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On Dominating Sets and Independent Sets of Graphs

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For a graph G on vertex set $V = \{1, \ldots, n\}$ let $\mathbf{k} = (k_1, \ldots, k_n)$ be an integral vector such that $1 \leqslant k_i \leqslant d_i$ for $i \in V$, where d_i is the degree of the vertex i in G. A $\mathbf{k}$-dominating set is a set $D_\mathbf{k} \subseteq V$ such that every vertex $i \in V \backslash D_\mathbf{k}$ has at least k_i neighbours in $D_\mathbf{k}$. The $\mathbf{k}$-domination number $\gamma_\mathbf{k}(G)$ of G is the cardinality of a smallest $\mathbf{k}$-dominating set of G.

For $k_1 = \cdots = k_n = 1$, $\mathbf{k}$-domination corresponds to the usual concept of domination. Our approach yields an improvement of an upper bound for the domination number found by N. Alon and J. H. Spencer.

If $k_i = d_i$ for $i = 1, \ldots, n$, then the notion of $\mathbf{k}$-dominating set corresponds to the complement of an independent set. A function $f_\mathbf{k}(\mathbf{p})$ is defined, and it will be proved that $\gamma_\mathbf{k}(G) = \min f_\mathbf{k}(\mathbf{p})$, where the minimum is taken over the n-dimensional cube $\mathbf{C}^n = \{\mathbf{p} = (p_1, \ldots, p_n) \mid p_i \in \mathbb{R}, 0 \leqslant p_i \leqslant 1, i = 1, \ldots, n\}$. An $\mathcal{O}(\Delta^2 2^\Delta n)$-algorithm is presented, where Δ is the maximum degree of G, with INPUT: $\mathbf{p} \in \mathbf{C}^n$ and OUTPUT: a $\mathbf{k}$-dominating set $D_\mathbf{k}$ of G with $|D_\mathbf{k}| \leqslant f_\mathbf{k}(\mathbf{p})$.

1. Introduction

We consider an undirected simple graph G on vertex set $V = \{1, \ldots, n\}$. A *dominating set* of G is a set $D \subseteq V$ such that every vertex $i \in V \backslash D$ has at least one neighbour in D. The *domination number* $\gamma(G)$ of G is the cardinality of a smallest dominating set of G. An *independent set* of G is a set $I \subseteq V$ such that none of the neighbours of a vertex $i \in I$ is in I. The *independence number* $\alpha(G)$ of G is the cardinality of a largest independent set of G. The *neighbourhood* $N(i)$ of a vertex $i \in V$ is the set of all neighbours of i in G. The *degree* of a vertex i is the cardinality of $N(i)$ and is denoted by d_i. Let δ and Δ be the *minimum degree* and the *maximum degree* of G, respectively. Furthermore, we assume that G has no isolated vertices, that is, $\delta > 0$.

We generalize the concept of domination. Let $\mathbf{k} = (k_1, \ldots, k_n)$ be an integral vector such that $1 \leqslant k_i \leqslant d_i$ for $i = 1, \ldots, n$. A $\mathbf{k}$-*dominating set* is a set $D_\mathbf{k} \subseteq V$ such that every vertex $i \in V \backslash D_\mathbf{k}$ has at least k_i neighbours in $D_\mathbf{k}$. The $\mathbf{k}$-*domination number* $\gamma_\mathbf{k}(G)$ of G is

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the cardinality of a smallest $\mathbf{k}$-dominating set of G. For $k_1 = \cdots = k_n = 1$ the definition of $\mathbf{k}$-domination reduces to the usual definition of domination, and with $\mathbf{1} = (1, \ldots, 1)$ it holds $\gamma_{\mathbf{1}}(G) = \gamma(G)$. If $k_i = d_i$ for $i = 1, \ldots, n$, then $I = V \backslash D_{\mathbf{k}}$ is an independent set, and $\gamma_{\mathbf{d}}(G) = n - \alpha(G)$ with $\mathbf{d} = (d_1, \ldots, d_n)$.

The DOMINATING SET and INDEPENDENT SET problems are known to be NP-complete (see [9], p. 190 and p. 194).

