COMPOSITIO MATHEMATICA

Quantum mirrors of log Calabi–Yau surfaces and higher-genus curve counting

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Abstract

Gross, Hacking and Keel have constructed mirrors of log Calabi–Yau surfaces in terms of counts of rational curves. Using q-deformed scattering diagrams defined in terms of higher-genus log Gromov–Witten invariants, we construct deformation quantizations of these mirrors and we produce canonical bases of the corresponding non-commutative algebras of functions.

Contents

1. Introduction

1.1 Context and motivations

1.1.1 *Mirror symmetry.* The Strominger–Yau–Zaslow (SYZ) [SYZ96] picture of mirror symmetry suggests an original way of constructing algebraic varieties: given a Calabi–Yau variety, its mirror geometry should be constructed in term of its enumerative geometry of holomorphic discs. This picture has been developed by Fukaya [Fuk05], Kontsevich and Soibelman [KS06], Gross and Siebert [GS11], Auroux [Aur07] and many others. In particular, Gross and Siebert have developed an algebraic approach in which the enumerative geometry of holomorphic discs is replaced by some genus-0 logarithmic Gromov–Witten invariants. Given the recent progress in logarithmic Gromov–Witten theory, in particular the definition of punctured invariants by Abramovich, Chen, Gross and Siebert [ACGS17], it is likely that this approach will lead to some general mirror symmetry construction in the algebraic setting, see Gross and Siebert [GS16] for an announcement.

1.1.2 *The work of Gross, Hacking and Keel.* An early version of this mirror construction has been used by Gross, Hacking and Keel [GHK15a] to construct mirror families of log Calabi–Yau surfaces, with non-trivial applications to the theory of surface singularities and, in particular, a proof of Looijenga's conjecture on smoothing of cusp singularities. More precisely, the construction of [GHK15a] applies to Looijenga pairs, that is, to pairs (Y, D) where Y is a smooth projective surface and D is some anticanonical singular nodal curve. The upshot is in general a formal flat family $\mathcal{X} \to S$ of surfaces over a formal completion, near some point s_0, the 'large volume limit of Y', of an algebraic approximation to the complexified Kähler cone of Y.

Furthermore, $\mathcal{X}$ is an affine Poisson formal variety with a canonical linear basis of so-called theta functions and the map $\mathcal{X} \to S$ is Poisson if S is equipped with the zero Poisson bracket. Under some positivity assumptions on (Y, D), this family can be in fact extended to an algebraic family over an algebraic base and the generic fiber is then a smooth algebraic symplectic surface. To simplify the exposition in this introduction, we assume for now that it is the case.

The first step of the construction involves defining the fiber $\mathcal{X}_{s_0}$, that is, the 'large complex structure limit' of the family $\mathcal{X}$. This step is essentially combinatorial and can be reduced to some toric geometry: $\mathcal{X}_{s_0}$ is a reducible union of toric varieties.

The second step is to construct $\mathcal{X}$ by smoothing of $\mathcal{X}_{s_0}$. This construction is based on the consideration of an algebraic object, a scattering diagram, a notion introduced by Kontsevich and Soibelman [KS06] and further developed by Gross and Siebert [GS11], whose definition encodes genus-0 log Gromov–Witten invariants[1] of (Y, D). The key non-trivial property to check is the so-called consistency of the scattering diagram. In [GHK15a], the consistency relies on the work of Gross, Pandharipande and Siebert [GPS10], which itself relies on connections with tropical geometry [Mik05, NS06]. Once the consistency of the scattering diagram is guaranteed, some combinatorial objects, the broken lines [CPS10], are well defined and can be used to construct the algebra of functions $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ with its linear basis of theta functions.

1.1.3 *Quantization.*[2] The variety $\mathcal{X}$ being a Poisson variety over S, it is natural to ask about its quantization, for example in the sense of deformation quantization. As $\mathcal{X}$ and S are affine, the deformation quantization problem takes its simplest form: to construct a structure of non-commutative $H^0(S, \mathcal{O}_S)[\![\hbar]\!]$-algebra on $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \otimes \mathbb{C}[\![\hbar]\!]$ whose commutator is given at the linear order in $\hbar$ by the Poisson bracket on $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. There are general existence results [Kon01, Yek05] for deformation quantizations of smooth affine Poisson varieties. A useful reference on deformation quantization of algebraic symplectic varieties is [BK04]. In fact, on its smooth locus, the map $\mathcal{X} \to S$ is symplectic of relative dimension two and then the existence of a deformation is easy because the obstruction space vanishes for dimension reasons. But there are no known general results which would guarantee *a priori* the existence of a deformation quantization of $\mathcal{X}$ over S because $\mathcal{X} \to S$ is singular, for example over $s_0 \in S$ to start with. Specific examples of deformation quantization of such geometries usually involve some situation-specific representation theory or geometry; see, for example, [Obl04, EOR07, EG10, AK17].

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[1] In [GHK15a], an *ad hoc* definition of genus-0 Gromov–Witten invariants is used, which was supposed to coincide with genus-0 log Gromov–Witten invariants. This fact follows from the remark at the end of [Bou19, § 4]. In the present paper, we use log Gromov–Witten theory systematically.

[2] The existence of theta functions is related to the geometric quantization of the real integrable system formed by a Calabi–Yau manifold with an SYZ fibration. We do *not* refer to this quantization story. In this paper, quantization always means deformation quantization of a holomorphic symplectic/Poisson variety.

1.2 Main results

The main result of the present paper is a construction of a deformation quantization of $\mathcal{X} \to S$. Our construction follows along the lines of Gross, Hacking and Keel [GHK15a] except that, rather than using only genus-0 log Gromov–Witten invariants, we use higher-genus log Gromov–Witten invariants, the genus parameter playing the role of the quantization parameter $\hbar$ on the mirror side.

We construct a quantum version of a scattering diagram and we prove its consistency using the main result of [Bou18], which itself relies on the connection with refined tropical geometry [Bou19]. Once the consistency of the quantum scattering diagram is guaranteed, a quantum version of the broken lines is well defined and can be used to construct a deformation quantization of $H^0(\mathcal{X}, \mathcal{O}_\mathcal{X})$. In fact, it follows from the use of [Bou18] that the dependence on the deformation parameter $\hbar$ is algebraic in $q = e^{i\hbar}$,[3] something which in general cannot be obtained from some general deformation-theoretic argument. In other words, the main result of the present paper can be phrased in the following slightly vague terms (see Theorems 2.1–2.3 for precise statements).

THEOREM 1.1. *The Gross–Hacking–Keel [GHK15a] Poisson family $\mathcal{X} \to S$, the mirror of a Looijenga pair (Y, D), admits a deformation quantization, which can be constructed in a synthetic way from the higher-genus log Gromov–Witten theory of (Y, D). Furthermore, the dependence on the deformation quantization parameter $\hbar$ is algebraic in $q = e^{i\hbar}$.*

The notion of quantum scattering diagram is suggested at the end of [KS06, §11.8] and is used by Soibelman [Soi09] to construct non-commutative deformations of non-archimedean K3 surfaces. The connection with quantization, for example in the context of cluster varieties [FG09a, FG09b], was expected, and quantum broken lines have been studied by Mandel [Man15]. The key novelty of the present paper, building on the previous work [Bou19, Bou18] of the author, is the connection between these algebraic/combinatorial q-deformations and the geometric deformation given by higher-genus log Gromov–Witten theory.

This connection between higher-genus Gromov–Witten theory and quantization is perhaps a little surprising, even if similar-looking statements are known or expected. In §7 we explain that Theorem 1.1 should be viewed as an example of a higher-genus mirror symmetry relation, the deformation quantization being a two-dimensional reduction of the three-dimensional higher-genus B-model (Bershadsky–Cecotti–Ooguri–Vafa theory). We also comment on the relation with expectations from string theory, in a way parallel to [Bou18, §8].

In the context of mirror symmetry, there is a well-known symplectic interpretation of some non-commutative deformations on the B-side, involving deformation of the complexified symplectic form which does not preserve the Lagrangian nature of the fibers of the SYZ fibration. An example of this phenomenon has been studied by Auroux, Katzarkov and Orlov [AKO06] in the context of mirror symmetry for del Pezzo surfaces. There is some work in progress by Sheridan and Pascaleff about generalizing this approach to study non-commutative deformations of mirrors of log Calabi–Yau varieties. This approach remains entirely in the traditional realm of genus-0 holomorphic curves and so is completely different from our approach using higher-genus curves. The compatibility of these two approaches can be understood via a chain of string-theoretic dualities.

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[3] Because in general $\mathcal{X}$ is already a formal object, this claim has to be stated more precisely; see Theorem 2.3. It is correct in the most naive sense if (Y, D) is positive enough and $\mathcal{X}$ is then really an algebraic family.

It is natural to ask how the deformation quantization given by Theorem 1.1 is related to previously known examples of quantization. In § 6 we treat a simple example and we recover a well-known description of the A_2 quantum $\mathcal{X}$-cluster variety [FG09a].

For Y a cubic surface in $\mathbb{P}^3$ and D a triangle of lines on Y, the quantum scattering diagram can be explicitly computed and so, using techniques similar to those developed in [GHK19], one should be able to show that the deformation quantization given by Theorem 1.1 coincides with the one constructed by Oblomkov [Obl04] using Cherednik algebras (double affine Hecke algebras). We leave this verification, and the general relation to quantum $\mathcal{X}$-cluster varieties, to future work.

Similarly, if Y is a del Pezzo surface of degree 1, 2 or 3 and D a nodal cubic, it would be interesting to compare Theorem 1.1 with the construction of Etingof, Oblomkov and Rains [EOR07] using Cherednik algebras. In these cases, the quantum scattering diagrams are extremely complicated and new ideas are probably required.

1.3 Plan of the paper

In § 2 we set up our notation and we give precise versions of the main results. In § 3 we describe the formalism of quantum scattering diagrams and quantum broken lines. In § 4 we explain how to associate to every Looijenga pair (Y, D) a canonical quantum scattering diagram constructed in terms of higher-genus log Gromov–Witten invariants of (Y, D). The key result in our construction is Theorem 4.1 establishing the consistency of the canonical quantum scattering diagram. The proof of Theorem 4.1 follows the reduction steps used by Gross, Hacking and Keel [GHK15a] in the genus-0 case. In the final step, we use the main result of [Bou18] in place of the main result of [GPS10]. In § 5 we finish the proofs of the main theorems. In § 6 we work out an explicit example. Finally, in § 7, we discuss the relation of our main result, Theorem 1.1, with higher-genus mirror symmetry and some string-theoretic arguments.

2. Basics and main results

2.1 Looijenga pairs

Let (Y, D) be a Looijenga pair:[4] Y is a smooth projective complex surface and D is a singular reduced normal crossings anticanonical divisor on Y. Writing the irreducible components

$$D = D_1 + \cdots + D_r,$$

D is a cycle of r irreducible smooth rational curves D_j if $r \geqslant 2$, or an irreducible nodal rational curve if $r = 1$. The complement $U := Y - D$ is a non-compact Calabi–Yau surface, equipped with a holomorphic symplectic form Ω_U, defined up to non-zero scaling and having first-order poles along D. We refer to [Loo81, Fri15, GHK15a, GHK15b] for more background on Looijenga pairs.

There are two basic operations on Looijenga pairs:

- *Corner blow-up.* If (Y, D) is a Looijenga pair, then the blow-up $\tilde{Y}$ of Y at one of the corners of D, equipped with the preimage $\tilde{D}$ of D, is a Looijenga pair.
- *Boundary blow-up.* If (Y, D) is a Looijenga pair, then the blow-up $\tilde{Y}$ of Y at a smooth point of D, equipped with the strict transform $\tilde{D}$ of D, is a Looijenga pair.

A corner blow-up does not change the interior U of a Looijenga pair (Y, D). An interior blow-up changes the interior of a Looijenga pair: if $(\tilde{Y}, \tilde{D})$ is an interior blow-up of (Y, D), then,

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[4] We follow the terminology of Gross, Hacking and Keel [GHK15a].

for example, we have

$$e(\tilde{U}) = e(U) + 1,$$

where U is the interior of (Y, D), $\tilde{U}$ is the interior of $(\tilde{Y}, \tilde{D})$, and $e(-)$ denotes the topological Euler characteristic.

If $\bar{Y}$ is a smooth toric variety and $\bar{D}$ is its toric boundary divisor, then $(\bar{Y}, \bar{D})$ is a Looijenga pair, of interior $U = (\mathbb{C}^*)^2$. In particular, we have $e(U) = e((\mathbb{C}^*)^2) = 0$. Such Looijenga pairs are called toric. A Looijenga pair (Y, D) is toric if and only if its interior $U = Y - D$ has a vanishing Euler topological characteristic: $e(U) = 0$.

A toric model of a Looijenga pair (Y, D) is a toric Looijenga pair $(\bar{Y}, \bar{D})$ such that (Y, D) is obtained from $(\bar{Y}, \bar{D})$ by successively applying a finite number of boundary blow-ups.

If (Y, D) is a Looijenga pair, then, by [GHK15a, Proposition 1.3], there exists a Looijenga pair $(\tilde{Y}, \tilde{D})$, obtained from (Y, D) by successively applying a finite number of corner blow-ups, which admits a toric model. In particular, we have $e(U) \geqslant 0$, where U is the interior of (Y, D).

Let $(\bar{Y}, \bar{D})$ be a toric model of a Looijenga pair (Y, D) of interior U. Let $\bar{\omega}$ be a torus invariant real symplectic form on $(\mathbb{C}^*)^2 = \bar{Y} - \bar{D}$. Then the corresponding moment map for the torus action gives $\bar{Y}$ the structure of toric fibration, whose restriction to U is a smooth fibration in Lagrangian tori. By definition of a toric model, we have a map $p \colon (Y, D) \to (\bar{Y}, \bar{D})$, a composition of successive boundary blow-ups. Let E_j denote the exceptional divisors, $j = 1, \ldots, e(U)$. Then for small enough positive real numbers ϵ_j, there exists a symplectic form ω in the class

$$p^*[\bar{\omega}] - \sum_{j=1}^{e(U)} \epsilon_j E_j$$

with respect to which Y admits an almost toric fibration, whose restriction to U is a fibration in Lagrangian tori with $e(U)$ nodal fibers [AAK16].

Toric models of a given Looijenga pair are very far from being unique but are always related by sequences of corner blow-ups/blow-downs and boundary blow-ups/blow-downs. The corresponding almost toric fibrations are related by nodal trades [Sym03].

Following [GHK15a, §6.3], we say that (Y, D) is positive if one of the following equivalent conditions is satisfied.

– There exist positive integers $a_1, \ldots, a_r$ such that, for all $1 \leqslant k \leqslant r$, we have

$$\left(\sum_{j=1}^{r} a_j D_j \right) \cdot D_k > 0.$$

– U is deformation equivalent to an affine surface.

– U is the minimal resolution of $\mathrm{Spec}\,(H^0(U, \mathcal{O}_U))$, which is an affine surface with at worst Du Val singularities.

2.2 Tropicalization of Looijenga pairs

We refer to [GHK15a, §§ 1.2 and 2.1] and [GHKS16, § 1] for details. Let (Y, D) be a Looijenga pair. Let $D_1, \ldots, D_r$ be the component of D, ordered in a cyclic order, the index j of D_j being considered modulo r. For every j modulo r, we consider an integral affine cone $\sigma_{j,j+1} = (\mathbb{R}_{\geqslant 0})^2$, of edges ρ_j and ρ_{j+1}. We abstractly glue together the cones $\sigma_{j-1,j}$ and $\sigma_{j,j+1}$ along the edge ρ_j. We obtain a topological space B, homeomorphic to $\mathbb{R}^2$, equipped with a cone decomposition Σ in two-dimensional cones $\sigma_{j,j+1}$, all meeting at a point that we call $0 \in B$, and pairwise meeting

along one-dimensional cones ρ_j. The pair (B, Σ) is the dual intersection complex of (Y, D). We define an integral linear structure on $B_0 = B - \{0\}$ by the charts

$$\psi_j \colon U_j \to \mathbb{R}^2,$$

where $U_j := \mathrm{Int}(\sigma_{j-1,j} \cup \sigma_{j,j+1})$ and ψ_j is defined on the closure of U_j by

$$\psi_j(v_{j-1}) = (1, 0), \quad \psi_j(v_j) = (0, 1), \quad \psi_j(v_{j+1}) = (-1, -D_j^2),$$

where v_j is a primitive generator of ρ_j and ψ_j is defined linearly on the two-dimensional cones. Let Λ be the sheaf of integral tangent vectors of B_0. It is a locally constant sheaf on B_0 of fibers $\mathbb{Z}^2$.

The integral linear structure on B_0 extends to B through 0 if and only if (Y, D) is toric. In this case, B can be identified with $\mathbb{R}^2$ as an integral linear manifold and Σ is simply the fan of the toric variety Y. In general, the integral linear structure is singular at 0, with a non-trivial monodromy along a loop going around 0.

As B_0 is an integral linear manifold, its set $B_0(\mathbb{Z})$ of integral points is well defined. We denote $B(\mathbb{Z}) := B_0(\mathbb{Z}) \cup \{0\}$. If (Y, D) is toric, with $Y - D = (\mathbb{C}^*)^2$, then $B(\mathbb{Z})$ is the lattice of cocharacters of $(\mathbb{C}^*)^2$, that is, the lattice of one-parameter subgroups $\mathbb{C}^* \to (\mathbb{C}^*)^2$. Thus, intuitively, a point of $B_0(\mathbb{Z})$ is a way to go to infinity in $(\mathbb{C}^*)^2$. This intuition remains true in the non-toric case: a point in $B_0(\mathbb{Z})$ is a way to go to infinity in the interior U of the pair (Y, D).

More precisely, if we equip (Y, D) with its divisorial log structure, then $p \in B(\mathbb{Z})$ defines a tangency condition along D for a marked point x on a stable log curve $f \colon C \to (Y, D)$. If $p = 0$, then $f(x) \notin D$. If $p = m_j v_j$, $m_j \in \mathbb{N}$, then $f(x) \in D_j$ with tangency order m_j along D_j and tangency order zero along D_{j-1} and D_{j+1}. If $p = m_j v_{j+1} + m_{j+1} v_{j+1}$, $m_j, m_{j+1} \in \mathbb{N}$, then $f(x) \in D_j \cap D_{j+1}$ with tangency order m_j along D_j and tangency order m_{j+1} along D_{j+1}.[5]

Let P be a toric monoid and P^{gp} be its group completion, a finitely generated abelian group. Denote $P_{\mathbb{R}}^{\mathrm{gp}} := P^{\mathrm{gp}} \otimes_{\mathbb{Z}} \mathbb{R}$, a finite-dimensional $\mathbb{R}$-vector space. Let φ be a convex $P_{\mathbb{R}}^{\mathrm{gp}}$-valued multivalued Σ-piecewise linear function on B_0. Let Λ_j be the fiber of the sheaf Λ of integral tangent vectors over the chart U_j. Let $n_{j-1,j}, n_{j,j+1} \in \Lambda_i^\vee \otimes P^{\mathrm{gp}}$ be the slopes of $\varphi|_{\sigma_{j-1,j}}$ and $\varphi|_{\sigma_{j,j+1}}$. Let Λ_{ρ_j} be the fiber of the sheaf of integral tangent vectors to the ray ρ_j. Let $\delta_j \colon \Lambda_j \to \Lambda_j / \Lambda_{\rho_j} \simeq \mathbb{Z}$ be the quotient map. We fix signs by requiring δ_j to be non-negative on tangent vectors pointing from ρ_j to $\sigma_{j,j+1}$. Then $(n_{j,j+1} - n_{j-1,j})(\Lambda_{\rho_j}) = 0$ and hence there exists $\kappa_{\rho_j, \varphi} \in P$ with

$$n_{j,j+1} - n_{j-1,j} = \delta_j \kappa_{\rho_j, \varphi},$$

called the kink of φ along ρ_j.

Let $\mathbb{B}_{0,\varphi}$ be the $P_{\mathbb{R}}^{\mathrm{gp}}$-torsor, which is set-theoretically $B_0 \times P_{\mathbb{R}}^{\mathrm{gp}}$ but with an integral affine structure twisted by φ: for each chart $\psi_j \colon U_j \to \mathbb{R}^2$ of B_0, we define a chart on $\mathbb{B}_{0,\varphi}$ by

$$(x, p) \mapsto \begin{cases} (\psi_j(x), p) & \text{if } x \in \sigma_{j-1,j}, \\ (\psi_j(x), p + \tilde{\delta}_j(x) \kappa_{\rho_j, \varphi}) & \text{if } x \in \sigma_{j,j+1}, \end{cases}$$

where $\tilde{\delta}_j \colon \sigma_{j,j+1} \to \mathbb{R}_{\geqslant 0}$ is the integral affine map of differential δ_j. By definition, φ can be viewed as a section of the projection $\pi \colon \mathbb{B}_{0,\varphi} \to B_0$. Then $\mathcal{P} := \varphi^* \Lambda_{\mathbb{B}_{0,\varphi}}$ is a locally constant sheaf on $\mathbb{B}_{0,\varphi}$, of fiber $\mathbb{Z}^2 \oplus P^{\mathrm{gp}}$, and the projection $\pi \colon \mathbb{B}_{0,\varphi} \to B_0$ induces a short exact sequence

$$0 \to \underline{P}^{\mathrm{gp}} \to \mathcal{P} \xrightarrow{\mathrm{r}} \Lambda \to 0$$

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[5] This makes sense precisely because we are using log geometry.

of locally constant sheaves on B_0, where $\underline{P}^{\mathrm{gp}}$ is the constant sheaf on B_0 of fiber P^{gp}, and where $\mathfrak{r}$ is the derivative of π.

The sheaf Λ is naturally a sheaf of symplectic lattices: we have a skew-symmetric non-degenerate form

$$\langle -, - \rangle \colon \Lambda \otimes \Lambda \to \mathbb{Z}.$$

We extend $\langle -, - \rangle$ to a skew-symmetric form on $\mathcal{P}$ of kernel $\underline{P}^{\mathrm{gp}}$.

Let P be a toric monoid and let $\eta \colon NE(Y) \to P$ be a morphism of monoids. Then there exists a unique (up to a linear function) convex $P_{\mathbb{R}}^{\mathrm{gp}}$-valued multivalued Σ-piecewise linear function φ on B_0 with kinks $\kappa_{\rho_j,\varphi} = \eta([D_j])$.

2.3 Algebras and quantum algebras

When we write 'A is an R-algebra', we mean that A is an associative algebra with unit over a commutative ring with unit R. In particular, R is naturally contained in the center of A.

We fix $\Bbbk$ an algebraically closed field of characteristic zero and $i \in \Bbbk$ a square root of -1.

For every monoid[6] M equipped with a skew-symmetric bilinear form

$$\langle -, - \rangle \colon M \times M \to \mathbb{Z},$$

we denote by $\Bbbk[M]$ the monoid algebra of M, consisting of monomials z^m, $m \in M$, such that $z^m \cdot z^{m'} = z^{m+m'}$. It is a Poisson algebra, of Poisson bracket determined by

$$\{z^m, z^{m'}\} = \langle m, m' \rangle z^{m+m'}.$$

We denote by $\Bbbk_q := \Bbbk[q^{\pm 1/2}]$ and $\Bbbk_q[M]$ the possibly non-commutative $\Bbbk_q$-algebra structure on $\Bbbk[M] \otimes_{\Bbbk} \Bbbk_q$ such that

$$\hat{z}^m \cdot \hat{z}^{m'} = q^{(1/2)\langle m,m' \rangle} \hat{z}^{m+m'}.$$

We denote $\Bbbk_\hbar := \Bbbk[\![\hbar]\!]$. We view $\Bbbk_\hbar$ as a complete topological ring for the $\hbar$-adic topology and, in particular, we will use the operation of completed tensor product $\hat{\otimes}$ with $\Bbbk_\hbar$:

$$(-) \, \hat{\otimes}_{\Bbbk} \, \Bbbk_\hbar := \varprojlim_j \, (-) \otimes_{\Bbbk} (\Bbbk[\hbar]/\hbar^j).$$

We view $\Bbbk_\hbar$ as a $\Bbbk_q$-module by the change of variables

$$q = e^{i\hbar} = \sum_{k \geq 0} \frac{(i\hbar)^k}{k!}.$$

We denote $\Bbbk_\hbar[M] := \Bbbk_q[M] \, \hat{\otimes}_{\Bbbk_q} \, \Bbbk_\hbar$. The possibly non-commutative algebra $\Bbbk_\hbar[M]$ is a deformation quantization of the Poisson algebra $\Bbbk[M]$ in the sense that $\Bbbk_\hbar[M]$ is flat as $\Bbbk_\hbar$-module, we recover $\Bbbk[M]$ in the limit $\hbar \to 0$, $q \to 1$, and the linear term in $\hbar$ of the commutator $[\hat{z}^m, \hat{z}^{m'}]$ in $\Bbbk_\hbar[M]$ is determined by the Poisson bracket $\{z^m, z^{m'}\}$ in $\Bbbk[M]$:

$$[\hat{z}^m, \hat{z}^{m'}] = (q^{(1/2)\langle m,m' \rangle} - q^{-(1/2)\langle m,m' \rangle}) \hat{z}^{m+m'} = \langle m, m' \rangle i\hbar \hat{z}^{m+m'} + \mathcal{O}(\hbar^2).$$

We will often apply the constructions $\Bbbk[M]$ and $\Bbbk_\hbar[M]$ to M a fiber of the locally constant sheaves Λ or $\mathcal{P}$.

