A FAITHFUL MATRIX REPRESENTATION FOR CERTAIN CENTRE-BY-METABELIAN GROUPS

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(Received 20 August 1968)

To Bernhard Hermann Neumann on his 60th birthday

Communicated by G. B. Preston

1. Introduction

It is a well-known result of W. Magnus [3] that there is a faithful matrix representation for metabelian groups i.e. the groups satisfying the law $[x, y; u, v]$. The work in this paper arose in an attempt by the author to find a faithful matrix representation for centre-by-metabelian groups i.e. the groups satisfying the law $[x, y; u, v; w]$.

Let F be the free group of finite or countable infinite rank and let U and V respectively denote the verbal subgroups of F generated by the words

$$L_1 \qquad [x, y; u, v; w]$$

and

$$L_2 \quad [x^{-1}, y^{-1}; u, v][x^{-1}, v^{-1}; y, u][x^{-1}, u^{-1}; v, y] \cdot [v^{-1}, y^{-1}; x, u][y^{-1}, u^{-1}; x, v][u^{-1}, v^{-1}; x, y].$$

In this paper, we show that F/UV is isomorphic to a group M of 3×3 matrices over a commutative ring R. There is a 4-generator group which satisfies L_1 but not L_2 (C. K. Gupta [2]), so that the variety defined by L_1 and L_2 is a proper-subvariety of the variety of centre-by-metabelian groups. However, it follows from the text that the representation is faithful for the free centre-by-metabelian group of rank 3. In particular, every 3-generator centre-by-metabelian group satisfies the law L_2.

This work is a part of the author's Ph. D. thesis prepared at the Australian National University, Canberra (1967). The author wishes to thank her Supervisor, Dr. M. F. Newman for suggesting the topic of this research and for his many useful suggestions.

2. Construction of the group M

Let G be the free abelian group freely generated by a set $X = \{x_1, x_2, \cdots\}$ and let F be the free group freely generated by a corre-

https://doi.org/10.1017/S1446788700007710 Published online by Cambridge University Press

sponding set $X = \{x_1, x_2, \cdots\}$. Thus, if ξ is the homomorphism of F onto G given by $x_i \to \boldsymbol{x}_i$, then $\operatorname{Ker} \xi = F'$, the derived group of F. Let ZG denote the group-ring of G over integers. As usual, in ZG we identify $\boldsymbol{w} \in G$ with $1 \cdot \boldsymbol{w}$ in ZG and $j \in Z$ with $j \cdot \boldsymbol{e}$ in ZG, where $\boldsymbol{e}$ is the identity element of G. Let $\Lambda = \{\lambda^{(k)}_{i,i-1} | i \in \{2, 3\}, k \in \{1, 2, \cdots\}\}$ be a set of independent and commuting indeterminates which also commute with every element of ZG and let $R = ZG[\Lambda]$ denote the polynomial ring in $\lambda^{(k)}_{i,i-1}$'s with coefficients from ZG. Let M be the group generated by all 3×3 triangular matrices (over R)

$$\begin{pmatrix} \boldsymbol{e} & 0 & 0 \\ \lambda^{(k)}_{21} & \boldsymbol{x}_k & 0 \\ 0 & \lambda^{(k)}_{32} & \boldsymbol{e} \end{pmatrix} = \langle x_k \rangle$$

together with the identity matrix

$$\begin{pmatrix} \boldsymbol{e} & 0 & 0 \\ 0 & \boldsymbol{e} & 0 \\ 0 & 0 & \boldsymbol{e} \end{pmatrix} = \langle e \rangle.$$

For any arbitrary word $w = x^{\varepsilon(1)}_{i(1)} \cdots x^{\varepsilon(l)}_{i(l)}$ $(l \geqq 0, \varepsilon(1), \cdots, \varepsilon(l) \in \{1, -1\})$ in F, we define $\langle w \rangle = \langle x_{i(1)} \rangle^{\varepsilon(1)} \cdots \langle x_{i(l)} \rangle^{\varepsilon(l)}$. We show that M is the required group by proving the following,

THEOREM. *Let η denote the homomorphism of F onto M given by $x_k \to \langle x_k \rangle$. Then* $\operatorname{Ker} \eta = UV$.

PROOF. To ease the calculations, we introduce certain mappings $\alpha_{ij} (3 \geqq i > j \geqq 1)$ of F into R defined as, $\alpha_{ij}(w) = ij$-entry of the matrix $\langle w \rangle$.

As a consequence of the above definition we have,

LEMMA 1.

(1.1) $\alpha_{ij}(e) = 0$ *for* $3 \geqq i > j \geqq 1$;

(1.2) $\alpha_{i,i-1}(x_k) = \lambda^{(k)}_{i,i-1}$ *for* $i \in \{2, 3\}$ *and* $\alpha_{31}(x_k) = 0$ *for all* k;

(1.3) $\alpha_{21}(w_1 w_2) = \alpha_{21}(w_1) + \boldsymbol{w}_1 \alpha_{21}(w_2)$,
$\alpha_{32}(w_1 w_2) = \boldsymbol{w}_2 \alpha_{32}(w_1) + \alpha_{32}(w_2)$, *and*
$\alpha_{31}(w_1 w_2) = \alpha_{31}(w_1) + \alpha_{31}(w_2) + \alpha_{32}(w_1)\alpha_{21}(w_2)$
for all words w_1, w_2 in F;

(1.4) $\alpha_{i,i-1}(w^{-1}) = -\boldsymbol{w}^{-1}\alpha_{i,i-1}(w)$, $\alpha_{31}(w^{-1}) = -\alpha_{31}(w) + \boldsymbol{w}^{-1}\alpha_{32}(w)\alpha_{21}(w)$ *for all words w in F.*

NOTATION.

$[a, b] = a^{-1}b^{-1}ab$; $[[a, b], c] = [a, b, c]$; $[[a, b], [c, d]] = [a, b; c, d]$; $[[a, b; c, d], f] = [a, b; c, d; f]$ for all a, b, c, d, f in a group H.

Using Lemma 1, direct calculations give

LEMMA 2. *If w_1, w_2 are arbitrary words in F, then*

(2.1) $\alpha_{21}([w_1, w_2]) = \boldsymbol{w}_1^{-1}(-1+\boldsymbol{w}_2^{-1})\alpha_{21}(w_1)-\boldsymbol{w}_2^{-1}(-1+\boldsymbol{w}_1^{-1})\alpha_{21}(w_2)$;

(2.2) $\alpha_{32}([w_1, w_2]) = (-1+\boldsymbol{w}_2)\alpha_{32}(w_1)-(-1+\boldsymbol{w}_1)\alpha_{32}(w_2)$; *and*

(2.3)
$$\begin{aligned}\alpha_{31}([w_1, w_2]) = {} & \boldsymbol{w}_2^{-1}(-1+\boldsymbol{w}_1^{-1})\alpha_{32}(w_1)\alpha_{21}(w_2)\\ & -\boldsymbol{w}_2^{-1}(-1+\boldsymbol{w}_1)\alpha_{32}(w_2)\alpha_{21}(w_2)\\ & -\boldsymbol{w}_1^{-1}(-1+\boldsymbol{w}_2^{-1})\alpha_{32}(w_1)\alpha_{21}(w_1)\\ & +\alpha_{32}(w_1)\alpha_{21}(w_2)-\boldsymbol{w}_2^{-1}\alpha_{32}(w_2)\alpha_{21}(w_1).\end{aligned}$$