2. Results

Let $\mathbb{R}$ be the set of real numbers and let $f_{\mathbf{k}} : \mathbf{C}^n = \{\mathbf{p} = (p_1, \ldots, p_n) \mid p_i \in \mathbb{R}, 0 \leqslant p_i \leqslant 1, i = 1, \ldots, n\} \rightarrow \mathbb{R}$ be the function defined by

$$f_{\mathbf{k}}(\mathbf{p}) = \sum_{i=1}^{n} p_i + \sum_{i=1}^{n} (1 - p_i) \left(\sum_{l=0}^{k_i - 1} \sum_{\{i_1, \ldots, i_l\} \subset N(i)} \prod_{m \in \{i_1, \ldots, i_l\}} p_m \prod_{m \in N(i) \backslash \{i_1, \ldots, i_l\}} (1 - p_m) \right).$$

Theorem 2.1.

$$\gamma_{\mathbf{k}}(G) = \min_{\mathbf{p} \in \mathbf{C}^n} f_{\mathbf{k}}(\mathbf{p}).$$

Proof. We form a set $X \subseteq V$ by random and independent choice of $i \in V$, where $P(i \in X) = p_i$ with $0 \leqslant p_i \leqslant 1$ denotes the probability that the vertex i belongs to X. With $Y = \{i \in V \mid i \notin X \text{ and } |N(i) \cap X| \leqslant k_i - 1\}$ the set $D = X \cup Y$ is a $\mathbf{k}$-dominating set of G. We obtain $E(|D|) = E(|X|) + E(|Y|)$ because of the linearity of the expectation, and, consequently,

$$\begin{aligned} E(|D|) &= \sum_{i=1}^{n} P(i \in X) + \sum_{i=1}^{n} P(i \in Y) \\ &= \sum_{i=1}^{n} p_i + \sum_{i=1}^{n} (1 - p_i) \left(\sum_{l=0}^{k_i - 1} P(|N(i) \cap X| = l) \right) \\ &= \sum_{i=1}^{n} p_i + \sum_{i=1}^{n} (1 - p_i) \left(\sum_{l=0}^{k_i - 1} \sum_{\{i_1, \ldots, i_l\} \subset N(i)} \prod_{m \in \{i_1, \ldots, i_l\}} p_m \prod_{m \in N(i) \backslash \{i_1, \ldots, i_l\}} (1 - p_m) \right). \end{aligned}$$

The expectation being an average value, there is a $\mathbf{k}$-dominating set $D_{\mathbf{k}}$ such that the cardinality of $D_{\mathbf{k}}$ is at most $E(|D|)$. Hence, $\gamma_{\mathbf{k}}(G) \leqslant \min_{\mathbf{p} \in \mathbf{C}^n} f_{\mathbf{k}}(\mathbf{p})$. Let $D_{\mathbf{k}}^*$ be a $\mathbf{k}$-dominating set of G of cardinality $\gamma_{\mathbf{k}}(G)$. For $\mathbf{p}^* = (p_1^*, \ldots, p_n^*)$ with $p_i^* = 1$ if $i \in D_{\mathbf{k}}^*$ and $p_i^* = 0$ otherwise, $\gamma_{\mathbf{k}}(G) = f_{\mathbf{k}}(\mathbf{p}^*)$ holds. Theorem 2.1 follows. □

Considering the case $k_1 = \cdots = k_n = 1$, we obtain the following result.

Corollary 2.2. $$\gamma(G) = \min_{(p_1,\ldots,p_n)\in \mathbf{C}^n} \sum_{i=1}^{n} \left(p_i + (1-p_i) \prod_{m\in N(i)} (1-p_m) \right). \qquad \square$$

Using $\sum_{l=0}^{d_i-1} P(|N(i)\cap X| = l) = 1 - P(|N(i)\cap X| = d_i)$ (see the proof of Theorem 2.1) and $\alpha(G) = n - \gamma_{\mathbf{d}}(G)$ we have the following result.

