV problem generates a free cash flow. The performance of the portfolio is unaffected by the removal of this free cash flow. In other words, there exists a better way to manage the cash to further improve the portfolio. Therefore, investigating the time-consistent MV (TC-MV) ALM problem with cointegration represents important and interesting research. A well-received framework to overcome time inconsistency is based on the Hamilton–Jacobi–Bellman (HJB) approach, independently developed by Basak and Chabakauri [4] and Bjork et al. [5]. According to this approach, the decision is made by the investor in a game that he plays with all of his future-selves. After his future-self makes a decision, his current-self reacts to locally optimise the objective function. This induces a backward sequential game among the current-self and all the future-selves. The current decision becomes a subgame perfect Nash equilibrium that preserves the DPP. This equilibrium control framework has been applied to ALM in a regime-switching economy without cointegration [26], cointegration pairs-trading without a liability [13], mortality hedging [27], index hedging [7] and pairs-trading in a regime-switching cointegration economy [7]. Typically, Chiu and Wong [13] prove mathematically that the equilibrium MV strategy offers a statistical arbitrage opportunity in the cointegration financial market. However, they do not take into account the real situation in which institutional investors usually encounter uncontrollable random liability and are concerned primarily with their surplus. To the best of our knowledge, the TC-MV ALM with high-dimensional cointegrated risky assets and an uncontrollable liability is yet to be considered. Although a recent work [8] also considers ALM for the same market setting, the objective is rather different, because that work concentrates on exponential utility optimisation.

The TC-MV ALM problem with cointegration is mathematically challenging, because the risky asset prices' dynamic contains a high-dimensional mean reversion term that causes the corresponding stochastic differential equation (SDE) to have unbounded coefficients. Hu et al. [21] also investigate TC-MV problems with backward SDE approach. Their approach requires that all coefficients in the state process and hence in the SDE of risky asset prices are bounded. The case of cointegration clearly violates such a regularity condition. In fact, removing the boundedness condition is not obvious, as shown in the case of stochastic volatility by Zhou and Li [29]. It is, however, even more challenging with the ALM problem, because the stochastic liability is typically assumed not to be controllable for infeasible debt financing. In the literature, the liability risk is projected to the controllable stock portfolio through correlation between the stock market and the liability. Such a projection becomes nontrivial once cointegration occurs in the stock market. The investor has to strike a balance between profiting from the cointegrated pairs trading and ensuring sufficient cash to settle the liability at the end of the investment horizon.

Therefore, this paper contributes to the literature by introducing the TC-MV HJB framework to the ALM problem with cointegrated risky assets and an uncontrollable liability. We derive a closed-form explicit solution to the equilibrium ALM strategy and the MV efficient frontier describing the trade-off between the expected surplus and the variance of the surplus. The mathematical contribution is the solution to a complicated system of high-dimensional matrix differential equations (see Section 4).

The rest of the paper is organised as follows. Section 2 reviews the basic concept of cointegration and gives details of the formulation of the problem. The corresponding TC-MV ALM problem is completely solved in Section 3. The analytical result is used to study the efficient frontier of the ALM problem, and, finally, Section 5 concludes the paper.

2. Problem formulation

2.1. Cointegration The error-correction model is the basis for constructing our cointegrated asset-return dynamics in continuous time. By Granger's representation theorem [18], the cointegrated vector time series can be expressed as an error correction model. Cointegration is based on the belief that many nonstationary time series may form certain long-run relationships.

In a discrete-time model with constant parameters, an error correction dynamic for the m-component asset price time series with k ($1 \le k \le m$) cointegrating factors is defined as follows:

$$\ln S_{i,t} - \ln S_{i,t-1} = \mu + \sum_{j=1}^{k} \delta_{ij} z_{j,t-1} + \sigma_{i,t}\epsilon_{i,t} \quad \text{for } i = 1, \dots, m,$$

$$z_{j,t} = a_j + b_j t + \sum_{i=1}^{m} c_{ij} \ln S_{i,t} \quad \text{for } j = 1, \dots, k,$$

https://doi.org/10.1017/S1446181120000164 Published online by Cambridge University Press

where $S_{i,t}$ is the price of asset i at time t for $i = 1, \ldots, m$; $(c_{1j}, \ldots, c_{nj})$ are linearly independent vectors for $j = 1, \ldots, k$; and the random vector $[\epsilon_{1,t}, \ldots, \epsilon_{m,t}]$ follows a multivariate normal distribution with mean zero and constant correlation coefficient matrix. In the error correction model, the vector of k cointegrating factors, $[z_{1,t}, \ldots, z_{k,t}]$, should be a stationary time series such that each $z_{j,t}$ has a bounded variance at all time points. A stationary time series can easily be identified within an autoregressive (AR) model. In the case where $k = 1$, an AR(1) stationary time series of $\{z_{1,t}\}$ takes the form

$$z_{1,t} = \alpha z_{1,t-1} + \sigma_z \epsilon_t,$$

where $\alpha < 1$. If $k = m$, then the vector of log-asset prices has already formed a stationary vector AR(1) time series and hence $z_{j,t} = \ln S_{j,t}$ for all $j = 1, \ldots, m$.

In a continuous-time economy, the cointegration model [24] becomes

$$d \ln S_{i,t} = \Big(\mu + \sum_{j=1}^{k} \delta_{ij} \mathbf{z}_{j,t}\Big) dt + \sigma_i \, d\hat{W}_{i,t} \quad i = 1, \ldots, m, \tag{2.1}$$

$$\mathbf{z}_{j,t} = a_j + b_j t + \sum_{i=1}^{m} c_{ij} \ln S_{i,t} \quad j = 1, \ldots, k, \tag{2.2}$$

where $\hat{W}_t = (\hat{W}_t^1, \hat{W}_t^2, \ldots, \hat{W}_t^n)'$ is a vector of correlated Wiener process, and $n \geq m$, representing a possibly incomplete market. When $k = 1$, $\mathbf{z}_{1,t}$ in (2.2) follows the Ornstein–Uhlenbeck process,

$$d\mathbf{z}_{1,t} = (\mu_1 - \rho \mathbf{z}_{1,t}) \, dt + \sigma_{\mathbf{z}1 \, d\hat{W}_t},$$

where μ_1 is a real constant, and ρ and $\sigma_{\mathbf{z}1}$ are positive constants. In general, the vector of k cointegrating factors, $\mathbf{z}_t = [\mathbf{z}_{1,t}, \ldots, \mathbf{z}_{k,t}]$, satisfies the stochastic differential equation (SDE)

$$d\mathbf{z}_t = (\mu_{\mathbf{z}} - J \cdot \mathbf{z}_t) \, dt + \sigma_{\mathbf{z}} \, d\hat{W}_t, \tag{2.3}$$

where J is a $k \times k$ positive diagonal matrix, $\sigma_{\mathbf{z}} \sigma_{\mathbf{z}}'$ is a positive definite matrix of size k, and μ_z is a k-dimensional vector. Note that a vector of correlated Wiener processes $\hat{W}_t$ in (2.3) can be expressed as the product of a correlation matrix and the vector of independent Wiener processes, W_t.

To simplify the notation, we substitute (2.2) into (2.1) and write

$$d \ln S_t = (\theta(t) - \mathcal{A} \cdot \ln S_t) \, dt + \sigma_A \, dW_t,$$

where $\ln S_t$ is a vector that contains log-prices of m assets, $\theta(t)$ is an m-dimensional vector with each component linear in t, W_t is a vector of uncorrelated Wiener processes, $\sigma_A \sigma_A{}'$ represents the $m \times m$ variance-covariance matrix, and $\mathcal{A}$ is the $m \times m$ coefficient matrix of cointegration.

2.2. The market Consider a financial market in which $m + 1$ assets are traded continuously within the time interval $[0, T]$. These assets are labelled by S_i for

https://doi.org/10.1017/S1446181120000164 Published online by Cambridge University Press

$i = 0, 1, 2 \ldots, m$, with the 0th asset being risk-free. The risk-free asset satisfies the differential equation

$$dS_0(t) = r(t)S_0(t)\,dt,$$
$$S_0(0) = R_0 > 0,$$

where $r(t)$ is the time deterministic risk-free rate. The risky assets are defined through their log-price processes $X_1(t), \ldots, X_m(t)$, where

$$X_j(t) = \ln S_j(t). \tag{2.4}$$

The log-prices vector, $X(t)$, satisfies the SDE

$$dX(t) = [\theta(t) - \mathcal{A}X(t)]\,dt + \sigma_A(t)\,dW_t \quad t \in [0, T], \tag{2.5}$$

where $W_t = (W_t^1, \ldots, W_t^n)'$ is a standard $\mathcal{F}_{t\geq 0}$-adapted n-dimensional Wiener process on a fixed filtered complete probability space $(\Omega, \mathcal{F}, \mathcal{P}, \mathcal{F}_{t\geq 0})$, W_t^i and W_t^j are mutually independent for all $i \neq j$, $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is the filtration generated by W_t augmented by the null sets of $\mathcal{P}$, $\mathcal{A}$ is an $m \times m$ constant matrix of cointegration coefficients, and $\sigma_A(t)\sigma_A(t)'$ is the covariance matrix of assets defined in the Banach space of $\mathbb{R}^{m\times m}$-valued continuous functions on $[0, T]$. In line with the literature [30], we assume that the nondegeneracy condition of $\sigma_A(t)\sigma_A(t)' \geq \delta I_m$ holds for all $t \in [0, T]$ and for some $\delta > 0$. We also assume that $r(t)$, $\theta(t)$ and $\sigma_A(t)$ are measurable and uniformly bounded in $[0, T]$.

The risky asset dynamics in (2.5) have appealing financial interpretations and embrace several interesting classical models as special cases. When $\mathcal{A} \equiv 0$, the S_i with $i = 1, 2, \ldots, m$ in (2.4) are reduced to the geometric Brownian motions, and the corresponding MV portfolio problem has been fully analysed in the literature. If $\mathcal{A}$ is a full rank positive diagonal matrix, then all of the individual assets are stationary and exhibit mean reversion.

These m risky assets are said to have k cointegrating factors if $\text{rank}(\mathcal{A}) = k$, $\mathcal{A}$ has exactly k positive eigenvalues, and $\theta(t)$ is linear in t. In such a situation, there exists an invertible matrix P such that $\mathcal{A} = P^{-1}\hat{J}P$, where $\hat{J}$ is a Jordan normal form of $\mathcal{A}$. As $\mathcal{A}$ has k positive eigenvalues, its Jordan normal form is

$$\hat{J} = \begin{pmatrix} J & 0_{k\times(m-k)} \\ 0_{(m-k)\times k} & 0_{(m-k)\times(m-k)} \end{pmatrix},$$

where $J \in \mathbb{R}^{k\times k}$ is the Jordan matrix with a positive diagonal. Let $Z_t = P \cdot X_t$. From (2.5),

$$dZ_t = (P\theta(t) - \hat{J}Z_t)\,dt + P\sigma_A(t)\,dW_t \quad t \in [0, T].$$

Let z_t be a k-dimensional vector collecting the first k components of Z_t. Then, z_t has the stochastic process of the form of (2.3) and represents the k cointegrating factors of the m risky assets.

https://doi.org/10.1017/S1446181120000164 Published online by Cambridge University Press

2.3. Mean–variance ALM Consider an investor with an initial capital of w_0 in the specified financial market, with cointegration and initial liability of l_0. The liability price process is postulated as

$$\begin{aligned} dl(t) &= l(t)\tilde{\alpha}_L(t)\,dt + l(t)\sigma_L(t)\,dW_t, \\ l(0) &= l_0, \end{aligned} \tag{2.6}$$

where $\tilde{\alpha}_L(t)$ is the appreciation rate of the value of liabilities; and $\sigma_L(t)$ is the volatility, which belongs to $C([0,T];\mathbb{R}^{n\times 1})$, a Banach space of $\mathbb{R}^{n\times 1}$-valued continuous functions on $[0,T]$, and satisfies the nondegeneracy condition. The investor is allowed to continuously adjust his portfolio over a time period $[0,T]$. Denote $Y(t) = w(t) - l(t)$ as the surplus. The investor aims to minimise the variance of the terminal surplus, $Y(T)$, and maximise the expected final surplus.

This paper focuses on the situation where the random liability is uncontrollable, meaning that the investor cannot trade the liability in the financial market. This consideration agrees with those in the literature [9, 12, 26].