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[6] All the monoids considered will be commutative and with an identity element.

In particular, considering the toric monoid P with the zero skew-symmetric form, we denote

$$R := \Bbbk[P]$$

and

$$R^\hbar := \Bbbk_\hbar[P] = R \,\hat{\otimes}_\Bbbk\, \Bbbk_\hbar.$$

For every monomial ideal I of R, we denote

$$R_I := R/I,$$

$$R_I^q := R/I \otimes_\Bbbk \Bbbk_q = R_I[q^{\pm 1/2}]$$

and

$$R_I^\hbar := R^\hbar/I = R_I \,\hat{\otimes}_\Bbbk\, \Bbbk_\hbar = R_I[\![\hbar]\!].$$

Observe that the algebras $R^\hbar$, R_I^q and $R_I^\hbar$ are commutative.

2.4 Ore localization

As should be clear from the previous section, we will be dealing with non-commutative rings. Unlike what happens for commutative rings, it is not possible in general to localize with respect to an arbitrary multiplicative subset of a non-commutative ring, because of left–right issues. These left–right issues are absent by definition if the multiplicative subset satisfies the so-called Ore conditions.

We refer, for example, to [Kap98, § 2.1] and [Gin98, § 1.3] for short presentations of these elementary notions of non-commutative algebra. A multiplicative subset $S \subset A - \{0\}$ of an associative ring A is said to satisfy the Ore conditions if:

- for all $a \in A$ and $s \in S$, there exist $b \in A$ and $t \in S$ such that[7] $ta = bs$;
- for all $a \in A$, if there exists $s \in S$ such that $as = 0$, then there exists $t \in S$ such that $ta = 0$;
- for all $b \in A$ and $t \in S$, there exists $a \in A$ and $s \in S$ such that[8] $ta = bs$;
- for all $a \in A$, if there exists $s \in S$ such that $sa = 0$, then there exists $t \in S$ such that $at = 0$.

If S is a multiplicative subset of an associative ring A and if S satisfies the Ore conditions, then there is a well-defined localized ring $A[S^{-1}]$.

Let R be a commutative ring. Denote $R^\hbar := R[\![\hbar]\!]$.

LEMMA 2.1. *Let A be an $R^\hbar$-algebra such that $A_0 := A/\hbar A$ is a commutative R-algebra. Assume that A is $\hbar$-nilpotent, that is, that there exists j such that $\hbar^j A = 0$. Denote by $\pi\colon A \to A_0$ the natural projection. Let $\overline{S} \subset A_0 - \{0\}$ be a multiplicative subset. Then the multiplicative subset $S := \pi^{-1}(\overline{S})$ of A satisfies the Ore conditions.*

Proof. See the proof of [Kap98, Proposition 2.1.5]. □

DEFINITION 2.1. Let A be an $R^\hbar$-algebra such that $A_0 := A/\hbar A$ is a commutative R-algebra. Assume that A is $\hbar$-complete, that is, that $A = \varprojlim_j A/\hbar^j A$. By Lemma 2.1, each $A/\hbar^j A$ defines

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[7] Informally, $as^{-1} = t^{-1}b$, that is, every fraction with a denominator on the right can be rewritten as a fraction with a denominator on the left.

[8] Informally, $t^{-1}b = as^{-1}$, that is, every fraction with a denominator on the left can be rewritten as a fraction with a denominator on the right.

a sheaf of algebras on $X_0 := \operatorname{Spec} A_0$, which we denote by $\mathcal{O}_{X_0}^{\hbar}/\hbar^j$. We define

$$\mathcal{O}_{X_0}^{\hbar} := \varprojlim_j \mathcal{O}_{X_0}^{\hbar}/\hbar^j,$$

which is a sheaf in $R^{\hbar}$-algebras over X_0, such that $\mathcal{O}_{X_0}^{\hbar}/\hbar = \mathcal{O}_{X_0}$.

Definition 2.1 gives us a systematic way to turn certain non-commutative algebras into sheaves of non-commutative algebras.

2.5 The Gross–Hacking–Keel mirror family

We fix a Looijenga pair (Y, D). Let $NE(Y)_{\mathbb{R}} \subset A_1(Y, \mathbb{R})$ be the cone generated by curve classes and let $NE(Y)$ be the monoid $NE(Y)_{\mathbb{R}} \cap A_1(Y, \mathbb{Z})$.

Let $\sigma_P \subset A_1(Y, \mathbb{R})$ be a strictly convex polyhedral cone containing $NE(Y)_{\mathbb{R}}$. Let $P := \sigma_P \cap A_1(Y, \mathbb{Z})$ be the associated monoid and let $R := \Bbbk[P]$ be the corresponding $\Bbbk$-algebra. We denote by t^{β} the monomial in R defined by $\beta \in P$. Let $\mathfrak{m}_R$ be the maximal monomial ideal of R. For every monomial ideal I of R with radical $\mathfrak{m}_R$, we denote $R_I := R/I$ and $S_I := \operatorname{Spec} R_I$.

Let $T^D := \mathbb{G}_m^r$ be the torus whose character group has a basis e_{D_j} indexed by the irreducible components D_j of D. The map

$$\beta \mapsto \sum_{j=1}^{r} (\beta \cdot D_j) e_{D_j}$$

induces an action of T^D on S_I.

[GHK15a, Theorem 0.1] gives the existence of a flat T^D-equivariant morphism

$$X_I \to S_I,$$

with X_I affine. The algebra of functions of X_I is given as R_I-module by

$$H^0(X_I, \mathcal{O}_{X_I}) = A_I := \bigoplus_{p \in B(\mathbb{Z})} R_I \vartheta_p.$$

The algebra structure on $H^0(X_I, \mathcal{O}_{X_I})$ is determined by genus-0 log Gromov–Witten invariants of (Y, D).

By [GHK15a, Theorem 0.2], there exists a unique smallest radical monomial ideal $J_{\min} \subset R$ such that the following statements hold.

- For every monomial ideal I of R of radical containing $J_{\min}$, there is a finitely generated R_I-algebra structure on A_I compatible with the $R_{I+\mathfrak{m}^N}$-algebra structure on $A_{I+\mathfrak{m}^N}$ given by [GHK15a, Theorem 0.1] for all $N > 0$.
- The zero locus $V(J_{\min}) \subset \operatorname{Spec} R$ contains the union of the closed toric strata corresponding to faces F of σ_P such that there exists $1 \leqslant j \leqslant r$ such that $[D_j] \notin F$. If (Y, D) is positive, then $J_{\min} = 0$ and $V(J_{\min}) = \operatorname{Spec} R$.
- Let $\hat{R}^{J_{\min}}$ denote the $J_{\min}$-adic completion of R. The algebras A_I determine a T^D-equivariant formal flat family of affine surfaces $\mathcal{X}^{J_{\min}} \to \operatorname{Spf} \hat{R}^{J_{\min}}$. The theta functions ϑ_p determine a canonical embedding $\mathcal{X}^{J_{\min}} \subset \mathbb{A}^{\max(r,3)} \times \operatorname{Spf} \hat{R}^{J_{\min}}$. In particular, if (Y, D) is positive, then we get an algebraic family $\mathcal{X} \to \operatorname{Spec} R$ and the theta functions ϑ_p determine a canonical embedding $\mathcal{X} \subset \mathbb{A}^{\max(r,3)} \times \operatorname{Spec} R$.

2.6 Deformation quantization

We now discuss the notion of deformation quantization. There are two technical aspects to keep in mind: first, we work relative to a non-trivial base; and second, we work in general with formal schemes. We refer to [Kon01, Yek05, BK04], for general facts about deformation quantization in algebraic geometry.

DEFINITION 2.2. A Poisson scheme over a scheme S is a scheme $\pi\colon X \to S$ over S, equipped with a $\pi^{-1}\mathcal{O}_S$-bilinear Poisson bracket, that is, a $\pi^{-1}\mathcal{O}_S$-bilinear skew-symmetric map of sheaves

$$\{-,-\}\colon \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X,$$

which is a biderivation

$$\{a,bc\} = \{a,b\}c + \{a,c\}b,$$

and a Lie bracket

$$\{a,\{b,c\}\} + \{b,\{c,a\}\} + \{c,\{a,b\}\} = 0.$$

The two definitions below give two notions of deformation quantization of a Poisson scheme.

DEFINITION 2.3. Let $\pi\colon (X,\{-,-\}) \to S$ be a Poisson scheme over a scheme S. A *deformation quantization* of $(X,\{-,-\})$ over S is a sheaf $\mathcal{O}_X^\hbar$ of associative flat $\pi^{-1}\mathcal{O}_S \,\hat{\otimes}\, \Bbbk_\hbar$-algebras on X, complete in the $\hbar$-adic topology, equipped with an isomorphism $\mathcal{O}_X^\hbar/\hbar\mathcal{O}_X^\hbar \simeq \mathcal{O}_X$, such that for every f and g in $\mathcal{O}_X$, and lifts $\tilde{f}$ and $\tilde{g}$ of f and g in $\mathcal{O}_X^\hbar$, we have

$$[\tilde{f},\tilde{g}] = i\hbar\{f,g\} \bmod \hbar^2,$$

where $[\tilde{f},\tilde{g}] := \tilde{f}\tilde{g} - \tilde{g}\tilde{f}$ is the commutator in $\mathcal{O}_X^\hbar$.

DEFINITION 2.4. Let $\pi\colon (X,\{-,-\}) \to S$ be a Poisson scheme over a scheme S. Assume that both X and S are affine. A *deformation quantization* of $(X,\{-,-\})$ over S is a flat $H^0(S, \mathcal{O}_S)\,\hat{\otimes}\,\Bbbk_\hbar$-algebra A, complete in the $\hbar$-adic topology, equipped with an isomorphism $A/\hbar A \simeq H^0(X,\mathcal{O}_X)$, such that for every f and g in $H^0(X,\mathcal{O}_X)$, and lifts $\tilde{f}$ and $\tilde{g}$ of f and g in A, we have

$$[\tilde{f},\tilde{g}] = i\hbar\{f,g\} \bmod \hbar^2,$$

where $[\tilde{f},\tilde{g}] := \tilde{f}\tilde{g} - \tilde{g}\tilde{f}$ is the commutator in A.

The compatibility of these two definitions is guaranteed by the following lemma.

LEMMA 2.2. *When both X and S are affine, the notions of deformation quantization given by Definitions 2.3 and 2.4 are equivalent.*

Proof. One goes from a sheaf quantization to an algebra quantization by taking global sections. One goes from an algebra quantization to a sheaf quantization by Ore localization; see § 2.4. □

Definitions 2.3 and 2.4 and Lemma 2.2 have obvious analogues if one replaces schemes by formal schemes.[9]

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[9] Or, in fact, any locally ringed space.

2.7 Main results

We fix a Looijenga pair (Y, D) and we use notation introduced in §2.5. Our main result, Theorem 2.1, is the construction of a deformation quantization of the Gross–Hacking–Keel mirror family by a higher-genus deformation of the Gross–Hacking–Keel construction.

THEOREM 2.1. *Let I be a monomial ideal of R with radical $\mathfrak{m}_R$. Then there exists a flat T^D-equivariant $R_I^\hbar$-algebra $A_I^\hbar$, such that $A_I^\hbar$ is a deformation quantization over S_I of the Gross–Hacking–Keel mirror family $X_I \to S_I$, and $A_I^\hbar$ is given as $R_I^\hbar$-module by*

$$A_I^\hbar = \bigoplus_{p \in B(\mathbb{Z})} R_I^\hbar \hat{\vartheta}_p,$$

where the algebra structure is determined by higher-genus log Gromov–Witten invariants of (Y, D), with genus expansion parameter identified with the quantization parameter $\hbar$.

Taking the limit over all monomial ideals I of R with radical $\mathfrak{m}_R$, we get a deformation quantization of the formal family

$$\varinjlim_I X_I \to \varinjlim_I S_I.$$

The following theorem is a quantum version of [GHK15a, Theorem 0.2].

THEOREM 2.2. *There is a unique smallest radical monomial $J_{\min}^\hbar \subset R$ such that the following statements hold.*

- *For every monomial ideal I of R of radical containing $J_{\min}^\hbar$, there is a finitely generated $R_I^\hbar$-algebra structure on*

$$A_I^\hbar = \bigoplus_{p \in B(\mathbb{Z})} R_I^\hbar \hat{\vartheta}_p,$$

  compatible with the $R_{I+\mathfrak{m}_R^k}^\hbar$-algebra structure on $A_{I+\mathfrak{m}_R^k}^\hbar$ given by Theorem 2.1 for all $k > 0$.
- *The zero locus $V(J_{\min}) \subset R$ contains the union of the closed toric strata corresponding to faces F of σ_P such that there exists $1 \leqslant j \leqslant r$ such that $[D_j] \notin F$. If (Y, D) is positive, then $J_{\min}^\hbar = 0$, that is, $V(J_{\min}^\hbar) = \operatorname{Spec} R$ and $A_0^\hbar$ is a deformation quantization of the mirror family $\mathcal{X} \to \operatorname{Spec} R$.*

The following result controls the dependence in $\hbar$ of the deformation quantization given by Theorem 2.2: this dependence is algebraic in $q = e^{i\hbar}$.

THEOREM 2.3. *Let I be a monomial ideal of R with radical containing $J_{\min}^\hbar$. Then there exists a flat R_I^q-algebra A_I^q such that*

$$A_I^\hbar = A_I^q \,\hat{\otimes}_{\Bbbk_q} \Bbbk_\hbar,$$

where $\Bbbk_\hbar$ is viewed as a $\Bbbk_q$-module via $q = e^{i\hbar}$.

The proof of Theorems 2.1–2.3 is given in §§ 3–5. In §3 we explain how a consistent quantum scattering diagram can be used as input to a construction of quantum modified Mumford degeneration, giving a deformation quantization of the modified Mumford degeneration of [GHK15a, GHKS16], constructed from a classical scattering diagram. In §4 we explain how to construct a quantum scattering diagram from higher-genus log Gromov–Witten theory of a Looijenga pair and we prove its consistency using the main result of [Bou18]. We finish the proof of Theorems 2.1–2.3 in §5.

3. Quantum modified Mumford degenerations

In this section we explain how to construct a quantization of the mirror family of a given Looijenga pair (Y, D) starting from its tropicalization (B, Σ) and a consistent quantum scattering diagram.

In § 3.1 we describe the rings $R^{\hbar}_{\sigma,I}$ and $R^{\hbar}_{\rho,I}$ involved in the construction of the quantum version of modified Mumford degenerations. In § 3.2 we review the notion of quantum scattering diagrams. In § 3.4 we explain how a consistent quantum scattering diagram gives a way to glue together the rings $R^{\hbar}_{\sigma,I}$ and $R^{\hbar}_{\rho,I}$ to produce a quantum modified Mumford degeneration. In § 3.6 we review the notions of quantum broken lines and theta functions and we use them in § 3.7 to prove that the quantum modified Mumford degeneration is indeed a deformation quantization of the modified Mumford degeneration of [GHK15a]. In § 3.8 we express the structure constants of the quantum algebra of global sections in terms of quantum broken lines.

3.1 Building blocks

The goal of this section is to define non-commutative deformations $R^{\hbar}_{\sigma,I}$ and $R^{\hbar}_{\rho,I}$ of the rings $R_{\sigma,I}$ and $R_{\rho,I}$ defined in [GHK15a, §§ 2.1 and 2.2]. The way to go from $R_{\sigma,I}$ to $R^{\hbar}_{\sigma,I}$ is fairly obvious. The deformation $R^{\hbar}_{\rho,I}$ of $R_{\rho,I}$ is perhaps not so obvious.

We fix a Looijenga pair (Y, D), its tropicalization (B, Σ), a toric monoid P, a radical monomial ideal J of P, and a $P^{\mathrm{gp}}_{\mathbb{R}}$-valued multivalued convex Σ-piecewise linear function φ on B.

For any locally constant sheaf $\mathcal{F}$ on B_0 and any simply connected subset τ of B_0, we write $\mathcal{F}_{\tau}$ for the stalk of this local system at any point of τ. We will constantly use this notation for τ a cone of Σ.

If τ is a cone of Σ, we define the localized fan $\tau^{-1}\Sigma$ as being the fan in $\Lambda_{\mathbb{R},\tau}$ defined as follows.

– If τ is two-dimensional, then $\tau^{-1}\Sigma$ consists just of the entire space $\Lambda_{\mathbb{R},\tau}$.
– If τ is one-dimensional, then $\tau^{-1}\Sigma$ consists of the tangent line of τ in $\Lambda_{\mathbb{R},\tau}$ along with the two half-planes with boundary this tangent line.

For each τ cone of Σ, the Σ-piecewise P-convex function $\varphi \colon B_0 \to \mathbb{B}_{0,\varphi}$ determines a $\tau^{-1}\Sigma$-piecewise linear P-convex function $\varphi_{\tau} \colon \Lambda_{\mathbb{R},\tau} \to \mathcal{P}_{\mathbb{R},\tau}$: if we choose a two-dimensional cone σ of Σ containing τ, we have an identification $\mathcal{P}_{\mathbb{R},\tau} \simeq \Lambda_{\mathbb{R},\tau} \oplus P_{\mathbb{R}}$, and we define

$$\varphi_{\tau} \colon \Lambda_{\mathbb{R},\tau} \to \mathcal{P}_{\mathbb{R},\tau} = \Lambda_{\mathbb{R},\tau} \oplus P_{\mathbb{R}}$$
$$m \mapsto (m, \varphi|_{\sigma}(m)).$$

It follows from the definition of $\mathcal{P}$ given in § 2.2 that φ_{τ} is well defined, that is, independent of the choice of σ. By construction, $\varphi_{\tau} \colon \Lambda_{\mathbb{R},\tau} \to \mathcal{P}_{\mathbb{R},\tau}$ is a section of the natural projection map $\mathfrak{r} \colon \mathcal{P}_{\mathbb{R},\tau} \to \Lambda_{\mathbb{R},\tau}$ discussed in § 2.2.

We define the toric monoid $P_{\varphi_{\tau}} \subset \mathcal{P}_{\tau}$ by

$$P_{\varphi_{\tau}} := \{s \in \mathcal{P}_{\tau} \,|\, s = p + \varphi_{\tau}(m) \text{ for some } p \in P,\, m \in \Lambda_{\tau}\}.$$

If ρ is a one-dimensional cone of Σ, bounding the two-dimensional cones σ_+ and σ_- of Σ, we have $P_{\varphi_{\rho}} \subset P_{\varphi_{\sigma_+}}$, $P_{\varphi_{\rho}} \subset P_{\varphi_{\sigma_-}}$, and

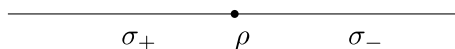$$P_{\varphi_{\sigma_+}} \cap P_{\varphi_{\sigma_-}} = P_{\varphi_{\rho}}.$$
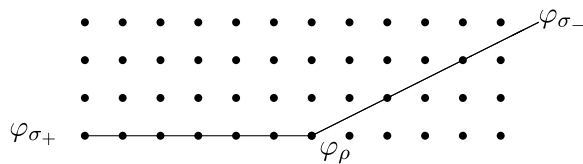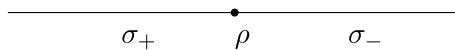
371

FIGURE 1. The monoid $P_{\varphi_{\sigma_+}}$.
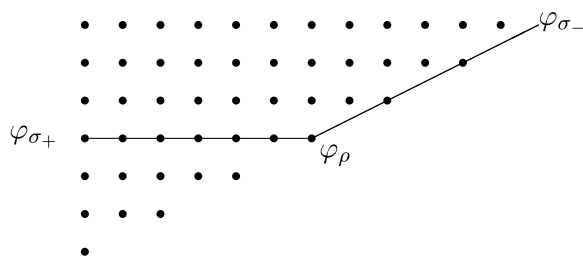
FIGURE 2. The monoid $P_{\varphi_{\sigma_-}}$.
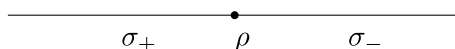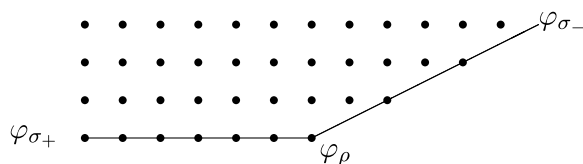
FIGURE 3. The monoid P_{φ_ρ}.

The monoids $P_{\varphi_{\sigma_+}}$, $P_{\varphi_{\sigma_-}}$ and P_{φ_ρ} are represented in Figures 1–3.

For every σ two-dimensional cone of Σ, we define $R^\hbar_{\sigma,I} := \Bbbk_\hbar[P_{\varphi_\sigma}]/I$, a deformation quantization of $R_{\sigma,I} := \Bbbk[P_{\varphi_\sigma}]/I$. We have a natural trivialization $P_{\varphi_\sigma} = P \oplus \Lambda_\sigma$ and so $R^\hbar_{\sigma,I}$ is simply the algebra of functions on a trivial family of two-dimensional quantum tori parametrized by Spec R_I.

Let ρ be a one-dimensional cone of Σ. Let $\kappa_{\rho,\varphi} \in P$ be the kink of φ across ρ, so that $z^{\kappa_{\rho,\varphi}} \in R_I$. Let X be an invertible formal variable. We fix elements $\hat{f}_{\rho^{\mathrm{out}}} \in R^\hbar_I[X^{-1}]$ and $\hat{f}_{\rho^{\mathrm{in}}} \in R^\hbar_I[X]$.

372

Let $R_{\rho,I}^{\hbar}$ be the $R_I^{\hbar}$-algebra generated by formal variables X_+, X_- and X, with X invertible, and with relations

$$XX_+ = qX_+X,$$
$$XX_- = q^{-1}X_-X,$$
$$X_+X_- = q^{(1/2)D_\rho^2}\hat{z}^{\kappa_{\rho,\varphi}}\hat{f}_{\rho^{\mathrm{out}}}(q^{-1}X)\hat{f}_{\rho^{\mathrm{in}}}(X)X^{-D_\rho^2},$$
$$X_-X_+ = q^{-(1/2)D_\rho^2}\hat{z}^{\kappa_{\rho,\varphi}}\hat{f}_{\rho^{\mathrm{out}}}(X)\hat{f}_{\rho^{\mathrm{in}}}(qX)X^{-D_\rho^2},$$

where $q = e^{i\hbar}$. The $R_I^{\hbar}$-algebra $R_{\rho,I}^{\hbar}$ is flat as $R_I^{\hbar}$-module and so is a deformation quantization of

$$R_{\rho,I} := R_I[X_+, X_-, X^{\pm}]/(X_+X_- - z^{\kappa_{\rho,\varphi}}X^{-D_\rho^2}f_{\rho^{\mathrm{out}}}(X)f_{\rho^{\mathrm{in}}}(X)).$$

Let σ_+ and σ_- be the two-dimensional cones of Σ bounding ρ, and let ρ_+ and ρ_- be the other boundary rays of σ_+ and σ_- respectively, such that ρ_-, ρ and ρ_+ are in anticlockwise order.

The precise form of $R_{\rho,I}^{\hbar}$ is justified by the following proposition.