Next, we prove

LEMMA 3. *If w_1', w_2' are in F', then*

(3.1) $\alpha_{i,i-1}([w_1', w_2']) = 0$ *for* $i \in \{2, 3\}$;

(3.2) $\alpha_{31}([w_1', w_2']) = \alpha_{32}(w_1')\alpha_{21}(w_2')-\alpha_{32}(w_2')\alpha_{21}(w_1')$;

(3.3) $\alpha_{31}([x_1, x_2; x_3, x_4]) \neq 0$; *and*

(3.4) $\alpha_{31}([[w_1', w_2'], z]]) = 0$ *for all z in F.*

Since $\boldsymbol{w}_1 = \boldsymbol{w}_2 = \boldsymbol{e}$, the proof of (3.1) and (3.2) follow from (2.1), (2.2) and (2.3). Next, we note that (2.3) together with (3.1) imply (3.4). To prove (3.3), we use (3.2) to write

$$\alpha_{31}([x_1, x_2; x_3, x_4]) = \alpha_{32}([x_1, x_2])\alpha_{21}([x_3, x_4])-\alpha_{32}([x_3, x_4])\alpha_{21}([x_1, x_2]),$$

which on using (2.1), (2.2) and (1.2) reduces to

$$\begin{aligned}&(\{(\boldsymbol{x}_2-1)\lambda_{32}^{(1)}-(\boldsymbol{x}_1-1)\lambda_{32}^{(2)}\}\cdot\{\boldsymbol{x}_3^{-1}(\boldsymbol{x}_4^{-1}-1)\lambda_{21}^{(3)}-\boldsymbol{x}_4^{-1}(\boldsymbol{x}_3^{-1}-1)\lambda_{21}^{(4)}\})\\ &\quad-(\{(\boldsymbol{x}_4-1)\lambda_{32}^{(3)}-(\boldsymbol{x}_3-1)\lambda_{32}^{(4)}\}\cdot\{\boldsymbol{x}_1^{-1}(\boldsymbol{x}_2^{-1}-1)\lambda_{21}^{(1)}-\boldsymbol{x}_2^{-1}(\boldsymbol{x}_1^{-1}-1)\lambda_{21}^{(2)}\});\end{aligned}$$

and is obviously non-zero.

LEMMA 4. *For f_1, f_2, f_3, f_4 in F, let*

$$\begin{aligned}f^* = {} & [f_1^{-1}, f_2^{-1}; f_3, f_4][f_1^{-1}, f_4^{-1}; f_2, f_3]\cdot[f_1^{-1}, f_3^{-1}; f_4, f_2]\\ & \cdot[f_4^{-1}, f_2^{-1}; f_1, f_3][f_2^{-1}, f_3^{-1}; f_1, f_4][f_3^{-1}, f_4^{-1}; f_1, f_2].\end{aligned}$$

Then $\alpha_{31}(f^*) = 0$.

Using (1.3) and (3.1), we note that

$$\begin{aligned}\alpha_{31}(f^*) = &\ \alpha_{31}([f_1^{-1}, f_2^{-1}; f_3, f_4])+\alpha_{31}([f_1^{-1}, f_4^{-1}; f_2, f_3])\\ &+\alpha_{31}([f_1^{-1}, f_3^{-1}; f_4, f_2])+\alpha_{31}([f_4^{-1}, f_2^{-1}; f_1, f_3])\\ &+\alpha_{31}([f_2^{-1}, f_3^{-1}; f_1, f_4])+\alpha_{31}([f_3^{-1}, f_4^{-1}; f_1, f_2]);\end{aligned}$$

and making use of (3.2), (2.1), (2.2) and (1.4) we obtain

$$\begin{aligned}\alpha_{31}([f_1^{-1}, f_2^{-1}; f_3, f_4]) = &\ (\mathbf{f}_1^{-1}\alpha_{32}(f_1)+\mathbf{f}_1^{-1}\mathbf{f}_2^{-1}\alpha_{32}(f_2)-\mathbf{f}_1^{-1}\mathbf{f}_2^{-1}\alpha_{32}(f_1)-\mathbf{f}_2^{-1}\alpha_{32}(f_2))\\ &\cdot\ (-\mathbf{f}_3^{-1}\alpha_{21}(f_3)-\mathbf{f}_3^{-1}\mathbf{f}_4^{-1}\alpha_{21}(f_4)+\mathbf{f}_3^{-1}\mathbf{f}_4^{-1}\alpha_{21}(f_3)+\mathbf{f}_4^{-1}\alpha_{21}(f_4))\\ &-(-\mathbf{f}_3\mathbf{f}_3^{-1}\alpha_{32}(f_3)-\mathbf{f}_3\mathbf{f}_4\mathbf{f}_4^{-1}\alpha_{32}(f_4)+\mathbf{f}_4\alpha_{32}(f_3)+\alpha_{32}(f_4))\\ &\cdot\ (\alpha_{21}(f_1)+\mathbf{f}_1\alpha_{21}(f_2)-\mathbf{f}_1^{-1}\mathbf{f}_1\mathbf{f}_2\alpha_{21}(f_1)-\mathbf{f}_2^{-1}\mathbf{f}_2\alpha_{21}(f_2)).\end{aligned}$$

Writing out the corresponding expressions for the other five terms in the same way and adding them all give the desired result.

It now follows from (3.4) and Lemma 4 that $UV \subseteqq \operatorname{Ker} \eta$.

To prove the other inclusion, we take

$$w = x_{i(1)}^{\varepsilon(1)} \cdots x_{i(l)}^{\varepsilon(l)}\ (l \geqq 0,\ \varepsilon(1), \cdots, \varepsilon(l) \in \{1, -1\})$$

to be an arbitrary word in F such that $\alpha_{ij}(w) = 0$ $(3 \geqq i > j \geqq 1)$, and we proceed to show that $w \in UV$. First, we note

LEMMA 5. (W. Magnus). *For* $i = 2, 3$ $\alpha_{i,i-1}(w) = 0$, *if and only if* $w \in F''$, *the second derived group of* F.

(See, R. H. Fox [1] for a proof; an alternative proof is given in C. K. Gupta [2]).

The following Lemma on the commutator properties of centre-by-metabelian groups is of independent interest and shall be found useful in the proof of the theorem.

LEMMA 6. *Let* H *be a centre-by-metabelian group. If* $d, d_1, d_2, \cdots$ *are in* H' *and* $a, a_1, a_2, \cdots, b, b_1, b_2, \cdots$ *are in* H *then*

(6.1) $[d^k, \prod_{i=1}^{r} d_i] = \prod_{i=1}^{r} [d, d_i]^k$ *for all integers* k.

(6.2) $[d, a; d_1] = [d; d_1, a^{-1}]$

(6.3) $[d; d_1, a_1, \cdots, a_r] = [d; d_1, a_{1\sigma}, \cdots, a_{r\sigma}]$ $(r \geqq 0)$,
where σ *is any permutation of* $\{1, \cdots, r\}$.

(6.4) $[d; a_1, a_2, a_3] = [d; a_1, a_3, a_2][d; a_3, a_2, a_1]$

(6.5) $[d; a_1, a_2, a_2^{-1}] = [d; a_1, a_2]^{-1}[d; a_1, a_2^{-1}]^{-1}$

(6.6) $[a_1, b^{-1}; a_2, b^{-1}, b_1, \cdots, b_r] = [a_1, b; a_2, b, b_1, \cdots, b_r]$ $(r \geqq 0)$.