Let $u_i(t)$ be the amount invested in asset i, and let $N_i(t)$ be the number of asset i in the portfolio of the investor. The wealth of the investor at time t is then defined as $w(t) = \sum_{i=0}^m u_i(t) = \sum_{i=0}^m N_i(t)S_i(t)$. The portfolio

$$u(t) = (u_1(t), u_2(t), \ldots, u_m(t))'$$

is said to be admissible if $u(t)$ is a nonanticipating and $\mathcal{F}_t$-adapted process such that $\int_0^t u(\tau)'u(\tau)\,d\tau < \infty$ almost surely. Note that here we focus on self-financing portfolios. This means that no money will be withdrawn from or input into the portfolio. Any change in the value of the portfolio is due only to changes in the prices of the assets. Applying Itô's lemma to $w(t)$ with respect to the cointegrating dynamics (2.5), the wealth process is given by

$$\begin{aligned} dw(t) &= [r(t)w(t) + u(t)'\alpha_A(t)]\,dt + u(t)'\sigma_A(t)\,dW_t, \\ w(0) &= w_0, \end{aligned} \tag{2.7}$$

where $\mathbf{1}$ is the column vector with all elements 1, and

$$\alpha_A(t) = \theta(t) - \mathcal{A}X(t) + \tfrac{1}{2}\mathcal{D}(\sigma_A(t)\sigma_A(t)')\mathbf{1} - r(t)\mathbf{1}, \tag{2.8}$$

in which $\mathcal{D}(\sigma_A(t)\sigma_A(t)')$ is the diagonal matrix with all diagonal elements equal to those of $\sigma_A(t)\sigma_A(t)'$. By subtracting (2.6) from (2.7), the SDE for the surplus can be derived as

$$\begin{aligned} dY(t) &= [r(t)Y(t) + u(t)'\alpha_A(t) - \alpha_L(t)l(t)]\,dt + [u(t)'\sigma_A(t) - \sigma_L(t)l(t)]\,dW_t, \\ Y(0) &= w_0 - l_0, \end{aligned} \tag{2.9}$$

where $\alpha_L(t) = \tilde{\alpha}_L(t) - r(t)$. In this paper, a closed-form and explicit solution to the equilibrium TC-MV ALM strategy is derived, subject to cointegration that embraces mean reversion as its special case.

PROBLEM 2.1. *The original MV ALM problem under cointegration is formulated as*

$$
\begin{aligned}
(P(\lambda)) \qquad & \min_{u(\cdot)} \qquad \operatorname{Var}(Y(T)) - 2\lambda E[Y(T)] \\
& \textit{subject to} \qquad u(\cdot) \in \mathcal{L}^2_{\mathcal{F}_T}([0,T],\mathbb{R}^m), \\
& \qquad\qquad \textit{surplus process with cointegration } (2.9), \\
& \qquad\qquad \textit{liabilities process } (2.6).
\end{aligned}
$$

The space $\mathcal{L}^2_{\mathcal{F}_T}([0,T],\mathbb{R}^m)$ represents the set of all $\mathbb{R}^m$-valued, $\mathcal{F}_T$-measurable random variables with finite second moments. For any $\lambda > 0$, this problem is equivalent to a standard utility maximisation problem in which the utility function is given by

$$
\mathrm{E}[Y(T)] - \frac{1}{2\lambda}\operatorname{Var}(Y(T)).
$$

Thus, the reciprocal of 2λ reflects the risk aversion of the utility function. However, (P(λ)) in Problem 2.1 is not time-consistent because of the nonseparability of the variance operator.

3. Time consistency and solution

3.1. Time-consistent HJB

DEFINITION 3.1. A control law u^* is said to be an equilibrium control or time consistency control if for every admissible u, a fixed real number $\tau > 0$ and a fixed and arbitrarily chosen initial point (t, y, x, l), a new control law u_τ related to u^* by

$$
u_\tau(s,y,x,l) = \begin{cases} u(s,y,x,l) & \text{for } t \le s < t+\tau \\ u^*(s,y,x,l) & \text{for } t+\tau \le s \le T \end{cases}
$$

has the property that

$$
\limsup_{\tau \to 0^+} \frac{U(t,y,x,l,u^*) - U(t,y,x,l,u_\tau)}{\tau} \le 0 \quad \text{for all } (t,y,x,l) \in [0,T]\times\mathbb{R}^3,
$$

where

$$
\begin{aligned}
U(t,y,x,l,u(\cdot)) = {} & \operatorname{Var}(Y^u(T)|Y(t)=y,\ X(t)=x,\ l(t)=l), \\
& - 2\lambda\mathbb{E}[Y^u(T)|Y(t)=y,\ X(t)=x,\ l(t)=l],
\end{aligned}
$$

and $Y^u(\cdot)$ means the surplus process adopting the trading strategy $u(\cdot)$.

Definition 3.1 extracted from Bjork [5] asserts that the feedback control minimises the objective function $U(\cdot)$ locally. When the risk aversion λ is a constant, this definition of a TC-MV problem is consistent with the definition given by Basak and Chabakauri [4]. This paper adopts the presentation of Basak and Chabakauri [4], and readers interested in the connection between the two approaches are referred to Bjork [5].

https://doi.org/10.1017/S1446181120000164 Published online by Cambridge University Press

For any given constant $\lambda > 0$, consider the utility function

$$U_t = \mathrm{Var}(Y(T)|\mathcal{F}_t) - 2\lambda \mathrm{E}[Y(T)|\mathcal{F}_t].$$

Using the law of iterated expectations, Basak and Chabakauri [4] showed that a portfolio policy satisfying

$$U_t = \mathrm{E}[U_{t+\tau}|\mathcal{F}_t] + \mathrm{Var}(\mathrm{E}[Y(T)|\mathcal{F}_{t+\tau}]|\mathcal{F}_t) \quad t \geq 0,\ \tau > 0 \tag{3.1}$$

is a time-consistent policy and obeys the DPP. Hence, we define the time-consistent solution space of cointegrated assets as

$$\mathcal{U}(t,T) = \{u \in \mathcal{L}^2_{\mathcal{F}_T}([t,T],\mathbb{R}^n) \mid \text{conditions (2.9), (2.6) and (3.1) hold}\}. \tag{3.2}$$

The time-consistent versions of P(λ) are respectively revised as

$$\min_{u\in\mathcal{U}(0,T)} \mathrm{Var}(Y(T)) - 2\lambda \mathrm{E}[Y(T)].$$

To derive a HJB equation for the value function subject to time consistency, consider the value function

$$J(t,Y(t),X(t),l(t)) = \min_{u\in\mathcal{U}(t,T)} \mathrm{Var}(Y(T)|\mathcal{F}_t) - 2\lambda \mathrm{E}[Y(T)|\mathcal{F}_t], \tag{3.3}$$

where $\mathcal{U}(t,T)$ is mentioned in (3.2). Notice that $J^* = J(0, w_0 - l_0, X(0), l_0)$ is the optimal value function ($P(\lambda)$) in Problem 2.1.

Following Basak and Chabakauri [4], an application of Itô's lemma to $e^{\int_t^T r(s)\,ds}Y(t)$ yields

$$d(e^{\int_t^T r(s)\,ds}Y(t)) = e^{\int_t^T r(s)\,ds}[(\alpha_A(t)'u(t) - \alpha_L(t)l(t))\,dt + (u(t)'\sigma_A(t) - l(t)\sigma_L(t))\,dW_t], \tag{3.4}$$

which implies

$$\begin{aligned}
Y(T) &= e^{\int_t^T r(s)\,ds}Y(t) + \int_t^T e^{\int_s^T r(\tau)\,d\tau}(\alpha_A(s)'u(s) - \alpha_L(s)l(s))\,ds \\
&\quad + \int_t^T e^{\int_s^T r(\tau)\,d\tau}(u(s)'\sigma_A(s) - l(s)\sigma_L(s))\,dW_s, \\
\mathrm{E}[Y(T)|\mathcal{F}_t] &= e^{\int_t^T r(s)\,ds}Y(t) + \mathrm{E}\Big[\int_t^T e^{\int_s^T r(\tau)\,d\tau}(\alpha_A(s)'u(s) - \alpha_L(s)l(s))\,ds|\mathcal{F}_t\Big], \\
\mathrm{Var}(Y(T)|\mathcal{F}_t) &= \mathrm{Var}\Big(\int_t^T e^{\int_s^T r(\tau)\,d\tau}(\alpha_A(s)'u(s) - \alpha_L(s)l(s))\,ds \\
&\quad + \int_t^T e^{\int_s^T r(\tau)\,d\tau}(u(s)'\sigma_A(s) - l(s)\sigma_L(s))\,dW_s|\mathcal{F}_t\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned} J(t, Y(t), X(t), l(t)) = \min_{u\in\mathcal{U}(t,T)} \Big\{ &\mathrm{Var}\Big(\int_t^T e^{\int_s^T r(\tau)\,d\tau}(\alpha_A(s)'u(s) - \alpha_L(s)l(s))\,ds \\ &+ \int_t^T e^{\int_s^T r(\tau)\,d\tau}(u(s)'\sigma_A(s) - l(s)\sigma_L(s))\,dW_s|\mathcal{F}_t\Big) \\ &- 2\lambda \mathrm{E}\Big[\int_t^T e^{\int_s^T r(\tau)\,d\tau}(\alpha_A(s)'u(s) - \alpha_L(s)l(s))\,ds|\mathcal{F}_t\Big]\Big\} \\ &- 2\lambda e^{\int_t^T r(s)\,ds}Y(t), \end{aligned} \tag{3.5}$$

which shows the separable structure of $e^{\int_t^T r(s)\,ds}Y(t)$, and hence the equilibrium $u^*(s)$ does not depend on $Y(t)$ for $s \geq t$. Owing to the Markovian nature of the economy, $u^*(t)$ only depends on $X(t)$, $l(t)$ and t for a given constant λ. Then, we define

$$J(t, Y(t), X(t), l(t)) = J_1(t, X(t), l(t)) - 2\lambda e^{\int_t^T r(s)\,ds}Y(t) \quad \text{and} \tag{3.6}$$

$$\Gamma(t, X(t), l(t)) = \mathrm{E}\Big[\int_t^T e^{\int_s^T r(\tau)\,d\tau}(\alpha_A(s)'u^*(s) - \alpha_L(s)l(s))\,ds|\mathcal{F}_t\Big], \tag{3.7}$$

where $J_1(t, X(t), l(t))$ is the first term of equation (3.5).

LEMMA 3.2. *The value function $J(t, Y(t), X(t), l(t))$ defined in (3.3) satisfies*

$$\min_{u(t)} \{E[dJ|\mathcal{F}_t] + \mathrm{Var}(d(e^{\int_t^T r(s)\,ds}Y) + d\Gamma|\mathcal{F}_t)\} = 0, \tag{3.8}$$

where Γ is defined in equation (3.7). Hence, the optimal time-consistent trading strategy $(P(\lambda))$ in Problem 2.1 is

$$\begin{aligned} u^*(t, X(t), l(t)) = e^{-\int_t^T r(s)\,ds}\Big[&\lambda(\sigma_A(t)\sigma_A(t)')^{-1}\alpha_A(t) - \frac{\partial\Gamma}{\partial X}(t, X(t), l(t)) \\ &+ (\sigma_A(t)\sigma_A(t)')^{-1}\sigma_A(t)\sigma_L(t)'l(t)\Big(e^{\int_t^T r(s)\,ds} - \frac{\partial\Gamma}{\partial l}(t, X(t), l(t))\Big)\Big], \end{aligned} \tag{3.9}$$

where

$$\alpha_A(t) = \theta(t) - \mathcal{A}X(t) + \tfrac{1}{2}\mathcal{D}(\sigma_A(t)\sigma_A(t)')\mathbf{1} - r(t)\mathbf{1}.$$