PROPOSITION 3.1. *The map of $R_I^{\hbar}$-algebras*

$$\tilde{\psi}_{\rho,-}\colon R_I^{\hbar}\langle X_+, X_-, X^{\pm}\rangle \to R_{\sigma_-,I}^{\hbar}$$

defined by

$$\tilde{\psi}_{\rho,-}(X) = \hat{z}^{\varphi_\rho(m_\rho)},$$
$$\tilde{\psi}_{\rho,-}(X_-) = \hat{z}^{\varphi_\rho(m_{\rho_-})},$$
$$\tilde{\psi}_{\rho,-}(X_+) = \hat{f}_{\rho^{\mathrm{in}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\varphi_\rho(m_{\rho_+})}\hat{f}_{\rho^{\mathrm{out}}}(\hat{z}^{\varphi_\rho(m_\rho)})$$
$$= \hat{z}^{\varphi_\rho(m_{\rho_+})}\hat{f}_{\rho^{\mathrm{in}}}(q\hat{z}^{\varphi_\rho(m_\rho)})\hat{f}_{\rho^{\mathrm{out}}}(\hat{z}^{\varphi_\rho(m_\rho)})$$

induces a map of $R_I^{\hbar}$-algebras

$$\hat{\psi}_{\rho,-}\colon R_{\rho,I}^{\hbar} \to R_{\sigma_-,I}^{\hbar}.$$

The map of $R_I^{\hbar}$-algebras

$$\tilde{\psi}_{\rho,+}\colon R_I^{\hbar}\langle X_+, X_-, X^{\pm}\rangle \to R_{\sigma_+,I}^{\hbar}$$

defined by

$$\tilde{\psi}_{\rho,+}(X) = \hat{z}^{\varphi_\rho(m_\rho)},$$
$$\tilde{\psi}_{\rho,+}(X_-) = \hat{f}_{\rho^{\mathrm{out}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\varphi_\rho(m_{\rho_-})}\hat{f}_{\rho^{\mathrm{in}}}(\hat{z}^{\varphi_\rho(m_\rho)})$$
$$= \hat{z}^{\varphi_\rho(m_{\rho_-})}\hat{f}_{\rho^{\mathrm{out}}}(q^{-1}\hat{z}^{\varphi_\rho(m_\rho)})\hat{f}_{\rho^{\mathrm{in}}}(\hat{z}^{\varphi_\rho(m_\rho)}),$$
$$\tilde{\psi}_{\rho,+}(X_+) = \hat{z}^{\varphi_\rho(m_{\rho_+})}$$

induces a map of $R_I^{\hbar}$-algebras

$$\hat{\psi}_{\rho,+}\colon R_{\rho,I}^{\hbar} \to R_{\sigma_+,I}^{\hbar}.$$

Proof. We have to check that $\tilde{\psi}_{\rho,-}$ and $\tilde{\psi}_{\rho,+}$ map the relations defining $R_{\rho,I}^{\hbar}$ to zero.

We have $\langle m_\rho, m_{\rho_+}\rangle = 1$ and $\langle m_\rho, m_{\rho_-}\rangle = -1$. It follows that

$$\tilde{\psi}_{\rho,-}(XX_+ - qX_+X) = 0,$$
$$\tilde{\psi}_{\rho,+}(XX_+ - qX_+X) = 0,$$

373

and

$$\tilde{\psi}_{\rho,-}(XX_- - q^{-1}X_-X) = 0,$$
$$\tilde{\psi}_{\rho,+}(XX_- - q^{-1}X_-X) = 0.$$

Furthermore, we have

$$m_{\rho_-} + D_\rho^2 m_\rho + m_{\rho_+} = 0$$

so

$$\langle m_{\rho_+}, m_{\rho_-} \rangle = D_\rho^2$$

and

$$\varphi_\rho(m_{\rho_-}) + \varphi_\rho(m_{\rho_+}) = \kappa_{\rho,\varphi} - D_\rho^2 \varphi_\rho(m_\rho).$$

It follows that

$$\begin{aligned}
\tilde{\psi}_{\rho,-}(X_+X_-) &= \hat{f}_{\rho^{\mathrm{in}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\varphi_\rho(m_{\rho_+})}\hat{f}_{\rho^{\mathrm{out}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\varphi_\rho(m_{\rho_-})} \\
&= q^{(1/2)D_\rho^2}\hat{f}_{\rho^{\mathrm{in}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{f}_{\rho^{\mathrm{out}}}(q^{-1}\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\kappa_{\rho,\varphi}-D_\rho^2\varphi_\rho(m_\rho)} \\
&= q^{(1/2)D_\rho^2}\hat{z}^{\kappa_{\rho,\varphi}}\tilde{\psi}_{\rho,-}(\hat{f}_{\rho^{\mathrm{in}}}(X)\hat{f}_{\rho^{\mathrm{out}}}(q^{-1}X)X^{-D_\rho^2}), \\
\tilde{\psi}_{\rho,+}(X_+X_-) &= \hat{z}^{\varphi_\rho(m_{\rho_+})}\hat{f}_{\rho^{\mathrm{out}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\varphi_\rho(m_{\rho_-})}\hat{f}_{\rho^{\mathrm{in}}}(\hat{z}^{\varphi_\rho(m_\rho)}) \\
&= q^{(1/2)D_\rho^2}\hat{f}_{\rho^{\mathrm{in}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{f}_{\rho^{\mathrm{out}}}(q^{-1}\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\kappa_{\rho,\varphi}-D_\rho^2\varphi_\rho(m_\rho)} \\
&= q^{(1/2)D_\rho^2}\hat{z}^{\kappa_{\rho,\varphi}}\tilde{\psi}_{\rho,+}(\hat{f}_{\rho^{\mathrm{in}}}(X)\hat{f}_{\rho^{\mathrm{out}}}(q^{-1}X)X^{-D_\rho^2}),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\psi}_{\rho,-}(X_-X_+) &= \hat{z}^{\varphi_\rho(m_{\rho_-})}f_{\rho^{\mathrm{in}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\varphi_\rho(m_{\rho_+})}f_{\rho^{\mathrm{out}}}(\hat{z}^{\varphi_\rho(m_\rho)}) \\
&= q^{-(1/2)D_\rho^2}\hat{z}^{\kappa_{\rho,\varphi}-D_\rho^2\varphi_\rho(m_\rho)}f_{\rho^{\mathrm{in}}}(\hat{z}^{q\varphi_\rho(m_\rho)})f_{\rho^{\mathrm{out}}}(\hat{z}^{\varphi_\rho(m_\rho)}) \\
&= q^{-(1/2)D_\rho^2}\hat{z}^{\kappa_{\rho,\varphi}}\tilde{\psi}_{\rho,-}(\hat{f}_{\rho^{\mathrm{in}}}(qX)\hat{f}_{\rho^{\mathrm{out}}}(X)X^{-D_\rho^2}), \\
\tilde{\psi}_{\rho,+}(X_-X_+) &= \hat{f}_{\rho^{\mathrm{out}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\varphi_\rho(m_{\rho_-})}f_{\rho^{\mathrm{in}}}(\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\varphi_\rho(m_{\rho_+})} \\
&= q^{-(1/2)D_\rho^2}\hat{f}_\rho(\hat{z}^{\varphi_\rho(m_\rho)})f_{\rho^{\mathrm{in}}}(q\hat{z}^{\varphi_\rho(m_\rho)})\hat{z}^{\kappa_{\rho,\varphi}-D_\rho^2\varphi_\rho(m_\rho)} \\
&= q^{-(1/2)D_\rho^2}\hat{z}^{\kappa_{\rho,\varphi}}\tilde{\psi}_{\rho,+}(\hat{f}_{\rho^{\mathrm{in}}}(qX)\hat{f}_{\rho^{\mathrm{out}}}(X)X^{-D_\rho^2}). \qquad \square
\end{aligned}$$

Remark.

– In the special case where $D_\rho^2 = 0$ and $\hat{f}_{\rho^{\mathrm{in}}} = 1$, our description of $R_{\rho,I}^\hbar$ by generators and relations coincides with the description given by Soibelman in [Soi09, § 7.5] of a local model for deformation quantization of a neighborhood of a focus–focus singularity.

– The algebra $R_{\sigma,I}^\hbar$ is a deformation quantization of $R_{\sigma,I}$, and $R_{\rho,I}^\hbar$ is a deformation quantization of $R_{\rho,I}$. The maps $\hat{\psi}_{\rho,+}$ and $\hat{\psi}_{\rho,-}$ are quantizations of the maps $\psi_{\rho,-}$ and $\psi_{\rho,+}$ defined by [GHK15a, formula (2.8)]. Following [GHK15a], we denote $U_{\sigma,I} := \mathrm{Spec}\, R_{\sigma,I}$ and $U_{\rho,I} := \mathrm{Spec}\, R_{\rho,I}$. If ρ is a one-dimensional cone of Σ, and σ_+ and σ_- are the two-dimensional cones of Σ bounding ρ, then the maps $\psi_{\rho,-}$ and $\psi_{\rho,+}$ induce open immersions

$$U_{\sigma-,I} \hookrightarrow U_{\rho,I}$$

and

$$U_{\sigma+,I} \hookrightarrow U_{\rho,I}.$$

374

Using Ore localization (see Definition 2.1), we can produce from $R^\hbar_{\sigma,I}$ and $R^\hbar_{\rho,I}$ some sheaves of flat $\Bbbk_\hbar$-algebras $\mathcal{O}^\hbar_{U_{\sigma,I}}$ and $\mathcal{O}^\hbar_{U_{\rho,I}}$ on $U_{\sigma,I}$ and $U_{\rho,I}$, such that

$$\mathcal{O}^\hbar_{U_{\sigma,I}}/\hbar\mathcal{O}^\hbar_{U_{\sigma,I}} \simeq \mathcal{O}_{U_{\sigma,I}}$$

and

$$\mathcal{O}^\hbar_{U_{\rho,I}}/\hbar\mathcal{O}^\hbar_{U_{\rho,I}} \simeq \mathcal{O}_{U_{\rho,I}},$$

respectively.

3.2 Quantum scattering diagrams

Quantum scattering diagrams have been studied by Filippini and Stoppa [FS15] in dimension two and by Mandel [Man15] in higher dimensions. Mandel [Man15] also studied quantum broken lines and quantum theta functions. Both [FS15] and [Man15] work with smooth integral affine manifolds. We need to make some changes to include the case we care about, where the integral affine manifold is the tropicalization B of a Looijenga pair and has a singularity at the origin with a non-trivial monodromy around it.

As in the previous section, we fix a Looijenga pair (Y, D), its tropicalization (B, Σ), a toric monoid P, a radical monomial ideal J of P, and a $P^{\text{gp}}_\mathbb{R}$-valued multivalued convex Σ-piecewise linear φ function on B. Recall from §2.2 that we then have an exact sequence

$$0 \to \underline{P}^{\text{gp}} \to \mathcal{P} \xrightarrow{\mathfrak{r}} \Lambda \to 0$$

of locally constant sheaves on B_0.

We explained in §3.1 how to define, for every cone τ of Σ, a toric monoid P_{φ_τ}. We denote by

$$\Bbbk_\hbar\widehat{[P_{\varphi_\tau}]}$$

the J-adic completion of the $\Bbbk_\hbar$-algebra $\Bbbk_\hbar[P_{\varphi_\tau}]$. The map $\mathfrak{r}\colon \mathcal{P} \to \Lambda$ induces a morphism of monoids $\mathfrak{r}\colon P_{\varphi_\tau} \to \Lambda_\tau$.

DEFINITION 3.1. A *quantum scattering diagram* $\hat{\mathfrak{D}}$ for the data (B, Σ), P, J and φ is a set

$$\hat{\mathfrak{D}} = \{(\mathfrak{d}, \hat{H}_\mathfrak{d})\},$$

where $\mathfrak{d} \subset B$ is a ray of rational slope in B with endpoint the origin $0 \in B$.

– Let $\tau_\mathfrak{d}$ be the smallest cone of Σ containing $\mathfrak{d}$ and let $m_\mathfrak{d} \in \Lambda_{\tau_\mathfrak{d}}$ be the primitive generator of $\mathfrak{d}$ pointing away from the origin. Then we have either

$$\hat{H}_\mathfrak{d} = \sum_{\substack{p \in P_{\varphi_{\tau_\mathfrak{d}}} \\ \mathfrak{r}(p) \in \mathbb{Z}_{<0} m_\mathfrak{d}}} H_p \hat{z}^p \in \Bbbk_\hbar\widehat{[P_{\varphi_{\tau_\mathfrak{d}}}]}$$

or

$$\hat{H}_\mathfrak{d} = \sum_{\substack{p \in P_{\varphi_{\tau_\mathfrak{d}}} \\ \mathfrak{r}(p) \in \mathbb{Z}_{>0} m_\mathfrak{d}}} H_p \hat{z}^p \in \Bbbk_\hbar\widehat{[P_{\varphi_{\tau_\mathfrak{d}}}]}.$$

In the first case, we say that the ray $(\mathfrak{d}, \hat{H}_\mathfrak{d})$ is *outgoing*, and in the second case, we say that the ray $(\mathfrak{d}, \hat{H}_\mathfrak{d})$ is *ingoing*.

– Let $\tau_{\mathfrak{d}}$ be the smallest cone of Σ containing $\mathfrak{d}$. If $\dim \tau_{\mathfrak{d}} = 2$, or if $\dim \tau_{\mathfrak{d}} = 1$ and $\kappa_{\tau_{\mathfrak{d}},\varphi} \notin J$, then $\hat{H}_{\mathfrak{d}} = 0 \bmod J$.

– For any ideal $I \subset P$ of radical J, there are only finitely many rays $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ such that $\hat{H}_{\mathfrak{d}} \neq 0 \bmod I$.

Given a ray $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ of a quantum scattering diagram, we call $\hat{H}_{\mathfrak{d}}$ the Hamiltonian attached to ρ. This terminology is justified by § 3.3, where we attach to $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ the automorphism $\hat{\Phi}_{\hat{H}_{\mathfrak{d}}}$ given by the time-one evolution according to the quantum Hamiltonian $\hat{H}_{\mathfrak{d}}$.

3.3 Quantum automorphisms

Let $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ be a ray of a quantum scattering diagram $\hat{\mathfrak{D}}$ for the data (B, Σ), P, J and φ. Let $\tau_{\mathfrak{d}}$ be the smallest cone of Σ containing $\mathfrak{d}$ and let $m_{\mathfrak{d}} \in \Lambda_{\tau_{\mathfrak{d}}}$ be the primitive generator of $\mathfrak{d}$ pointing away from the origin. Denote $m(\hat{H}_{\mathfrak{d}}) = m_{\mathfrak{d}}$ if $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ is outgoing and $m(\hat{H}_{\mathfrak{d}}) = -m_{\mathfrak{d}}$ if $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ is ingoing. Writing

$$\hat{H}_{\mathfrak{d}} = \sum_{p \in P_{\varphi_{\tau_{\mathfrak{d}}}}} H_p \hat{z}^p \in \Bbbk_\hbar \widehat{[P_{\varphi_{\tau_{\mathfrak{d}}}}]},$$

we denote

$$\hat{f}_{\mathfrak{d}} := \exp\left(\sum_{\substack{p \in P_{\varphi_{\tau_{\mathfrak{d}}}} \\ \mathfrak{r}(p) = \ell m(\hat{H}_{\mathfrak{d}})}} (q^\ell - 1) H_p \hat{z}^p \right) \in \Bbbk_\hbar \widehat{[P_{\varphi_{\tau_{\mathfrak{d}}}}]},$$

where $q = e^{i\hbar}$. Remark that, by our definition of $m(\hat{H}_{\mathfrak{d}})$, we have $\ell \leqslant 0$ when writing $\mathfrak{r}(p) = \ell m(\hat{H}_{\mathfrak{d}})$.

We write

$$\hat{f}_{\mathfrak{d}} = \sum_{p \in P_{\varphi_{\tau_{\mathfrak{d}}}}} f_p \hat{z}^p.$$

For every $j \in \mathbb{Z}$, we define

$$\hat{f}_{\mathfrak{d}}(q^j \hat{z}) := \sum_{\substack{p \in P_{\varphi_{\tau_{\mathfrak{d}}}} \\ \mathfrak{r}(p) = \ell m(\hat{H}_{\mathfrak{d}})}} q^{\ell j} f_p \hat{z}^p \in \Bbbk_\hbar \widehat{[P_{\varphi_{\tau_{\mathfrak{d}}}}]},$$

where $q = e^{i\hbar}$.

LEMMA 3.1. *The automorphism $\hat{\Phi}_{\hat{H}_{\mathfrak{d}}}$ of $\Bbbk_\hbar \widehat{[P_{\varphi_{\tau_{\mathfrak{d}}}}]}$ given by conjugation by $\exp(\hat{H}_{\mathfrak{d}})$,*

$$\hat{z}^p \mapsto \exp(\hat{H}_{\mathfrak{d}}) \hat{z}^p \exp(-\hat{H}_{\mathfrak{d}}),$$

is equal to

$$\hat{z}^p \mapsto \begin{cases} \hat{z}^p \displaystyle\prod_{j=0}^{\langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p) \rangle - 1} \hat{f}_{\mathfrak{d}}(q^j \hat{z}) & \text{if } \langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p) \rangle \geqslant 0, \\ \hat{z}^p \displaystyle\prod_{j=0}^{|\langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p) \rangle| - 1} \hat{f}_{\mathfrak{d}}(q^{-j-1} \hat{z})^{-1} & \text{if } \langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p) \rangle < 0. \end{cases}$$

376

Proof. Using $\hat{z}^{p'}\hat{z}^p = q^{\langle \mathfrak{r}(p'),\mathfrak{r}(p)\rangle}\hat{z}^p\hat{z}^{p'}$, we get

$$\exp(\hat{H}_\mathfrak{d})\hat{z}^p\exp(-\hat{H}_\mathfrak{d}) = \hat{z}^p\exp\left(\sum_{\substack{p'\in P_{\varphi_{\tau_\mathfrak{d}}} \\ \mathfrak{r}(p')=\ell m(\hat{H}_\mathfrak{d})}} (q^{\ell\langle m(\hat{H}_\mathfrak{d}),\mathfrak{r}(p)\rangle}-1)H_{p'}\hat{z}^{p'}\right).$$

If $\langle m(\hat{H}_\mathfrak{d}),\mathfrak{r}(p)\rangle \geqslant 0$, this can be written

$$\hat{z}^p\exp\left(\sum_{\substack{p'\in P_{\varphi_{\tau_\mathfrak{d}}} \\ \mathfrak{r}(p')=\ell m(\hat{H}_\mathfrak{d})}} \frac{1-q^{\ell\langle m(\hat{H}_\mathfrak{d}),\mathfrak{r}(p)\rangle}}{1-q^\ell}(q^\ell-1)H_{p'}\hat{z}^{p'}\right)$$

$$= \hat{z}^p\exp\left(\sum_{\substack{p'\in P_{\varphi_{\tau_\mathfrak{d}}} \\ \mathfrak{r}(p')=\ell m(\hat{H}_\mathfrak{d})}} \sum_{j=0}^{\langle m(\hat{H}_\mathfrak{d}),\mathfrak{r}(p)\rangle-1} q^{\ell j}(q^\ell-1)H_{p'}\hat{z}^{p'}\right)$$

$$= \hat{z}^p\prod_{j=0}^{\langle m(\hat{H}_\mathfrak{d}),\mathfrak{r}(p)\rangle-1}\hat{f}_\mathfrak{d}(q^j\hat{z}).$$

If $\langle m(\hat{H}_\mathfrak{d}),\mathfrak{r}(p)\rangle < 0$, this can be written

$$\hat{z}^p\exp\left(-\sum_{\substack{p'\in P_{\varphi_{\tau_\mathfrak{d}}} \\ \mathfrak{r}(p')=\ell m(\hat{H}_\mathfrak{d})}} \frac{1-q^{-\ell|\langle m(\hat{H}_\mathfrak{d}),\mathfrak{r}(p)\rangle|}}{1-q^{-\ell}}q^{-\ell}(q^\ell-1)H_{p'}\hat{z}^{p'}\right)$$

$$= \hat{z}^p\exp\left(-\sum_{\substack{p'\in P_{\varphi_{\tau_\mathfrak{d}}} \\ \mathfrak{r}(p')=\ell m(\hat{H}_\mathfrak{d})}} \sum_{j=0}^{|\langle m(\hat{H}_\mathfrak{d}),\mathfrak{r}(p)\rangle|-1} (q^{-j-1})^\ell(q^\ell-1)H_{p'}\hat{z}^{p'}\right)$$

$$= \hat{z}^p\prod_{j=0}^{|\langle m(\hat{H}_\mathfrak{d}),\mathfrak{r}(p)\rangle|-1}\hat{f}_\mathfrak{d}(q^{-j-1}\hat{z})^{-1}. \qquad \square$$

Remark. One can equivalently write $\hat{\Phi}_{\hat{H}_\mathfrak{d}}$ as

$$\hat{z}^p \mapsto \begin{cases} \left(\displaystyle\prod_{j=0}^{\langle m(\hat{H}_\mathfrak{d}),\mathfrak{r}(p)\rangle-1}\hat{f}_\rho(q^{-j-1}z)\right)\hat{z}^p & \text{if } \langle m(\hat{H}_\mathfrak{d}),\mathfrak{r}(p)\rangle \geqslant 0, \\[2em] \left(\displaystyle\prod_{j=0}^{|\langle m(\hat{H}_\mathfrak{d}),\mathfrak{r}(p)\rangle|-1}\hat{f}_\rho(q^j z)^{-1}\right)\hat{z}^p & \text{if } \langle m(\hat{H}_\mathfrak{d}),\mathfrak{r}(p)\rangle < 0. \end{cases}$$

A direct application of the definition of $\hat{f}_\mathfrak{d}$ gives the following lemma.

LEMMA 3.2. *If*

$$\hat{H} = i\sum_{\ell\geqslant 1}\frac{(-1)^{\ell-1}}{\ell}\frac{\hat{z}^{-\ell\varphi(m_\mathfrak{d})}}{2\sin(\ell\hbar/2)} = -\sum_{\ell\geqslant 1}\frac{(-1)^{\ell-1}}{\ell}\frac{\hat{z}^{-\ell\varphi(m_\mathfrak{d})}}{q^{\ell/2}-q^{-\ell/2}},$$

where $q = e^{i\hbar}$, we have $m(\hat{H}) = m_{\mathfrak{d}}$ and

$$\hat{f} = \exp\left(-\sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{\ell} \frac{q^{-\ell} - 1}{q^{\ell/2} - q^{-\ell/2}} \hat{z}^{-\ell\varphi(m_{\mathfrak{d}})}\right) = 1 + q^{-1/2}\hat{z}^{-\varphi(m_{\mathfrak{d}})}.$$

If

$$\hat{H} = i\sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{\ell} \frac{\hat{z}^{\ell\varphi(m_{\mathfrak{d}})}}{2\sin(\ell\hbar/2)} = -\sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{\ell} \frac{\hat{z}^{\ell\varphi(m_{\mathfrak{d}})}}{q^{\ell/2} - q^{-\ell/2}},$$

where $q = e^{i\hbar}$, we have $m(\hat{H}) = -m_{\mathfrak{d}}$ and

$$\hat{f} = \exp\left(-\sum_{\ell \geq 1} \frac{(-1)^{\ell-1}}{\ell} \frac{q^{-\ell} - 1}{q^{\ell/2} - q^{-\ell/2}} \hat{z}^{\ell\varphi(m_{\mathfrak{d}})}\right) = 1 + q^{-1/2}\hat{z}^{\varphi(m_{\mathfrak{d}})}.$$

3.4 Gluing

We fix a quantum scattering diagram $\hat{\mathfrak{D}}$ for the data (B, Σ), P, J and φ, and an ideal I of radical J.

Let ρ be a one-dimensional cone of Σ, bounding the two-dimensional cones σ_+ and σ_-, such that σ_-, ρ, σ_+ are in anticlockwise order. Identifying X with $\hat{z}^{\varphi_\rho(m_\rho)}$, we define $\hat{f}_{\rho^{\mathrm{out}}} \in R_I^\hbar[X^{-1}]$ by

$$\hat{f}_{\rho^{\mathrm{out}}} := \prod_{\substack{\mathfrak{d} \in \hat{\mathfrak{D}}, \mathfrak{d} = \rho \\ \text{outgoing}}} \hat{f}_{\mathfrak{d}} \bmod I,$$

where the product is over the outgoing rays of $\hat{\mathfrak{D}}$ of support ρ, and we define $\hat{f}_{\rho^{\mathrm{in}}} \in R_I^\hbar[X]$ by

$$\hat{f}_{\rho^{\mathrm{in}}} := \prod_{\substack{\mathfrak{d} \in \hat{\mathfrak{D}}, \mathfrak{d} = \rho \\ \text{ingoing}}} \hat{f}_{\mathfrak{d}} \bmod I,$$

where the product is over the ingoing rays of $\hat{\mathfrak{D}}$ of support ρ.