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$$(6.7)\quad \begin{aligned}&[a_1^{-1}, a_2^{-1}; a_3, a_4, a_5, \cdots, a_{i-1}, a_i, a_{i+1}, \cdots, a_r]\\ &\quad\cdot [a_1^{-1}, a_i^{-1}; a_3, a_4, a_5, \cdots, a_{i-1}, a_{i+1}, \cdots, a_r, a_2]^{-1}\\ &\quad\cdot [a_3^{-1}, a_2^{-1}; a_1, a_4, a_5, \cdots, a_{i-1}, a_i, a_{i+1}, \cdots, a_r]^{-1}\\ &\quad\cdot [a_3^{-1}, a_i^{-1}; a_1, a_4, a_5, \cdots, a_{i-1}, a_{i+1}, \cdots, a_r, a_2]\\ &= [a_2^{-1}, a_i^{-1}; a_1, a_3, a_4, a_5, \cdots, a_{i-1}, a_{i+1}, \cdots, a_r] \quad (r \geqq 4).\end{aligned}$$

$$(6.8)\quad \begin{aligned}&[a_1^{-1}, a_2^{-1}; a_3, a_4, \cdots, a_{i-1}, a_i, a_{i+1}, \cdots, a_r]\\ &\quad\cdot [a_2^{-1}, a_i^{-1}; a_3, a_4, \cdots, a_{i-1}, a_{i+1}, \cdots, a_r, a_1]\\ &= [a_1^{-1}, a_i^{-1}; a_3, a_4, \cdots, a_{i-1}, a_{i+1}, \cdots, a_r, a_2].\end{aligned}$$

The proof of (6.1) is immediate. For (6.2) we have,

$$\begin{aligned}[d, a; d_1] &= [d^{-1}d^a, d_1] = [d^{-1}, d_1][d^a, d_1] = [d, d_1]^{-1}[d, d_1^{a^{-1}}]\\ &= [d, d_1]^{-1}[d, d_1[d_1, a^{-1}]] = [d, d_1]^{-1}[d; d_1, a^{-1}][d, d_1]\\ &= [d; d_1, a^{-1}].\end{aligned}$$

For (6.3), we use the fact that $[d_1, a_1, \cdots, a_r] = [d_1, a_{1\sigma}, \cdots, a_{r\sigma}]$ modulo F''. Further, since $[a_1, a_2, a_3] = [a_1, a_3, a_2][a_3, a_2, a_1]$ modulo F'', we get (6.4). Also, $[a_1, a_2, a_2^{-1}] = [a_1, a_2]^{-1}[a_1, a_2^{-1}]^{-1}$ gives (6.5). Finally, the standard commutator identities give

$$\begin{aligned}[a_1, b^{-1}; a_2, b^{-1}, b_1, \cdots, b_r] &= [[a_1, b]^{-b^{-1}}; [a_2, b]^{-b^{-1}}, b_1, \cdots, b_r]\\ &= [a_1, b; a_2, b, b_1^b, \cdots, b_r^b]\\ &= [a_1, b; a_2, b, b_1[b_1, b], \cdots, b_r[b_r, b]]\\ &= [a_1, b; a_2, b, b_1, \cdots, b_r]\end{aligned}$$

proving (6.6). Now careful applications of the identities (6.1) to (6.4) give (6.7) and (6.8) (details are omitted).

Since $w \in F''$ (by Lemma 5), it can be written as

$$w = C_1^{\delta_1}C_2^{\delta_2}\cdots C_m^{\delta_m}w'' \qquad (m \geqq 1, \delta_i \in \{1, -1\}),$$

where each C_i is a commutator in $F''\setminus[F'', F]([F'', F] = U)$ with entries from $X \cup X^{-1}$, $w'' \in [F'', F]$ and weight $C_1 \geqq \cdots \geqq$ weight C_m. As an application of Lemma 6, we observe that if C is a commutator in $F''\setminus[F'', F]$, then modulo $[F'', F]$, C can be written as a power product of commutators of the form

$$(7)\qquad [u_1^{-1}, u_2^{-1}; u_3, u_4, \cdots, u_s] \qquad (s \geqq 4, u_i \in X \cup X^{-1}),$$

satisfying the following three properties:

$$(7.1)\qquad \{u_3, u_4, \cdots, u_s\} \cap \{u_3^{-1}, u_4^{-1}, \cdots, u_s^{-1}\} = \emptyset,$$

$$(7.2)\qquad \{u_1^{-1}, u_2^{-1}\} \cap \{u_5, \cdots, u_s\} = \emptyset, \text{ and}$$

$$(7.3)\qquad \{u_1^{-1}, u_2^{-1}\} \neq \{u_3, u_4\}.$$

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DEFINITION. A commutator in $F''\setminus[F'', F]$ of the form (7) is called a *special commutator* if it satisfies the properties (7.1) to (7.3).

DEFINITION. If $C = [v_1^{-1}, v_2^{-1}; v_3, v_4, \cdots, v_s]$ ($s \geqq 4$, $v_i \in X \cup X^{-1}$) is a special commutator then we say that C is of Category I (in notation $C \in \text{Cat.}\, I$), if

(i) $v_i \neq v_j$ for $i \neq j$, (ii) $\{v_1, \cdots, v_s\} \cap \{v_1^{-1}, \cdots, v_s^{-1}\} = \emptyset$;

$C \in$ Cat. II, if

(i) $v_1, \cdots, v_s$ are not all distinct, (ii) $\{v_1^{-1}, v_2^{-1}\} \cap \{v_3, v_4\} = \emptyset$;

$C \in$ Cat. III, if

(i) $\{v_1^{-1}, v_2^{-1}\} \cap \{v_3, v_4\} = v_1^{-1}$ or v_2^{-1}, (ii) $v_i \neq v_j$ for $i \neq j$;

and $C \in$ Cat. IV, if

(i) $\{v_1^{-1}, v_2^{-1}\} \cap \{v_3, v_4\} = v_1^{-1}$ or v_2^{-1} (ii) $v_i = v_j$ for some $i \neq j$.

It is clear that a special commutator belongs to one and only one of the above four categories.

We now write w as

$$w = w_1 w_2 w'',$$

where w_1 is a power product of special commutators all of maximum weight (say) r^*, w_2 is a power product of special commutators all of weight strictly less than r^* and $w'' \in [F'', F]$. The aim is to prove that there is a representation of w in which w_1 is either empty or is in V; this then implies that $w \in UV$.

We suppose that w_1 is non-empty and write

$$w = w_{11} w_{12} w_{13} w_{14} w_2 w''' \quad (w''' \in [F'', F]),$$

where w_{11}, w_{12}, w_{13} and w_{14} are respectively power products of commutators in Cat. I, Cat. II, Cat. III and Cat. IV. In step I, we shall show that if w_{11} is non-empty then $w_{11} \in V$. Next, we shall prove in three separate steps that w_{12}, w_{13}, and w_{14} are all empty.