PROOF. To simplify the notation, we denote $J(t) = J(t, Y(t), X(t), l(t))$ and $\Gamma(t) = \Gamma(t, X(t), l(t))$. The definition of J implies that

$$\begin{aligned} J(t) &= \min_{u\in\mathcal{U}(t,T)} \{\mathrm{Var}(Y(T)|\mathcal{F}_t) - 2\lambda\mathrm{E}[Y(T)|\mathcal{F}_t]\} \\ &= \min_{u\in\mathcal{U}(t,T)} (\mathrm{E}[\mathrm{Var}(Y(T)|\mathcal{F}_{t+\tau})|\mathcal{F}_t] + \mathrm{Var}(\mathrm{E}[Y(T)|\mathcal{F}_{t+\tau}]|\mathcal{F}_t)) - 2\lambda\mathrm{E}[\mathrm{E}[Y(T)|\mathcal{F}_{t+\tau}]|\mathcal{F}_t] \\ &= \min_{u\in\mathcal{U}(t,T)} \mathrm{E}[\mathrm{Var}(Y(T)|\mathcal{F}_{t+\tau}) - 2\lambda\mathrm{E}[Y(T)|\mathcal{F}_{t+\tau}]|\mathcal{F}_t] + \mathrm{Var}(\mathrm{E}[Y(T)|\mathcal{F}_{t+\tau}]|\mathcal{F}_t) \\ &= \min_{u\in\mathcal{U}(t,t+\tau)} \mathrm{E}[J(t+\tau)|\mathcal{F}_t] + \mathrm{Var}(\mathrm{E}[Y^*(T)|\mathcal{F}_{t+\tau}]|\mathcal{F}_t) \end{aligned} \tag{3.10}$$

https://doi.org/10.1017/S1446181120000164 Published online by Cambridge University Press

$$= \min_{u\in\mathcal{U}(t,t+\tau)} \mathrm{E}[J(t+\tau)|\mathcal{F}_t] + \mathrm{Var}(e^{\int_{t+\tau}^T r(s)\,ds}Y(t+\tau) + \Gamma(t+\tau)|\mathcal{F}_t) \tag{3.11}$$

$$= \min_{u\in\mathcal{U}(t,t+\tau)} \mathrm{E}[J(t+\tau)|\mathcal{F}_t] + \mathrm{Var}(e^{\int_{t+\tau}^T r(s)\,ds}Y(t+\tau) + \Gamma(t+\tau) + (\Gamma(t) - \Gamma(t))|\mathcal{F}_t)$$

$$\begin{aligned} &= \min_{u\in\mathcal{U}(t,t+\tau)} \{\mathrm{Var}(e^{\int_{t+\tau}^T r(s)\,ds}Y(t+\tau) - e^{\int_t^T r(s)\,ds}Y(t) + \Gamma(t+\tau) - \Gamma(t)|\mathcal{F}_t) \\ &\quad + \mathrm{E}[J(t+\tau)|\mathcal{F}_t]\} \end{aligned} \tag{3.12}$$

$$\begin{aligned} 0 &= \min_{u\in\mathcal{U}(t,t+\tau)} \{\mathrm{Var}((e^{\int_{t+\tau}^T r(s)\,ds}Y(t+\tau) - e^{\int_t^T r(s)\,ds}Y(t)) + (\Gamma(t+\tau) - \Gamma(t))|\mathcal{F}_t) \\ &\quad + \mathrm{E}[(J(t+\tau) - J(t))|\mathcal{F}_t]\}. \end{aligned} \tag{3.13}$$

Letting $\tau \to 0$ in equation (3.13),

$$\min_{u(t)}\{\mathrm{E}[dJ|\mathcal{F}_t] + \mathrm{Var}(d(e^{\int_t^T r(s)\,ds}Y) + d\Gamma|\mathcal{F}_t)\} = 0.$$

Note that equality (3.10) holds if the trading strategy $u^*(\cdot)$ is time consistent. Equalities (3.11) and (3.12) hold owing to the definition of Γ. This is because

$$\begin{aligned} \Gamma(t, X(t), l(t)) &= \mathrm{E}\Big[\int_t^T e^{\int_s^T r(\tau)\,d\tau}(\alpha_A(s)'u^*(s) - \alpha_L(s)l(s))\,ds|\mathcal{F}_t\Big], \\ &= \mathrm{E}[Y^*(T)|\mathcal{F}_t] - e^{\int_t^T r(s)\,ds}Y(t), \end{aligned}$$

where $Y^*(T) = Y(T)|_{u=u^*}$. To obtain the time-consistent trading strategy $u^*(t)$, equation (3.8) must be solved. According to (3.6),

$$\begin{aligned} dJ &= -2\lambda d(e^{\int_t^T r(s)\,ds}Y(t)) + dJ_1 \\ \mathrm{E}[dJ|\mathcal{F}_t] &= \mathrm{E}[-2\lambda e^{\int_t^T r(s)\,ds}(\alpha_A(t)'u(t) - \alpha_L(t)l(t))\,dt + dJ_1|\mathcal{F}_t]. \end{aligned} \tag{3.14}$$

Applying Itô's lemma to Γ produces the SDE of Γ as follows:

$$\begin{aligned} d\Gamma &= \left\{\frac{\partial\Gamma}{\partial t} + \Big(\frac{\partial\Gamma}{\partial X}\Big)'(\theta - \mathcal{A}X) + \frac{\partial\Gamma}{\partial l}\tilde{\alpha}_L l + \frac{1}{2}tr\left(\begin{pmatrix}\sigma_A' & l\sigma_L'\end{pmatrix}\begin{pmatrix}\frac{\partial^2\Gamma}{\partial X^2} & \frac{\partial^2\Gamma}{\partial X\partial l} \\ \frac{\partial^2\Gamma}{\partial l\partial X} & \frac{\partial^2\Gamma}{\partial l^2}\end{pmatrix}\begin{pmatrix}\sigma_A \\ l\sigma_L\end{pmatrix}\right)\right\} dt \\ &\quad + \Big[\Big(\frac{\partial\Gamma}{\partial X}\Big)'\sigma_A + \frac{\partial\Gamma}{\partial l}\sigma_L l\Big]\,dW_t. \end{aligned} \tag{3.15}$$

Thus,

$$\begin{aligned} &\mathrm{Var}(d(e^{\int_t^T r(s)\,ds}Y(t)) + d\Gamma|\mathcal{F}_t) \\ &\quad = \mathrm{Var}\Big(\Big(e^{\int_t^T r(s)\,ds}(u'\sigma_A - l\sigma_L) + \Big(\frac{\partial\Gamma}{\partial X}\Big)'\sigma_A + \frac{\partial\Gamma}{\partial l}\sigma_L l\Big)\,dW_t|\mathcal{F}_t\Big) \end{aligned}$$

https://doi.org/10.1017/S1446181120000164 Published online by Cambridge University Press

$$= \mathrm{E}\Big[e^{2\int_t^T r(s)\,ds}(u'\sigma_A\sigma_A' u - 2lu'\sigma_A\sigma_L') + 2e^{\int_t^T r(s)\,ds}u'\Big(\sigma_A\sigma_A'\frac{\partial\Gamma}{\partial X} + \sigma_A\sigma_L' l\frac{\partial\Gamma}{\partial l}\Big) + \Psi(t, X, l)|\mathcal{F}_t\Big]\,dt, \tag{3.16}$$

where Ψ is a function that only depends on t, $X(t)$ and $l(t)$.

By substituting the terms in (3.14) and (3.16) into equation (3.8), the right-hand side of (3.8) represents a minimisation of a quadratic form in u. This implies that u^* satisfies the first-order condition. Hence,

$$u^*(t, X(t), l(t)) = e^{-\int_t^T r(s)\,ds}\Big[\lambda(\sigma_A(t)\sigma_A(t)')^{-1}\alpha_A(t) - \frac{\partial\Gamma}{\partial X}(t, X(t), l(t)) + (\sigma_A(t)\sigma_A(t)')^{-1}\sigma_A(t)\sigma_L(t)' l(t)\Big(e^{\int_t^T r(s)\,ds} - \frac{\partial\Gamma}{\partial l}(t, X(t), l(t))\Big)\Big].$$ □

Although the above lemma offers a representation of the TC-MV ALM strategy, it lacks an explicit expression of the function $\Gamma(t, X(t), l(t))$. This is addressed in the following subsection.

3.2. The function $\Gamma(t, X(t), l(t))$

LEMMA 3.3. *The function $\Gamma(t, X(t), l(t))$ defined in (3.7) takes the form*

$$\Gamma(t, X(t), l(t)) = \lambda\hat{\mathcal{H}}(t, \alpha_A(t)) - \mathcal{G}(t, \alpha_A(t), l(t)), \tag{3.17}$$

where

$$\hat{\mathcal{H}}(t, \alpha_A(t)) = E_t^{\hat{\mathcal{P}}}\Big[\int_t^T \alpha_A(s)'(\sigma_A(s)\sigma_A(s))^{-1}\alpha_A(s)\,ds\Big], \tag{3.18}$$

$$\mathcal{G}(t, \alpha_A(t), l(t)) = E_t^{\hat{\mathcal{P}}}\Big[\int_t^T l(s)e^{\int_s^T r(\tau)\,d\tau}(\alpha_L(s) - \alpha_A(s)'(\sigma_A(s)\sigma_A(s))^{-1}\sigma_A(s)\sigma_L(s)')\,ds\Big], \tag{3.19}$$

and the probability $\hat{\mathcal{P}}$ is equivalent to $\mathcal{P}$ as follows:

$$\frac{d\hat{\mathcal{P}}}{d\mathcal{P}} = \exp\Big\{\int_t^T -\frac{1}{2}\alpha_A(s)'(\sigma_A(s)\sigma_A(s)')^{-1}\alpha_A(s)\,ds - \int_t^T \alpha_A(s)'(\sigma_A(s)\sigma_A(s)')^{-1}\sigma_A(s)\,dW_s\Big\}.$$

PROOF. Substituting u^* from (3.9) into the definition of Γ in (3.7), we obtain

$$\Gamma = \mathrm{E}\Big[\int_t^T \Big(\lambda\alpha_A'(\sigma_A\sigma_A')^{-1}\alpha_A - \Big(\frac{\partial\Gamma}{\partial X}\Big)'\alpha_A - \Big(\frac{\partial\Gamma}{\partial l}\Big)'\sigma_L\sigma_A'(\sigma_A\sigma_A')^{-1}\alpha_A l + le^{\int_s^T r(\tau)\,d\tau}(\alpha_A'(\sigma_A\sigma_A')^{-1}\sigma_A\sigma_L' - \alpha_L)\Big)\,ds|\mathcal{F}_t\Big]. \tag{3.20}$$

https://doi.org/10.1017/S1446181120000164 Published online by Cambridge University Press

Note that $\Gamma = 0$ at $t = T$. From the SDE of Γ in (3.15),

$$-\Gamma = \mathrm{E}_t\left[\int_t^T\left(\frac{\partial\Gamma}{\partial t} + \left(\frac{\partial\Gamma}{\partial X}\right)'(\theta - \mathcal{A}X) + \frac{\partial\Gamma}{\partial l}\tilde{\alpha}_L l\right.\right.$$
$$\left.\left. + \frac{1}{2}tr\left(\begin{pmatrix}\sigma_A' & l\sigma_L'\end{pmatrix}\begin{pmatrix}\frac{\partial^2\Gamma}{\partial X^2} & \frac{\partial^2\Gamma}{\partial X\partial l}\\ \frac{\partial^2\Gamma}{\partial l\partial X} & \frac{\partial^2\Gamma}{\partial l^2}\end{pmatrix}\begin{pmatrix}\sigma_A\\ l\sigma_L\end{pmatrix}\right)\right) ds\right] - \mathrm{E}_t[\Gamma(T, X(T), l(T))]$$
$$= \mathrm{E}_t\left[\int_t^T\left\{\frac{\partial\Gamma}{\partial t} + \left(\frac{\partial\Gamma}{\partial X}\right)'(\theta - \mathcal{A}X) + \frac{\partial\Gamma}{\partial l}\tilde{\alpha}_L l\right.\right.$$
$$\left.\left. + \frac{1}{2}tr\left(\begin{pmatrix}\sigma_A' & l\sigma_L'\end{pmatrix}\begin{pmatrix}\frac{\partial^2\Gamma}{\partial X^2} & \frac{\partial^2\Gamma}{\partial X\partial l}\\ \frac{\partial^2\Gamma}{\partial l\partial X} & \frac{\partial^2\Gamma}{\partial l^2}\end{pmatrix}\begin{pmatrix}\sigma_A\\ l\sigma_L\end{pmatrix}\right)\right\} ds\right]. \tag{3.21}$$