By § 3.1, we then have $R_I^\hbar$-algebras $R_{\sigma_+,I}^\hbar$, $R_{\sigma_-,I}^\hbar$, $R_{\rho,I}^\hbar$.

Let $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ be a ray of $\hat{\mathfrak{D}}$ such that $\tau_{\mathfrak{d}} = \sigma$ is a two-dimensional cone of Σ. Let $m_{\mathfrak{d}} \in \Lambda_{\tau_{\mathfrak{d}}}$ be the primitive generator of $\mathfrak{d}$ pointing away from the origin. Let γ be a path in B_0 which crosses $\mathfrak{d}$ transversally at time t_0. We define

$$\hat{\theta}_{\gamma,\mathfrak{d}} \colon R_{\sigma,I}^\hbar \to R_{\sigma,I}^\hbar,$$
$$\hat{z}^p \mapsto \hat{\Phi}_{\hat{H}_{\mathfrak{d}}}^\epsilon(\hat{z}^p),$$

where $\epsilon \in \{\pm 1\}$ is the sign of $-\langle m(\hat{H}_{\mathfrak{d}}), \gamma'(t_0)\rangle$.

Let $\hat{\mathfrak{D}}_I \subset \hat{\mathfrak{D}}$ be the finite set of rays $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ with $\hat{H}_{\mathfrak{d}} \neq 0 \bmod I$, that is, $\hat{f}_{\mathfrak{d}} \neq 1 \bmod I$. If γ is a path in B_0 entirely contained in the interior of a two-dimensional cone σ of Σ, and crossing elements of $\mathfrak{D}_I$ transversally, we define

$$\hat{\theta}_{\gamma,\hat{\mathfrak{D}}_I} := \hat{\theta}_{\gamma,\mathfrak{d}_n} \circ \cdots \circ \hat{\theta}_{\gamma,\mathfrak{d}_1},$$

where γ crosses the elements $\mathfrak{d}_1, \ldots, \mathfrak{d}_n$ of $\hat{\mathfrak{D}}_I$ in the given order.

For every σ two-dimensional cone of Σ, bounded by rays ρ_R and ρ_L, such that ρ_R, σ, ρ_L are in anticlockwise order, we choose a path $\gamma_\sigma \colon [0, 1] \to B_0$ whose image is entirely contained in the

interior of σ, with $\gamma(0)$ close to ρ_R and $\gamma(1)$ close to ρ_L, such that γ_σ crosses every ray of $\hat{\mathfrak{D}}_I$ contained in σ transversally exactly once. Let

$$\hat{\theta}_{\gamma_\sigma, \hat{\mathfrak{D}}_I} \colon R^\hbar_{\sigma, I} \to R^\hbar_{\sigma, I}$$

be the corresponding automorphism. In the classical limit, $\hat{\theta}_{\gamma, \hat{\mathfrak{D}}_I}$ induces an automorphism $\theta_{\gamma, \mathfrak{D}_I}$ of $U_{\sigma, I}$. Gluing together the open sets $U_{\sigma, I} \subset U_{\rho_R, I}$ and $U_{\sigma, I} \subset U_{\rho_L, I}$ along these automorphisms, we get the scheme $X^\circ_{I, \mathfrak{D}}$ defined in [GHK15a].

Recall from the end of §3.1 that by Ore localization the algebras $R^\hbar_{\sigma, I}$ and $R^\hbar_{\rho, I}$ produce sheaves $\mathcal{O}^\hbar_{U_{\sigma, I}}$ and $\mathcal{O}^\hbar_{U_{\rho, I}}$ on $U_{\sigma, I}$ and $U_{\rho, I}$, respectively. Using $\hat{\theta}_{\gamma_\sigma, \hat{\mathfrak{D}}_I}$, we can glue together the sheaves $\mathcal{O}^\hbar_{U_{\rho, I}}$ to get a sheaf of $R^\hbar_I$-algebras $\mathcal{O}^\hbar_{X^\circ_{I, \mathfrak{D}}}$ on $X^\circ_{I, \mathfrak{D}}$.

From the fact that the sheaves $\mathcal{O}^\hbar_{U_{\rho, I}}$ are deformation quantizations of $U_{\rho, I}$, we deduce that the sheaf $\mathcal{O}^\hbar_{X^\circ_{I, \mathfrak{D}}}$ is a deformation quantization of $X^\circ_{I, \mathfrak{D}}$. In particular, we have $\mathcal{O}^\hbar_{X^\circ_{I, \mathfrak{D}}} / \hbar \mathcal{O}^\hbar_{X^\circ_{I, \mathfrak{D}}} = \mathcal{O}_{X^\circ_{I, \mathfrak{D}}}$ and $\mathcal{O}^\hbar_{X^\circ_{I, \mathfrak{D}}}$ is a sheaf a flat $R^\hbar_I$-algebras.

Remark. Let ρ be a one-dimensional cone of Σ. Let σ_+ and σ_- be the two two-dimensional cones of Σ bounding ρ, and let ρ_+ and ρ_- be the other boundary rays of σ_+ and σ_- respectively, such that ρ_-, ρ and ρ_+ are in anticlockwise order. According to [GHK15a, Remark 2.6], we have, in $U_{\rho, I}$,

$$U_{\rho_-, I} \cap U_{\rho_+, I} \simeq (\mathbb{G}_m)^2 \times \operatorname{Spec}(R_I)_{z^{\kappa_\rho, \varphi}},$$

where $(R_I)_{z^{\kappa_\rho, \varphi}}$ is the localization of R_I defined by inverting $z^{\kappa_\rho, \varphi}$. Similarly, the restriction of $\mathcal{O}^\hbar_{X^\circ_{I, \mathfrak{D}}}$ to $U_{\rho_-, I} \cap U_{\rho_+, I}$ is the Ore localization of $\Bbbk_\hbar[M] \hat{\otimes} (R_I)_{z^{\kappa_\rho, \varphi}}$, where $M = \mathbb{Z}^2$ is the character lattice of $(\mathbb{G}_m)^2$, equipped with the standard unimodular integral symplectic pairing. We have a natural identification $M = \Lambda_\rho$. Restricted to $\Bbbk_\hbar[M] \hat{\otimes} (R_I)_{z^{\kappa_\rho, \varphi}}$, and assuming that $\hat{f}_{\rho^{\mathrm{in}}} = 1 \bmod \hat{z}^{\kappa_\rho, \varphi}$ and $\hat{f}_{\rho^{\mathrm{out}}} = 1 \bmod \hat{z}^{\kappa_\rho, \varphi}$, the expression $\hat{\psi}_{\rho_+} \circ \hat{\psi}_{\rho_-}^{-1}$ makes sense[10] and is given by

$$(\hat{\psi}_{\rho_+} \circ \hat{\psi}_{\rho_-}^{-1})(\hat{z}^{\varphi_\rho(m_\rho)}) = \hat{z}^{\varphi_\rho(m_\rho)},$$
$$(\hat{\psi}_{\rho_+} \circ \hat{\psi}_{\rho_-}^{-1})(\hat{z}^{\varphi_\rho(m_{\rho_-})}) = \hat{f}_{\rho^{\mathrm{out}}}(\hat{z}^{\varphi_\rho(m_\rho)}) \hat{z}^{\varphi_\rho(m_{\rho_-})} \hat{f}_{\rho^{\mathrm{in}}}(\hat{z}^{\varphi_\rho(m_\rho)}),$$
$$(\hat{\psi}_{\rho_+} \circ \hat{\psi}_{\rho_-}^{-1})(\hat{z}^{\varphi_\rho(m_{\rho_+})}) = \hat{f}_{\rho^{\mathrm{in}}}^{-1}(\hat{z}^{\varphi_\rho(m_\rho)}) \hat{z}^{\varphi_\rho(m_{\rho_+})} \hat{f}_{\rho^{\mathrm{out}}}^{-1}(\hat{z}^{\varphi_\rho(m_\rho)}).$$

As $\langle m_\rho, m_{\rho_-} \rangle = -1$ and $\langle m_\rho, m_{\rho_+} \rangle = 1$, this implies that $\hat{\psi}_{\rho_+} \circ \hat{\psi}_{\rho_-}^{-1}$ coincides with the transformation

$$\hat{\theta}_{\gamma, \rho} = \prod_{\mathfrak{d} \in \mathfrak{D}, \mathfrak{d} = \rho} \hat{\theta}_{\gamma, \mathfrak{d}},$$

where $\hat{\theta}_{\gamma, \mathfrak{d}}$ is defined by the same formulas as above and with γ a path intersecting ρ at a single point and going from σ_- to σ_+.

3.5 Result of the gluing for $I = J$

Assume $r \geqslant 3$ and $\kappa_{\rho, \varphi} \in J$ for every ρ one-dimensional cone of Σ. Lemma 3.3 below gives an explicit description of $\mathcal{O}^\hbar_{X^\circ_{I, \mathfrak{D}}}$ for $I = J$.

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[10] Without restriction, $\hat{\psi}_{\rho_-}$ is not invertible and so $\hat{\psi}_{\rho_-}^{-1}$ does not make sense.

Denote by $\Bbbk[\Sigma]$ the $\Bbbk$-algebra with a $\Bbbk$-basis $\{z^m \mid m \in B(\mathbb{Z})\}$ with multiplication given by

$$z^m \cdot z^{m'} = \begin{cases} z^{m+m'} & \text{if } m \text{ and } m' \text{ lie in a common cone of } \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Let 0 be the closed point of $\operatorname{Spec} \Bbbk[\Sigma]$ whose ideal is generated by $\{z^m \mid m \neq 0\}$. Denote $R_J[\Sigma] := R_J \otimes_{\Bbbk} \Bbbk[\Sigma]$. According to [GHK15a, Lemma 2.12], we have

$$X_J^\circ \simeq (\operatorname{Spec} R_J[\Sigma]) - ((\operatorname{Spec} R_J) \times \{0\}).$$

Denote by $\Bbbk_\hbar[\Sigma]$ the $\Bbbk_\hbar$-algebra with a $\Bbbk_\hbar$-basis $\{\hat{z}^m \mid m \in B(\mathbb{Z})\}$ with multiplication given by

$$\hat{z}^m \cdot \hat{z}^{m'} = \begin{cases} q^{(1/2)\langle m, m' \rangle} \hat{z}^{m+m'} & \text{if } m \text{ and } m' \text{ lie in a common cone of } \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Denote $R_J^\hbar[\Sigma] := R_J \hat{\otimes}_{\Bbbk} \Bbbk_\hbar[\Sigma]$.

LEMMA 3.3. *Assume $r \geqslant 3$ and $\kappa_{\rho,\varphi} \in J$ for every ρ one-dimensional cone of Σ. Then $\Gamma(X_{J,\mathfrak{D}}^\circ, \mathcal{O}_{X_{J,\mathfrak{D}}^\circ}^\hbar) = R_J^\hbar[\Sigma]$, and the sheaf $\mathcal{O}_{X_{J,\mathfrak{D}}^\circ}^\hbar$ is the restriction to X_J° of the Ore localization (see § 2.4) of $R_J^\hbar[\Sigma]$ over $\operatorname{Spec} R_J[\Sigma]$.*

Proof. By definition of a quantum scattering diagram, if $\mathfrak{d}$ is contained in the interior of a two-dimensional cone of Σ, we have $\hat{H}_{\mathfrak{d}} = 0 \bmod J$ and so the corresponding automorphism $\hat{\Phi}_{\hat{H}_{\mathfrak{d}}}$ is the identity. As we are assuming $\kappa_{\rho,\varphi} \in J$, $R_{\rho,J}^\hbar$ is the $R_J^\hbar$-algebra generated by formal variables X_+, X_- and X, with X invertible, and with relations

$$\begin{aligned} XX_+ &= qXX_+, \\ XX_- &= q^{-1}X_-X, \\ X_+X_- &= X_-X_+ = 0, \end{aligned}$$

where $q = e^{i\hbar}$. Let σ_+ and σ_- be the two two-dimensional cones of Σ bounding ρ, and let ρ_+ and ρ_- be the other boundary rays of σ_+ and σ_- respectively, such that ρ_-, ρ and ρ_+ are in anticlockwise order.

From $\varphi_\rho(m_{\rho_-}) + \varphi_\rho(m_{\rho_+}) = \kappa_{\rho,\varphi} - D_\rho^2 \varphi_\rho(m_\rho)$ and $\kappa_{\rho,\varphi} \in J$, we deduce that $\hat{z}^{\varphi_\rho(m_{\rho_-})} \hat{z}^{\varphi_\rho(m_{\rho_+})}$ $= 0$ in $R_{\rho,I}^\hbar$, $R_{\sigma_-I}^\hbar$ and $R_{\sigma_+,I}^\hbar$. As $\hat{z}^{\varphi_\rho(m_{\rho_-})}$ is invertible in $R_{\sigma_-I}^\hbar$, we have $\hat{z}^{\varphi_\rho(m_{\rho_+})} = 0$ in $R_{\sigma_-I}^\hbar$. Similarly, as $\hat{z}^{\varphi_\rho(m_{\rho_+})}$ is invertible in $R_{\sigma_+I}^\hbar$, we have $\hat{z}^{\varphi_\rho(m_{\rho_-})} = 0$ in $R_{\sigma_+I}^\hbar$.

So the map $\hat{\psi}_{\rho,-} \colon R_{\rho,J}^\hbar \to R_{\sigma_-,J}^\hbar$ is given by $\hat{\psi}_{\rho,-}(X) = \hat{z}^{\varphi_\rho(m_\rho)}$, $\hat{\psi}_{\rho,-}(X_-) = \hat{z}^{\varphi_\rho(m_{\rho_-})}$, $\hat{\psi}_{\rho,-}(X_+) = 0$. Similarly, the map $\hat{\psi}_{\rho,+} \colon R_{\rho,J}^\hbar \to R_{\sigma_+,J}^\hbar$ is given by $\hat{\psi}_{\rho,+}(X) = \hat{z}^{\varphi_\rho(m_\rho)}$, $\hat{\psi}_{\rho,+}(X_-) = 0$, $\hat{\psi}_{\rho,+}(X_+) = \hat{z}^{\varphi_\rho(m_{\rho_+})}$. The result follows. $\square$

3.6 Quantum broken lines and theta functions

We fix a Looijenga pair (Y, D), its tropicalization (B, Σ), a toric monoid P, a radical monomial ideal J of P, a $P_{\mathbb{R}}^{\mathrm{gp}}$-valued multivalued convex Σ-piecewise linear function φ on B, and a quantum scattering diagram $\hat{\mathfrak{D}}$ for the data (B, Σ), P, J and φ.

Quantum broken lines and quantum theta functions have been studied by Mandel [Man15], for smooth integral affine manifolds. We make below the easy combination of the notion of quantum broken lines and theta functions used by [Man15] with the notion of classical broken lines and theta functions used in [GHK15a, § 2.3] for the tropicalization B of a Looijenga pair.

DEFINITION 3.2. A *quantum broken line* of charge $p \in B_0(\mathbb{Z})$ with endpoint Q in B_0 is a proper continuous piecewise integral affine map

$$\gamma \colon (-\infty, 0] \to B_0$$

with only finitely many domains of linearity, together with, for each $L \subset (-\infty, 0]$ a maximal connected domain of linearity of γ, a choice of monomial $m_L = c_L \hat{z}^{p_L}$ where $c_L \in \Bbbk_\hbar^*$ and $p_L \in \Gamma(L, \gamma^{-1}(\mathcal{P})|_L)$, such that the following statements hold.

- For each L and $t \in L$, we have $-\mathfrak{r}(p_L) = \gamma'(t)$, that is, the direction of the line is determined by the monomial attached to it.
- We have $\gamma(0) = Q \in B_0$.
- For the unique unbounded domain of linearity L, $\gamma|_L$ goes off for $t \to -\infty$ to infinity in the cone σ of Σ containing p and $m_L = \hat{z}^{\varphi_\sigma(p)}$, that is, the charge p is the asymptotic direction of the broken line.
- Let $t \in (-\infty, 0)$ be a point at which γ is not linear, passing from the domain of linearity L to the domain of linearity L'. Let τ be the cone of Σ containing $\gamma(t)$. Let $(\mathfrak{d}_1, \hat{H}_{\mathfrak{d}_1}), \ldots,$ $(\mathfrak{d}_N, \hat{H}_{\mathfrak{d}_N})$ be the rays of $\hat{\mathfrak{D}}$ that contain $\gamma(t)$. Then γ passes from one side of these rays to the other side at time t.

  Expand the product of

$$\prod_{\substack{1 \leqslant k \leqslant N \\ \langle m(H_{\mathfrak{d}_k}), \mathfrak{r}(p_L) \rangle > 0}} \prod_{j=0}^{\langle m(H_k), \mathfrak{r}(p_L) \rangle - 1} \hat{f}_{\mathfrak{d}_k}(q^j \hat{z})$$

and

$$\prod_{\substack{1 \leqslant k' \leqslant N \\ \langle m(H_{\mathfrak{d}_{k'}}), \mathfrak{r}(p_L) \rangle < 0}} \prod_{j'=0}^{|\langle m(H_{k'}), \mathfrak{r}(p_L) \rangle| - 1} \hat{f}_{\mathfrak{d}_{k'}}(q^{-j'-1} \hat{z}),$$

as a formal power series in $\Bbbk_\hbar \widehat{[P_{\varphi_\tau}]}$. Then there is a term $c \hat{z}^s$ in this sum with

$$m_{L'} = m_L \cdot (c \hat{z}^s).$$

Let $Q \in B - \mathrm{Supp}_I(\hat{\mathfrak{D}})$ be in the interior of a two-dimensional cone σ of Σ. Let γ be a quantum broken line with endpoint Q. We denote by $\mathrm{Mono}(\gamma) \in \Bbbk_\hbar[P_{\varphi_\sigma}]$ the monomial attached to the last domain of linearity of γ.

The following finiteness result is formally identical to [GHK15a, Lemma 2.25].

LEMMA 3.4. *Let $Q \in B - \mathrm{Supp}_I(\hat{\mathfrak{D}})$ be in the interior of a two-dimensional cone σ of Σ. Fix $p \in B_0(\mathbb{Z})$. Let I be an ideal of radical J. Assume that $\kappa_{\rho,\varphi} \in J$ for at least one ray ρ of Σ. Then the following statements hold.*

- *The collection of quantum broken lines γ of charge p with endpoint Q and such that $\mathrm{Mono}(\gamma) \notin I \Bbbk_\hbar[P_{\varphi_\sigma}]$ is finite.*
- *If one boundary ray of the connected component of $B - \mathrm{Supp}_I(\hat{\mathfrak{D}})$ containing Q is a ray ρ of Σ, then for every quantum broken line γ of charge p with endpoint Q, we have $\mathrm{Mono}(\gamma) \in \Bbbk_\hbar[P_{\varphi_\rho}]$.*

Proof. Identical to the proof of [GHK15a, Lemma 2.25]. □

381

Let $Q \in B - \text{Supp}_I(\hat{\mathfrak{D}})$ be in the interior of a two-dimensional cone σ of Σ. Fix $p \in B_0(\mathbb{Z})$. Let I be an ideal of radical J. We define

$$\text{Lift}_Q(p) := \sum_\gamma \text{Mono}(\gamma) \in \Bbbk_\hbar[P_{\varphi_\sigma}]/I,$$

where the sum is over all the quantum broken lines γ of charge p with endpoint Q. According to Lemma 3.4, there are only finitely many such γ with $\text{Mono}(\gamma) \notin I\Bbbk_\hbar[P_{\varphi_\sigma}]$ and so $\text{Lift}_Q(p)$ is well defined.

The following definition is formally identical to [GHK15a, Definition 2.26].

DEFINITION 3.3. Assume that $\kappa_{\rho,\varphi} \in J$ for at least one one-dimensional cone ρ of Σ. We say that a quantum scattering diagram $\hat{\mathfrak{D}}$ for the data (B, Σ), P, J and φ is *consistent* if for every ideal I of P of radical J and for all $p \in B_0(\mathbb{Z})$, the following holds. Let $Q \in B_0$ be chosen so that the line joining the origin and Q has irrational slope, and $Q' \in B_0$ similarly.

- If Q and Q' are contained in a common two-dimensional cone σ of Σ, then we have

$$\text{Lift}_{Q'}(p) = \hat{\theta}_{\gamma,\hat{\mathfrak{D}}_I}(\text{Lift}_Q(p))$$

  in $R^\hbar_{\sigma,I}$, for every γ path in the interior of σ connecting Q and Q', and intersecting transversely the rays of $\hat{\mathfrak{D}}$.

- If Q_- is contained in a two-dimensional cone σ_- of Σ, and Q_+ is contained in a two-dimensional cone σ_+ of Σ, such that σ_+ and σ_- intersect along a one-dimensional cone ρ of Σ, and furthermore Q_- and Q_+ are contained in connected components of $B - \text{Supp}_I(\hat{\mathfrak{D}})$ whose closures contain ρ, then $\text{Lift}_{Q_+}(p) \in R^\hbar_{\sigma_+,I}$ and $\text{Lift}_{Q_-}(p) \in R^\hbar_{\sigma_-,I}$ are both images under $\hat{\psi}_{\rho,+}$ and $\hat{\psi}_{\rho,-}$ respectively of a single element $\text{Lift}_\rho(p) \in R^\hbar_{\rho,I}$.

The following construction is formally identical to [GHK15a, Construction 2.27]. Suppose that D has $r \geqslant 3$ irreducible components, and that $\hat{\mathfrak{D}}$ is a consistent quantum scattering diagram for the data (B, Σ), P, J and φ. Assume that $\kappa_{\rho,\varphi} \in J$ for all one-dimensional cones ρ of Σ. Let I be an ideal of P of radical J. We construct below an element

$$\hat{\vartheta}_p \in \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}^\hbar_{X^\circ_{I,\mathfrak{D}}})$$

for each $p \in B(\mathbb{Z}) = B_0(\mathbb{Z}) \cup \{0\}$.

We first define $\hat{\vartheta}_0 := 1$. Let $p \in B_0(\mathbb{Z})$. Recall that $X^\circ_{I,\mathfrak{D}}$ is defined by gluing together schemes $U_{\rho,I}$, indexed by ρ rays of Σ, and that $\mathcal{O}^\hbar_{X^\circ_{I,\mathfrak{D}}}$ is defined by gluing together sheaves $\mathcal{O}^\hbar_{U_{\rho,I}}$ on $U_{\rho,I}$, such that $\Gamma(U_{\rho,I}, \mathcal{O}^\hbar_{X^\circ_{I,\mathfrak{D}}}) = R^\hbar_{\rho,I}$. So, to define $\hat{\vartheta}_p$, it is enough to define elements of $R^\hbar_{\rho,I}$ compatible with the gluing functions. But, by definition, the consistency of $\hat{\mathfrak{D}}$ gives us such elements $\text{Lift}_\rho(p) \in R^\hbar_{\rho,I}$.

The quantum theta functions $\hat{\vartheta}_p \in \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}^\hbar_{X^\circ_{I,\mathfrak{D}}})$ reduce in the classical limit to the theta functions $\vartheta_p \in \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}_{X^\circ_{I,\mathfrak{D}}})$ defined in [GHK15a].

3.7 Deformation quantization of the mirror family

Suppose D has $r \geqslant 3$ irreducible components, and let φ be a $P^{\text{gp}}_{\mathbb{R}}$-valued convex Σ-piecewise linear function on B such that $\kappa_{\rho,\varphi} \in J$ for all one-dimensional cones ρ of Σ. Let $\hat{\mathfrak{D}}$ be a consistent quantum scattering diagram for the data (B, Σ), P, J and φ. Let I be an ideal of P of radical J.

Denote by

$$X_{I,\mathfrak{D}} := \operatorname{Spec} \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}_{X^\circ_{I,\mathfrak{D}}})$$

the affinization of $X^\circ_{I,\mathfrak{D}}$ and $j\colon X^\circ_{I,\mathfrak{D}} \to X_{I,\mathfrak{D}}$ the affinization morphism. It is proved in [GHK15a, Theorem 2.28] that j is an open immersion, that $j_*\mathcal{O}_{X^\circ_{I,\mathfrak{D}}} = \mathcal{O}_{X_{I,\mathfrak{D}}}$, and that X_I is flat over R_I. More precisely, the R_I-algebra

$$A_I := \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}_{X^\circ_{I,\mathfrak{D}}}) = \Gamma(X_{I,\mathfrak{D}}, \mathcal{O}_{X_{I,\mathfrak{D}}})$$

is free as R_I-module and the set of theta functions ϑ_p, $p \in B(\mathbb{Z})$ is an R_I-module basis of A_I.