First, we record another consequence of Lemma 6 as,

LEMMA 8. *If* $C = [v_1^{-1}, v_2^{-1}; v_3, v_4, \cdots, v_s]$ ($s \geqq 5$, $v_i \in X \cup X^{-1}$) *is a special commutator then for any pair* v_i, v_j *for* $i \neq j$, C *can be written (module* $[F'', F]$) *as a power product of special commutators of the form* $[v_1'^{-1}, v_2'^{-1}; v_3', v_4', \cdots, v_{s(1)}']$, *where* $s(1) \leqq s$, $v_1', \cdots, v_{s(1)}' \in \{v_1, \cdots, v_s\}$ *and either* (i) $v_i \in \{v_1', v_2'\}$ *and* $v_j \in \{v_3', v_4'\}$ *or* (ii) $v_j \in \{v_1', v_2'\}$ *and* $v_i \in \{v_3', v_4'\}$.

STEP I. Suppose w_{11} is non-empty, then w_{11} is a power product of special commutators each of weight $t(1) (= r^*)$.

The Case when $r^* = 4$. By hypothesis, there is a factor say $C_1^{\delta_1} = f$ in w_{11} of the form

$$f = [x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(3)}^{\varepsilon(3)}, x_{i(4)}^{\varepsilon(4)}]^{\delta_1},$$

with $i(1)$, $i(2)$, $i(3)$, $i(4)$ all distinct. By (3.2),

$$\alpha_{31}(f) = \alpha_{32}([x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{-\varepsilon(2)}])\alpha_{21}([x_{i(3)}^{\varepsilon(3)}, x_{i(4)}^{\varepsilon(4)}]) - \alpha_{32}([x_{i(3)}^{\varepsilon(3)}, x_{i(4)}^{\varepsilon(4)}])\alpha_{21}([x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{-\varepsilon(2)}]);$$

and (by (2.1), (2.2)) the coefficient of $\lambda_{32}^{(i(3))}\lambda_{21}^{(i(1))}$ $(=\alpha_{32}(x_{i(3)})\alpha_{21}(x_{i(1)})$ in $\alpha_{31}(f)$ is

$$\delta_1\varepsilon(1)\varepsilon(3)x_{i(1)}^{\frac{1}{2}(\varepsilon(1)-1)}x_{i(3)}^{\frac{1}{2}(\varepsilon(3)-1)}(-1+x_{i(2)}^{\varepsilon(2)})(-1+x_{i(4)}^{\varepsilon(4)}).$$

Since, $\alpha_{31}(w) = \sum_{i=1}^{m}\delta_i\alpha_{31}(C_i)$ (by (1.3) and (3.1)) and by hypothesis $\alpha_{31}(w) = 0$, it follows that the sum of the coefficients of $\lambda_{32}^{(i(3))}\lambda_{21}^{(i(1))}$ in $\alpha_{31}(w)$ is zero. Thus, there is a factor in w whose coefficient of $\lambda_{32}^{(i(3))}\lambda_{21}^{(i(1))}$ is

$$-\delta_1\varepsilon(1)\varepsilon(3)x_{i(1)}^{\frac{1}{2}(\varepsilon(1)-1)}x_{i(3)}^{\frac{1}{2}(\varepsilon(3)-1)}(-1+x_{i(2)}^{\varepsilon(2)})(-1+x_{i(4)}^{\varepsilon(4)}).$$

But, the only factor with this property other than $C_1^{-\delta_1}$ is

$$[x_{i(1)}^{-\varepsilon(1)}, x_{i(4)}^{-\varepsilon(4)}; x_{i(3)}^{\varepsilon(3)}, x_{i(2)}^{\varepsilon(2)}]^{-\delta_1},$$

which is again in Cat. I. In this way (i.e. by considering the coefficients of $\lambda_{32}^{(i(4))}\lambda_{21}^{(i(1))}$ etc. using C_1 and/or any other factor so obtained) we observe that w_{11} contains the power product

$$\begin{aligned}&[x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(3)}^{\varepsilon(3)}, x_{i(4)}^{\varepsilon(4)}]^{\delta_1}[x_{i(1)}^{-\varepsilon(1)}, x_{i(4)}^{-\varepsilon(4)}; x_{i(3)}^{\varepsilon(3)}, x_{i(2)}^{\varepsilon(2)}]^{-\delta_1}\\ &\cdot[x_{i(1)}^{-\varepsilon(1)}, x_{i(3)}^{-\varepsilon(3)}; x_{i(2)}^{\varepsilon(2)}, x_{i(4)}^{\varepsilon(4)}]^{-\delta_1}[x_{i(4)}^{-\varepsilon(4)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(3)}^{\varepsilon(3)}, x_{i(1)}^{\varepsilon(1)}]^{-\delta_1}\\ &\cdot[x_{i(3)}^{-\varepsilon(3)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(1)}^{\varepsilon(1)}, x_{i(4)}^{\varepsilon(4)}]^{-\delta_1}[x_{i(3)}^{-\varepsilon(3)}, x_{i(4)}^{-\varepsilon(4)}; x_{i(1)}^{\varepsilon(1)}, x_{i(2)}^{\varepsilon(2)}]^{\delta_1},\end{aligned}$$

which lies in V, as was to be proved.

The Case when $r^* \geqq 5$. Here we have in w_{11},

$$C_1^{\delta_1} = [x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(3)}^{\varepsilon(3)}, x_{i(4)}^{\varepsilon(4)}, \cdots, x_{i(t(1))}^{\varepsilon(t(1))}]^{\delta_1},$$

where C_1 is a special commutator of Cat. I and $t(1) = r^* \geqq 5$. By Lemma 8, about the remaining factors of w (if any), we can make the following assumption:

(9) If $$[u_1^{-1}, u_2^{-1}; u_3, u_4, \cdots, u_s]^{\delta} \qquad (s \geqq 5)$$

is a factor of w such that

$$\{x_{i(1)}^{\varepsilon'(1)}, x_{i(3)}^{\varepsilon'(3)}\} \subset \{u_1, u_2, \cdots, u_s\} \qquad (\varepsilon'(1), \varepsilon'(3) \in \{1, -1\})$$

then either (i) $x_{i(1)}^{\varepsilon'(1)} \in \{u_1, u_2\}$ and $x_{i(3)}^{\varepsilon'(3)} \in \{u_3, u_4\}$ or (ii) $x_{i(3)}^{\varepsilon'(3)} \in \{u_1, u_2\}$ and $x_{i(1)}^{\varepsilon'(1)} \in \{u_3, u_4\}$. Thus, there is a representation of w as power product of special commutators such that the factors of $w_{11}w_{12}w_{13}w_{14}w_2$ satisfy (9). Among such representations of w we choose one in which w_{11} consists of least number of factors and we write

$$w_{11} = C_{11}^{\delta_{11}}C_{21}^{\delta_{21}}\cdots C_{m(1)1}^{\delta_{m(1)1}} \; (m(1) \geqq 1, \delta_{11}, \cdots, \delta_{m(1)1} \in \{1, -1\}),$$

where $C_{11}^{\delta_{11}} = C_1^{\delta_1}$.