Corollary 2.3.

$$\begin{aligned} \alpha(G) &= n - \min_{(p_1,\ldots,p_n)\in \mathbf{C}^n} \sum_{i=1}^{n} \left(p_i + (1-p_i)(1 - \prod_{m\in N(i)} p_m) \right) \\ &= \max_{(p_1,\ldots,p_n)\in \mathbf{C}^n} \sum_{i=1}^{n} (1-p_i) \prod_{m\in N(i)} p_m. \qquad \square \end{aligned}$$

Theorem 2.4. *There is an $\mathcal{O}(\Delta^2 2^\Delta n)$-algorithm that constructs a* $\mathbf{k}$*-dominating set $D_{\mathbf{k}}$ of G with $|D_{\mathbf{k}}| \leqslant f_{\mathbf{k}}(\mathbf{p})$ for any given $\mathbf{p} \in \mathbf{C}^n$.*

Proof of Theorem 2.4. For given $\mathbf{p} \in \mathbf{C}^n$ let $f^* = f_{\mathbf{k}}(\mathbf{p})$. First we present an algorithm that constructs a set $D \subset V$. For the current $\mathbf{p} \in \mathbf{C}^n$ of this algorithm let

$$\begin{aligned} A_j &= \sum_{l=0}^{k_j-1} \sum_{\{j_1,\ldots,j_l\}\subseteq N(j)} \prod_{m\in\{j_1,\ldots,j_l\}} p_m \prod_{m\in N(j)\setminus\{j_1,\ldots,j_l\}} (1-p_m), \\ B_j &= \sum_{i\in N(j)} (1-p_i) \sum_{\substack{\{i_1,\ldots,i_l\}\subseteq N(i)\setminus\{j\} \\ l=k_i-1}} \prod_{m\in\{i_1,\ldots,i_l\}} p_m \prod_{m\in N(i)\setminus\{i_1,\ldots,i_l\}\setminus\{j\}} (1-p_m), \end{aligned}$$

for $j = 1, \ldots, n$.

Algorithm.

INPUT: $\mathbf{p} \in \mathbf{C}^n$
OUTPUT: D

(1) For $j = 1,\ldots,n$ do if $1 - A_j - B_j \geqslant 0$ then $p_j := 0$, else $p_j := 1$.

(2) For $j = 1,\ldots,n$ do if $A_j = 1$ then $p_j = 1$.

(3) $D := \{j \in \{1,\ldots,n\} \mid p_j = 1\}$.

END

It is easy to see that A_j and B_j can be calculated in $\mathcal{O}(\Delta^2 2^\Delta)$ time. Thus the algorithm works in $\mathcal{O}(\Delta^2 2^\Delta n)$ time.

Now we want to show that the cardinality of the resulting set D does not exceed the value f^* and that D is a $\mathbf{k}$-dominating set.

Claim 1. $|D| \leqslant f^*$

Let us consider step (1) of the algorithm. Note that the function $f_{\mathbf{k}}((p_1,\ldots,p_n))$ is linear in each variable, and that we have

$$\begin{aligned}
&\frac{\partial f_{\mathbf{k}}((p_1,\ldots,p_n))}{\partial p_j}\\
&= 1 - A_j\\
&\quad + \sum_{i\in N(j)}(1-p_i)\sum_{l=1}^{k_i-1}\left(\sum_{\substack{\{i_1,\ldots,i_l\}\subset N(i)\\ j\in\{i_1,\ldots,i_l\}}}\ \prod_{\substack{m\in\{i_1,\ldots,i_l\}\\ m\neq j}} p_m \prod_{m\in N(i)\setminus\{i_1,\ldots,i_l\}}(1-p_m)\right)\\
&\quad - \sum_{i\in N(j)}(1-p_i)\sum_{l=0}^{k_i-1}\left(\sum_{\substack{\{i_1,\ldots,i_l\}\subset N(i)\\ j\in N(i)\setminus\{i_1,\ldots,i_l\}}}\ \prod_{m\in\{i_1,\ldots,i_l\}} p_m \prod_{\substack{m\in N(i)\setminus\{i_1,\ldots,i_l\}\\ m\neq j}}(1-p_m)\right)\\
&= 1 - A_j\\
&\quad + \sum_{i\in N(j)}(1-p_i)\sum_{l=0}^{k_i-2}\left(\sum_{\{i_1,\ldots,i_l\}\subset N(i)\setminus\{j\}}\ \prod_{m\in\{i_1,\ldots,i_l\}} p_m \prod_{m\in N(i)\setminus\{i_1,\ldots,i_l\}\setminus\{j\}}(1-p_m)\right)\\
&\quad - \sum_{i\in N(j)}(1-p_i)\sum_{l=0}^{k_i-1}\left(\sum_{\{i_1,\ldots,i_l\}\subset N(i)\setminus\{j\}}\ \prod_{m\in\{i_1,\ldots,i_l\}} p_m \prod_{m\in N(i)\setminus\{i_1,\ldots,i_l\}\setminus\{j\}}(1-p_m)\right)\\
&= 1 - A_j - B_j.
\end{aligned}$$