THEOREM 3.1. *Suppose D has $r \geqslant 3$ irreducible components, and let φ be a $P^{\mathrm{gp}}_{\mathbb{R}}$-valued convex Σ-piecewise linear function on B such that $\kappa_{\rho,\varphi} \in J$ for all one-dimensional cones ρ of Σ. Let $\hat{\mathfrak{D}}$ be a consistent quantum scattering diagram for the data (B,Σ), P, J and φ. Let I be an ideal of P of radical J. Then the following statements hold.*

- *The sheaf $\mathcal{O}^\hbar_{X_{I,\mathfrak{D}}} := j_*\mathcal{O}^\hbar_{X^\circ_{I,\mathfrak{D}}}$ of $R^\hbar_I$-algebras is a deformation quantization of $X_{I,\mathfrak{D}}$ over R_I in the sense of Definition 2.3.*
- *The $R^\hbar_I$-algebra*

$$A^\hbar_I := \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}^\hbar_{X^\circ_{I,\mathfrak{D}}}) = \Gamma(X_{I,\mathfrak{D}}, \mathcal{O}^\hbar_{X_{I,\mathfrak{D}}})$$

  is a deformation quantization of $X_{I,\mathfrak{D}}$ over R_I in the sense of Definition 2.4.
- *The $R^\hbar_I$-algebra $A^\hbar_I$ is free as $R^\hbar_I$-module.*
- *The set of quantum theta functions*

$$\{\vartheta^\hbar_p | \, p \in B(\mathbb{Z})\}$$

  is an $R^\hbar_I$-module basis for $A^\hbar_I$.

Proof. We follow the structure of the proof of [GHK15a, Theorem 2.28].

We first prove the result for $I = J$. As $r \geqslant 3$ and $\kappa_{\rho,\varphi} \in J$ for all one-dimensional cones ρ of Σ, the only broken line contributing to $\mathrm{Lift}_Q(p)$, for every Q in B_0 and $p \in B_0(\mathbb{Z})$, is the straight line of endpoint Q and direction p, and this provides a non-zero contribution only if Q and p lie in the same two-dimensional cone of Σ. Combined with Lemma 3.3, this implies that the map

$$\bigoplus_{p \in B(\mathbb{Z})} R^\hbar_J \, \hat{\vartheta}_p \to A^\hbar_J := \Gamma(X^\circ_{J,\mathfrak{D}}, \mathcal{O}^\hbar_{X^\circ_{J,\mathfrak{D}}}) = R^\hbar_J[\Sigma]$$

is given by

$$\hat{\vartheta}_p \mapsto \hat{z}^p$$

and so is an isomorphism.

We now treat the case of a general ideal I of P of radical J. By construction, $\mathcal{O}^\hbar_{X^\circ_{I,\mathfrak{D}}}$ is a deformation quantization of $X^\circ_{I,\mathfrak{D}}$ over R_I. In particular, $\mathcal{O}^\hbar_{X^\circ_{I,\mathfrak{D}}}$ is a sheaf in flat $R^\hbar_I$-algebras. As used in [GHK15a], the fibers of $X_{J,\mathfrak{D}} \to \operatorname{Spec} R_J$ satisfy Serre's condition S_2 by [Ale02]. We have $\mathcal{O}^\hbar_{X_{J,\mathfrak{D}}} \simeq \mathcal{O}_{X_{J,\mathfrak{D}}} \hat{\otimes} \Bbbk_\hbar$ as $\Bbbk_\hbar$-module and so it follows that $j_*j^*\mathcal{O}^\hbar_{X_{J,\mathfrak{D}}} = \mathcal{O}^\hbar_{X_{J,\mathfrak{D}}}$. The existence of quantum theta functions $\hat{\vartheta}_p$ guarantees that the natural map

$$\mathcal{O}^\hbar_{X_{I,\mathfrak{D}}} := j_*\mathcal{O}^\hbar_{X^\circ_{I,\mathfrak{D}}} \to j_*j^*\mathcal{O}^\hbar_{X_{J,\mathfrak{D}}} = \mathcal{O}^\hbar_{X_{J,\mathfrak{D}}}$$

is surjective. So the result follows from the following lemma, analogous to [GHK15a, Lemma 2.29]. □

383

Lemma 3.5. *Let X_0/S_0 be a flat family of surfaces such whose fibers satisfy Serre's condition S_2. Let $j \colon X_0^\circ \subset X_0$ be the inclusion of an open subset such that the complement has finite fiber. Let $S_0 \subset S$ be an infinitesimal thickening of S_0, and X/S a flat deformation of X_0/S_0, inducing a flat deformation X°/S of X_0°/S_0. Let $\mathcal{O}^\hbar_{X_0}$ be a deformation quantization of X_0/S_0 such that $\mathcal{O}^\hbar_{X_0} \simeq \mathcal{O}_{X_0} \hat{\otimes} \Bbbk_\hbar$ as $\mathcal{O}_{S_0} \hat{\otimes} \Bbbk_\hbar$-module, and so $j_* j^* \mathcal{O}^\hbar_{X_0} = \mathcal{O}^\hbar_{X_0}$ by the relative S_2 condition satisfied by X_0/S_0. Let $\mathcal{O}^\hbar_{X^\circ}$ be a deformation quantization of X°/S, restricting to $j^* \mathcal{O}^\hbar_{X_0}$ over X_0°. If the natural map*

$$\mathcal{O}^\hbar_X := j_* \mathcal{O}^\hbar_{X^\circ} \to j_* j^* \mathcal{O}^\hbar_{X_0} = \mathcal{O}^\hbar_{X_0}$$

is surjective, then $\mathcal{O}^\hbar_X$ is a deformation quantization of X/S.

Proof. We have to prove that $\mathcal{O}^\hbar_X$ is flat over $\mathcal{O}_S \hat{\otimes} \Bbbk_\hbar$.

Let $\mathcal{I} \subset \mathcal{O}_S$ be the nilpotent ideal defining $S_0 \subset S$. Let X_n, X_n°, S_n be the nth-order infinitesimal thickening of X_0, X_0°, S_0 in S, that is, $\mathcal{O}_{X_n} = \mathcal{O}_X/\mathcal{I}^{n+1}$, $\mathcal{O}_{X_n^\circ} = \mathcal{O}_{X^\circ}/\mathcal{I}^{n+1}$ and $\mathcal{O}_{S_n} = \mathcal{O}_S/\mathcal{I}^{n+1}$.

We define $\mathcal{O}^\hbar_{X_n} := j_* \mathcal{O}^\hbar_{X_n^\circ}$. We show by induction on n that $\mathcal{O}^\hbar_{X_n}$ is flat over $\mathcal{O}_{S_n} \hat{\otimes} \Bbbk_\hbar$.

For $n = 0$, we have $j_* \mathcal{O}^\hbar_{X_0^\circ} = j_* j^* \mathcal{O}^\hbar_{X_0} = \mathcal{O}^\hbar_{X_0}$, which is flat over $\mathcal{O}_{S_0} \hat{\otimes} \Bbbk_\hbar$ by assumption.

Assume that the induction hypothesis is true for $n - 1$. Since $\mathcal{O}^\hbar_{X_n^\circ}$ is flat over $\mathcal{O}_{S_n} \hat{\otimes} \Bbbk_\hbar$, we have an exact sequence

$$0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \otimes \mathcal{O}^\hbar_{X_0^\circ} \to \mathcal{O}^\hbar_{X_n^\circ} \to \mathcal{O}^\hbar_{X_{n-1}^\circ} \to 0.$$

Applying j_*, we get an exact sequence

$$0 \to j_*(\mathcal{I}^n/\mathcal{I}^{n+1} \otimes j^* \mathcal{O}^\hbar_{X_0}) \to \mathcal{O}^\hbar_{X_n} \to \mathcal{O}^\hbar_{X_{n-1}}.$$

We have $j_*(\mathcal{I}^n/\mathcal{I}^{n+1} \otimes j^* \mathcal{O}^\hbar_{X_0}) = \mathcal{I}^n/\mathcal{I}^{n+1} \otimes \mathcal{O}^\hbar_{X_0}$.

By assumption, the natural map $\mathcal{O}^\hbar_X \to j_* j^* \mathcal{O}^\hbar_{X_0} = \mathcal{O}^\hbar_{X_0}$ is surjective. By the induction hypothesis, we have $\mathcal{O}^\hbar_{X_{n-1}}/\mathcal{I} = \mathcal{O}^\hbar_{X_0}$. As $\mathcal{I}$ is nilpotent, it follows that the map $\mathcal{O}^\hbar_{X_n} \to \mathcal{O}^\hbar_{X_{n-1}}$ is surjective. So we have an exact sequence

$$0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \otimes \mathcal{O}^\hbar_{X_0} \to \mathcal{O}^\hbar_{X_n} \to \mathcal{O}^\hbar_{X_{n-1}} \to 0,$$

implying that $\mathcal{O}^\hbar_{X_n}$ is flat over $\mathcal{O}_{S_n} \hat{\otimes} \Bbbk_\hbar$. $\qquad\square$

3.8 The algebra structure

This section is a q-deformed version of [GHK15a, § 2.4].

We saw in the previous Section that the $R^\hbar_I$-algebra

$$A^\hbar_I := \Gamma(X^\circ_{I,\mathfrak{D}}, \mathcal{O}^\hbar_{X^\circ_{I,\mathfrak{D}}})$$

is free as $R^\hbar_I$-module, admitting a basis of quantum theta functions $\hat{\vartheta}_p$, $p \in B(\mathbb{Z})$. Theorem 3.2 below gives a combinatorial expression for the structure constants of the algebra $A^\hbar_I$ in the basis of quantum theta functions.

If γ is a quantum broken line of endpoint Q in a cone τ of Σ, we can write the monomial $\text{Mono}(\gamma)$ attached to the segment ending at Q as

$$\text{Mono}(\gamma) = c(\gamma) \hat{z}^{\varphi_\tau(s(\gamma))}$$

with $c(\gamma) \in \Bbbk_\hbar[P_{\varphi_\tau}]$ and $s(\gamma) \in \Lambda_\tau$.

THEOREM 3.2. *Let $p \in B(\mathbb{Z})$ and let $z \in B - \mathrm{Supp}_I(\hat{\mathfrak{D}}^{\mathrm{can}})$ be very close to p. For every p_1, $p_2 \in B(\mathbb{Z})$, the structure constants $C^p_{p_1,p_2} \in R^\hbar_I$ in the product expansion*

$$\hat{\vartheta}_{p_1} \hat{\vartheta}_{p_2} = \sum_{p \in B(\mathbb{Z})} C^p_{p_1,p_2} \hat{\vartheta}_p$$

are given by

$$C^p_{p_1,p_2} = \sum_{\gamma_1,\gamma_2} c(\gamma_1) c(\gamma_2) q^{(1/2)\langle s(\gamma_1), s(\gamma_2)\rangle},$$

where the sum is over all broken lines γ_1 and γ_2 of asymptotic charges p_1 and p_2, satisfying $s(\gamma_1) + s(\gamma_2) = p$, and both ending at the point $z \in B_0$.

Proof. Let τ be the smallest cone of Σ containing p. Working in the algebra $\Bbbk_\hbar[P_{\varphi_\tau}]/I$, we have

$$\mathrm{Lift}_z(p_1)\,\mathrm{Lift}_z(p_2) = \sum_{p \in B(\mathbb{Z})} C^p_{p_1,p_2}\,\mathrm{Lift}_z(p).$$

By definition, we have

$$\mathrm{Lift}_z(p_1) = \sum_{\gamma_1} c(\gamma_1) \hat{z}^{\varphi_\tau(s(\gamma_1))}$$

and

$$\mathrm{Lift}_z(p_2) = \sum_{\gamma_2} c(\gamma_2) \hat{z}^{\varphi_\tau(s(\gamma_2))}.$$

As p and z belong to the cone τ, the only quantum broken line of charge p ending at z is the straight line $z + \mathbb{R}_{\geqslant 0}$ equipped with the monomial $\hat{z}^{\varphi_\tau(p)}$, and so we have

$$\mathrm{Lift}_z(p) = \hat{z}^{\varphi_\tau(p)}.$$

The result then follows from the multiplication rule

$$\hat{z}^{\varphi_\tau(s(\gamma_1))} \hat{z}^{\varphi_\tau(s(\gamma_2))} = q^{(1/2)\langle s(\gamma_1), s(\gamma_2)\rangle} \hat{z}^{\varphi_\tau(p)}. \qquad \square$$

Remark. In the formula given by the previous theorem, the non-commutativity of the product of the quantum theta functions comes from the twist by the power of q,

$$q^{(1/2)\langle s(\gamma_1), s(\gamma_2)\rangle},$$

which is obviously not symmetric in γ_1 and γ_2 as $\langle -, -\rangle$ is skew-symmetric.

Taking the classical limit $\hbar \to 0$, we get an explicit formula for the Poisson bracket of classical theta functions, which could have been written and proved in [GHK15a].

COROLLARY 3.1. *Let $p \in B(\mathbb{Z})$ and let $z \in B - \mathrm{Supp}_I(\mathfrak{D}^{\mathrm{can}})$ be very close to p. For every $p_1, p_2 \in B(\mathbb{Z})$, the Poisson bracket of the classical theta functions ϑ_{p_1} and ϑ_{p_2} is given by*

$$\{\vartheta_{p_1}, \vartheta_{p_2}\} = \sum_{p \in B(\mathbb{Z})} P^p_{p_1,p_2} \vartheta_p,$$

where

$$P^p_{p_1,p_2} := \sum_{\gamma_1,\gamma_2} \langle s(\gamma_1), s(\gamma_2)\rangle c(\gamma_1) c(\gamma_2),$$

where the sum is over all broken lines γ_1 and γ_2 of asymptotic charges p_1 and p_2, satisfying $s(\gamma_1) + s(\gamma_2) = p$, and both ending at the point $z \in B_0$.

4. The canonical quantum scattering diagram

In this section we construct a quantum deformation of the canonical scattering diagram constructed in [GHK15a, § 3] and we prove its consistency. In § 4.1 we give the definition of a family of higher-genus log Gromov–Witten invariants of a Looijenga pair. In § 4.2 we use these invariants to construct the quantum canonical scattering diagram of a Looijenga pair and we state its consistency in Theorem 4.1. The proof of Theorem 4.1 occupies §§ 4.4–4.8, and follows the general structure of the proof given in the classical case by [GHK15a], the use of [GPS10] being replaced by the use of [Bou18].

4.1 Log Gromov–Witten invariants

We fix a Looijenga pair (Y, D), its tropicalization (B, Σ), a toric monoid P and a morphism $\eta \colon NE(Y) \to P$ of monoids. Let φ be the unique (up to addition of a linear function) $P_{\mathbb{R}}^{\mathrm{gp}}$-valued multivalued convex Σ-piecewise linear function on B such that $\kappa_{\rho, \varphi} = \eta([D_\rho])$ for every one-dimensional cone ρ of Σ, where $[D_\rho] \in NE(Y)$ is the class of the divisor D_ρ dual to ρ.

Let $\mathfrak{d} \subset B$ be a ray with endpoint the origin and with rational slope. Let $\tau_{\mathfrak{d}} \in \Sigma$ be the smallest cone containing $\mathfrak{d}$ and let $m_{\mathfrak{d}} \in \Lambda_{\tau_{\mathfrak{d}}}$ be the primitive generator of $\mathfrak{d}$ pointing away from the origin.

Let us first assume that $\tau = \sigma$ is a two-dimensional cone of Σ. The ray $\mathfrak{d}$ is then contained in the interior of σ. Let ρ_R and ρ_L be the two rays of Σ bounding σ. Let $m_{\rho_R}, m_{\rho_L} \in \Lambda_\sigma$ be primitive generators of ρ_R, ρ_L pointing away from the origin. As σ is isomorphic as integral affine manifold to the standard positive quadrant $(\mathbb{R}_{\geq 0})^2$ of $\mathbb{R}^2$, there exists a unique decomposition

$$m_{\mathfrak{d}} = n_R m_{\rho_R} + n_L m_{\rho_L}$$

with n_R and n_L positive integers. Let $NE(Y)_{\mathfrak{d}}$ be the set of classes $\beta \in NE(Y)$ such that there exists a positive integer ℓ_β such that

$$\begin{aligned}
\beta \cdot D_{\rho_R} &= \ell_\beta n_R, \\
\beta \cdot D_{\rho_L} &= \ell_\beta n_L, \\
\beta \cdot D_\rho &= 0,
\end{aligned}$$

for every one-dimensional cone ρ of Σ distinct of ρ_R and ρ_L.

If $\tau = \rho$ is a one-dimensional cone of Σ, we define $NE(Y)_{\mathfrak{d}}$ as being the set of classes $\beta \in NE(Y)$ such that there exists a positive integer ℓ_β such that

$$\beta \cdot D_\rho = \ell_\beta,$$

and

$$\beta \cdot D_{\rho'} = 0$$

for every one-dimensional cone ρ' of Σ distinct from ρ.

The upshot of the preceding discussion is that, for any ray $\mathfrak{d}$ with endpoint the origin and of rational slope, we have defined a subset $NE(Y)_{\mathfrak{d}}$ of $NE(Y)$.

We equip Y with the divisorial log structure defined by the normal crossing divisor D. The resulting log scheme is log smooth. As reviewed in § 2.2, integral points $p \in B(\mathbb{Z})$ of the tropicalization naturally define tangency conditions for stable log maps to Y.

For every $\beta \in NE(Y)_{\mathfrak{d}}$, let $\overline{M}_g(Y/D, \beta)$ be the moduli space of genus-g stable log maps to (Y, D), of class β, and satisfying the tangency condition $\ell_\beta m_{\mathfrak{d}} \in B(\mathbb{Z})$. By the work of Gross and

Siebert [GS13] and of Abramovich and Chen [Che14, AC14], $\overline{M}_g(Y/D, \beta)$ is a proper Deligne–Mumford stack of virtual dimension g and admits a virtual fundamental class

$$[\overline{M}_g(Y/D, \beta)]^{\mathrm{virt}} \in A_g(\overline{M}_g(Y/D, \beta), \mathbb{Q}).$$

If $\pi \colon \mathcal{C} \to \overline{M}_g(Y/D, \beta)$ is the universal curve, of relative dualizing sheaf ω_π, then the Hodge bundle

$$\mathbb{E} := \pi_* \omega_\pi$$

is a rank-g vector bundle over $\overline{M}_g(Y/D, \beta)$. Its Chern classes are classically [Mum83] called the lambda classes,

$$\lambda_j := c_j(\mathbb{E}),$$

for $j = 0, \ldots, g$. We define genus-g log Gromov–Witten invariants of (Y, D) by

$$N_{g,\beta}^{Y/D} := \int_{[\overline{M}_g(Y/D,\beta)]^{\mathrm{virt}}} (-1)^g \lambda_g \in \mathbb{Q}.$$

4.2 Definition

Using the higher-genus log Gromov–Witten invariants defined in the previous section, we can define a natural deformation of the canonical scattering diagram defined in [GHK15a, §3.1].

DEFINITION 4.1. We define $\hat{\mathfrak{D}}^{\mathrm{can}}$ as being the set of pairs $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$, where $\mathfrak{d}$ is a ray of rational slope in B with endpoint the origin, and, by denoting $\tau_{\mathfrak{d}}$ the smallest cone of Σ containing $\mathfrak{d}$, and $m_{\mathfrak{d}} \in \Lambda_{\tau_{\mathfrak{d}}}$ the primitive generator of $\mathfrak{d}$ pointing away from the origin, $\hat{H}_{\mathfrak{d}}$ is given by

$$\hat{H}_{\mathfrak{d}} := \left(\frac{i}{\hbar} \right) \sum_{\beta \in NE(Y)_{\mathfrak{d}}} \left(\sum_{g \geqslant 0} N_{g,\beta}^{Y/D} \hbar^{2g} \right) \hat{z}^{\eta(\beta) - \varphi_{\tau_{\mathfrak{d}}}(\ell_\beta m_{\mathfrak{d}})} \in \Bbbk_\hbar \widehat{[P_{\varphi_{\tau_{\mathfrak{d}}}}]}.$$

The following lemma is formally almost identical to [GHK15a, Lemma 3.5].

LEMMA 4.1. *Let J be a radical ideal of P. Suppose that the map $\eta \colon NE(Y) \to P$ satisfies the following conditions.*

- *If $\mathfrak{d}$ is contained in the interior of a two-dimensional cone of Σ, then $\eta(\beta) \in J$ for every $\beta \in NE(Y)_{\mathfrak{d}}$ such that $N_{g,\beta} \neq 0$ for some g.*
- *If $\mathfrak{d}$ is a ray ρ of Σ and $\kappa_{\rho,\varphi} \notin J$, then $\eta(\beta) \in J$ for every $\beta \in NE(Y)_{\mathfrak{d}}$ such that $N_{g,\beta} \neq 0$ for some g.*
- *For any ideal I in P of radical J, there are only finitely may classes $\beta \in NE(Y)$ such that $N_{g,\beta} \neq 0$ for some g and such that $\eta(\beta) \notin I$.*

Then $\hat{\mathfrak{D}}^{\mathrm{can}}$ is a quantum scattering diagram for the data (B, Σ), P, J and φ. Furthermore, the quantum scattering diagram $\hat{\mathfrak{D}}^{\mathrm{can}}$ has only outgoing rays.

Proof. The assumptions guarantee the finiteness requirements in the definition of a quantum scattering diagram; see §3.2. The ray $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ is outgoing because

$$\mathfrak{r}(\eta(\beta) - \varphi_{\tau_{\mathfrak{d}}}(\ell_\beta m_{\mathfrak{d}})) = -\ell_\beta m_{\mathfrak{d}} \in \mathbb{Z}_{<0} m_{\mathfrak{d}}. \qquad \square$$

LEMMA 4.2. *The classical limit of the canonical quantum scattering diagram $\hat{\mathfrak{D}}^{\mathrm{can}}$ is the canonical scattering diagram defined in [GHK15a, §3.1].*

Proof. It follows from an analogue of the cycle arguments detailed in [Bou19, Proposition 11] and [Bou18, Lemma 15], and from the log birational invariance of logarithmic Gromov–Witten invariants [AW18], that the relative genus-0 Gromov–Witten invariants of non-compact surfaces used in [GHK15a] coincide with the genus-0 log Gromov–Witten invariants $N_{0,\beta}^{Y/D}$. More precisely, given a stable log map $f \colon C \to Y$ defining a point in $\overline{M}_0(Y/D, \beta)$, we claim that no component of C is mapped inside D. Indeed, if it were not the case, one could argue at the tropical level: knowing the asymptotic behavior of the tropical map to the tropicalization B of Y imposed by the tangency condition $\ell_\beta m_{\mathfrak{d}}$, and repeatedly using the tropical balancing condition [GS13, Proposition 1.15], we would get that C needs to contain a cycle of components mapping surjectively to D, contradicting the genus-0 assumption.

By Lemma 3.1, the quantum automorphism $\hat{\Phi}_{\hat{H}_{\mathfrak{d}}}$ coincides in the classical limit $\hbar \to 0$, $q = e^{i\hbar} \to 1$ with the automorphism

$$z^p \mapsto z^p f_{\mathfrak{d}}^{\langle m(\hat{H}_{\mathfrak{d}}), \mathfrak{r}(p) \rangle}$$

of [GHK15a], where, using $\mathfrak{r}(\eta(\beta) - \varphi_{\tau_{\mathfrak{d}}}(\ell_\beta m_{\mathfrak{d}})) = -\ell_\beta m_{\mathfrak{d}}$, we have

$$f_{\mathfrak{d}} = \lim_{\hbar \to 0} \hat{f}_{\mathfrak{d}} = \lim_{\hbar \to 0} \exp\left(\left(\frac{i}{\hbar} \right) \sum_{\beta \in NE(Y)_{\mathfrak{d}}} (e^{-i\ell_\beta \hbar} - 1) \left(\sum_{g \geqslant 0} N_{g,\beta}^{Y/D} \hbar^{2g} \right) \hat{z}^{\eta(\beta) - \varphi_{\tau_{\mathfrak{d}}}(\ell_\beta m_{\mathfrak{d}})} \right)$$

that is,

$$f_{\mathfrak{d}} = \exp\left(\sum_{\beta \in NE(Y)_{\mathfrak{d}}} \ell_\beta N_{0,\beta}^{Y/D} \hat{z}^{\eta(\beta) - \varphi_{\tau_{\mathfrak{d}}}(\ell_\beta m_{\mathfrak{d}})} \right),$$

which coincides with [GHK15a, Definition 3.3]. $\square$

4.3 Consistency

The following result states that the quantum scattering diagram $\hat{\mathfrak{D}}^{\mathrm{can}}$, defined in §4.2, is consistent in the sense of §3.6.