Combining (3.20) and (3.21) produces a partial differential equation (PDE) of Γ. Specifically,

$$\frac{\partial\Gamma}{\partial t} + \left(\frac{\partial\Gamma}{\partial X}\right)'(\theta - \mathcal{A}X - \alpha_A) + \frac{\partial\Gamma}{\partial l}(\tilde{\alpha}_L - \sigma_L\sigma_A'(\sigma_A\sigma_A')^{-1}\alpha_A)l$$
$$+ \frac{1}{2}tr\left(\sigma_A'\frac{\partial^2\Gamma}{\partial X^2}\sigma_A\right) + \frac{1}{2}\frac{\partial^2\Gamma}{\partial l^2}\sigma_L\sigma_L' l^2 + tr\left(\sigma_A'\frac{\partial^2\Gamma}{\partial X\partial l}\sigma_L l\right)$$
$$+ \lambda\alpha_A'(\sigma_A\sigma_A')^{-1}\alpha_A + le^{\int_s^T r(\tau)\,d\tau}(\alpha_A'(\sigma_A\sigma_A')^{-1}\sigma_A\sigma_L' - \alpha_L) = 0, \tag{3.22}$$

with $\Gamma(T, X(T), L(T)) = 0$. In fact, $\partial\Gamma/\partial X$ and $\partial\Gamma/\partial l$ are the gradient vectors of Γ with respect to X and l, respectively. The matrix

$$\begin{pmatrix}\frac{\partial^2\Gamma}{\partial X^2} & \frac{\partial^2\Gamma}{\partial X\partial l}\\ \frac{\partial^2\Gamma}{\partial l\partial X} & \frac{\partial^2\Gamma}{\partial l^2}\end{pmatrix}$$

denotes the Hessian matrix of Γ, and $tr(\cdot)$ is the trace of a square matrix. By the Feynman–Kač formula,

$$\Gamma(t, X(t), l(t)) = \lambda\hat{\mathcal{H}}(t, \alpha_A(t)) - \mathcal{G}(t, \alpha_A(t), l(t)),$$

where

$$\hat{\mathcal{H}}(t,\alpha_A(t)) = \mathrm{E}_t^{\hat{\mathcal{P}}}\Big[\int_t^T \alpha_A(s)'(\sigma_A(s)\sigma_A(s))^{-1}\alpha_A(s)\,ds\Big],$$

$$\mathcal{G}(t,\alpha_A(t),l(t)) = \mathrm{E}_t^{\hat{\mathcal{P}}}\Big[\int_t^T l e^{\int_s^T r(\tau)\,d\tau}(\alpha_L - \alpha_A'(\sigma_A\sigma_A)^{-1}\sigma_A\sigma_L')\,ds\Big],$$

$$\alpha_A(t) = \theta(t) - \mathcal{A}X(t) + \tfrac{1}{2}\mathcal{D}(\sigma_A(t)\sigma_A(t)')\mathbf{1} - r(t)\mathbf{1},$$

and the equivalent probability measure $\hat{\mathcal{P}}$ is defined as

$$\frac{d\hat{\mathcal{P}}}{d\mathcal{P}} = \exp\Big\{\int_t^T -\frac{1}{2}\alpha_A(s)'(\sigma_A\sigma_A')^{-1}\alpha_A\,ds - \int_t^T \alpha_A(s)'(\sigma_A\sigma_A')^{-1}\sigma_A\,dW_s\Big\}. \qquad \square$$

Although the above lemma shows an expectation representation for Γ, it is still not explicit. The next step finds explicit forms for $\hat{\mathcal{H}}$ and $\mathcal{G}$.

LEMMA 3.4. *The expectation $\hat{\mathcal{H}}(t,\alpha_A(t))$ in (3.18) is equal to a quadratic form of $\alpha_A(t)$:*

$$\hat{\mathcal{H}}(t,\alpha_A(t)) = \tfrac{1}{2}\alpha_A(t)'\hat{K}(t)\alpha_A(t) + \hat{N}'(t)\alpha_A(t) + \hat{M}(t),$$

where

$$\hat{K}(t) = 2\int_t^T (\sigma_A(s)\sigma_A(s)')^{-1}\,ds, \tag{3.23}$$

$$\hat{N}(t)' = 2\int_t^T \Theta(s)'\int_s^T (\sigma_A(\tau)\sigma_A(\tau)')^{-1}\,d\tau\,ds, \tag{3.24}$$

$$\begin{aligned}\hat{M}(t) = \int_t^T \Big[2\int_s^T \Theta(\tau)'\Big(\int_\tau^T (\sigma_A(\xi)\sigma_A(\xi)')^{-1}d\xi\Big)\,d\tau\,\Theta(s) \\ + tr\Big(\sigma_A(s)'\mathcal{A}'\int_s^T (\sigma_A(\tau)\sigma_A(\tau)')^{-1}\,d\tau\mathcal{A}\sigma_A(s)\Big)\Big]\,ds,\end{aligned} \tag{3.25}$$

$$\Theta = \dot{\theta} + \tfrac{1}{2}(\dot{\mathcal{D}} + \mathcal{A}\mathcal{D})\mathbf{1} - (\dot{r} + \mathcal{A}r)\mathbf{1}, \tag{3.26}$$

and $0_{i\times j}$ is an $(i \times j)$-matrix with all entries 0.

PROOF. By Itô's lemma, the dynamic of $\alpha_A(t)$ under the $\hat{\mathcal{P}}$-measure is given by

$$d\alpha_A(t) = \Theta\,dt - \mathcal{A}\sigma_A\,d\hat{W}_t, \tag{3.27}$$

where Θ is given in (3.26). Applying the Feynman–Kač formula to $\hat{\mathcal{H}}(t,\alpha_A)$ in (3.18) with respect to (3.27), a PDE governing $\hat{\mathcal{H}}$ is obtained:

$$\begin{aligned}&\frac{\partial\hat{\mathcal{H}}}{\partial t} + \Big(\frac{\partial\hat{\mathcal{H}}}{\partial\alpha_A}\Big)'\Theta + \frac{1}{2}tr\Big(\sigma_A'\mathcal{A}'\frac{\partial^2\hat{\mathcal{H}}}{\partial{\alpha_A}^2}\mathcal{A}\sigma_A\Big) + \alpha_A'(\sigma_A\sigma_A')^{-1}\alpha_A = 0,\\ &\hat{\mathcal{H}}(T,\alpha_A) = 0.\end{aligned} \tag{3.28}$$

https://doi.org/10.1017/S1446181120000164 Published online by Cambridge University Press

Let $\hat{h}(t,\alpha_A) = (1/2)\alpha_A(t)'\hat{K}(t)\alpha_A(t) + \hat{N}(t)'\alpha_A(t) + \hat{M}(t)$, where $\hat{K}$, $\hat{N}$ and $\hat{M}$ are defined in (3.23), (3.24) and (3.25), respectively. Therefore, $\hat{h}(T,\alpha_A(T)) = 0 = \hat{H}(T,\alpha_A(T))$, and

$$\begin{aligned}\dot{\hat{K}} + 2(\sigma_A\sigma_A')^{-1} &= 0_{m\times m}, \quad \hat{K}(T) = 0_{m\times m},\\ \dot{\hat{N}}' + \Theta'\hat{K} &= 0_{1\times m}, \quad \hat{N}(T) = 0_{m\times 1},\\ \dot{\hat{M}} + \hat{N}'\Theta + \tfrac{1}{2}tr(\sigma_A'\mathcal{A}'\hat{K}\mathcal{A}\sigma_A) &= 0, \quad \hat{M}(T) = 0.\end{aligned}$$

Consider

$$\begin{aligned}&\frac{\partial\hat{h}}{\partial t} + \Big(\frac{\partial\hat{h}}{\partial\alpha_A}\Big)'\Theta + \frac{1}{2}tr\Big(\sigma_A'\mathcal{A}'\frac{\partial^2\hat{h}}{\partial{\alpha_A}^2}\mathcal{A}\sigma_A\Big) + \alpha_A'(\sigma_A\sigma_A')^{-1}\alpha_A\\ &\quad= \tfrac{1}{2}\alpha_A'\dot{\hat{K}}\alpha_A + \dot{\hat{N}}'\alpha_A + \dot{\hat{M}} + (\hat{K}\alpha_A + \hat{N})'\Theta + \tfrac{1}{2}tr(\sigma_A'\mathcal{A}'\hat{K}\mathcal{A}\sigma_A) + \alpha_A'(\sigma_A\sigma_A')^{-1}\alpha_A\\ &\quad= \tfrac{1}{2}\alpha_A'(\dot{\hat{K}} + 2(\sigma_A\sigma_A')^{-1})\alpha_A + (\dot{\hat{N}}' + \Theta'\hat{K})\alpha_A + \dot{\hat{M}} + \hat{N}'\Theta + \tfrac{1}{2}tr(\sigma_A'\mathcal{A}'\hat{K}\mathcal{A}\sigma_A) = 0.\end{aligned}$$

Hence, $\hat{h}(t,\alpha_A)$ is the solution of PDE (3.28), and

$$\hat{\mathcal{H}}(t,\alpha_A(t)) = \tfrac{1}{2}\alpha_A(t)'\hat{K}(t)\alpha_A(t) + \hat{N}'(t)\alpha_A(t) + \hat{M}(t). \qquad \square$$

LEMMA 3.5. *The function $\mathcal{G}(t,\alpha_A(t),l(t))$ in (3.19) has the following explicit form:*

$$\mathcal{G}(t,\alpha_A(t),l(t)) = l(t)e^{\int_t^T r(s)\,ds}[e^{\int_t^T \alpha_L(s)\,ds}e^{p(t)'\alpha_A(t)+q(t)} - 1],$$

where

$$p(t) = -\int_t^T (\sigma_A(s)\sigma_A(s)')^{-1}\sigma_A(s)\sigma_L(s)'\,ds. \tag{3.29}$$

$$q(t) = \int_t^T [p(s)'(\Theta(s) - \mathcal{A}\sigma_A(s)\sigma_L(s)') + \tfrac{1}{2}p(s)'\mathcal{A}\sigma_A(s)\sigma_A(s)'\mathcal{A}'p(s)]\,ds. \tag{3.30}$$

PROOF. Under $\hat{\mathcal{P}}$-measure, the dynamics of $l(t)$ and $e^{\int_t^T r(s)\,ds}l(t)$ are given by

$$dl(t) = l(t)[(\tilde{\alpha}_L(t) - \alpha_A(t)'(\sigma_A(s)\sigma_A(s)')^{-1}\sigma_A(t)\sigma_L(t)')\,dt + \sigma_L(t)\,d\hat{W}_t], \tag{3.31}$$

$$\begin{aligned}d(e^{\int_t^T r(s)\,ds}l(t)) = e^{\int_t^T r(s)\,ds}&l(t)[(\alpha_L(t) - \alpha_A(t)'(\sigma_A(t)\sigma_A(t)')^{-1}\sigma_A(t)\sigma_L{}'(t))\,dt\\ &+ \sigma_L(t)\,d\hat{W}_t],\end{aligned} \tag{3.32}$$

respectively. After integrating (3.32) from t to T and then taking the expectation of both sides of (3.32), we have

$$\mathrm{E}^{\hat{\mathcal{P}}}[l(T)|\mathcal{F}_t] - e^{\int_t^T r(s)\,ds}l(t) = \mathrm{E}_t^{\hat{\mathcal{P}}}\Big[\int_t^T e^{\int_s^T r(\tau)\,d\tau}l(s)(\alpha_L - \alpha_A'(\sigma_A\sigma_A')^{-1}\sigma_A\sigma_L{}')\,ds\Big].$$

This is equal to $\mathcal{G}(t,\alpha_A(t),l(t))$ according to the expression in (3.19). We conclude that

$$\mathcal{G}(t,\alpha_A(t),l(t)) = \mathrm{E}^{\hat{\mathcal{P}}}[l(T)|\mathcal{F}_t] - e^{\int_t^T r(s)\,ds}l(t). \tag{3.33}$$

Next, we find the explicit form of $\mathrm{E}^{\hat{\mathcal{P}}}[l(T)|\mathcal{F}_t]$.

https://doi.org/10.1017/S1446181120000164 Published online by Cambridge University Press

According to the SDE in (3.31), we have

$$l(T) = l(t)e^{\int_t^T [\tilde{\alpha}_L - \alpha'_A(\sigma_A\sigma'_A)^{-1}\sigma_A\sigma'_L - \sigma_L\sigma'_L/2]\,ds + \int_t^T \sigma_L\,d\hat{W}_s}.$$