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For simplicity of notation, we write $C_1^{\delta_1}$ as

$$f_1^* = \begin{bmatrix} -1 & -1 & \\ i(1), i(2); & i(3), i(4), \cdots, i(t(1)) \end{bmatrix}^{\delta_1} \quad (i(1), \cdots, i(t(1))$$

are all distinct). As in the previous case, the coefficient of $\lambda_{32}^{(i(4))}\lambda_{21}^{(i(2))}$ in $\alpha_{31}(f_1^*)$ is

$$\delta_1 \varepsilon(2)\varepsilon(4) x_{i(2)}^{\frac{1}{2}(\varepsilon(2)-1)} \cdot x_{i(4)}^{\frac{1}{2}(\varepsilon(4)-1)}(-1+x_{i(1)}^{\varepsilon(1)})(-1+x_{i(3)}^{\varepsilon(3)})(-1+x_{i(5)}^{\varepsilon(5)}) \cdots (-1+x_{i(t(1))}^{\varepsilon(t(1))});$$

so that there is a factor in w which is different from $C_1^{-\delta_1}$ and whose coefficient of $\lambda_{32}^{(i(4))}\lambda_{21}^{(i(2))}$ is

$$-\delta_1 \varepsilon(2)\varepsilon(4) x_{i(2)}^{\frac{1}{2}(\varepsilon(2)-1)} \cdot x_{i(4)}^{\frac{1}{2}(\varepsilon(4)-1)}(-1+x_{i(1)}^{\varepsilon(1)})(-1+x_{i(3)}^{\varepsilon(3)})(-1+x_{i(5)}^{\varepsilon(5)}) \cdots (-1+x_{i(t(1))}^{\varepsilon(t(1))}).$$

However, the only factor with this property in w is

$$\begin{bmatrix} -1 & -1 & \\ i(3), i(2); & i(1), i(4), \cdots, i(t(1)) \end{bmatrix}^{-\delta_1} = f_2^* \text{ (say)},$$

which is again in w_{11}. In the same way, considering the coefficient of $\lambda_{32}^{(i(4))}\lambda_{21}^{(i(1))}$ in $\alpha_{31}(f_1^*)$ shows that in w_{11} there is another factor

$$f_3^* = \begin{bmatrix} -1 & -1 & \\ i(1), i(2\sigma_1); & i(3), i(4), i(5\sigma_1), \cdots, i(t(1)\sigma_1) \end{bmatrix}^{-\delta_1},$$

where σ_1 is a permutation of $\{2, 5, \cdots, t(1)\}$ and $2\sigma_1 \neq 2$. Similarly, considering the coefficients of $\lambda_{32}^{(i(4))}\lambda_{21}^{(i(2\sigma_1))}$ and $\lambda_{32}^{(i(3))}\lambda_{21}^{(i(2\sigma_1))}$ in $\alpha_{31}(f_3^*)$ implies that in w_{11} there are factors

$$f_4^* = \begin{bmatrix} -1 & -1 & \\ i(3), i(2\sigma_1); & i(1), i(4), i(5\sigma_1), \cdots, i(t(1)\sigma_1) \end{bmatrix}^{\delta_1}$$

and

$$f_5^* = \begin{bmatrix} -1 & -1 & \\ i(1), i(2\sigma_1); & i(3), i(4\sigma_2), i(5\sigma_1\sigma_2), \cdots, i(t(1)\sigma_1\sigma_2) \end{bmatrix}^{\delta_1},$$

where σ_2 is a permutation of $\{4, 5\sigma_1, \cdots, t(1)\sigma_1\}$ and $4\sigma_2 \neq 4$. Further, considering the coefficients of $\lambda_{32}^{(i(4\sigma_2))}\lambda_{21}^{(i(2\sigma_1))}$ and $\lambda_{32}^{(i(4\sigma_2))}\lambda_{21}^{(i(1))}$ in $\alpha_{31}(f_5^*)$ gives as before

$$f_6^* = \begin{bmatrix} -1 & -1 & \\ i(3), i(2\sigma_1); & i(1), i(4\sigma_2), i(5\sigma_1\sigma_2), \cdots, i(t(1)\sigma_1\sigma_2) \end{bmatrix}^{-\delta_1}$$

and

$$f_7^* = \begin{bmatrix} -1 & -1 & \\ i(1), i(2\sigma_1\sigma_3); & i(3), i(4\sigma_2), i(5\sigma_1\sigma_2\sigma_3), \cdots, i(t(1)\sigma_1\sigma_2\sigma_3) \end{bmatrix}^{-\delta_1},$$

where σ_3 is a permutation of $\{2\sigma_1, 5\sigma_1\sigma_2, \cdots, t(1)\sigma_1\sigma_2\}$ and $\lambda_{32}^{(i(4\sigma_2))}\lambda_{21}^{(i(2\sigma_1\sigma_3))}$ in $\alpha_{31}(f_7^*)$ gives

$$f_8^* = \begin{bmatrix} -1 & -1 & \\ i(3), & i(2\sigma_1\sigma_3); & i(1), i(4\sigma_2), i(5\sigma_1\sigma_2\sigma_3), \cdots, i(t(1)\sigma_1\sigma_2\sigma_3) \end{bmatrix}^{\delta_1}.$$

Thus, in the minimal representation of w_{11}, we have obtained a product factor $f_1^* \cdots f_8^*$. Now, we shall obtain a contradiction to the choice of w_{11} by proving that this product $f_1^* \cdots f_8^*$ can be replaced by a product with smaller number of factors of w_{11}. By (6.7),

$$f_1^* \cdots f_4^* = \begin{bmatrix} -1 & -1 & \\ i(2), & i(2\sigma_1); & i(1), i(3), i(5\sigma_1), \cdots, i(t(1)\sigma_1) \end{bmatrix}^{\delta_1}$$

and

$$f_5^* \cdots f_8^* = \begin{bmatrix} -1 & -1 & \\ i(2\sigma_1), & i(2\sigma_1\sigma_3); & i(1), i(3), i(4\sigma_2), i(5\sigma_1\sigma_2), \cdots, i(t(1)\sigma_1\sigma_2) \end{bmatrix}^{\delta_1}.$$

Hence,

$$f_1^* \cdots f_8^*$$
$$= \begin{bmatrix} -1 & -1 & \\ i(2), & i(2\sigma_1\sigma_3); & i(1), i(3), i(2\sigma_1), i(4\sigma_1), i(5\sigma_1\sigma_2), \cdots, i(t(1)\sigma_1\sigma_2) \end{bmatrix}^{\delta_1}$$

by (6.8)

$$= \begin{bmatrix} -1 & -1 & \\ i(2), & i(j); & i(1), i(3), i(4), \cdots, i(j-1), i(j+1), \cdots, i(t(1)) \end{bmatrix}^{\delta_1}$$

for some $j \in \{4, 5, \cdots, t(1)\}$,

$$= \begin{bmatrix} -1 & -1 & \\ i(1), & i(2); & i(3), i(4), \cdots, i(t(1)) \end{bmatrix}^{\delta_1} \begin{bmatrix} -1 & -1 & \\ i(3), & i(2); & i(1), i(4), \cdots, i(t(1)) \end{bmatrix}^{-\delta_1}$$
$$\cdot \begin{bmatrix} -1 & -1 & \\ i(1), & i(j); & i(3), i(4), \cdots, i(j-1), i(j+1), \cdots, i(t(1)) \end{bmatrix}^{-\delta_1}$$
$$\cdot \begin{bmatrix} -1 & -1 & \\ i(3), & i(j); & i(1), i(4), \cdots, i(j-1), i(j+1), \cdots, i(t(1)) \end{bmatrix}^{\delta_1}$$

by (6.7), as was required.