THEOREM 4.1. *Suppose that:*

- *for any class $\beta \in NE(Y)$ such that $N_{g,\beta} \neq 0$ for some g, we have $\eta(\beta) \in J$;*
- *for any ideal I of P of radical J, there are only finitely many classes $\beta \in NE(Y)$ such that $N_{g,\beta} \neq 0$ for some g and $\eta(\beta) \notin I$;*
- *$\eta([D_\rho]) \in J$ for at least one boundary component $D_\rho \subset D$.*

Then the canonical quantum scattering diagram $\hat{\mathfrak{D}}^{\mathrm{can}}$ is consistent.

Let us review the various steps taken by [GHK15a] to prove the consistency of the canonical scattering diagram in the classical case.

Step I We can replace (Y, D) by a corner blow-up of (Y, D).

Step II Changing the monoid P.

Step III Reduction to the Gross–Siebert locus.

Step IV Pushing the singularities at infinity.

Step V $\bar{\mathfrak{D}}$ satisfies the required compatibility condition.

Step I (see [GHK15a, Proposition 3.10]) is easy in the classical case. The quantum case is similar: the scattering diagram changes only in a trivial way under corner blow-up and we will not say more.

Step II (see [GHK15a, Proposition 3.12]) is more subtle and involves some regrouping of monomials in the comparison of the broken lines for two different monoids. Exactly the same regrouping operation deals with the quantum case, too.

Step III in [GHK15a] requires an understanding of genus-0 multicover contributions of exceptional divisors of a toric model. We explain in §4.4 how the quantum analogue is obtained from the knowledge of higher-genus multicover contributions.

Step IV in [GHK15a] is the reduction of the consistency of $\mathfrak{D}^{\mathrm{can}}$ to the consistency of a scattering diagram $\nu(\mathfrak{D}^{\mathrm{can}})$ on an integral affine manifold without singularities. We explain in §§4.5, 4.7 and 4.8 how the consistency of the quantum scattering diagram $\hat{\mathfrak{D}}^{\mathrm{can}}$ can be reduced to the consistency of a quantum scattering diagram $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$ on an integral affine manifold without singularities.

Step V in [GHK15a] is the proof of consistency of $\nu(\mathfrak{D}^{\mathrm{can}})$ and ultimately relies on the main result of [GPS10]. We explain in §4.6 how its q-analogue, the consistency of $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$, ultimately relies on the main result of [Bou18].

4.4 Reduction to the Gross–Siebert locus

We start by recalling some notation from [Bou18].

Let $\mathfrak{m} = (m_1, \ldots, m_n)$ be an n-tuple of primitive non-zero vectors of $M = \mathbb{Z}^2$. We add extra rays to the fan given by the rays $-\mathbb{R}_{\geqslant 0} m_1, \ldots, -\mathbb{R}_{\geqslant 0} m_n$ such that the resulting fan defines a smooth projective toric surface $\bar{Y}_{\mathfrak{m}}$. The choice of the added rays will be irrelevant for us (ultimately because of the log birational invariance result in logarithmic Gromov–Witten theory proved in [AW18]) and so is not included in the notation. Denote by $\partial \bar{Y}_{\mathfrak{m}}$ the anticanonical toric divisor of $\bar{Y}_{\mathfrak{m}}$, and let $D_{m_1}, \ldots, D_{m_n}$ be the irreducible components of $\partial \bar{Y}_{\mathfrak{m}}$ dual to the rays $-\mathbb{R}_{\geqslant 0} m_1, \ldots, -\mathbb{R}_{\geqslant 0} m_n$.

For every $j = 1, \ldots, n$, we blow up a point x_j in general position on the toric divisor D_{m_j}. Remark that it is possible to have $\mathbb{R}_{\geqslant 0} m_j = \mathbb{R}_{\geqslant 0} m_{j'}$, and so $D_{m_j} = D_{m_{j'}}$, for $j \neq j'$, and that in this case we blow up several distinct points on the same toric divisor. We denote by $Y_{\mathfrak{m}}$ the resulting projective surface and $\pi \colon Y_{\mathfrak{m}} \to \bar{Y}_{\mathfrak{m}}$ the blow-up morphism. Let $E_j := \pi^{-1}(x_j)$ be the exceptional divisor over x_j. We denote $\partial Y_{\mathfrak{m}}$ the strict transform of $\partial \bar{Y}_{\mathfrak{m}}$.

Using Steps I and II and the deformation invariance of log Gromov–Witten invariants in log smooth families, we can make the following assumptions (see [GHK15a, Assumptions 3.13]).

- There exists an n-tuple $\mathfrak{m} = (m_1, \ldots, m_n)$ of primitive non-zero vectors of $M = \mathbb{Z}^2$ such that $(Y, D) = (Y_{\mathfrak{m}}, \partial Y_{\mathfrak{m}})$.
- The map $\eta \colon NE(Y) \to P$ is an inclusion and $P^{\times} = \{0\}$.
- There is an ample divisor H on $\overline{Y}$ such that there is a face of P whose intersection with $NE(Y)$ is the face $NE(Y) \cap (p^*H)^{\perp}$ generated by the classes $[E_j]$ of exceptional divisors. Let G be the prime monomial ideal of R generated by the complement of this face.
- $J = P - \{0\}$.

Following [GHK15a, Definition 3.14], the Gross–Siebert locus is the open torus orbit T^{gs} of the toric face $\operatorname{Spec} \Bbbk[P]/G$ of $\operatorname{Spec} \Bbbk[P]$.

PROPOSITION 4.1. *For each ray ρ of Σ, with primitive generator $m_\rho \in \Lambda_\rho$ pointing away from the origin, the Hamiltonian $\hat{H}_\rho$ attached to ρ in the scattering diagram $\hat{\mathfrak{D}}^{\mathrm{can}}$ satisfies*

$$\hat{H}_\rho = i \sum_{j, D_{m_j} = D_\rho} \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin(\ell\hbar/2)} \hat{z}^{\ell[E_j] - \ell\varphi_\rho(m_\rho)} \bmod G.$$

Proof. The only contributions to $\hat{H}_\rho \bmod G$ come from the multiple covers of the exceptional divisors E_j. The result then follows from [Bou18, Lemma 23], which relies on the study of Gromov–Witten theory of local curves done by Bryan and Pandharipande in [BP05]. $\quad\square$

PROPOSITION 4.2. *The canonical quantum scattering diagram $\hat{\mathfrak{D}}^{\mathrm{can}}$ is a scattering diagram for the data (B, Σ), P, G and φ. Concretely, for every ideal I of P of radical G, there are only finitely many rays such $(\mathfrak{d}, \hat{H}_\mathfrak{d})$ such that $\hat{H}_\mathfrak{d} \neq 0 \bmod I$.*

Proof. This follows from the argument given in the proof of [GHK15a, Corollary 3.16]. It is a geometric argument about curve classes, and the genus of the curves plays no role. $\quad\square$

PROPOSITION 4.3. *If $\hat{\mathfrak{D}}^{\mathrm{can}}$ is consistent as a quantum scattering diagram for the data (B, Σ), P, G and φ, then $\hat{\mathfrak{D}}^{\mathrm{can}}$ is consistent as a quantum scattering diagram for the data (B, Σ), P, J and φ.*

Proof. Identical to the proof of [GHK15a, Theorem 3.17]. $\quad\square$

Following [GHK15a, Remark 3.18], we denote by $E \subset P^{\mathrm{gp}}$ the sublattice generated by the face $P \setminus G$. We naturally have $T^{\mathrm{gs}} = \mathrm{Spec}\,\Bbbk[E] \subset \mathrm{Spec}\,\Bbbk[P]$. Denote $\mathfrak{m}_{P+E} = (P + E) \setminus E$. The following lemma is formally identical to [GHK15a, Lemma 3.19].

LEMMA 4.3. *If $\hat{\mathfrak{D}}^{\mathrm{can}}$, viewed as a quantum scattering diagram for the data (B, Σ), $P + E$, φ and $\mathfrak{m}_{P+E}$, is consistent, then $\hat{\mathfrak{D}}^{\mathrm{can}}$, viewed as a quantum scattering diagram for the data (B, Σ), P, φ and G, is consistent.*

Proof. Identical to the proof of [GHK15a, Lemma 3.19]. $\quad\square$

It follows that we can replace P by $P + E$, and so from now on we assume that $P^* = E$ and $G = P \setminus E$. Concretely, this means that it is enough to check the consistency of $\hat{\mathfrak{D}}^{\mathrm{can}}$ by working in rings in which the monomials $\hat{z}^{[E_j] - \varphi_\rho(m_\rho)}$ are invertible.

4.5 Pushing the singularities at infinity

We first recall the notation introduced at the beginning of [GHK15a, Step IV].

We denote by $M = \mathbb{Z}^2$ the lattice of cocharacters of the torus acting on the toric surface $(\bar{Y}_\mathfrak{m}, \partial\bar{Y}_\mathfrak{m})$. Let $(\bar{B}, \bar{\Sigma})$ be the tropicalization of $(\bar{Y}_\mathfrak{m}, \partial\bar{Y}_\mathfrak{m})$. The affine manifold $\bar{B}$ has no singularity at the origin and so is naturally isomorphic to $M_\mathbb{R} = \mathbb{R}^2$. The cone decomposition $\bar{\Sigma}$ of $M_\mathbb{R} = \mathbb{R}^2$ is simply the fan of $\bar{Y}$. Let $\bar{\varphi}$ be the single-valued $P_\mathbb{R}^{\mathrm{gp}}$-valued on $\bar{B}$ such that

$$\kappa_{\bar{\rho}, \bar{\varphi}} = \pi^*[\bar{D}_{\bar{\rho}}],$$

for every $\bar{\rho}$ one-dimensional cone of $\bar{\Sigma}$ and where $\bar{D}_{\bar{\rho}}$ is the toric divisor dual to $\bar{\rho}$. Since $\bar{\varphi}$ is single-valued and $\bar{B}$ has no singularities, the sheaf $\bar{\mathcal{P}}$, as defined in §2.2, is constant with fiber $P^{\mathrm{gp}} \oplus M$.

There is a canonical piecewise linear map $\nu\colon B \to \bar{B}$ which restricts to an integral affine isomorphism $\nu|_\sigma\colon \sigma \to \bar{\sigma}$ from each two-dimensional cone σ of Σ to the corresponding two-dimensional cone $\bar{\sigma}$ of $\bar{\Sigma}$. This map naturally identifies $B(\mathbb{Z})$ with $\bar{B}(\mathbb{Z})$. Restricted to each two-dimensional cone σ of Σ, the derivative ν_* of ν induces a identification $\Lambda_{B,\sigma} \simeq \Lambda_{\bar{B},\bar{\sigma}}$, an isomorphism of monoids

$$\tilde{\nu}_\sigma\colon P_{\varphi_\sigma} \to P_{\bar{\varphi}_{\bar{\sigma}}}$$
$$p + \varphi_\sigma(m) \mapsto p + \bar{\varphi}_{\bar{\sigma}}(\nu_*(m)),$$

for $p \in P$ and $m \in \Lambda_\sigma$, and so an identification of algebras of $\Bbbk_\hbar[P_{\varphi_\sigma}]$ and $\Bbbk_\hbar[P_{\bar{\varphi}_{\bar{\sigma}}}]$.

If ρ is a one-dimensional cone of Σ, then ν_* is only defined on the tangent space to ρ (not on the full Λ_ρ because ν is only piecewise linear) and so gives an identification

$$\tilde{\nu}_\rho\colon \{p + \varphi_\rho(m) \mid m \text{ tangent to } \rho,\ p \in P\} \to \{p + \bar{\varphi}_{\bar{\rho}}(m) \mid m \text{ tangent to } \bar{\rho},\ p \in P\}$$
$$p + \varphi_\rho(m) \mapsto p + \bar{\varphi}_{\bar{\rho}}(\nu_*(m)).$$

We define below a quantum scattering diagram $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$ for the data $(\bar{B}, \bar{\Sigma})$, P, $\bar{\varphi}$ and G.

– For every ray $(\mathfrak{d}, \hat{H}_{\mathfrak{d}})$ of $\hat{\mathfrak{D}}^{\mathrm{can}}$ contained in the interior of a two-dimensional cone of Σ, the quantum scattering diagram $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$ contains the ray

$$(\nu(\mathfrak{d}), \tilde{\nu}_{\tau_\sigma}(\hat{H}_{\mathfrak{d}})),$$

which is outgoing.

– For every ray $(\rho, \hat{H}_\rho)$, with ρ a one-dimensional cone of Σ, and so by Proposition 4.1,

$$\hat{H}_\rho = \hat{G}_\rho + i \sum_{j,\, D_{m_j} = D_\rho} \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin(\ell\hbar/2)} \hat{z}^{\ell[E_j] - \ell\varphi_\rho(m_\rho)},$$

with $\hat{G}_\rho = 0 \bmod G$, the quantum scattering diagram $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$ contains two rays:

$$(\bar{\rho}, \tilde{\nu}_{\tau_{\mathfrak{d}}}(\hat{G}_\rho)),$$

which is outgoing, and

$$\left(\bar{\rho},\ i \sum_{j,\, D_{m_j} = D_\rho} \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin(\ell\hbar/2)} \hat{z}^{\ell\bar{\varphi}(m_\rho) - \ell[E_j]}\right),$$

which is ingoing.

Remark. In going from $\hat{\mathfrak{D}}^{\mathrm{can}}$ to $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$, we invert $\hat{z}^{\ell[E_j] - \ell\bar{\varphi}_{\bar{\rho}}(m_\rho)}$, which becomes $\hat{z}^{\ell\bar{\varphi}_{\bar{\rho}}(m_\rho) - \ell[E_j]}$. This makes sense because we are assuming $P^* = E$.

4.6 Consistency of $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$

Let $\hat{\mathfrak{D}}_{\mathfrak{m}}$ be the quantum scattering diagram for the data $(\bar{B}, \bar{\Sigma})$, P, $\bar{\varphi}$ and G, having, for each one-dimensional cone $\bar{\rho}$ of $\bar{\Sigma}$, a ray $(\bar{\rho}, \hat{H}_{\bar{\rho}})$ where

$$\hat{H}_{\bar{\rho}} := i \sum_{j,\, D_{m_j} = D_\rho} \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2\sin(\ell\hbar/2)} \hat{z}^{\ell\bar{\varphi}(m_\rho) - \ell[E_j]}.$$

Writing $\ell\bar{\varphi}(m_\rho) - \ell[E_j] = (\ell m_\rho, \bar{\varphi}(\ell m_\rho) - \ell[E_j])$, it is clear that $\hat{H}_{\bar{\rho}} \in \Bbbk_\hbar\widehat{[P_{\bar{\varphi}}]}$, where the monoid

$$P_{\overline{\varphi}} = \{(m, \bar{\varphi}(m) + p) \mid m \in M, p \in P\}$$

is independent of $\bar{\rho}$.

For such quantum scattering diagram $\hat{\mathfrak{D}}$, with all Hamiltonians valued in the same ring, it makes sense to define an automorphism $\hat{\theta}_{\gamma,\hat{\mathfrak{D}}}$ of this ring, as in §3.4, but for γ an arbitrary path in $\bar{B}_0$ transverse to the rays of the diagram. By [KS06, Theorem 6], there exists another scattering diagram $S(\hat{\mathfrak{D}})$ containing $\hat{\mathfrak{D}}$, such that $S(\hat{\mathfrak{D}}) - \hat{\mathfrak{D}}$ consists only of outgoing rays and $\hat{\theta}_{\gamma,S(\hat{\mathfrak{D}})}$ is the identity for γ a loop in $\bar{B}_0$ going around the origin. We can assume that there is at most one ray of $S(\hat{\mathfrak{D}}) - \hat{\mathfrak{D}}$ in each possible outgoing direction.

The scattering diagram $S(\hat{\mathfrak{D}}_\mathfrak{m})$ is the main object of study of [Bou18].[11]

For every $m \in M - \{0\}$, let P_m be the subset of $p = (p_1, \ldots, p_n) \in \mathbb{N}^n$ such that $\sum_{j=1}^n p_j m_j$ is positively collinear with m:

$$\sum_{j=1}^n p_j m_j = \ell_p m$$

for some $\ell_p \in \mathbb{N}$. Given $p \in P_m$, we defined in [Bou18] a curve class $\beta_p \in A_1(Y, \mathbb{Z})$.

Recall that if $\mathfrak{d} \subset \bar{B}$ is a ray with endpoint the origin and rational slope, we denote by $m_\mathfrak{d} \in M$ the primitive generator of $\mathfrak{d}$ pointing away from the origin.

The following proposition expresses $S(\hat{\mathfrak{D}}_\mathfrak{m})$ in terms of the log Gromov–Witten invariants $N_{g,\beta}^{Y_\mathfrak{m}/\partial Y_\mathfrak{m}}$ defined in §4.1 and entering in the definition of $\hat{\mathfrak{D}}^{\mathrm{can}}$.

PROPOSITION 4.4. *The Hamiltonian $\hat{H}_\mathfrak{d}$ attached to an outgoing ray $\mathfrak{d}$ of $S(\hat{\mathfrak{D}}_\mathfrak{m}) - \hat{\mathfrak{D}}_\mathfrak{m}$ is given by*

$$\hat{H}_\mathfrak{d} = \left(\frac{i}{\hbar}\right) \sum_{p \in P_{m_\mathfrak{d}}} \left(\sum_{g \geqslant 0} N_{g,\beta_p}^{Y_\mathfrak{m}/\partial Y_\mathfrak{m}} \hbar^{2g}\right) \hat{z}^{(-\ell_\beta m_\mathfrak{d}, \beta_p - \bar{\varphi}(\ell_\beta m_\mathfrak{d}))},$$

where $(-\ell_\beta m_\mathfrak{d}, \beta_p - \bar{\varphi}(\ell_\beta m_\mathfrak{d})) \in P_{\bar{\varphi}}$.

Proof. This is the main result of [Bou18]. $\qquad\square$

PROPOSITION 4.5. *We have $S(\hat{\mathfrak{D}}_\mathfrak{m}) = \nu(\hat{\mathfrak{D}}^{\mathrm{can}})$.*

Proof. We compare the explicit description of $S(\hat{\mathfrak{D}}_\mathfrak{m})$ given by Proposition 4.4 with the explicit description of $S(\hat{\mathfrak{D}}_\mathfrak{m})$ obtained from its definition in §4.5 and from the definition of $\hat{\mathfrak{D}}^{\mathrm{can}}$ in §4.2.

The ingoing rays obviously coincide.

Let $\mathfrak{d}$ be an outgoing ray. The corresponding Hamiltonian in $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$ involves the log Gromov–Witten invariants $N_{g,\beta}^{Y_\mathfrak{m}/\partial Y_\mathfrak{m}}$ for

$$\beta \in NE(Y)_\mathfrak{d} \cap G,$$

whereas the corresponding Hamiltonian in $S(\hat{\mathfrak{D}}_\mathfrak{m})$ involves the log Gromov–Witten invariants $N_{g,\beta_p}^{Y_\mathfrak{m}/\partial Y_\mathfrak{m}}$ for $p \in P_{m_\mathfrak{d}}$. The only thing to show is that $N_{g,\beta}^{Y_\mathfrak{m}/\partial Y_\mathfrak{m}} = 0$ if $\beta \in NE(Y)_\mathfrak{d} \cap G$ is not of the form β_p for some $p \in P_{m_\mathfrak{d}}$.

---

[11] Comparing the conventions of the present paper and [Bou18], the notions of outgoing and ingoing rays are exchanged. This implies that a global sign must be included in comparing the Hamiltonians of the present paper and [Bou18].

Recall that we have the blow-up morphism $\pi \colon Y_{\mathfrak{m}} \to \bar{Y}_{\mathfrak{m}}$. Let $\beta \in NE(Y)_{\mathfrak{d}} \cap G$. We can uniquely write $\beta = \pi^* \pi_* \beta - \sum_{j=1}^n p_j E_j$ for some $p_j \in \mathbb{Z}$, $j = 1, \ldots, n$. If $p_j \geqslant 0$ for every $j = 1, \ldots, n$, then $p = (p_1, \ldots, p_n) \in \mathbb{N}^n$ and $\beta = \beta_p$.

Assume that there exists $1 \leqslant j \leqslant n$ such that $p_j < 0$. Then $\beta \cdot E_j = p_j < 0$ and so every stable log map $f \colon C \to Y_{\mathfrak{m}}$ of class β has a component dominating E_j. If $\mathfrak{d} \neq -\mathbb{R}_{\geqslant 0} m_j$, then we can employ an analogue of the cycle argument of [Bou19, Proposition 11] and [Bou18, Lemma 15]. Knowing the asymptotic behavior of the tropical map to the tropicalization B of $Y_{\mathfrak{m}}$, imposed by the tangency condition $\ell_\beta m_{\mathfrak{d}}$, and repeatedly using the balancing condition, we get that C needs to contain a cycle of components mapping surjectively to $\partial Y_{\mathfrak{m}}$. Vanishing properties of the lambda class (see, for example, [Bou19, Lemma 8]) then imply that $N_{g,\beta}^{Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}}} = 0$. If $\mathfrak{d} = -\mathbb{R}_{\geqslant 0} m_j$ for some j, then the same argument implies the vanishing of $N_{g,\beta}^{Y_{\mathfrak{m}}/\partial Y_{\mathfrak{m}}}$, unless β is a multiple of some E_j, which is not the case by the assumption $\beta \in G$. $\qquad\square$

The following proposition is the quantum version of [GHK15a, Theorem 3.30].

Proposition 4.6. *Let I be an ideal of P of radical G. If Q and Q' are two points in general position in $M_{\mathbb{R}} - \mathrm{Supp}(S(\hat{\mathfrak{D}}_{\mathfrak{m}}))_I$, and γ is a path connecting Q and Q' for which $\hat{\theta}_{\gamma, S(\hat{\mathfrak{D}}_{\mathfrak{m}})_I}$ is defined, then*

$$\mathrm{Lift}_{Q'}(p) = \hat{\theta}_{\gamma, S(\hat{\mathfrak{D}}_{\mathfrak{m}})_I}(\mathrm{Lift}_Q(p))$$

as elements of $\Bbbk_\hbar[P_{\bar{\varphi}}]/I$.

Proof. The key input is that, by construction, $\hat{\theta}_{\gamma, S(\hat{\mathfrak{D}}_{\mathfrak{m}})}$ is the identity for a loop γ in $\bar{B}_0$ going around the origin. Proofs of the classical statement can be found in [CPS10], [Gro11, §5.4] and §3.2 of the first arXiv version of [GHK15a]. Putting hats everywhere, the same argument proves the quantum version, without extra complication. $\qquad\square$

4.7 Comparing $\hat{\mathfrak{D}}^{\mathrm{can}}$ and $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$

In order to obtain the consistency of $\hat{\mathfrak{D}}^{\mathrm{can}}$ from some properties of $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$, we need to compare the rings $R_{\sigma,I}^\hbar$, $R_{\rho,I}^\hbar$ coming from (B, Σ), φ, with the corresponding rings $\bar{R}_{\sigma,I}^\hbar$, $\bar{R}_{\rho,I}^\hbar$ coming from $(\bar{B}, \bar{\Sigma})$, $\bar{\varphi}$. Such comparison is done in the following lemma.

Lemma 4.4. *There are isomorphisms $p_\rho \colon R_{\rho,I}^\hbar \to \bar{R}_{\bar{\rho},I}^\hbar$ and $p_\sigma \colon R_{\sigma,I}^\hbar \to \bar{R}_{\bar{\sigma},I}^\hbar$, intertwining*

- *the maps $\hat{\psi}_{\rho,-} \colon R_{\rho,I}^\hbar \to R_{\sigma_-,I}^\hbar$ and $\hat{\psi}_{\bar{\rho},-} \colon \bar{R}_{\bar{\rho},I}^\hbar \to \bar{R}_{\bar{\sigma}_-,I}^\hbar$,*
- *the maps $\hat{\psi}_{\rho,+} \colon R_{\rho,I}^\hbar \to R_{\sigma_+,I}^\hbar$ and $\hat{\psi}_{\bar{\rho},+} \colon \bar{R}_{\bar{\rho},I}^\hbar \to \bar{R}_{\bar{\sigma}_+,I}^\hbar$,*
- *the automorphisms $\hat{\theta}_{\gamma, \hat{\mathfrak{D}}^{\mathrm{can}}} \colon R_{\sigma,I}^\hbar \to R_{\sigma,I}^\hbar$ and $\hat{\theta}_{\bar{\gamma}, \nu(\hat{\mathfrak{D}}^{\mathrm{can}})} \colon \bar{R}_{\bar{\sigma},I}^\hbar \to \bar{R}_{\bar{\sigma},I}^\hbar$, where γ is a path in σ for which $\hat{\theta}_{\gamma, \mathfrak{D}^{\mathrm{can}}}$ is defined and $\bar{\gamma} = \nu \circ \gamma$.*

Proof. This is a quantum version of [GHK15a, Lemma 3.27]. The isomorphism p_σ simply comes from the isomorphism of monoids $\tilde{\nu}_\sigma \colon P_{\varphi_\sigma} \to P_{\bar{\varphi}_{\bar{\sigma}}}$.