This implies that

$$\begin{aligned}
\mathrm{E}^{\hat{\mathcal{P}}}[l(T)|\mathcal{F}_t] &= l(t)\mathrm{E}^{\hat{\mathcal{P}}}[e^{\int_t^T [\tilde{\alpha}_L - \alpha'_A(\sigma_A\sigma'_A)^{-1}\sigma_A\sigma'_L - \sigma_L\sigma'_L/2]\,ds + \int_t^T \sigma_L\,d\hat{W}_s}|\mathcal{F}_t] \\
&= l(t)\mathrm{E}^{Q}[e^{\int_t^T \tilde{\alpha}_L - \alpha'_A(\sigma_A\sigma'_A)^{-1}\sigma_A\sigma'_L\,ds}|\mathcal{F}_t] \\
&= l(t)e^{\int_t^T \tilde{\alpha}_L\,ds}\mathrm{E}^{Q}[e^{-\int_t^T \alpha'_A(\sigma_A\sigma'_A)^{-1}\sigma_A\sigma'_L\,ds}|\mathcal{F}_t] \\
&= l(t)e^{\int_t^T \tilde{\alpha}_L\,ds}\mathcal{G}_1(t, \alpha_A(t)),
\end{aligned}$$

where

$$\begin{aligned}
\frac{dQ}{d\hat{\mathcal{P}}} &= \exp\bigg\{\int_t^T -\frac{1}{2}\sigma_L(s)\sigma_L(s)'\,ds + \int_t^T \sigma_L(s)\,d\hat{W}_s\bigg\} \text{ and} \\
\mathcal{G}_1(t, \alpha_A(t)) &= \mathrm{E}^{Q}[e^{-\int_t^T \alpha'_A(\sigma_A\sigma'_A)^{-1}\sigma_A\sigma'_L\,ds}|\mathcal{F}_t].
\end{aligned}$$

By Itô's lemma, the dynamic of $\alpha_A(t)$ under Q-measure is given by

$$d\alpha_A = (\Theta - \mathcal{A}\sigma_A\sigma'_L)\,dt - \mathcal{A}\sigma_A d\tilde{W}_t, \tag{3.34}$$

where $d\tilde{W}_t = d\hat{W}_t - \sigma_L(t)'\,dt$. Applying the Feynman–Kač formula to $\mathcal{G}_1(t, \alpha_A(t))$ with respect to (3.34), a PDE governing $\mathcal{G}_1$ is obtained as follows:

$$\begin{aligned}
&\frac{\partial \mathcal{G}_1}{\partial t} + \Big(\frac{\partial \mathcal{G}_1}{\partial \alpha_A}\Big)'(\Theta - \mathcal{A}\sigma_A\sigma'_L) + \frac{1}{2}tr\Big(\sigma_A\mathcal{A}'\frac{\partial^2 \mathcal{G}_1}{\partial {\alpha_A}^2}\mathcal{A}\sigma_A\Big) - \alpha'_A(\sigma_A\sigma'_A)^{-1}\sigma_A\sigma'_L\mathcal{G}_1 = 0; \\
&\mathcal{G}_1(T, \alpha_A) = 1.
\end{aligned} \tag{3.35}$$

Consider an exponential form $\mathcal{G}_1(t, \alpha_A(t)) = e^{p(t)'\alpha_A(t)+q(t)}$, where $p(t)$ and $q(t)$ are given in (3.29) and (3.30). Simple differentiation shows that

$$\frac{\partial \mathcal{G}_1}{\partial t} = \mathcal{G}_1(\dot{p}'\alpha_A + \dot{q}), \quad \frac{\partial \mathcal{G}_1}{\partial \alpha_A} = \mathcal{G}_1 p, \quad \frac{\partial^2 \mathcal{G}_1}{\partial {\alpha_A}^2} = \mathcal{G}_1 pp',$$

$$\dot{p} - (\sigma_A\sigma'_A)^{-1}\sigma_A\sigma'_l = 0, \quad p(T) = 0;$$

$$\dot{q} + p'(\Theta - \mathcal{A}\sigma_A\sigma'_L) + \tfrac{1}{2}p'\mathcal{A}\sigma_A\sigma'_A\mathcal{A}'p = 0, \quad q(T) = 0.$$

Substituting these formulas into (3.35) yields

$$\begin{aligned}
&\mathcal{G}_1(\dot{p}'\alpha_A + \dot{q}) + \mathcal{G}_1 p'(\Theta - \mathcal{A}\sigma_A\sigma'_L) \\
&\qquad + \tfrac{1}{2}tr(\sigma'_A\mathcal{A}'\mathcal{G}_1 pp'\mathcal{A}\sigma_A) - \alpha'_A(\sigma_A\sigma'_A)^{-1}\sigma_A\sigma'_L\mathcal{G}_1 = 0. \\
&\mathcal{G}_1[\alpha'_A(\dot{p} + (\sigma_A\sigma'_A)^{-1}\sigma_A\sigma'_L) + (\dot{q} + p'(\Theta - \mathcal{A}\sigma_A\sigma'_L) + \tfrac{1}{2}p'\mathcal{A}\sigma_A\sigma'_A\mathcal{A}'p)] = 0.
\end{aligned}$$

https://doi.org/10.1017/S1446181120000164 Published online by Cambridge University Press

This verifies that the exponential affine form satisfies the governing equation of (3.35). In addition, the values of $p(t)$ and $q(t)$ at $t = T$ ensure that $\mathcal{G}_1(T, \alpha_A) = 1$. From (3.33),

$$\begin{aligned}\mathcal{G}(t, \alpha_A(t), l(t)) &= \mathrm{E}^{\hat{\mathcal{P}}}[l(T)|\mathcal{F}_t] - e^{\int_t^T r(s)\,ds} l(t)\\ &= l(t)e^{\int_t^T \tilde{\alpha}_L\,ds}\mathcal{G}_1(t, \alpha_A(t)) - e^{\int_t^T r(s)\,ds} l(t)\\ &= l(t)e^{\int_t^T \tilde{\alpha}_L(s)\,ds} e^{p(t)'\alpha_A(t)+q(t)} - e^{\int_t^T r(s)\,ds} l(t)\\ &= l(t)e^{\int_t^T r(s)\,ds}[e^{\int_t^T \alpha_L(s)\,ds} e^{p(t)'\alpha_A(t)+q(t)} - 1].\end{aligned} \tag{3.36}$$

This completes the proof. □

THEOREM 3.6. *The function $\Gamma(t, X(t), l(t))$ defined in (3.7) takes the form*

$$\begin{aligned}\Gamma(t, X(t), l(t)) = \lambda[\tfrac{1}{2}\alpha_A(t)'\hat{K}(t)\alpha_A(t) + \hat{N}'(t)\alpha_A(t) + \hat{M}(t)]\\ - l(t)e^{\int_t^T r(s)\,ds}[e^{\int_t^T \alpha_L(s)\,ds} e^{p(t)'\alpha_A(t)+q(t)} - 1],\end{aligned}$$

where $\alpha_A(t)$, $\hat{K}(t)$, $\hat{N}(t)$, $\Theta(t)$, $p(t)$ and $q(t)$ are defined in (2.8), (3.23), (3.24), (3.26), (3.29) and (3.30), respectively. Besides, the time-consistent trading strategy ($P(\lambda)$) in Problem 2.1 is given by

$$\begin{aligned}&u^*(t, X(t), l(t))\\ &\quad = \lambda e^{-\int_t^T r(s)\,ds}[((\sigma_A(t)\sigma_A(t)')^{-1} + \mathcal{A}\hat{K}(t))\alpha_A(t) + \mathcal{A}'\hat{N}(t)]\\ &\qquad + l(t)e^{\int_t^T \alpha_L(s)\,ds + p(t)'\alpha_A(t)+q(t)}[(\sigma_A(t)\sigma_A(t)')^{-1}\sigma_A(t)\sigma_L(t)' - \mathcal{A}'p(t)].\end{aligned} \tag{3.37}$$

PROOF. Combining the results in Lemmas 3.4 and 3.5,

$$\begin{aligned}\Gamma(t, X(t), l(t)) &= \lambda\hat{\mathcal{H}}(t, \alpha_A(t)) - \mathcal{G}(t, \alpha_A(t), l(t))\\ &= \lambda[\tfrac{1}{2}\alpha_A(t)'\hat{K}(t)\alpha_A(t) + \hat{N}'(t)\alpha_A(t) + \hat{M}(t)]\\ &\quad - l(t)e^{\int_t^T r(s)\,ds}[e^{\int_t^T \alpha_L(s)\,ds} e^{p(t)'\alpha_A(t)+q(t)} - 1].\end{aligned}$$

Taking derivatives of $\Gamma(t, X(t), l(t))$ with respect to X and l,

$$\begin{aligned}\frac{\partial\Gamma}{\partial X} &= -\lambda\mathcal{A}'(\hat{K}\alpha_A + \hat{N}) + l(t)e^{\int_t^T \alpha_L(s)\,ds + p'\alpha_A + q}\mathcal{A}'p \quad \text{and}\\ \frac{\partial\Gamma}{\partial l} &= e^{\int_t^T r(s)\,ds}(1 - e^{\int_t^T \alpha_L(s)\,ds + p'\alpha_A + q}).\end{aligned}$$

Substituting the above derivatives into the expression for u^* in (3.9),

$$\begin{aligned}u^*(t, X(t), l(t)) &= \lambda e^{-\int_t^T r(s)\,ds}[((\sigma_A(t)\sigma_A(t)')^{-1} + \mathcal{A}\hat{K}(t))\alpha_A(t) + \mathcal{A}'\hat{N}(t)]\\ &\quad + l(t)e^{\int_t^T \alpha_L(s)\,ds + p(t)'\alpha_A(t)+q(t)}[(\sigma_A(t)\sigma_A(t)')^{-1}\sigma_A(t)\sigma_L(t)' - \mathcal{A}'p(t)].\end{aligned}$$ □

https://doi.org/10.1017/S1446181120000164 Published online by Cambridge University Press

The optimal ALM strategy given in (3.37) shows that a pairs-trader has to adjust the allocation on the stock market by observing the variability of her stochastic liability. When either the liability value or the liability volatility is zero, the stock allocation strategy is consistent with the literature [13]. However, when the investor has a stochastic liability, that is, both $l(t)$ and σ_L are nonzero, an adjustment should be made based on the sensitivity of Γ with respect to the change in the level of liability. In our time-consistent setting, the function Γ detailed in Section 3.2 reflects the change in the objective function associated with the time-consistency constraint. Therefore, the TC-MV criterion implies a nontrivial adjustment to stochastic liability through the function Γ, which also captures the feature of cointegration.

4. TC-MV ALM efficient frontier

An important topic in MV problems is the “efficient frontier”, which describes the trade-off between the expected final surplus and the variance of the final surplus in the equilibrium strategy.

LEMMA 4.1. *Consider the surplus process (2.9) for $Y(t)$ and the liability process (2.6) for $l(t)$. If the investor employs the equilibrium ALM strategy $u^*(t, X(t), l(t))$, then the expected surplus and the variance of the surplus are given as follows:*

$$\begin{aligned}\mathrm{E}[Y^*(T)] &= e^{\int_0^T r(s)\,ds}Y(0) + \lambda E^{\hat{\mathcal{P}}}\Big[\int_0^T \alpha_A(s)'(\sigma_A(s)\sigma_A(s)')^{-1}\alpha_A(s)\,ds\Big]\\ &\quad - (E^{\hat{\mathcal{P}}}[l(T)] - e^{\int_0^T r(s)\,ds}l_0),\end{aligned} \tag{4.1}$$

$$\begin{aligned}\mathrm{Var}(Y^*(T)) &= \lambda^2 E\Big[\int_0^T \alpha_A(s)'(\sigma_A(s)\sigma_A(s)')^{-1}\alpha_A(s)\,ds\Big]\\ &\quad + E\Big[\int_0^T e^{\int_s^T 2\tilde{\alpha}_L(\tau)\,d\tau + 2p(s)'\alpha_A(s)+2q(s)}\\ &\quad \times l^2(s)\sigma_L(s)'(I_m - \sigma_A(s)'(\sigma_A(s)\sigma_A(s)')^{-1}\sigma_L(s)')\,ds\Big].\end{aligned} \tag{4.2}$$

PROOF. According to the definition of Γ,

$$\mathrm{E}[Y^*(T)] = e^{\int_0^T r(s)\,ds}Y(0) + \Gamma(0, X(0), l_0). \tag{4.3}$$

Substituting formulas (3.17), (3.18) and (3.33) into (4.3) yields

$$\begin{aligned}\mathrm{E}[Y^*(T)] &= e^{\int_0^T r(s)\,ds}Y(0) + \lambda \mathrm{E}^{\hat{\mathcal{P}}}\Big[\int_0^T \alpha_A(s)'(\sigma_A(s)\sigma_A(s)')^{-1}\alpha_A(s)\,ds\Big]\\ &\quad - (\mathrm{E}^{\hat{\mathcal{P}}}[l(T)] - e^{\int_0^T r(s)\,ds}l_0).\end{aligned}$$

https://doi.org/10.1017/S1446181120000164 Published online by Cambridge University Press