Therefore, for fixed $(p_1,\ldots,p_{j-1},p_{j+1},\ldots,p_n)$ we always choose p_j in such a way that the value of the function $f_{\mathbf{k}}(\mathbf{p})$ is not increased. Then, $p_i \in \{0,1\}$ for $i = 1,\ldots,n$ after step (1) of the algorithm.

Lemma. *Let $p_i \in \{0,1\}$ for $i = 1,\ldots,n$ and $D = \{i \in \{1,\ldots,n\} \mid p_i = 1\}$. Then for $j = 1,\ldots,n$ we have $A_j \in \{0,1\}$ and $A_j = 1$ if and only if there is no subset $S \subseteq N(j)\cap D$ with $|S| \geqslant k_j$.*

Proof of Lemma. First, assume there is a set $S \subseteq N(j)\cap D$ with $|S| \geqslant k_j$. Then

$$\prod_{m\in N(j)\setminus\{j_1,\ldots,j_l\}}(1-p_m) = 0$$

for all subsets $\{j_1,\ldots,j_l\} \subset N(j)$ with $l < k_j$, implying $A_j = 0$.

Second, assume there is no set $S \subseteq N(j)\cap D$ with $|S| \geqslant k_j$. Define $S' := N(j)\cap D$. Clearly, $|S'| \leqslant k_j - 1$. Note that, for any subset $\{j_1,\ldots,j_l\} \subseteq N(j)$,

$$\prod_{m\in\{j_1,\ldots,j_l\}} p_m \prod_{m\in N(j)\setminus\{j_1,\ldots,j_l\}}(1-p_m) = 1$$

if $\{j_1,\ldots,j_l\} = S'$ and 0 otherwise. Thus $A_j = 1$.

This completes the proof of the lemma. □

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Now consider step (2) of the algorithm setting $p_j = 1$ if $A_j = 1$. Note that

$$f_{\mathbf{k}}(\mathbf{p}) = \sum_{i=1}^{n} p_i + \sum_{i=1}^{n}(1 - p_i)A_i.$$

If p_j was already 1 before this step, nothing is changed.

If $p_j = 0$ before step (2), then p_j is added to D. Note that $\sum_{i=1}^{n} p_i$ increases by 1 but $(1 - p_j)A_j$ decreases by 1. For all $i \neq j$ the term A_i cannot increase by this operation because of the above lemma.

Thus the value of the function $f_{\mathbf{k}}$ after step (2) does not exceed the value of that function before this step. Consequently, the value of $f_{\mathbf{k}}$ is not increased by the algorithm. Furthermore, $p_j = 0$ at the end of the algorithm implies $A_j = 0$ because of step (2). Therefore, the cardinality of D equals the value of the function $f_{\mathbf{k}}$ at the end of the algorithm; it follows that $|D| \leqslant f^*$, proving Claim 1.

Claim 2. *D is a* **k***-dominating set.*

Let $j \in \{1, \dots, n\}$ with $p_j = 0$. Then, because of step (2) of the algorithm, $A_j \neq 1$. Using the above lemma we have $A_j = 0$, implying that there is a subset $S \subseteq N(j) \cap D$ with $|S| \geqslant k_j$. Hence, $D = \{j \in \{1, \dots, n\} \mid p_j = 1\}$ is a **k**-dominating set and Claim 2 is proved.

With $D_{\mathbf{k}} = D$ the proof of Theorem 2.4 is complete. □

Let us remark that for $\mathbf{k} = \mathbf{1}$ or $\mathbf{k} = \mathbf{d}$ the algorithm of Theorem 2.4 runs in $\mathcal{O}(\Delta^2 n)$ time.