Recall from §3.1 that the rings $R_{\rho,I}^\hbar$ and $\bar{R}_{\bar{\rho},I}^\hbar$ are generated by variables X_+, X_-, X and $\bar{X}_+$, $\bar{X}_-$, $\bar{X}$ respectively, and we define p_ρ as the morphism of $R_I^\hbar$-algebras such that $p_\rho(X_+) = \bar{X}_+$, $p_\rho(X_-) = \bar{X}_-$, $p_\rho(X) = \bar{X}$. We have to check that p_ρ is compatible with the relations defining $R_{\rho,I}^\hbar$ and $\bar{R}_{\bar{\rho},I}^\hbar$.

393

We have $\hat{f}_{\rho^{\mathrm{in}}} = 1$. Using Proposition 4.1 and Lemma 3.2, we can write

$$\hat{f}_{\rho^{\mathrm{out}}}(X) = \hat{g}_\rho(X) \prod_{j, D_{m_j} = D_\rho} (1 + q^{-1/2} \hat{z}^{[E_j]} X^{-1}),$$

for some $\hat{g}_\rho(X) = 1 \bmod G$. Using the definition of $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$ given in §4.5, and Lemma 3.2, we have

$$\hat{f}_{\bar{\rho}^{\mathrm{in}}}(\bar{X}) = \prod_{j, D_{m_j} = D_\rho} (1 + q^{-1/2} \hat{z}^{-[E_j]} \bar{X})$$

and

$$\hat{f}_{\bar{\rho}^{\mathrm{out}}}(\bar{X}) = \hat{g}_\rho(\bar{X}).$$

We need to check that

$$p_\rho(q^{(1/2)D_\rho^2} \hat{z}^{[D_\rho]} \hat{f}_{\rho^{\mathrm{in}}}(X) \hat{f}_{\rho^{\mathrm{out}}}(q^{-1}X) X^{-D_\rho^2}) = q^{(1/2)D_{\bar{\rho}}^2} \hat{z}^{[D_{\bar{\rho}}]} \hat{f}_{\bar{\rho}^{\mathrm{in}}}(\bar{X}) \hat{f}_{\bar{\rho}^{\mathrm{out}}}(q^{-1}\bar{X}) \bar{X}^{-D_{\bar{\rho}}^2}.$$

We have $D_\rho^2 = D_{\bar{\rho}}^2 - a_\rho$, where a_ρ is the number of j such that $D_{m_j} = D_\rho$, and

$$[D_\rho] = [D_{\bar{\rho}}] - \sum_{j, D_{m_j} = D_\rho} [E_j].$$

Thus, the desired identity follows from

$$(1 + q^{-1/2} \hat{z}^{[E_j]} (q^{-1}X)^{-1}) = (1 + q^{1/2} \hat{z}^{[E_j]} X^{-1}) = q^{1/2} \hat{z}^{[E_j]} X^{-1} (1 + q^{-1/2} \hat{z}^{-[E_j]} X).$$

Similarly, the relation

$$p_\rho(q^{-(1/2)D_\rho^2} \hat{z}^{[D_\rho]} \hat{f}_{\rho^{\mathrm{out}}}(X) \hat{f}_{\rho^{\mathrm{in}}}(qX) X^{-D_\rho^2}) = q^{-(1/2)D_{\bar{\rho}}^2} \hat{z}^{[D_{\bar{\rho}}]} \hat{f}_{\bar{\rho}^{\mathrm{out}}}(X) \hat{f}_{\bar{\rho}^{\mathrm{in}}}(qX) X^{-D_{\bar{\rho}}^2}$$

follows from

$$
\begin{aligned}
(1 + q^{-1/2} \hat{z}^{[E_j]} X^{-1}) &= q^{-1/2} \hat{z}^{[E_j]} X^{-1} (1 + q^{1/2} \hat{z}^{-[E_j]} X) \\
&= q^{-1/2} \hat{z}^{[E_j]} X^{-1} (1 + q^{-1/2} \hat{z}^{-[E_j]} (qX)). \qquad \square
\end{aligned}
$$

LEMMA 4.5. *The piecewise linear map $\nu \colon B \to \bar{B}$ induces a bijection between broken lines of $\hat{\mathfrak{D}}^{\mathrm{can}}$ and broken lines of $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$.*

Proof. This is a quantum version of [GHK15a, Lemma 3.28].

It is enough to compare bending and attached monomials of broken lines near a one-dimensional cone ρ of Σ. Indeed, away from such ρ, ν is linear and so the claim is obvious.

Let ρ be a one-dimensional cone of Σ. Let σ_+ and σ_- be the two-dimensional cones of Σ bounding ρ, and let ρ_+ and ρ_- be the other boundary one-dimensional cones of σ_+ and σ_- respectively, such that ρ_-, ρ and ρ_+ are in anticlockwise order. Let m_ρ be the primitive generator of ρ pointing away from the origin. We continue to use the notation introduced in the proof of Lemma 4.4.

Let γ be a quantum broken line in B_0, passing from σ_- to σ_+ across ρ. Let $c\hat{z}^s$, $s \in P_{\varphi_{\sigma_-}}$, be the monomial attached to the domain of linearity of γ preceding the crossing with ρ. Without loss of generality, we can assume $s = \varphi_{\sigma_-}(m_{\rho_-})$. Indeed, the pairing $\langle -, - \rangle$ is trivial on P, $\mathfrak{r}(s)$ is a linear combination of m_ρ and m_{ρ_-}, and $\hat{z}^{\varphi_{\sigma_-}(m_\rho)}$ transforms trivially across ρ.

394

By the definition of a quantum broken line (Definition 3.2), we have to show that

$$p_{\sigma_+}(\hat{z}^{\varphi_{\sigma_-}(m_{\rho_-})} \hat{f}_{\rho^{\mathrm{out}}}(q^{-1}X)) = \hat{z}^{\bar{\varphi}_{\sigma_-}(m_{\bar{\rho}_-})} \hat{f}_{\bar{\rho}^{\mathrm{out}}}(q^{-1}\bar{X}) \hat{f}_{\bar{\rho}^{\mathrm{in}}}(\bar{X}).$$

From the relations

$$\hat{z}^{\varphi_\rho(m_{\rho_+})} \hat{z}^{\varphi_\rho(m_{\rho_-})} = q^{(1/2)D_\rho^2} \hat{z}^{[D_\rho]} X^{-D_\rho^2}$$

in $\Bbbk[P_{\varphi_\rho}]$,

$$\hat{z}^{\bar{\varphi}_{\bar{\rho}}(m_{\rho_+})} \hat{z}^{\bar{\varphi}_{\bar{\rho}}(m_{\bar{\rho}_-})} = q^{(1/2)D_{\bar{\rho}}^2} \hat{z}^{[D_{\bar{\rho}}]} \bar{X}^{-D_{\bar{\rho}}^2}$$

in $\Bbbk[P_{\bar{\varphi}_{\bar{\rho}}}]$, and using $D_\rho^2 = D_{\bar{\rho}}^2 - a_\rho$ and $D_\rho = D_{\bar{\rho}} - \sum_{j,D_{m_j}=D_\rho}[E_j]$, we get

$$p_{\sigma_+}(\hat{z}^{\varphi_\rho(m_{\rho_-})}) = \hat{z}^{\bar{\varphi}_{\bar{\rho}}(m_{\rho_-})} \prod_{j,D_{m_j}=D_\rho} (q^{-1/2}\bar{X}\hat{z}^{-[E_j]}).$$

The result follows from the identity

$$q^{-1/2}\bar{X}\hat{z}^{-[E_j]}(1 + q^{-1/2}\hat{z}^{[E_j]}(q^{-1}\bar{X})^{-1}) = 1 + q^{-1/2}\hat{z}^{-[E_j]}\overline{X}. \qquad \square$$

LEMMA 4.6. *Let σ be a two-dimensional cone of Σ. For every $Q \in \sigma$ and every $p \in B_0(\mathbb{Z})$, we have*

$$p_\sigma(\mathrm{Lift}_Q(p)) = \mathrm{Lift}_{\nu(Q)}(\nu(p)).$$

Proof. This is a direct consequence of Lemma 4.5. $\qquad \square$

4.8 Conclusion of the proof of Theorem 4.1

This section is parallel to the end of the proof of Theorem 3.8 given at the end of [GHK15a, § 3.3]. We have to show that $\hat{\mathfrak{D}}^{\mathrm{can}}$ satisfies the two conditions in the definition of consistency of a quantum scattering diagram (Definition 3.3).

– Let Q and Q' be generic points in B_0 contained in a common two-dimensional cone σ of Σ, and let γ be a path in the interior of σ connecting Q and Q', and transversely intersecting the rays of $\hat{\mathfrak{D}}$. We have to show that

$$\mathrm{Lift}_{Q'}(p) = \hat{\theta}_{\gamma,\hat{\mathfrak{D}}^{\mathrm{can}}}(\mathrm{Lift}_Q(p)).$$

By Lemmas 4.4 and 4.6, it is enough to show that

$$\mathrm{Lift}_{\nu(Q')}(\nu(p)) = \hat{\theta}_{\nu(\gamma),\nu(\hat{\mathfrak{D}}^{\mathrm{can}})}(\mathrm{Lift}_{\nu(Q)}(\nu(p))),$$

which follows from the combination of Propositions 4.5 and 4.6.

– Let Q_- and Q_+ be two generic points in B_0, contained respectively in two-dimensional cones σ_- and σ_+ of Σ, such that σ_+ and σ_- intersect along a one-dimensional cone ρ of Σ. Assuming further that Q_- and Q_+ are contained in connected components of $B - \mathrm{Supp}_I(\hat{\mathfrak{D}})$ whose closures contain ρ, we have to show that $\mathrm{Lift}_{Q_+}(p) \in R^\hbar_{\sigma_+,I}$ and $\mathrm{Lift}_{Q_-}(p) \in R^\hbar_{\sigma_-,I}$ are both images under $\hat{\psi}_{\rho,+}$ and $\hat{\psi}_{\rho,-}$ respectively of a single element $\mathrm{Lift}_\rho(p) \in R^\hbar_{\rho,I}$. By Lemmas 4.4 and 4.6, it is enough to prove the corresponding statement after application of ν. This result follows from the combination of the remark at the end of § 3.4 and the second point of Lemma 3.4.

5. Conclusion of the proofs of the main results

In this section we finish the proof of Theorems 2.1–2.3.

We fix a Looijenga pair (Y, D). Let $\sigma_P \subset A_1(Y, \mathbb{R})$ be a strictly convex polyhedral cone containing $NE(Y)_\mathbb{R}$. Let $P := \sigma_P \cap A_1(Y, \mathbb{Z})$ be the associated monoid and let $R := \Bbbk[P]$ be the corresponding $\Bbbk$-algebra. We denote by $\eta \colon NE(Y) \to P$ the inclusion of $NE(Y)$ in P. For the maximal ideal monomial $J = \mathfrak{m}_R$ of R, the assumptions of Theorem 4.1 are satisfied and so the canonical quantum scattering diagram $\widehat{\mathfrak{D}}^{\mathrm{can}}$ constructed from (Y, D), P, η and J in § 4 is consistent.

If D has $r \geqslant 3$ irreducible components, then we can apply Theorem 3.1 to produce, for every ideal I of P of radical J, the $R_I^\hbar$-algebra $A_I^\hbar$, deformation quantization of $X_{I,\mathfrak{D}^{\mathrm{can}}}$. In § 5.1 we lift the torus action on $X_{I,\mathfrak{D}^{\mathrm{can}}}$ constructed in [GHK15a, § 5] to a torus action on $A_I^\hbar$. This finishes the proof of Theorem 2.1 if $r \geqslant 3$. In § 5.2 we finish the proof of Theorem 2.1 in general, that is, for any $r \geqslant 1$.

In § 5.3 we give an explicit description of the deformation quantization of the special fiber of the mirror family for $r = 1$ and $r = 2$. We finish the proof of Theorem 2.2 in § 5.4, and the proof of Theorem 2.3 in § 5.5.

5.1 Torus equivariance

Let I be an ideal of P of radical $J = \mathfrak{m}_R$. Recall from § 2.5 that $T^D := \mathbb{G}_m^r$ is the torus whose character group $\chi(T^D)$ has a basis e_{D_j} indexed by the irreducible components D_j of D, $1 \leqslant j \leqslant r$. The map

$$\beta \mapsto \sum_{j=1}^r (\beta \cdot D_j) e_{D_j}$$

induces an action of T^D on $S_I = \mathrm{Spec}\, R_I$.

Following [GHK15a, § 5], we consider

$$w \colon B \to \chi(T_D) \otimes \mathbb{R},$$

the unique piecewise linear map such that $w(0) = 0$ and $w(m_{\rho_j}) = e_{D_j}$ for all $1 \leqslant j \leqslant r$, where m_{ρ_j} is the primitive generator of the ray ρ_j, viewed as an element of $B_0(\mathbb{Z})$.

We assume that $r \geqslant 3$, so that $A_I^\hbar$ is defined by Theorem 3.1.

According to [GHK15a, Theorem 5.2], the T^D-action on $\mathrm{Spec}\, R_I$ has a natural lift to $X_{I,\mathfrak{D}^{\mathrm{can}}}$, such that the decomposition

$$H^0(X_{I,\mathfrak{D}^{\mathrm{can}}}, \mathcal{O}_{X_{I,\mathfrak{D}^{\mathrm{can}}}}) = A_I = \bigoplus_{p \in B(\mathbb{Z})} R_I \vartheta_p$$

as R_I-module is a weight decomposition where T^D acts on ϑ_p with weight $w(p)$.

We extend the action of T^D on R_I by $\Bbbk$-algebra automorphisms to an action of T^D on $R_I^\hbar$ by $\Bbbk_\hbar$-automorphism by assigning weight zero to $\hbar$.

PROPOSITION 5.1. *The T^D-action on A_I by $\Bbbk$-algebra automorphisms, equivariant for the structure of R_I-algebra, lifts to a T^D-action on $A_I^\hbar$ by $\Bbbk_\hbar$-automorphisms, equivariant for the structure of $R_I^\hbar$-algebra. Furthermore, the decomposition*

$$A_I^\hbar = \bigoplus_{p \in B(\mathbb{Z})} R_I^\hbar \hat{\vartheta}_p$$

as $R_I^\hbar$-module is a weight decomposition where T^D acts on $\hat{\vartheta}_p$ with weight $w(p)$.

396

Proof. This is a quantum deformation of the proof of [GHK15a, Theorem 5.2]. As

$$A_I^\hbar = \Gamma(X_{I,\mathfrak{D}^{\mathrm{can}}}^\circ, \mathcal{O}_{X_{I,\mathfrak{D}^{\mathrm{can}}}^\circ}^\hbar),$$

it is enough to define the T^D-action on $\mathcal{O}_{X_{I,\mathfrak{D}^{\mathrm{can}}}^\circ}^\hbar$.

Let ρ be a one-dimensional cone of Σ. Let σ_+ and σ_- be the two-dimensional cones of Σ bounding ρ, and let ρ_+ and ρ_- be the other boundary one-dimensional cones of σ_+ and σ_- respectively, such that ρ_-, ρ and ρ_+ are in anticlockwise order. According to §3.1, $R_{\rho,I}^\hbar$ is generated as $R_I^\hbar$-algebra by variables X_+, X_- and X. We define an action of T^D on $R_{\rho,I}^\hbar$ by lifting the action of T^D on $R_I^\hbar$ and by assigning weight $e_{D_{\rho_+}}$ to X_+, weight $e_{D_{\rho_-}}$ to X_-, and weight e_{D_ρ} to X. We have to check that this action is well defined, that is, preserves the relations between X_+, X_- and X defining $R_{\rho,I}^\hbar$.

The relation $XX_+ = qX_+X$ (respectively, $XX_- = q^{-1}X_-X$) is clearly T^D-invariant as both the left-hand side and right-hand side have weight $e_{D_\rho} + e_{D_{\rho_+}}$ (respectively, $e_{D_\rho} + e_{D_{\rho_-}}$).

Let us consider the remaining relations:

(i) $X_+X_- = q^{(1/2)D_\rho^2}\hat{z}^{[D_\rho]}\hat{f}_{\rho^{\mathrm{out}}}(q^{-1}X)\hat{f}_{\rho^{\mathrm{in}}}(X)X^{-D_\rho^2}$;

(ii) $X_-X_+ = q^{-(1/2)D_\rho^2}\hat{z}^{[D_\rho]}\hat{f}_{\rho^{\mathrm{out}}}(X)\hat{f}_{\rho^{\mathrm{in}}}(qX)X^{-D_\rho^2}$.

For the canonical quantum scattering diagram $\hat{\mathfrak{D}}^{\mathrm{can}}$, defined in Definition 4.1, we have $\hat{f}_{\rho^{\mathrm{in}}}(X) = 1$, and $\hat{f}_{\rho^{\mathrm{out}}}(X)$ is a power series of the form

$$\sum_{\beta \in NE(Y)_\rho} c_\beta(\hbar)\hat{z}^\beta X^{-\ell_\beta},$$

for some $c_\beta(\hbar) \in \mathbb{Q}[\![\hbar]\!]$. According to the definition of $NE(Y)_\rho$ (see §4.1), for $\beta \in NE(Y)_\rho$, we have $\beta \cdot D_\rho = \ell_\beta$ and $\beta \cdot D_{\rho'} = 0$ if $\rho' \neq \rho$. Thus, by the definition of the action of T^D on $R_I^\hbar$, T^D acts on $\hat{z}^\beta$ with weight $\ell_\beta e_{D_\rho}$. On the other hand, $X^{-\ell_\beta}$ has weight $-\ell_\beta e_{D_\rho}$. It follows that $\hat{f}_{\rho^{\mathrm{out}}}(X)$ has weight 0. On the other hand, by definition of the action of T^D on $R_I^\hbar$, $\hat{z}^{[D_\rho]}$ has weight

$$\sum_{j=1}^r (D_\rho \cdot D_j)e_{D_j} = e_{D_{\rho_+}} + e_{D_{\rho_-}} + D_\rho^2 e_{D_\rho}.$$

Thus, the above relations (i) and (ii) are indeed T^D-invariant: the left-hand sides (X_+X_- or X_-X_+) have weight

$$e_{D_{\rho_+}} + e_{D_{\rho_-}}$$

and the right-hand sides ($q^{(1/2)D_\rho^2}\hat{z}^{[D_\rho]}\hat{f}_{\rho^{\mathrm{out}}}(q^{-1}X)X^{-D_\rho^2}$ or $q^{-(1/2)D_\rho^2}\hat{z}^{[D_\rho]}\hat{f}_{\rho^{\mathrm{out}}}(X)X^{-D_\rho^2}$) have weight

$$(e_{D_{\rho_+}} + e_{D_{\rho_-}} + D_\rho^2 e_{D_\rho}) - D_\rho^2 e_{D_\rho}.$$

Having defined an action of T^D on the $R_I^\hbar$-algebras $R_{\rho,I}^\hbar$, in order to define an action of T^D on $\mathcal{O}_{X_{I,\mathfrak{D}^{\mathrm{can}}}^\circ}^\hbar$, it remains to check that the gluing transformations $\hat{\theta}_{\gamma_\sigma}^\hbar$ of §3.4 are T^D-equivariant. Let σ be a two-dimensional cone of Σ, bounded by rays ρ_L and ρ_R, such that ρ_L, σ, ρ_R are in anticlockwise order. It follows from Proposition 3.1 and the fact that $\hat{f}_{\rho_L^{\mathrm{out}}}$ and $\hat{f}_{\rho_R^{\mathrm{out}}}$ have weight 0, that the actions of T^D on $R_{\rho_L,I}^\hbar$ and $R_{\rho_R,I}^\hbar$ restrict to the same action

on $R_{\sigma,I}^\hbar$: for $p \in P_{\varphi_\sigma}$, one can uniquely write $p = \varphi_\sigma(\mathfrak{r}(p)) + p'$ for some $p' \in P$, one can uniquely write $\mathfrak{r}(p) = n_R m_{\rho_R} + n_L m_{\rho_L}$ with $m_L, m_R \in \mathbb{Z}$, and then T^D acts on $\hat{z}^p$ with weight $n_R e_{D_{\rho_R}} + n_L e_{D_{\rho_L}} + \sum_{j=1}^r (p' \cdot D_j) e_{D_j}$.

For every ray $\mathfrak{d}$ of $\hat{\mathfrak{D}}^{\mathrm{can}}$ contained in σ, the monomials appearing in $\hat{H}_\mathfrak{d}$ and so in $\hat{f}_\mathfrak{d}$ are of the form $\hat{z}^{\beta - \varphi_\sigma(\ell_\beta m_\mathfrak{d})}$ with $\beta \in NE(Y)_\mathfrak{d}$; see Definition 4.1. Writing $m_\mathfrak{d} = n_R m_{\rho_R} + n_L m_{\rho_L}$, we get that T^D acts on $\hat{z}^{\beta - \varphi_\sigma(\ell_\beta m_\mathfrak{d})}$ with weight

$$-\ell_\beta n_R e_{D_{\rho_R}} - \ell_\beta n_L e_{D_{\rho_L}} + \sum_{j=1}^r (\beta \cdot D_j) e_{D_j}.$$

But by definition of $NE(Y)_\mathfrak{d}$ (see § 4.1), the condition $\beta \in NE(Y)_\mathfrak{d}$ means that $\beta \cdot D_{\rho_R} = \ell_\beta n_R$, $\beta \cdot D_{\rho_L} = \ell_\beta n_L$, and $\beta \cdot D_{\rho'} = 0$ if $\rho' \neq \rho_R$ and $\rho' \neq \rho_L$. Thus, all the monomials $\hat{z}^{\beta - \varphi_\sigma(\ell_\beta m_\mathfrak{d})}$ have weight 0. It follows that the gluing transformations $\hat{\theta}_{\gamma_\sigma}^\hbar$ are T^D-equivariant.

The check that $\hat{\vartheta}_p$ is an eigenfunction of the T^D-action with weight $w(p)$ is now formally identical to the corresponding classical check given in the proof of [GHK15a, Theorem 5.2]. As the scattering automorphisms have weight 0, the weights of the monomials on the various domains of linearity of a broken line are identical and so it is enough to consider the unbounded domain of linearity. In this case, the monomial is $\hat{z}^{\varphi_{\tau_p}(p)}$, which has weight $w(p)$. □

5.2 Conclusion of the proof of Theorem 2.1

We fix a Looijenga pair (Y, D). Let $\sigma_P \subset A_1(Y, \mathbb{R})$ be a strictly convex polyhedral cone containing $NE(Y)_\mathbb{R}$. Let $P := \sigma_P \cap A_1(Y, \mathbb{Z})$ be the associated monoid and let $R := \Bbbk[P]$ be the corresponding $\Bbbk$-algebra. For the maximal ideal monomial $J = \mathfrak{m}_R$ of R, the assumptions of Theorem 4.1 are satisfied and so the canonical quantum scattering diagram $\hat{\mathfrak{D}}^{\mathrm{can}}$ is consistent.

If (Y, D) admits a toric model, then D has $r \geqslant 3$ irreducible components, and so we can apply Theorem 3.1. Combined with Proposition 5.1, this proves Theorem 2.1 in this case.

In general, it is proven in [GHK15a, § 6.2] that

$$H^0(X_I, \mathcal{O}_{X_I}) = A_I := \bigoplus_{p \in B(\mathbb{Z})} R_I \theta_p,$$

with the R_I-algebra structure determined by the classical version of the product formula given in Theorem 3.2. So Theorem 2.1 follows from the following result.