To calculate $\mathrm{Var}(Y^*(T))$, consider the SDE and PDE of Γ in (3.15) and (3.22):

$$\begin{aligned} d\Gamma &= \bigg[\frac{\partial \Gamma}{\partial t} + \Big(\frac{\partial \Gamma}{\partial X}\Big)'(\theta - \mathcal{A}X) + \frac{\partial \Gamma}{\partial l}\tilde{\alpha}_L l \\ &\quad + \frac{1}{2} tr\bigg\{ \begin{pmatrix} \sigma_A' & l\sigma_L' \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \Gamma}{\partial X^2} & \frac{\partial^2 \Gamma}{\partial X \partial l} \\ \frac{\partial^2 \Gamma}{\partial l \partial X} & \frac{\partial^2 \Gamma}{\partial l^2} \end{pmatrix} \begin{pmatrix} \sigma_A \\ l\sigma_L \end{pmatrix} \bigg\}\bigg] dt + \bigg[\Big(\frac{\partial \Gamma}{\partial X}\Big)'\sigma_A + \frac{\partial \Gamma}{\partial l}\sigma_L l\bigg] dW_t, \\ &= \bigg[\Big(\frac{\partial \Gamma}{\partial X}\Big)'\alpha_A + \frac{\partial \Gamma}{\partial l}\sigma_L\sigma_A'(\sigma_A\sigma_A')^{-1}\alpha_A l - \lambda\alpha_A'(\sigma_A\sigma_A')^{-1}\alpha_A \\ &\quad - le^{\int_s^T r(\tau)\,d\tau}(\alpha_A'(\sigma_A\sigma_A')^{-1}\sigma_A\sigma_L' - \alpha_L)\bigg] dt + \bigg[\Big(\frac{\partial \Gamma}{\partial X}\Big)'\sigma_A + \frac{\partial \Gamma}{\partial l}\sigma_L l\bigg] dW_t. \end{aligned}$$

Equation (3.4) shows that

$$d(e^{\int_t^T r(s)\,ds}Y^*) = e^{\int_t^T r(s)\,ds}[(\alpha_A' u^* - \alpha_L l)\,dt + (u^{*\prime}\sigma_A - l\sigma_L)\,dW_t].$$

By substituting u^* in (3.9) into the drift term of $e^{\int_t^T r(s)\,ds}Y^*$,

$$\begin{aligned} e^{\int_t^T r(s)\,ds}(\alpha_A' u^* - \alpha_L l) = -\bigg[&\Big(\frac{\partial \Gamma}{\partial X}\Big)'\alpha_A + \frac{\partial \Gamma}{\partial l}\sigma_L\sigma_A'(\sigma_A\sigma_A')^{-1}\alpha_A l \\ &- \lambda\alpha_A'(\sigma_A\sigma_A')^{-1}\alpha_A - le^{\int_s^T r(\tau)\,d\tau}(\alpha_A'(\sigma_A\sigma_A')^{-1}\sigma_A\sigma_L' - \alpha_L)\bigg], \end{aligned}$$

which is a negative value of the drift of Γ. After substituting u^* in (3.9) into the volatility of $e^{\int_t^T r(s)\,ds}Y^*$,

$$\begin{aligned} e^{\int_t^T r(s)\,ds}(u^{*\prime}\sigma_A - l\sigma_L) &= \lambda\alpha_A'(\sigma_A\sigma_A')^{-1}\sigma_A + le^{\int_s^T r(\tau)\,d\tau}(\sigma_L\sigma_A'(\sigma_A\sigma_A')^{-1}\sigma_A - \sigma_L) \\ &\quad - \Big(\frac{\partial \Gamma}{\partial X}\Big)'\sigma_A - \frac{\partial \Gamma}{\partial l}\sigma_L\sigma_A'(\sigma_A\sigma_A')^{-1}\sigma_A l. \end{aligned}$$

Hence,

$$\begin{aligned} &d(e^{\int_t^T r(s)\,ds}Y^*) + d\Gamma \\ &\quad = [\lambda\alpha_A'(\sigma_A\sigma_A')^{-1}\sigma_A - e^{\int_t^T \alpha_L(s)\,ds + p'\alpha_A + q}l\sigma_L(I_m - \sigma_A'(\sigma_A\sigma_A')^{-1}\sigma_A)]\,dW_t. \end{aligned}$$

Integrating both sides from 0 to T and applying the variance operator to them, the following result holds.

$$\begin{aligned} \mathrm{LHS} &= \mathrm{Var}(Y^*(T) - e^{\int_0^T r(s)\,ds}Y(0) + \Gamma(T, X(T), l(T)) - \Gamma(0, X(0), l_0)) \\ &= \mathrm{Var}(Y^*(T)). \end{aligned}$$

$$\begin{aligned} \mathrm{RHS} = \mathrm{Var}\bigg(\int_0^T &(\lambda\alpha_A(t)'(\sigma_A(t)\sigma_A(t)')^{-1}\sigma_A(t) \\ &- e^{\int_t^T \tilde{\alpha}_L(s)\,ds + p(t)'\alpha_A(t) + q(t)}l(t)\sigma_L(t)(I_m - \sigma_A(t)'(\sigma_A(t)\sigma_A(t)')^{-1}\sigma_A(t)))\,dW_t\bigg) \end{aligned}$$

$$= \lambda^2 \mathrm{E}\Big[\int_0^T \alpha_A(s)'(\sigma_A(s)\sigma_A(s)')^{-1}\alpha_A(s)\,ds\Big]$$
$$+ \mathrm{E}\Big[\int_0^T e^{\int_s^T 2\tilde{\alpha}_L(\tau)\,d\tau + 2p(s)'\alpha_A(s) + 2q(s)} l^2(s)\sigma_L(s)'$$
$$\times (I_m - \sigma_A(s)'(\sigma_A(s)\sigma_A(s)')^{-1}\sigma_L(s)')\,ds\Big]. \qquad \square$$

LEMMA 4.2. *The expectation* $E[\int_0^T \alpha_A(s)'(\sigma_A(s)\sigma_A(s)')^{-1}\alpha_A(s)\,ds]$ *takes the quadratic form of* $\alpha_A(0)$*:*

$$\tfrac{1}{2}\alpha_A(0)'K(0)\alpha_A(0) + N'(0)\alpha_A(0) + M(0),$$

where

$$K(t) = 2\int_t^T e^{\mathcal{A}'(t-s)/2}(\sigma_A(s)\sigma_A(s)')^{-1}e^{\mathcal{A}(t-s)/2}\,ds, \tag{4.4}$$

$$N(t)' = 2\int_t^T \Theta(s)'\Big(\int_s^T e^{\mathcal{A}'(s-\tau)/2}(\sigma_A(\tau)\sigma_A(\tau)')^{-1}e^{\mathcal{A}(s-\tau)/2}\,d\tau\Big)e^{\mathcal{A}(t-s)}\,ds, \tag{4.5}$$

$$M(t) = \int_t^T N(s)'\Theta(s) + \frac{1}{2}tr(\sigma_A(s)'\mathcal{A}'K(s)\mathcal{A}\sigma_A(t))\,ds. \tag{4.6}$$

PROOF. By Itô's lemma, the dynamic of $\alpha_A(t)$ under the $\mathcal{P}$-measure is given by

$$d\alpha_A(t) = (\Theta - \mathcal{A}\alpha_A)\,dt - \mathcal{A}\sigma_A\, d\hat{W}_t, \tag{4.7}$$

where Θ is given in (3.26). Applying the Feynman–Kač formula to $\mathcal{H}(t,\alpha_A)$, where

$$\mathcal{H}(t,\alpha_A) = \mathrm{E}\Big[\int_t^T \alpha_A(s)'(\sigma_A(s)\sigma_A(s)')^{-1}\alpha_A(s)\,ds|\mathcal{F}_t\Big],$$

with respect to (4.7), a PDE is obtained for $\mathcal{H}$ as

$$\frac{\partial\mathcal{H}}{\partial t} + \Big(\frac{\partial\mathcal{H}}{\partial\alpha_A}\Big)'(\Theta - \mathcal{A}\alpha_A) + \frac{1}{2}tr\Big(\sigma_A'\mathcal{A}'\frac{\partial^2\mathcal{H}}{\partial{\alpha_A}^2}\mathcal{A}\sigma_A\Big) + \alpha_A'(\sigma_A\sigma_A')^{-1}\alpha_A = 0, \tag{4.8}$$
$$\mathcal{H}(T,\alpha_A) = 0.$$

After differentiating (4.4–4.6) with respect to t, we have

$$\dot{K}(t) - K(t)\mathcal{A} - \mathcal{A}'K(t) + 2(\sigma_A(t)\sigma_A(t)')^{-1} = 0_{m\times m}, \quad K(T) = 0_{m\times m},$$
$$\dot{N}(t)' - N(t)'\mathcal{A} + \Theta(t)'K(t) = 0, \quad N(T) = 0_{m\times 1}$$
$$\dot{M}(t) + N(t)'\Theta(t) + \tfrac{1}{2}tr(\sigma_A(t)'\mathcal{A}'K(t)\mathcal{A}\sigma_A(t)) = 0, \quad M(T) = 0.$$

Consider the quadratic form

$$h(t,\alpha_A) = \tfrac{1}{2}\alpha_A(t)'K(t)\alpha_A(t) + N(t)'\alpha_A(t) + M(t). \tag{4.9}$$

https://doi.org/10.1017/S1446181120000164 Published online by Cambridge University Press

Clearly, $h(T,\alpha_A(T)) = 0 = \mathcal{H}(T,\alpha_A(T))$. Also, the differentiation shows that

$$\frac{\partial h}{\partial t} = \frac{1}{2}\alpha_A' \dot{K}\alpha_A + \dot{N}'\alpha_A + \dot{M}, \qquad \frac{\partial \mathcal{H}}{\partial \alpha_A} = K\alpha_A + \hat{N}, \qquad \frac{\partial^2 \mathcal{H}}{\partial {\alpha_A}^2} = K.$$

Besides,

$$\begin{aligned}
&\frac{\partial h}{\partial t} + \Big(\frac{\partial h}{\partial \alpha_A}\Big)'(\Theta - \mathcal{A}\alpha_A) + \frac{1}{2}tr\Big(\sigma_A'\mathcal{A}'\frac{\partial^2 h}{\partial {\alpha_A}^2}\mathcal{A}\sigma_A\Big) + \alpha_A'(\sigma_A\sigma_A')^{-1}\alpha_A \\
&\quad = \tfrac{1}{2}\alpha_A'\dot{K}\alpha_A + \dot{N}'\alpha_A + \dot{M} + (\hat{K}\alpha_A + \hat{N})'(\Theta - \mathcal{A}\alpha_A) \\
&\qquad + \tfrac{1}{2}tr(\sigma_A'\mathcal{A}'K\mathcal{A}\sigma_A) + \alpha_A'(\sigma_A\sigma_A')^{-1}\alpha_A \\
&\quad = \tfrac{1}{2}\alpha_A'(\dot{K} - \mathcal{A}'K - K\mathcal{A} + 2(\sigma_A\sigma_A')^{-1})\alpha_A + (\dot{N}' - N'\mathcal{A} + \Theta'K)\alpha_A \\
&\qquad + \dot{M} + N'\Theta + \tfrac{1}{2}tr(\sigma_A'\mathcal{A}'K\mathcal{A}\sigma_A) = 0.
\end{aligned}$$

This verifies that the quadratic form (4.9) satisfies the governing equation of (4.8). Therefore,

$$\mathcal{H}(t,\alpha_A(t)) = \tfrac{1}{2}\alpha_A(t)'K(t)\alpha_A(t) + N(t)'\alpha_A(t) + M(t). \qquad \square$$

THEOREM 4.3. *The variance of $Y^*(T)$ in (4.2) is a combination of a quadratic form and an exponential form of $\alpha_A(0)$:*

$$\mathrm{Var}(Y^*(T)) = \lambda^2(\tfrac{1}{2}\alpha_A(0)'K(0)\alpha_A(0) + N'(0)\alpha_A(0) + M(0)) + l_0^2\Lambda,$$

where

$$\begin{aligned}
\Lambda &= e^{\int_0^T 2\tilde{\alpha}_L(s)\,ds}\int_0^T e^{2(p(s)\tilde{\mu}(s)+p(s)'\tilde{\Sigma}(s)p(s)+q(s))+\int_0^s \sigma_L(\tau)\sigma_L(\tau)'\,d\tau} \\
&\qquad \times \sigma_L(s)(I_m - \sigma_A(s)'(\sigma_A(s)\sigma_A'(s))^{-1}\sigma_A(s))\sigma_L(s)'\,ds, &(4.10)\\
\tilde{\mu}(t) &= e^{-\mathcal{A}t}\alpha_A(0) + \int_0^t e^{-\mathcal{A}(t-s)}\tilde{\Theta}(s)\,ds, &(4.11)\\
\tilde{\Theta}(t) &= \Theta(t) - 2\mathcal{A}\sigma_A(t)\sigma_L(t)', &(4.12)\\
\tilde{\Sigma}(t) &= \int_0^t e^{-\mathcal{A}(t-s)}\mathcal{A}\sigma_A(s)\sigma_A(s)'\mathcal{A}'e^{-\mathcal{A}'(t-s)}\,ds, &(4.13)
\end{aligned}$$

$\Theta(t)$ is defined in (3.26), and $K(t)$, $N(t)$ and $M(t)$ are presented in Lemma 4.2.