STEP II. By the previous step we can assume that w_{11} is empty, so that $w = w_{12}w_{13}w_{14}w_2w'''$. Now we suppose that w_{12} is non-empty and arrive at a contradiction. If $C = [v_1^{-1}, v_2^{-1}; v_3, v_4, \cdots, v_s]$ $(s \geqq 5)$ is a special commutator of Cat. II, then, by definition, for some $j \neq k$, $v_j = v_k$. By Lemma 8, we can write C as a power product of special commutators in Cat. II of the kind $[v_1'^{-1}, v_2'^{-1}; v_3', v_4', \cdots, v_s']$, where v_1', v_2', v_3', v_4' are not all distinct and $\{v_1', \cdots, v_s'\} = \{v_1, \cdots, v_s\}$.

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Thus, if $C_1^{\delta_1}$ is a factor of w_{12} then

$$C_1^{\delta_1} = [x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(1)}^{\varepsilon(1)}, x_{i(3)}^{\varepsilon(3)}, (r_1-1)x_{i(1)}^{\varepsilon(1)}, r_2 x_{i(2)}^{\varepsilon(2)}, r_3 x_{i(3)}^{\varepsilon(3)}, (r_4+1)x_{i(4)}^{\varepsilon(4)}, \cdots, (r_{t(2)}+1)x_{i(t(2))}^{\varepsilon(t(2))}]^{\delta_1},$$

where $r_1 \geqq 1$, $r_j \geqq 0$ for $2 \leqq j \leqq t(2)$, $t(2) \geqq 3$, $i(j) \neq i(k)$ for $j \neq k$ and $\sum_{j=1}^{t(2)} (r_j+1) = r^*$; or

$$C_1^{\delta_1} = [x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(1)}^{\varepsilon(1)}, x_{i(2)}^{\varepsilon(2)}, (r_1-1)x_{i(1)}^{\varepsilon(1)}, (r_2-1)x_{i(2)}^{\varepsilon(2)}, (r_3+1)x_{i(3)}^{\varepsilon(3)}, \cdots, (r_{t(2)}+1)x_{i(t(2))}^{\varepsilon(t(2))}],$$

where $r_1, r_2 \geqq 1$, $r_j \geqq 0$ for $3 \leqq j \leqq t(2)$, $t(2) \geqq 2$, $i(j) \neq i(k)$ for $j \neq k$ and $\sum_{j=1}^{t(2)} (r_j+1) = r^*$.

About the remaining factors of w, again by Lemma 8, we can make the following assumption:

(10) If $$C^\delta = [v_1^{-1}, v_2^{-1}; v_3, v_4, \cdots, v_s]^\delta \qquad (s \geqq 4, \delta \in \{1, -1\})$$

is a factor of w such that $v_j = v_k = x_{i(1)}^{\varepsilon'(1)}$ for some $j \neq k$ then $x_{i(1)}^{\varepsilon'(1)} \in \{v_1, v_2\}$ and $x_{i(1)}^{\varepsilon'(1)} \in \{v_3, v_4\}$.

There is a representation of w as a power product of special commutators in which w_{11} is empty, w_{12} is a power product of special commutators of maximum weight in Cat. II and the factors of $w_{12}w_{13}w_{14}w_2$ satisfy (10). Of all such representations of w we choose one in which w_{12} contains least number of factors and we write

$$w_{12} = C_{12}^{\delta_{12}} C_{22}^{\delta_{22}} \cdots C_{m(2)2}^{\delta_{m(2)2}} (m(2) \geqq 1, \delta_{12}, \cdots, \delta_{m(2)2} \in \{1, -1\},$$

where $C_{12}^{\delta_{12}} = C_1^{\delta_1}$.

We first consider the case when

$$C_1^{\delta_1} = [x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(1)}^{\varepsilon(1)}, x_{i(2)}^{\varepsilon(2)}, (r_1-1)x_{i(1)}^{\varepsilon(1)}, (r_2-1)x_{i(2)}^{\varepsilon(2)}, (r_3+1)x_{i(3)}^{\varepsilon(3)}, \cdots, (r_{t(2)}+1)x_{i(t(2))}^{\varepsilon(t(2))}]^{\delta_1}.$$

The coefficient of $\lambda_{32}^{(i(2))}\lambda_{21}^{(i(2))}$ in $\alpha_{31}(C_1^{\delta_1})$ is

$$\delta_1 \boldsymbol{x}_{i(2)}^{\varepsilon(2)-1}(-1+\boldsymbol{x}_{i(1)}^{\varepsilon(1)})^{(r_1+1)}(-1+\boldsymbol{x}_{i(2)}^{\varepsilon(2)})^{(r_2-1)}(-1+\boldsymbol{x}_{i(3)}^{\varepsilon(3)})^{(r_3+1)} \cdots (-1+\boldsymbol{x}_{i(t(2))}^{\varepsilon(t(2))})^{(r_{t(2)}+1)}$$
$$-\delta_1 \boldsymbol{x}_{i(2)}^{-\varepsilon(2)-1}(-1+\boldsymbol{x}_{i(1)}^{-\varepsilon(1)})^{(r_1+1)}(-1+\boldsymbol{x}_{i(2)}^{-\varepsilon(2)})^{(r_2-1)}(-1+\boldsymbol{x}_{i(3)}^{-\varepsilon(3)})^{(r_3+1)} \cdots (-1+\boldsymbol{x}_{i(t(2))}^{-\varepsilon(t(2))})^{(r_{t(2)}+1)}.$$

Here, we see that the only factor of w other than $C_1^{-\delta_1}$ whose coefficient of $\lambda_{32}^{(i(2))}\lambda_{21}^{(i(2))}$ is negative of the above coefficient is

$$[x_{i(1)}^{\varepsilon(1)}, x_{i(2)}^{\varepsilon(2)}; x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{-\varepsilon(2)}, (r_1-1)x_{i(2)}^{-\varepsilon(1)}, (r_2-1)x_{i(2)}^{-\varepsilon(2)}, (r_3+1)x_{i(3)}^{-\varepsilon(3)}, \cdots, (r_{t(2)}+1)x_{i(t(2))}^{-\varepsilon(t(2))}]^{\delta_1};$$

which is same as $C_1^{-\delta_1}$ (by (6.2), (6.3) and since, $[a, b] = [b, a]^{-1}$). For the other case consider $\lambda_{32}^{i(3)}\lambda_{21}^{i(2)}$ to obtain $C_1^{-\delta_1}$.

Step III. By previous steps, w_{11} and w_{12} are both empty. Here we assume that w_{13} is non-empty, so that $C_1^{\delta_1}$ is a factor of w_{13}. Since $C_1 \in$ Cat. III, we write

$$C_1^{\delta_1} = [x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(1)}^{-\varepsilon(1)}, x_{i(3)}^{\varepsilon(3)}, \cdots, x_{i(t(3))}^{\varepsilon(t(3))}]^{\delta_1} \qquad (t(3) \geqq 3),$$

where $i(j) \neq i(k)$ for $j \neq k$ and $t(3)+1 = r^*$.