3. Concluding remarks

For $\mathbf{k} = \mathbf{1}$ we obtain results concerning the domination number $\gamma(G)$.

Let us consider the case $p_1 = \dots = p_n = p$. By Corollary 2.2 we have $\gamma(G) \leqslant \min_{p\in[0,1]} g(p)$ where $g(p) = np + \sum_{i=1}^{n}(1 - p)^{d_i+1}$.

Using the ideas of Alon and Spencer, their bound $\frac{n(1+\ln(\delta+1))}{\delta+1}$ on $\gamma(G)$ (*cf.* [1]) can be deduced from this result by the following steps. Since $(1 - p)^{d_i+1} \leqslant (1 - p)^{\delta+1}$ for all i, we obtain $\gamma(G) \leqslant \min_{p\in[0,1]} h(p)$ where $h(p) = (np + n(1 - p)^{\delta+1})$. Using $1 - p < e^{-p}$ and minimizing the function $np + ne^{-p(\delta+1)}$ for $p \in [0, 1]$, the bound of Alon and Spencer follows. However, with $p_1 = \cdots = p_n = \frac{ln(\delta+1)}{\delta+1}$, Theorem 2.4 yields a constructive proof of this bound.

An improvement of the result of Alon and Spencer is obtained by noticing that

$$\min_{p\in[0,1]} h(p) = h(\hat{p}) \quad \text{with } \hat{p} = 1 - \left(\frac{1}{\delta + 1}\right)^{\frac{1}{\delta}},$$

which implies $\gamma(G) \leqslant g(\hat{p})$; consequently, we have the following result.

Corollary 3.1. *With* $\mathbf{k} = \mathbf{1}$ *and* $p_1 = \cdots = p_n = 1 - (\frac{1}{\delta+1})^{\frac{1}{\delta}}$, *the above algorithm constructs*

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a dominating set of cardinality at most

$$n\left(1-\left(\frac{1}{\delta+1}\right)^{\frac{1}{\delta}}\right)+\sum_{i=1}^{n}\left(\frac{1}{\delta+1}\right)^{\frac{d_i+1}{\delta}}.$$

In order to find a small dominating set, it seems reasonable to assume that vertices of high degree should belong to the dominating set with higher probability than vertices of small degree. Thus, it may be advantageous to make the probability p_i depend on the degree d_i of the corresponding vertex. Hence, possible values $p_1,\ldots,p_n$ for the algorithm of Theorem 2.4 are $p_i = 1-(\frac{1}{d_i+1})^{\frac{1}{\delta}}$ for $i = 1,\ldots,n$, and we obtain the following.

Corollary 3.2.

$$\gamma(G) \leqslant \sum_{i=1}^{n}\left(1-\left(\frac{1}{d_i+1}\right)^{\frac{1}{\delta}}+\left(\frac{1}{d_i+1}\right)^{\frac{1}{\delta}}\prod_{j\in N(i)}\left(\frac{1}{d_j+1}\right)^{\frac{1}{\delta}}\right).$$

Our approach is also useful in the case when all components of $\mathbf{k}$ are equal to a constant greater than 1. For example, let c and s be integers, $1 < c \leqslant \delta$, $1 < s$, $\mathbf{c} = (c,\ldots,c)$ and $p_1 = \cdots = p_n = \frac{1}{s}$. Then (see proof of Theorem 2.1), we have

$$\gamma_{\mathbf{c}}(G) \leqslant \frac{n}{s}+\frac{s-1}{s}\sum_{i=1}^{n}P(|N(i)\cap X| \leqslant c-1).$$

For $\epsilon > 0$ we have $P(|N(i)\cap X| \leqslant c-1) < \epsilon$ for $i = 1,\ldots,n$ if $\delta - c$ is large enough. Other results for $\gamma_{\mathbf{c}}(G)$ can be found in [2], [4], [5], [6], [7], [8], [10] and [11].

We have seen that for $\mathbf{k} = \mathbf{d}$ our algorithm constructs the complement of an independent set. Unfortunately, we do not know appropriate values $p_1,\ldots,p_n$ for this algorithm to yield an independent set whose cardinality satisfies the Caro–Wei inequality (*cf.* [3])

$$\alpha(G) \geqslant \sum_{i\in V}\frac{1}{d_i+1}.$$

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