PROOF. Combining Lemma 4.1 and Lemma 4.2, we have

$$\begin{aligned}
\mathrm{Var}(Y^*(T)) &= \lambda^2\Big(\frac{1}{2}\alpha_A(0)'K(0)\alpha_A(0) + N'(0)\alpha_A(0) + M(0)\Big) \\
&\quad + \mathrm{E}\Big[\int_0^T e^{\int_s^T 2\tilde{\alpha}_L(\tau)\,d\tau + 2p(\tau)'\alpha_A(\tau) + 2q(\tau)\,d\tau} \\
&\quad \times l^2(s)\sigma_L(s)'(I_m - \sigma_A(s)'(\sigma_A(s)\sigma_A(s)')^{-1}\sigma_L(s)')\,ds\Big].
\end{aligned}$$

https://doi.org/10.1017/S1446181120000164 Published online by Cambridge University Press

Then, the rest of the proof obtains an explicit form of

$$\mathrm{E}\Big[\int_0^T e^{\int_s^T 2\tilde{\alpha}_L(\tau)\,d\tau+2p(\tau)'\alpha_A(\tau)+2q(\tau)\,d\tau} \times l^2(s)\sigma_L(s)'(I_m-\sigma_A(s)'(\sigma_A(s)\sigma_A(s)')^{-1}\sigma_L(s)')\,ds\Big].$$

From the SDE of $l(t)$ in (2.6), we have

$$l(t) = l_0 e^{\int_0^t \left(-\sigma_L\sigma_L'/2+\tilde{\alpha}_L\right)ds+\int_0^t \sigma_L\,dW_s} \quad \text{and}$$
$$l^2(t) = l_0^2 e^{\int_0^t(-\sigma_L\sigma_L'+2\tilde{\alpha}_L)\,ds+\int_0^t 2\sigma_L\,dW_s}.$$

Therefore,

$$\begin{aligned}
&\mathrm{E}\Big[\int_0^T e^{\int_s^T 2\tilde{\alpha}_L(\tau)\;d\tau+2(p(s)'\alpha_A(s)+q(s))} l^2(s)\sigma_L(s)'(I_m-\sigma_A(s)'(\sigma_A(s)\sigma_A(s)')^{-1}\sigma_L(s)')\,ds\Big]\\
&= l_0^2 E\Big[\int_0^T e^{\int_s^T 2\tilde{\alpha}_L(\tau)\;d\tau+2(p(s)'\alpha_A(s)+q(s))} e^{\int_0^s(-\sigma_L\sigma_L'+2\alpha_L)\,d\tau+\int_0^s 2\sigma_L\,dW_\tau}\\
&\quad \times \sigma_L(s)'(I_m-\sigma_A(s)'(\sigma_A(s)\sigma_A(s)')^{-1}\sigma_L(s)')\,ds\Big]\\
&= l_0^2 e^{\int_0^T 2\tilde{\alpha}_L\,ds} E\Big[\int_0^T e^{2(p(s)'\alpha_A(s)+q(s))} e^{\int_0^s -\sigma_L\sigma_L'\,d\tau+\int_0^s 2\sigma_L\,dW_\tau}\\
&\quad \times \sigma_L(s)'(I_m-\sigma_A(s)'(\sigma_A(s)\sigma_A(s)')^{-1}\sigma_L(s)')\,ds\Big].
\end{aligned} \tag{4.14}$$

By Girsanov's theorem, (4.14) can be simplified as follows.

$$l_0^2 e^{\int_0^T 2\tilde{\alpha}_L\,ds} E^{\hat{\hat{\mathcal{P}}}}\Big[\int_0^T e^{2(p'\alpha_A+q)+\int_0^s \sigma_L\sigma_L'\,d\tau}\sigma_L'(I_m-\sigma_A'(\sigma_A\sigma_A')^{-1}\sigma_L')\,ds\Big], \tag{4.15}$$

where $\hat{\hat{\mathcal{P}}}$ is a probability measure which is equivalent to $\mathcal{P}$ such that

$$\frac{d\hat{\hat{\mathcal{P}}}}{d\mathcal{P}} = \exp\Big\{\int_0^s -2\sigma_L(\tau)\sigma_L(\tau)'\,d\tau + \int_0^s 2\sigma_L(\tau)\;d\hat{W}_\tau\Big\} \tag{4.16}$$

with $d\hat{\hat{W}}_s = dW_s - 2\sigma_L(s)'\,ds$, and $\hat{\hat{W}}_s$ in (4.16) is the Wiener process under $\hat{\hat{\mathcal{P}}}$-measure. By Itô's lemma, the dynamic of $\alpha_A(t)$ under $\hat{\hat{\mathcal{P}}}$-measure is given by

$$\begin{aligned}
d\alpha_A &= (\Theta - \mathcal{A}\alpha_A - 2\mathcal{A}\sigma_A\sigma_L')\,dt - \mathcal{A}\sigma_A d\hat{\hat{W}}_t\\
&= (\tilde{\Theta} - \mathcal{A}\alpha_A)\,dt - \mathcal{A}\sigma_A d\hat{\hat{W}}_t,
\end{aligned}$$

where $\tilde{\Theta}(t) = \Theta - 2\mathcal{A}\sigma_A(t)\sigma_L(t)'$. Thus,

$$\alpha_A(t) = e^{-\mathcal{A}(t)}\alpha_A(0) + \int_0^t e^{-\mathcal{A}(t-s)}\tilde{\Theta}(s)\,ds - \int_0^t e^{-\mathcal{A}(t-s)}\mathcal{A}\sigma_A(s)d\hat{\hat{W}}_s, \tag{4.17}$$

https://doi.org/10.1017/S1446181120000164 Published online by Cambridge University Press

and $\alpha_A(t)$ follows a normal distribution with mean $\tilde{\mu}(t)$ and variance-covariance matrix $\tilde{\Sigma}(t)$, where

$$\tilde{\mu}(t) = e^{-\mathcal{A}t}\alpha_A(0) + \int_0^t e^{-\mathcal{A}(t-s)}\tilde{\Theta}(s)\,ds,$$
$$\tilde{\Sigma}(t) = \int_0^t e^{-\mathcal{A}(t-s)}\mathcal{A}\sigma_A(s)\sigma_A(s)'\mathcal{A}'e^{-\mathcal{A}'(t-s)}\,ds.$$

Hence, expression (4.15) becomes

$$\begin{aligned} & l_0^2 e^{\int_0^T 2\tilde{\alpha}_L\,ds}\int_0^T \mathrm{E}^{\hat{\hat{\mathcal{P}}}}[e^{2p'\alpha_A(s)}]e^{2q+\int_0^s \sigma_L\sigma_L'\,d\tau}\sigma_L'(I_m - \sigma_A'(\sigma_A\sigma_A')^{-1}\sigma_L')\,ds \\ &= l_0^2 e^{\int_0^T 2\tilde{\alpha}_L\,ds}\int_0^T e^{2p'\tilde{\mu}(s)+2p'\tilde{\Sigma}(s)p}e^{2q+\int_0^s \sigma_L\sigma_L'\,d\tau}\sigma_L'(I_m - \sigma_A'(\sigma_A\sigma_A')^{-1}\sigma_L')\,ds. \qquad \square \end{aligned}$$

Theorem 4.3 not only offers an explicit expression for the variance of the surplus subject to the optimal ALM strategy but also shows the difficulty of making statistical arbitrage profit when an investor faces uncontrollable stochastic liability. When the liability is absent, Chiu and Wong [13] rigorously prove that the TC-MV portfolio strategy leads to statistical arbitrage in the sense of Hogan et al. [20]: (i) there is a long-term expected profit; (ii) the ruin probability diminishes to zero in the long run; and (iii) the time-averaged portfolio variance diminishes to zero in the long run. However, Theorem 4.3 clearly shows that the variance will grow exponentially according to the volatility of the liability. This immediately rules out the possibility of diminishing time-average variance. In other words, statistical arbitrage fails for investors who have an uncontrollable stochastic liability. Actually, Chen et al. [7] also show that statistical arbitrage hardly occurs if the cointegration feature encounters regime switching. Here, we note that liability is also a factor preventing statistical arbitrage with cointegration.

THEOREM 4.4. *The TC-MV efficient frontier of the final surplus $Y(T)$ is*

$$\begin{aligned}\mathrm{Var}(Y^*(T)) - l_0^2\Lambda &= \frac{E[\int_0^T < \alpha_A(s), \alpha_A(s) > \,ds]}{(E^{\hat{\mathcal{P}}}[\int_0^T < \alpha_A(s), \alpha_A(s) > \,ds])^2} \\ &\quad \times (E[Y^*(T)] - e^{\int_0^T r(s)\,ds}Y_0 + \mathcal{G}(0, \alpha_A(0), l_0))^2,\end{aligned}$$

where $< \alpha_A, \alpha_A >= \alpha_A'(\sigma_A(s)\sigma_A(s)')^{-1}\alpha_A$, *and* $\mathcal{G}(0, \alpha_A(0), l_0)$, Λ *are defined in Lemma 3.5 and equation (4.10), respectively. Indeed,*

$$E\Big[\int_0^T < \alpha_A(s), \alpha_A(s) > \,ds\Big] = \frac{1}{2}\alpha_A(0)'K(0)\alpha_A(0) + N'(0)\alpha_A(0) + M(0),$$
$$E^{\hat{\mathcal{P}}}\Big[\int_0^T < \alpha_A(s), \alpha_A(s) > \,ds\Big] = \frac{1}{2}\alpha_A(0)'\hat{K}(0)\alpha_A(0) + \hat{N}'(0)\alpha_A(0) + \hat{M}(0),$$

where $(K(0), N(0), M(0))$ *and* $(\hat{K}(0), \hat{N}(0), \hat{M}(0))$ *are defined in Lemma* 4.2 *and Lemma* 3.4*, respectively.*

PROOF. The result follows by combining (4.1) and (4.2) through eliminating λ. □

Although the surplus variance is increased compared with the case of zero liability, Theorem 4.4 presents a good feature in that the efficient frontier when adopting the optimal ALM maintains a bullet-shaped curve. More specifically, the expected surplus and the surplus's variance form a quadratic relationship, retaining the feature of class MV results even in a cointegration economy.

We use an example to demonstrate the use of the analytical result and the form of the ALM efficient frontier.