By Lemma 8, about the remaining factors of w_{13}, we can make the following assumption:

(11) If $$[v_1^{-1}, v_2^{-1}; v_3, v_4, \cdots, v_s]^{\delta} \qquad (s \geqq 4)$$

is a factor of w_{13} such that $x_{i(2)}^{\varepsilon'(2)}, x_{i(3)}^{\varepsilon'(3)} \in \{v_1, v_2, \cdots, v_s\}$ for some $\varepsilon'(2), \varepsilon'(3) \in \{1, -1\}$, then either $x_{i(2)}^{\varepsilon'(2)} \in \{v_1, v_2\}$ and $x_{i(3)}^{\varepsilon'(3)} \in \{v_3, v_4\}$, or $x_{i(3)}^{\varepsilon'(3)} \in \{v_1, v_2\}$ and $x_{i(2)}^{\varepsilon'(2)} \in \{v_3, v_4\}$.

There is a representation of w as a power product of special commutators in which w_{11}, w_{12} are empty and the factors of w_{13} satisfy (11). Among all such representations of w we choose one in which w_{13} consists of least number of factors and we write

$$w_{13} = C_{13}^{\delta_{13}} C_{23}^{\delta_{23}} \cdots C_{m(3)3}^{\delta_{m(3)3}} \qquad (m(3) \geqq 1, \delta_{13}, \cdots, \delta_{m(3)3} \in \{1, -1\}),$$

where $C_{13}^{\delta_{13}} = C_1^{\delta_1}$.

First, we consider the case when $r^* \geqq 5$. Let

$$C_1^{\delta_1} = [x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(1)}^{-\varepsilon(1)}, x_{i(3)}^{\varepsilon(3)}, \cdots, x_{i(t(3))}^{\varepsilon(t(3))}] \qquad (t(3) \geqq 4).$$

The coefficient of $\lambda_{32}^{(i(1))}\lambda_{21}^{(i(1))}$ in $\alpha_{31}(C_1^{\delta_1})$ is

$$-\delta_1 \boldsymbol{x}_{i(1)}^{\frac{1}{2}(\varepsilon(1)-1)} \boldsymbol{x}_{i(1)}^{\frac{1}{2}(-\varepsilon(1)-1)} (-1+\boldsymbol{x}_{i(2)}^{\varepsilon(2)}) \cdots (-1+\boldsymbol{x}_{i(t(3))}^{\varepsilon(t(3))})$$
$$+\delta_1 \boldsymbol{x}_{i(1)}^{\frac{1}{2}(-\varepsilon(1)-1)} \boldsymbol{x}_{i(1)}^{\frac{1}{2}(\varepsilon(1)-1)} (-1+\boldsymbol{x}_{i(2)}^{-\varepsilon(2)}) \cdots (-1+\boldsymbol{x}_{i(t(3))}^{-\varepsilon(t(3))}).$$

The only factor of w other than $C_1^{-\delta_1}$ in which the above coefficient is comparable is

$$[x_{i(1)}^{-\varepsilon(1)}, x_{i(3)}^{-\varepsilon(3)}; x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{\varepsilon(2)}, x_{i(4)}^{\varepsilon(4)}, \cdots, x_{i(t(3))}^{\varepsilon(t(3))}]^{-\delta_1} = C^{-\delta_1} \text{ (say)};$$

but

https://doi.org/10.1017/S1446788700007710 Published online by Cambridge University Press

$$C_1^{\delta_1}C^{-\delta_1} = [x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{-\varepsilon(2)}, x_{i(4)}^{-\varepsilon(4)}; x_{i(1)}^{-\varepsilon(1)}, x_{i(3)}^{\varepsilon(3)}, \cdots, x_{i(t(3))}^{\varepsilon(t(3))}]^{\delta_1}$$
$$\cdot [x_{i(1)}^{-\varepsilon(1)}, x_{i(3)}^{-\varepsilon(3)}, x_{i(4)}^{-\varepsilon(4)}; x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{\varepsilon(2)}, \cdots, x_{i(t(3))}^{\varepsilon(t(3))}]^{-\delta_1}$$
by (6.2)

$$= [x_{i(1)}^{-\varepsilon(1)}, x_{i(4)}^{-\varepsilon(4)}; x_{i(2)}^{\varepsilon(2)}, x_{i(3)}^{\varepsilon(3)}, x_{i(1)}^{-\varepsilon(1)}, \cdots, x_{i(t(3))}^{\varepsilon(t(3))}]^{\delta_1}$$
$$\cdot [x_{i(4)}^{-\varepsilon(4)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(3)}^{\varepsilon(3)}, x_{i(1)}^{\varepsilon(1)}, \cdots, x_{i(t(3))}^{\varepsilon(t(3))}]^{\delta_1}$$
$$\cdot [x_{i(4)}^{-\varepsilon(4)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(3)}^{\varepsilon(3)}, x_{i(1)}^{-\varepsilon(1)}, \cdots, x_{i(t(3))}^{\varepsilon(t(3))}]^{\delta_1}$$
$$\cdot [x_{i(4)}^{-\varepsilon(4)}, x_{i(3)}^{-\varepsilon(3)}; x_{i(2)}^{\varepsilon(2)}, x_{i(1)}^{\varepsilon(1)}, \cdots, x_{i(t(3))}^{\varepsilon(t(3))}]^{-\delta_1}$$
$$\cdot [x_{i(4)}^{-\varepsilon(4)}, x_{i(3)}^{-\varepsilon(3)}; x_{i(2)}^{\varepsilon(2)}, x_{i(1)}^{-\varepsilon(1)}, \cdots, x_{i(t(3))}^{\varepsilon(t(3))}]^{-\delta_1}$$
by (6.4), (6.1), (6.3), (6.2) and (6.5)

$$= [x_{i(4)}^{-\varepsilon(4)}, x_{i(1)}^{\varepsilon(1)}; x_{i(2)}^{\varepsilon(2)}, x_{i(3)}^{\varepsilon(3)}, \cdots, x_{i(t(3))}^{\varepsilon(t(3))}]^{\delta_1}$$
$$\cdot [x_{i(4)}^{-\varepsilon(4)}, x_{i(1)}^{-\varepsilon(1)}; x_{i(2)}^{\varepsilon(2)}, x_{i(3)}^{\varepsilon(3)}, \cdots, x_{i(t(3))}^{\varepsilon(t(3))}]^{\delta_1}$$
$$\cdot [x_{i(4)}^{-\varepsilon(4)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(3)}^{\varepsilon(3)}, x_{i,1)}^{\varepsilon(1)}, \cdots, x_{i(t(3))}^{\varepsilon(t(3))}]^{\delta_1}$$
$$\cdot [x_{i(4)}^{-\varepsilon(4)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(3)}^{\varepsilon(3)}, x_{i(1)}^{-\varepsilon(1)}, \cdots, x_{i(t(3))}^{\varepsilon(t(3))}]^{\delta_1}$$
$$\cdot [x_{i(4)}^{-\varepsilon(4)}, x_{i)3)}^{-\varepsilon(3)}; x_{i(2)}^{\varepsilon(2)}, x_{i(1)}^{\varepsilon(1)}, \cdots, x_{i(t(3))}^{\varepsilon(t(3))}]^{-\delta_1}$$
$$\cdot [x_{i(4)}^{-\varepsilon(4)}, x_{i(3)}^{-\varepsilon(3)}; x_{i(2)}^{\varepsilon(2)}, x_{i(1)}^{-\varepsilon(1)}, \cdots, x_{i(t(3))}^{\varepsilon(t(3))}]^{-\delta_1}$$
by (6.2), (6.3), (6.5)

which is a power product of special commutators of weight strictly less than r^* and hence gives a representation of w_{13} with fewer factors — contrary to the assumption.