EXAMPLE 4.5. Consider two risky assets whose log-asset values at time t are $X(t) = [x_1(t)\ \ x_2(t)]'$ with initial values $[\ln 1\ \ \ln 2]$ and constant parameters

$$\theta = \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix}, \quad \sigma_A = \begin{pmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \end{pmatrix}, \quad \mathcal{A} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Clearly, $\mathbf{z}(t) = x_2(t) + x_1(t)$ is a cointegrating factor that exhibits mean reversion with a mean-reverting speed of 1. A risk-free asset is available in the market, which derives the risk-free interest rate to be 3%. It is assumed that the appreciation rate $\tilde{\alpha}_L$ and volatility σ_L of liability are 0.08 and $\begin{pmatrix} 0 & 0 & 0.3 \end{pmatrix}$, respectively. We are interested in the corresponding MV ALM problem with expected terminal wealth $\overline{Y}$, over an investment horizon of 6 months $(T = 1/2)$. We first find the values of $(\hat{K}(0), \hat{N}(0), \hat{M}(0))$ and $(K(0), N(0), M(0))$. According to Lemma 3.4,

$$\hat{K}(0) = 2(\sigma_A\sigma_A')^{-1}T = 25\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\hat{N}(0) = (\sigma_A\sigma_A')^{-1}\Theta T^2 = -0.125\begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$\hat{M}(0) = \frac{T^3}{3}\Theta'(\sigma_A\sigma_A{}')^{-1}\Theta + \frac{T^2}{2}tr\left(\sigma_A'\mathcal{A}'(\sigma_A\sigma_A')^{-1}\mathcal{A}\sigma_A\right) = 0.0279,$$

where

$$\Theta = \frac{1}{2}\mathcal{A}\left(\mathcal{D}(\sigma_A\sigma_A)\mathbf{1} - r\mathbf{1}\right) = -0.01\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

From Lemma 4.2,

$$\begin{aligned} K(0) &= 2\int_0^{0.5} e^{-\frac{1}{2}\mathcal{A}'s}(\sigma_A\sigma_A{}')^{-1}e^{-\frac{1}{2}\mathcal{A}s}\,ds \\ &= 25(1-e^{-0.5})\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + 12.5\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \end{aligned}$$

https://doi.org/10.1017/S1446181120000164 Published online by Cambridge University Press

$$
\begin{aligned}
N(0) &= 2\int_0^{0.5} \Theta'\Big(\int_s^{0.5} e^{\frac{1}{2}\mathcal{A}'(s-\tau)}(\sigma_A\sigma_A{}')^{-1}e^{\frac{1}{2}\mathcal{A}(s-\tau)}\,d\tau\Big)e^{-\mathcal{A}s}\,ds \\
&= -0.5(1-e^{-0.5})\begin{pmatrix}1\\1\end{pmatrix}, \\
M(0) &= \int_0^{0.5} N(s)'\Theta + \frac{1}{2}tr(\sigma_A{}'\mathcal{A}'K(s)\mathcal{A}\sigma_A)\,ds = 0.1067.
\end{aligned}
$$

Applying Theorem 4.4,

$$
\begin{aligned}
\mathrm{E}\Big[\int_0^T <\alpha_A(s),\alpha_A(s)>\ ds\Big] &= \frac{1}{2}\alpha_A(0)'K(0)\alpha_A(0) + N'(0)\alpha_A(0) + M(0) = 0.1851, \\
\mathrm{E}^{\hat{\mathcal{P}}}\Big[\int_0^T <\alpha_A(s),\alpha_A(s)>\ ds\Big] &= \frac{1}{2}\alpha_A(0)'\hat{K}(0)\alpha_A(0) + \hat{N}'(0)\alpha_A(0) + \hat{M}(0) = 0.1924,
\end{aligned}
$$

with $\alpha_A(0) = [0.09 - \ln(\sqrt{2}) \quad 0.19 - \ln(\sqrt{2})]'$. Substituting the values of α_A, α_L, σ_A and σ_L into the expressions for Λ and $\mathcal{G}$,

$$
\begin{aligned}
\mathcal{G}(0,\alpha_A(0),l_0) &= l_0e^{0.5(0.03)}[e^{0.5(0.08-0.03)}-1] = l_0(e^{0.04}-e^{0.015}), \\
\Lambda &= e^{\tilde{\alpha}_L}\int_0^{0.5} e^{(\sigma_L\sigma_L)'s}\sigma_L(I_3-\sigma_A{}'(\sigma_A\sigma_A{}')^{-1}\sigma_A)\sigma_L{}'\,ds \\
&= e^{0.08}(e^{0.045}-1).
\end{aligned}
$$

Suppose that the initial wealth and liability level are $x_0 = 10$ and $l_0 = 6$, respectively. Recalling $T = 1/2$, the efficient frontier is

$$
\begin{aligned}
\mathrm{Var}(Y_T)|_{u^*} = \sigma^2_{Y(T)} &= 0.6267[\overline{Y} - (e^{0.015}w_0 - e^{0.04}l_0)]^2 + 0.0499l_0^2 \\
&= 0.6267(\overline{Y} - 3.9063)^2 + 1.7964,
\end{aligned}
$$

which is clearly not a straight line on the plane of $(\sigma_{Y(T)}, \overline{Y})$ but shows a bullet-shaped curve. When the liability is absent, the problem is converted into the TC-MV portfolio problem with cointegrated assets. The corresponding efficient frontier appears as a quadratic relationship between the expected surplus and the variance of the surplus. Specifically,

$$
\begin{aligned}
\sigma^2_{Y(T)} &= 0.6267[\overline{Y} - e^{0.015}w_0]^2 \\
&= 0.6267(\overline{Y} - 10e^{0.015})^2,
\end{aligned}
$$

which is a straight line on the mean–standard-deviation plane. Note that

$$
\overline{Y} = 1.2632\sigma_{Y(T)} + 10e^{0.015}.
$$

Therefore, the ALM problem generates market incompleteness through the nontraded liability, making the efficient frontier no longer a straight line.

5. Conclusion

This paper derives a TC-MV strategy for asset-liability management, when the risky assets exhibit the cointegration property and the liability is uncontrollable. The derived closed-form solution enables us to understanding statistical arbitrage, subject to the liability risk. Using the TC-MV ALM strategy, statistical arbitrage hardly occurs, because the uncontrollable liability leads to exponential growth in the variance of the surplus. However, the efficient frontier for the surplus still displays a bullet-shape curve, showing consistency with the classic result in the MV portfolio management literature, even in an economy with cointegrated risky assets. A numerical example is given to demonstrate the applications of the TC-MV ALM strategy. Possible future research includes the incorporation of jump risk and an extension to open-loop controls with cointegration.

References

[1] C. Alexander, "Optimal hedging using cointegration", *Phil. Trans. R. Soc. A* **357** (1999) 2039–2058; doi:10.1098/rsta.1999.0416.

[2] C. Alexander, I. Giblin and W. Weddington, "Cointegration and asset allocation: a new active hedge fund strategy", *Res. Int. Bus. Finance* **16** (2002) 65–90; http://sro.sussex.ac.uk/id/eprint/40609/.

[3] R. Baillie and T. Bollerslev, "Common stochastic trends in a system of exchange rates", *J. Finance* **44** (1989) 137–151; doi:10.1111/j.1540-6261.1989.tb02410.x.

[4] S. Basak and G. Chabakauri, "Dynamic mean–variance asset allocation", *Rev. Financ. Stud.* **23** (2010) 2970–3016; doi:10.2139/ssrn.965926.

[5] T. Bjork, M. Khapko and A. Murgoc, "On time–inconsistent stochastic control in continuous time", *Finance Stoch.* **21** (2017) 331–360; doi:10.1007/s00780-017-0327-5.

[6] M. Cerchi and A. Havenner, "Cointegration and stock prices: the random walk on Wall Street revisited", *J. Econ. Dyn. Control* **12** (2017) 333–346; doi:10.1016/0165-1889(88)90044-9.

[7] K. Chen and H. Y. Wong, "Time-consistent mean–variance hedging of an illiquid asset with a cointegrated liquid asset", *Finance Res. Lett.* **29** (2019) 184–192; doi:10.1016/j.frl.2018.07.004.

[8] M. C. Chiu, "Asset-liability management in continuous-time: cointegration and exponential utility", in: *Optimization and control for systems in the big-data era*, Volume 252 of *Int. Ser. Oper. Res. Manag. Sci.* (Springer, Cham, 2017) 85–100; doi:10.1007/978-3-319-53518-0_6.

[9] M. C. Chiu and D. Li, "Continuous-time mean-variance optimization of assets and liabilities", *Insur. Math. Econ.* **39** (2006) 330–355; doi:10.1016/j.insmatheco.2006.03.006.

[10] M. C. Chiu and H.Y. Wong, "Mean-variance portfolio selection of cointegrated assets", *J. Econ. Dyn. Control* **35** (2011) 1369–1385; doi:10.1016/j.jedc.2011.04.003.

[11] M. C. Chiu and H. Y. Wong, "Mean–variance asset-liability management: cointegrated assets and insurance liabilities", *Eur. J. Oper. Res.* **223** (2012) 785–793; doi:10.1016/j.ejor.2012.07.009.

[12] M. C. Chiu and H. Y. Wong, "Mean–variance principle of managing cointegrated risky assets and random liabilities", *Oper. Res. Lett.* **41** (2013) 98–106; doi:10.1016/j.orl.2012.11.013.

[13] M. C. Chiu and H. Y. Wong, "Dynamic cointegrated pairs trading: mean-variance time-consistent strategies", *J. Comput. Appl. Math.* **290** (2015) 516–534; doi:10.1016/j.cam.2015.06.004.

[14] M. C. Chiu and H. Y. Wong, "Robust dynamic pairs trading with cointegration", *Oper. Res. Lett.* **46** (2018) 225–232; doi:10.1016/j.orl.2018.01.006.

[15] M. C. Chiu, H. Y. Wong and J. Zhao, "Commodity derivatives pricing with cointegration and stochastic covariances", *Eur. J. Oper. Res.* **246** (2015) 476–486; doi:10.1016/j.ejor.2015.05.012.

[16] X. Cui, D. Li, S. Wang and S. Zhu, "Better than dynamic mean-variance: time inconsistency and free cash flow stream", *Math. Finance* **22** (2012) 346–378; doi:10.1111/j.1467-9965.2010.00461.x.

[17] J. C. Duan and S. R. Pliska, "Option valuation with co-integrated asset prices", *J. Econ. Dyn. Control* **28** (2004) 727–754; doi:10.1016/S0165-1889(03)00042-3.
[18] R. Engle and C. Granger, "Co-integration and error correction: representation, estimation and testing", *Econometrica* **55** (1987) 251–276; doi:10.2307/1913236.
[19] C. Granger, "Some properties of time series data and their use in econometric model specification", *J. Econometrics* **23** (1981) 121–130; doi:10.1016/0304-4076(81)90079-8.
[20] S. Hogan, R. Jarrow, M. Teo and M. Warachkac, "Testing market efficiency using statistical arbitrage with applications to momentum and value strategies", *J. Financ. Econ.* **73** (2004) 525–565; doi:10.1016/j.jfineco.2003.10.004.
[21] Y. Hu, H. Jin and X. Y. Zhou, "Time-inconsistent stochastic linear–quadratic control", *SIAM J. Control Optim.* **50** (2012) 1548–1572; doi:10.1137/110853960.
[22] D. Li and W. L. Ng, "Optimal dynamic portfolio selection: multiperiod mean-variance formulation", *Math. Finance* **10** (2000) 387–406; doi:10.1111/1467-9965.00100.
[23] H. Markowitz, "Portfolio selection", *J. Finance* **7** (1952) 77–91; doi:10.1111/j.1540-6261.1952.tb01525.x.
[24] P. C. B. Phillips, "Error correction and long-run equilibrium in continuous time", *Econometrica* **59** (1991) 967–980; doi:10.2307/2938169.
[25] M. Taylor and M. I. Tonks, "The internationalisation of stock markets and the abolition of U.K. exchange control", *Rev. Econ. Stat.* **71** (1989) 332–336; doi:10.2307/1926980.
[26] J. Wei, K. C. Wong, S. C. P. Yam and S. P. Yung, "Markowitz's mean–variance asset-liability management with regime switching: a time-consistent approach", *Insur. Math. Econ.* **53** (2013) 281–291; doi:10.1016/j.insmatheco.2013.05.008.
[27] T. W. Wong, M. C. Chiu and H. Y. Wong, "Time-consistent mean-variance hedging of longevity risk: effect of cointegration", *Insur. Math. Econ.* **56** (2014) 56–67; doi:10.1016/j.insmatheco.2014.03.001.
[28] T. W. Wong, M. C. Chiu and H. Y. Wong, "Managing mortality risk with longevity bonds when mortality rates are cointegrated", *J. Risk Insur.* **84** (2017) 987–1023; doi:10.1111/jori.12110.
[29] T. Yan and H. Y. Wong, "Open-loop equilibrium strategy for mean-variance portfolio problem under stochastic volatility", *Automatica* **107** (2019) 211–223; doi:10.2139/ssrn.3246173.
[30] X. Zhou and D. Li, "Continuous-time mean-variance portfolio selection: a stochastic LQ framework", *Appl. Math. Optim.* **42** (2000) 19–33; doi:10.1007/s002450010003.