For the case $r^* = 4$, let $C_1^{\delta_1} = [x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(1)}^{-\varepsilon(1)}, x_{i(3)}^{\varepsilon(3)}]^{\delta_1}$. The coefficient of $\lambda_{32}^{(i(3))}\lambda_{21}^{(i(2))}$ in $\alpha_{31}(C_1^{\delta_1})$ is

$$-\delta_1\varepsilon(2)\varepsilon(3)\boldsymbol{x}_{i(2)}^{\frac{1}{2}(\varepsilon(2)-1)}\boldsymbol{x}_{i(3)}^{\frac{1}{2}(\varepsilon(3)-1)}(-1+\boldsymbol{x}_{i(1)}^{\varepsilon(1)})(-1+\boldsymbol{x}_{i(1)}^{-\varepsilon(1)});$$

and it compares only with the coefficient of $\lambda_{32}^{(i(3))}\lambda_{21}^{(i(2))}$ in

$$\alpha_{31}([x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(1)}^{-\varepsilon(1)}, x_{i(3)}^{\varepsilon(3)}])^{-\delta_1}.$$

This completes the proof that w_{13} is empty.

Step IV. We have shown in the previous steps that w_{11}, w_{12}, w_{13} are empty, so that $w = w_{14}w_2w'''$. Now we suppose that w_{14} is non-empty and arrive at a contradiction.

First of all, we note (by using (6.1) to (6.5)) that

(12) If $$C = [v_1^{-1}, v_2^{-1}; v_1^{-1}, v_3, v_4, \cdots, v_s] \qquad (s \geqq 4)$$

is a special commutator such that $v_i = v_j = v$ for some $i \neq j$; then C can be written as

$$C = [v_1^{-1}, v^{-1}; v_1^{-1}, v, v_2, \cdots, v_{i-1}, v_{i+1}, \cdots, v_{j-1}, v_{j+1}, \cdots, v_s]\pi$$

where π is a power product of commutators of weight strictly less than $s+1$.

Since $C_1 \in$ Cat. IV, by definition, the hypothesis of (12) is satisfied; so we can take

$$C_1^{\delta_1} = [x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{\varepsilon(2)}, (r_2-1)x_{i(2)}^{\varepsilon(2)}, (r_3+1)x_{i(3)}^{\varepsilon(3)}, \cdots, (r_{t(4)}+1)x_{i(t(4))}^{\varepsilon(t(4))}]^{\delta_1},$$

where $i(j) \neq i(k)$ for $j \neq k$, $r_2 \geqq 1$, $r_j \geqq 0$ for $j \neq 2$, $t(4) \geqq 2$ and $\sum_{j=2}^{t(4)} (r_j+1)+2 = r^*$.

About the remaining factors of w, by Lemma 8, we can make the following assumption:

(13) If $$[v_1^{-1}, v_2^{-1}; v_3, v_4, \cdots, v_s]^{\delta} \qquad (s \geqq 4)$$

is a factor of w such that $\{v_1, \cdots, v_s\}$ contains $x_{i(2)}^{\varepsilon'(2)}, x_{i(2)}^{\varepsilon''(2)}$ for some $\varepsilon'(2)$, $\varepsilon''(2) \in \{1, -1\}$ then either $x_{i(2)}^{\varepsilon'(2)} \in \{v_1, v_2\}$ and $x_{i(2)}^{\varepsilon''(2)} \in \{v_3, v_4\}$ or $x_{i(2)}^{\varepsilon''(2)} \in \{v_1, v_2\}$ and $x_{i(2)}^{\varepsilon'(2)} \in \{v_3, v_4\}$.

There is a representation of w as a power product of special commutators in which w_{11}, w_{12}, w_{13} are empty and the factors of w_{13}, w_2 satisfy (13). We take such a representation of w in which w_{14} consists of least number of factors and write

$$w_{14} = C_{14}^{\delta_{14}} C_{24}^{\delta_{24}} \cdots C_{m(4)4}^{\delta_{m(4)4}} \qquad (m(4) \geqq 1; \delta_{14}, \cdots, \delta_{m(4)4} \in \{1, -1\}),$$

where $C_{14}^{\delta_{14}} = C_1^{\delta_1}$. Let

$$C_1^{\delta_1} = [x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(1)}^{-\varepsilon(1)}, x_{i(2)}^{\varepsilon(2)}, (r_2-1)x_{i(2)}^{\varepsilon(2)}, (r_3+1)x_{i(3)}^{\varepsilon(3)}, \cdots, (r_{t(4)}+1)x_{i(t(4))}^{\varepsilon(t(4))}]^{\delta_1},$$

where $t(4) \geqq 2$. The coefficient of $\lambda_{32}^{(i(1))} \lambda_{21}^{(i(1))}$ in $\alpha_{31}(C_1^{\delta_1})$ is

$$-\delta_1 \mathbf{x}_{i(1)}^{\frac{1}{2}(\varepsilon(1)-1)} \mathbf{x}_{i(1)}^{\frac{1}{2}(-\varepsilon(1)-1)} (-1+\mathbf{x}_{i(2)}^{\varepsilon(2)})^{(r_2+1)} \cdots (-1+\mathbf{x}_{i(t(4))}^{\varepsilon(t(4))})^{(r_{t(4)}+1)}$$
$$+\delta_1 \mathbf{x}_{i(1)}^{\frac{1}{2}(-\varepsilon(1)-1)} \mathbf{x}_{i(1)}^{\frac{1}{2}(\varepsilon(1)-1)} (-1+\mathbf{x}_{i(2)}^{-\varepsilon(2)})^{(r_2+1)} \cdots (-1+\mathbf{x}_{i(t(4))}^{-\varepsilon(t(4))})^{(r_{t(4)}+1)}.$$

The only factor of w other than $C_1^{-\delta_1}$ whose coefficient of $\lambda_{32}^{(i(1))} \lambda_{21}^{(i(1))}$ is comparable with the above coefficient is

$$[x_{i(1)}^{-\varepsilon'(1)}, x_{i(2)}^{\varepsilon(2)}; x_{i(1)}^{-\varepsilon'(1)}, x_{i(2)}^{-\varepsilon(2)}, (r_2-1)x_{i(2)}^{-\varepsilon(2)}, (r_3+1)x_{i(3)}^{-\varepsilon(3)}, \cdots, (r_{t(4)}+1)x_{i(t(4))}^{-\varepsilon(t(4))}]^{\delta_1}$$

or

$$[x_{i(1)}^{-\varepsilon'(1)}, x_{i(2)}^{-\varepsilon(2)}; x_{i(1)}^{-\varepsilon'(1)}, x_{i(2)}^{\varepsilon(2)}, (r_2-1)x_{i(2)}^{\varepsilon(2)}, (r_3+1)x_{i(3)}^{\varepsilon(3)}, \cdots, (r_{t(4)}+1x_{i(t(4))}^{\varepsilon(t(4))}]^{-\delta_1}.$$

But each is equal to $C_1^{-\delta_1}$ by (6.2), (6.3) and (6.6). This completes the details of Step IV and hence also completes the proof of the theorem.

Remark. It is clear from the proof of the theorem that if w is a power product of special commutators of Cat. II, III and IV only, then $w \in [F'', F]$. Thus, it follows that for the free centre-by-metabelian group of rank 3, the matrix representation is faithful.

References

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https://doi.org/10.1017/S1446788700007710 Published online by Cambridge University Press