PROPOSITION 5.2. *For every monomial ideal I of R of radical $\mathfrak{m}_R$, the multiplication rule of Theorem 3.2 defines a structure of $R_I^\hbar$-algebra on the $R_I^\hbar$-module*

$$A_I^\hbar := \bigoplus_{p \in B(\mathbb{Z})} R_I^\hbar \hat{\vartheta}_p.$$

Proof. If (Y, D) admits a toric model, then D has $r \geqslant 3$ components and so the result follows from Theorem 3.2.

In general, there is a corner blow-up (Y', D') of (Y, D) admitting a toric model. The result for (Y', D') implies the result for (Y, D) as in [GHK15a, § 6.2]. □

398

5.3 Quantization of $\mathbb{V}_1$ and $\mathbb{V}_2$

By Proposition 5.2, for every monomial ideal I of R of radical $\mathfrak{m}_R$, we have a structure of $R_I^{\hbar}$-algebra on

$$A_I^{\hbar} = \bigoplus_{p \in B(\mathbb{Z})} R_I^{\hbar} \hat{\vartheta}_p.$$

In this section we explicitly describe this algebra for $I = \mathfrak{m}_R$.

In the classical limit $\hbar = 0$, we get a commutative R_I-algebra which, by [GHK15a], is the algebra of functions on the variety $\mathbb{V}_r$, where r is the number of irreducible components of D, and

- if $r \geqslant 3$, $\mathbb{V}_r$ is the r-cycle of coordinates planes in the affine space $\mathbb{A}^r$, $\mathbb{V}_r = \mathbb{A}^2_{x_1,x_2} \cup \mathbb{A}^2_{x_2,x_3} \cup \cdots \cup \mathbb{A}^2_{x_r,x_1} \subset \mathbb{A}^r_{x_1,\ldots,x_r}$;
- if $r = 2$, $\mathbb{V}_2$ is a union of two affine planes,[12]

$$\mathbb{V}_2 = \operatorname{Spec} \Bbbk[x,y,z]/(xyz - z^2),$$

  the affine cone over the union of the two rational curves $z = 0$ and $xy - z = 0$, intersecting in two points, embedded in the weighted projective plane $\mathbb{P}^{1,1,2}$;
- if $r = 1$, $\mathbb{V}_1 = \operatorname{Spec} \Bbbk[x,y,z]/(xyz - x^2 - z^3)$, the affine cone over a nodal curve embedded in the weighted projective plane $\mathbb{P}^{(3,1,2)}_{x,y,z}$.

When $r \geqslant 3$, the explicit description of $A_{\mathfrak{m}_R}^{\hbar}$ follows from the combination of § 3.5 and the beginning of the proof of Theorem 3.1: we have

$$\hat{\vartheta}_m \cdot \hat{\vartheta}_{m'} = \begin{cases} q^{(1/2)\langle m,m'\rangle} \hat{\vartheta}_{m+m'} & \text{if } m \text{ and } m' \text{ lie in a common cone of } \Sigma, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, denoting by $v_1, \ldots, v_r$ the primitive generators of the one-dimensional cones $\rho_1, \ldots, \rho_r$ of Σ, $A_{\mathfrak{m}_R}^{\hbar}$ is generated as $\Bbbk_{\hbar}$-algebra by $\hat{\vartheta}_{v_1}, \ldots, \hat{\vartheta}_{v_r}$.

For $r = 2$ and $r = 1$, computing $A_{\mathfrak{m}_R}^{\hbar}$ is slightly more subtle and the answer is given below in Propositions 5.3 and 5.4.

Both $\mathbb{V}_1$ and $\mathbb{V}_2$ are hypersurfaces in $\mathbb{A}^3_{x,y,z}$. Every hypersurface $F(x,y,z) = 0$ in $\mathbb{A}^3_{x,y,z}$ has a natural Poisson structure defined by

$$\{x,y\} = \frac{\partial F}{\partial z}, \quad \{y,z\} = \frac{\partial F}{\partial x}, \quad \{z,x\} = \frac{\partial F}{\partial y},$$

see [EG10] for example.

For $\mathbb{V}_2$ and $F(x,y,z) = z^2 - xyz$, we get

$$\{x,y\} = 2z - xy, \quad \{y,z\} = -yz, \quad \{z,x\} = -zx.$$

It follows from $\{y,z\} = -yz$ and $\{z,x\} = -zx$ that this bracket coincides with the one coming from the standard symplectic form on the two natural copies of $(\mathbb{G}_m)^2$ contained in $\mathbb{V}_2$.

For $\mathbb{V}_1$ and $F(x,y,z) = z^3 + x^2 - xyz$, we get

$$\{x,y\} = 3z^2 - xy, \quad \{y,z\} = 2x - yz, \quad \{z,x\} = -zx.$$

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[12] In [GHK15a], the description $\mathbb{V}_2 = \operatorname{Spec} \Bbbk[u,v,w]/(w^2 - u^2 v^2)$ is given. This is equivalent to our description via the change of variables $x = \sqrt{2}u$, $y = \sqrt{2}v$, $z = w + uv$.

It follows from $\{x, z\} = xz$ that the above Poisson structure is indeed the one induced by the standard symplectic form on the natural copy of $(\mathbb{G}_m)^2$ contained in $\mathbb{V}_1$.

We first explain how to recover the above Poisson brackets from the formula given by Corollary 3.1 in terms of classical broken lines. We then use the formula of Theorem 3.2 in terms of quantum broken lines to compute the q-commutators deforming these Poisson brackets.

For $\mathbb{V}_2$, the tropicalization B contains two two-dimensional cones σ_1, and σ_2, and two one-dimensional cones ρ_1 and ρ_2. Let v_1 and v_2 in $B(\mathbb{Z})$ be the primitive generators of ρ_1 and ρ_2. Cutting B along ρ_1, we can identify B with the union of two cones in $\mathbb{R}^2$. More precisely, we can find w, v_2, $w' \in \mathbb{Z}^2$ such that $\langle w, v_2 \rangle = \langle v_2, w' \rangle = 1$, and such that B can be viewed as the union of the two cones $\mathbb{R}_{\geqslant 0} w + \mathbb{R}_{\geqslant 0} v_2$ and $\mathbb{R}_{\geqslant 0} v_2 + \mathbb{R}_{\geqslant 0} w'$ with some identification of $\mathbb{R}_{\geqslant 0} w$ and $\mathbb{R}_{\geqslant 0} w'$ identifying w and w'. We have $x = \vartheta_{v_1} = \vartheta_w = \vartheta_{w'}$, $y = \vartheta_{v_2}$, $z = \vartheta_{w+v_2}$. The broken lines description of the product gives

$$xy = \vartheta_{v_1} \vartheta_{v_2} = \vartheta_{w+v_2} + \vartheta_{w'+v_2}$$

and

$$\vartheta_{w+v_2} \vartheta_{w'+v_2} = 0,$$

so $\vartheta_{w'+v_2} = xy - z$ and $(xy - z)z = 0$, which is indeed the equation defining $\mathbb{V}_2$. We have

$$\{x, y\} = \{\vartheta_{v_1}, \vartheta_{v_2}\} = \langle w, v_2 \rangle \vartheta_{w+v_2} + \langle w', v_2 \rangle \vartheta_{w'+v_2} = \vartheta_{w+v_2} - \vartheta_{w'+v_2}.$$

Using $\vartheta_{w'+v_2} = xy - z$, we get $\{x, y\} = 2z - xy$. We have

$$\{y, z\} = \{\vartheta_{v_2}, \vartheta_{w+v_2}\} = \langle v_2, w + v_2 \rangle \vartheta_{w+2v_2} = -\vartheta_{v_2} \vartheta_{w+v_2} = -yz.$$

Finally, we have

$$\{z, x\} = \langle w + v_2, w \rangle \vartheta_{2w+v_2} = -\vartheta_w \vartheta_{w+v_2} = -zx.$$

Using the formula of Theorem 3.2, we compute the q-commutators deforming the above Poisson brackets. We have

$$\hat{x}\hat{y} = \hat{\vartheta}_{v_1} \hat{\vartheta}_{v_2} = q^{1/2} \hat{\vartheta}_{w+v_2} + q^{-1/2} \hat{\vartheta}_{w'+v_2},$$

so $\hat{\vartheta}_{w'+v_2} = q^{1/2} \hat{x}\hat{y} - q\hat{z}^2$. On the other hand, we have

$$\hat{y}\hat{x} = \hat{\vartheta}_{v_2} \hat{\vartheta}_{v_1} = q^{-1/2} \hat{\vartheta}_{w'+v_2} + q^{1/2} \hat{\vartheta}_{w'+v_2},$$
$$q^{-1/2} \hat{y}\hat{x} = q^{-1} \hat{\vartheta}_{w'+v_2} + \hat{\vartheta}_{w'+v_2},$$

and so

$$q^{1/2} \hat{x}\hat{y} - q^{-1/2} \hat{y}\hat{x} = (q - q^{-1})\hat{z}^2.$$

We have

$$\hat{y}\hat{z} = \hat{\vartheta}_{v_2} \hat{\vartheta}_{w+v_2} = q^{-1/2} \hat{\vartheta}_{w+2v_2}$$

and

$$\hat{z}\hat{y} = \hat{\vartheta}_{w+v_2} \hat{\vartheta}_{v_2} = q^{1/2} \hat{\vartheta}_{w+2v_2},$$

so

$$q^{1/2} \hat{y}\hat{z} - q^{-1/2} \hat{z}\hat{y} = 0.$$

We have

$$\hat{z}\hat{x} = \hat{\vartheta}_{w+v_2} \hat{\vartheta}_w = q^{-1/2} \hat{\vartheta}_{2w+v_2}$$

and

$$\hat{x}\hat{z} = \hat{\vartheta}_w\hat{\vartheta}_{w+v_2} = q^{1/2}\hat{\vartheta}_{2w+v_2},$$

so

$$q^{1/2}\hat{z}\hat{x} - q^{-1/2}\hat{x}\hat{z} = 0.$$

Finally, we compute the q-deformation of the cubic relation $F = 0$:

$$\hat{x}\hat{y}\hat{z} = \hat{\vartheta}_w q^{-1/2}\hat{\vartheta}_{w+2v_2} = q^{1/2}\hat{z}^2.$$

In summary, we have proved the following proposition.

PROPOSITION 5.3. *The deformation quantization of* $\mathbb{V}_2$ *given by the product formula of Theorem 3.2 is the associative* $\Bbbk_\hbar$-*algebra generated by variables* $\hat{x}, \hat{y}, \hat{z}$ *and with relations*

$$q^{1/2}\hat{x}\hat{y} - q^{-1/2}\hat{y}\hat{x} = (q - q^{-1})\hat{z},$$
$$q^{1/2}\hat{y}\hat{z} - q^{-1/2}\hat{z}\hat{y} = 0,$$
$$q^{1/2}\hat{z}\hat{x} - q^{-1/2}\hat{x}\hat{z} = 0,$$
$$\hat{x}\hat{y}\hat{z} = q^{1/2}\hat{z}^2.$$

For $\mathbb{V}_1$, the tropicalization B contains one two-dimensional cone σ and one one-dimensional cone ρ. Let v in $B(\mathbb{Z})$ be the primitive generator of ρ. Cutting B along ρ, we can identify B as a quadrant in $\mathbb{R}^2$ with an identification of the two boundary rays. Denote $w = (1, 0)$ and $w' = (0, 1)$. The description of the product of classical theta functions by broken lines is given in [GHK15a, §6.2]. We have $x = \vartheta_{2w+w'}$, $y = \vartheta_v = \vartheta_w = \vartheta_{w'}$, $z = \vartheta_{w+w'}$. We have

$$\{x, y\} = \{\vartheta_{2w+w'}, \vartheta_v\} = \langle(2,1),(1,0)\rangle\vartheta_{3w+w'} + \langle(2,1),(0,1)\rangle\vartheta_{2w+2w'}$$
$$= -\vartheta_{3w+w'} + 2\vartheta_{2w+2w'}.$$

On the other hand, we have $xy = \vartheta_{3w+w'} + \vartheta_{2w+2w'}$ and $z^2 = \vartheta_{2w+2w'}$, and so $\{x, y\} = 3z^2 - xy$. We have

$$\{y, z\} = \{\vartheta_v, \vartheta_{w+w'}\} = \langle(1,0),(1,1)\rangle\vartheta_{2w+w'} + \langle(0,1),(1,1)\rangle\vartheta_{w+2w'}$$
$$= \vartheta_{2w+w'} - \vartheta_{w+2w'}.$$

On the other hand, we have $yz = \vartheta_{2w+w'} + \vartheta_{w+2w'}$ and $x = \vartheta_{2w+w'}$, and so $\{y, z\} = 2x - yz$. We have

$$\{z, x\} = \{\vartheta_{w+w'}, \vartheta_{2w+w'}\} = \langle(1,1),(2,1)\rangle\vartheta_{3w+2w'} = -\vartheta_{3w+2w'}.$$

On the other hand, we have $zx = \vartheta_{3w+2w'}$ and so $\{z, x\} = -zx$.

Using the formula of Theorem 3.2, we compute the q-commutators deforming the above Poisson brackets. We have

$$\hat{x}\hat{y} = \hat{\vartheta}_{2w+w'}\hat{\vartheta}_v = q^{-1/2}\hat{\vartheta}_{3w+w'} + q\hat{\vartheta}_{2w+2w'},$$

so

$$\hat{\vartheta}_{3w+w'} = q^{1/2}\hat{x}\hat{y} - q^{3/2}\hat{z}^2.$$

On the other hand, we have

$$\hat{y}\hat{x} = \hat{\vartheta}_v\hat{\vartheta}_{2w+w'} = q^{1/2}\hat{\vartheta}_{3w+w'} + q^{-1}\hat{\vartheta}_{2w+2w'},$$
$$q^{-1/2}\hat{y}\hat{x} = \hat{\vartheta}_{3w+w'} + q^{-3/2}\hat{z}^2,$$

401

and so
$$q^{1/2}\hat{x}\hat{y} - q^{-1/2}\hat{y}\hat{x} = (q^{3/2} - q^{-3/2})\hat{z}^2.$$

We have
$$\hat{y}\hat{z} = \hat{\vartheta}_v\hat{\vartheta}_{w+w'} = q^{1/2}\hat{\vartheta}_{2w+w'} + q^{-1/2}\hat{\vartheta}_{w+2w'},$$

so
$$\hat{\vartheta}_{w+2w'} = q^{1/2}\hat{y}\hat{z} - q\hat{x}.$$

On the other hand, we have
$$\hat{z}\hat{y} = \hat{\vartheta}_{w+w'}\hat{\vartheta}_v = q^{-1/2}\hat{\vartheta}_{2w+w'} + q^{1/2}\hat{\vartheta}_{w+2w'},$$
$$q^{-1/2}\hat{z}\hat{y} = q^{-1}\hat{x} + \hat{\vartheta}_{w+2w'},$$

and so
$$q^{1/2}\hat{y}\hat{z} - q^{-1/2}\hat{z}\hat{y} = (q - q^{-1})\hat{x}.$$

We have
$$\hat{z}\hat{x} = \hat{\vartheta}_{w+w'}\hat{\vartheta}_{2w+w'} = q^{-1/2}\hat{\vartheta}_{3w+2w'},$$
$$\hat{\vartheta}_{3w+2w'} = q^{1/2}\hat{z}\hat{x}.$$

On the other hand, we have
$$\hat{x}\hat{z} = \hat{\vartheta}_{2w+w'}\hat{\vartheta}_{w+w'} = q^{1/2}\hat{\vartheta}_{3w+2w'},$$

and so
$$q^{1/2}\hat{z}\hat{x} - q^{-1/2}\hat{x}\hat{z} = 0.$$

Finally, we compute the q-deformation of the cubic relation $F = 0$:
$$\hat{x}\hat{y}\hat{z} = \hat{\vartheta}_{2w+w'}(q^{1/2}\hat{\vartheta}_{2w+w'} + q^{-1/2}\hat{\vartheta}_{w+2w'}) = q^{1/2}\hat{\vartheta}_{2w+w'}^2 + q^{-1/2}q^{3/2}\hat{\vartheta}_{3w+3w'},$$
$$= q^{1/2}\hat{x}^2 + q\hat{z}^3.$$

In summary, we have proved the following proposition.

PROPOSITION 5.4. *The deformation quantization of $\mathbb{V}_1$ given by the product formula of Theorem 3.2 is the associative $\Bbbk_\hbar$-algebra generated by variables $\hat{x}, \hat{y}, \hat{z}$ and with relations*
$$q^{1/2}\hat{x}\hat{y} - q^{-1/2}\hat{y}\hat{x} = (q^{3/2} - q^{-3/2})\hat{z}^2,$$
$$q^{1/2}\hat{y}\hat{z} - q^{-1/2}\hat{z}\hat{y} = (q - q^{-1})\hat{x},$$
$$q^{1/2}\hat{z}\hat{x} - q^{-1/2}\hat{x}\hat{z} = 0,$$
$$\hat{x}\hat{y}\hat{z} = q^{1/2}\hat{x}^2 + q\hat{z}^3.$$

5.4 Conclusion of the proof of Theorem 2.2

In this section we finish the proof of Theorem 2.2, which is done by combination of Propositions 5.5 and 5.6. We follow [GHK15a, § 6.1].

For every monomial ideal I of P, we define the free $R_I^\hbar$-module
$$A_I^\hbar = \bigoplus_{p\in B(\mathbb{Z})} R_I^\hbar\hat{\vartheta}_p.$$

According to Proposition 5.2, if I has radical $\mathfrak{m}_R$, then there is a natural $R_I^\hbar$-algebra structure on $A_I^\hbar$.

Let $\Gamma \subset B(\mathbb{Z})$ be a finite collection of integral points such that the corresponding quantum theta functions $\hat{\vartheta}_p$ generate the $\Bbbk_\hbar$-algebra $A_{\mathfrak{m}_R}^\hbar$. Using the notation of § 5.3, we can take $\Gamma = \{v_1,\ldots,v_r\}$ if $r \geqslant 3$, $\Gamma = \{v_1, v_2, w + v_2\}$ if $r = 2$, and $\Gamma = \{v, w + w', 2w + w'\}$ if $r = 1$.

PROPOSITION 5.5. *There exists a unique minimal radical monomial ideal $J^\hbar_{\min}$ of P such that, for every monomial ideal I of P of radical containing $J^\hbar_{\min}$:*

- *there exists an $R^\hbar_I$-algebra structure on $A^\hbar_I$ such that, for every $k > 0$, the natural isomorphism of $R^\hbar_{I+\mathfrak{m}^k}$-modules $A^\hbar_I \otimes R^\hbar_{I+\mathfrak{m}^k} = A^\hbar_{I+\mathfrak{m}^k}$ is an isomorphism of $R^\hbar_{I+\mathfrak{m}^k}$-algebras;*
- *the quantum theta functions $\hat{\vartheta}_p$, $p \in \Gamma$, generate $A^\hbar_I$ as an $R^\hbar_I$-algebra.*

Proof. Follows as its classical version [GHK15a, Proposition 6.5]. □

As in [GHK15a, §6.1], the first point of Proposition 5.5 is equivalent to the fact that, for every $p_1, p_2 \in B(\mathbb{Z})$, at most finitely many terms $\hat{z}^\beta \hat{\vartheta}_p$ with $\beta \notin I$ appear in the expansion given by Theorem 3.2 for $\hat{\vartheta}_{p_1} \hat{\vartheta}_{p_2}$.

PROPOSITION 5.6. *Suppose that $F \subset \sigma_P$ is a face such that F does not contain the class of every component of D. Then $J^\hbar_{\min} \subset P - F$. If (Y, D) is positive, then $J^\hbar_{\min} = 0$.*

Proof. The proof is formally identical to the proof of its classical version, [GHK15a, Proposition 6.6]. The main input, the T^D-equivariance, is given in our case by Proposition 5.1. □

Remark. Let $J_{\min}$ be the ideal defined by [GHK15a, Proposition 6.5]. We obviously have $J_{\min} \subset J^\hbar_{\min}$, as the vanishing of all genus Gromov–Witten invariants includes the vanishing of genus-0 Gromov–Witten invariants. If (Y, D) is positive then $J_{\min} = J^\hbar_{\min} = 0$. In general, it is unclear if we always have $J_{\min} = J^\hbar_{\min}$ or if there are examples with $J_{\min} \neq J^\hbar_{\min}$. Geometrically, the question is whether or not some vanishing of genus-0 Gromov–Witten invariants implies a vanishing of all higher-genus Gromov–Witten invariants.

5.5 Conclusion of the proof of Theorem 2.3: q-integrality

The $R^\hbar_I$-algebra structure on

$$A^\hbar_I = \bigoplus_{p \in B(\mathbb{Z})} R^\hbar_I \hat{\vartheta}_p$$

is given by the product formula of Theorem 3.2,

$$\hat{\vartheta}_{p_1} \hat{\vartheta}_{p_2} = \sum_{p \in B(\mathbb{Z})} C^p_{p_1,p_2} \hat{\vartheta}_p.$$

A priori, we have $C^p_{p_1,p_2} \in R^\hbar_I = R_I[\![\hbar]\!]$. Theorem 2.3 follows from the following proposition.

PROPOSITION 5.7. *For every $p_1, p_2, p_3 \in B(\mathbb{Z})$, we have*

$$C^p_{p_1,p_2} \in R^q_I = R_I[q^{\pm 1/2}],$$

where $q = e^{i\hbar}$. More precisely, $C^p_{p_1,p_2}$ is the power series expansion around $\hbar = 0$ of a Laurent polynomial in $q^{1/2}$ after the change of variables $q = e^{i\hbar}$.

Proof. Recall that, if γ is a quantum broken line of endpoint Q in a cone τ of Σ, we write the monomial $\mathrm{Mono}(\gamma)$ attached to the segment ending at Q as

$$\mathrm{Mono}(\gamma) = c(\gamma) \hat{z}^{\varphi_\tau(s(\gamma))}$$

with $c(\gamma) \in \Bbbk_\hbar[P_{\varphi_\tau}]$ and $s(\gamma) \in \Lambda_\tau$.

403

By definition, we have

$$C^p_{p_1,p_2} = \sum_{\gamma_1,\gamma_2} c(\gamma_1)c(\gamma_2)q^{(1/2)\langle s(\gamma_1),s(\gamma_2)\rangle},$$

where the sum is over all broken lines γ_1 and γ_2, of asymptotic charges p_1 and p_2, satisfying $s(\gamma_1) + s(\gamma_2) = p$, and both ending at a given point $z \in B - \mathrm{Supp}_I(\mathfrak{D}^{\mathrm{can}})$ very close to p.

So it is enough to show that, for every γ quantum broken line of endpoint Q in a cone τ of Σ, we have $c(\gamma) \in \Bbbk_q[P_{\varphi_\tau}]$. We will show more generally that for every quantum broken line γ of $\hat{\mathfrak{D}}^{\mathrm{can}}$, and for every L domain of linearity of γ, the attached monomial $m_L = c_L \hat{z}^{p_L}$ satisfies $c_L \in \Bbbk_q$.

This is obviously true if L is the unbounded domain of linearity of γ since then $c_L = 1$. Given the formula in Definition 3.2 specifying the change of monomials when the quantum broken line bends, it is then enough to show that, for every ray $(\mathfrak{d}, \hat{H}_\mathfrak{d})$ of $\hat{\mathfrak{D}}^{\mathrm{can}}$, the corresponding $\hat{f}_\mathfrak{d}$ is in $\Bbbk_q[\widehat{P_{\varphi_{\tau_\mathfrak{d}}}}]$.

Given the argument used in [GHK15a, §6.2], we can assume that (Y, D) admits a toric model. Furthermore, by deformation invariance of log Gromov–Witten invariants in log smooth families, we can assume that (Y, D) is obtained from its toric model by blowing up distinct points, that is, that there exists $\mathfrak{m} = (m_1, \ldots, m_n)$ such that $(Y, D) = (Y_\mathfrak{m}, \partial Y_\mathfrak{m})$, as in §4.4. In §4.5 we introduced a quantum scattering diagram $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$. From the definition of $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$ and the explicit formulas given in the proof of Lemma 4.4 comparing $\hat{\mathfrak{D}}^{\mathrm{can}}$ and $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$, it is enough to prove the result for outgoing rays $\nu(\hat{\mathfrak{D}}^{\mathrm{can}})$.

By Proposition 4.5, we have $\nu(\hat{\mathfrak{D}}^{\mathrm{can}}) = S(\hat{\mathfrak{D}}_\mathfrak{m})$. So it remains to show that, for every outgoing ray $(\mathfrak{d}, \hat{H}_\mathfrak{d})$ of $S(\hat{\mathfrak{D}}_\mathfrak{m})$, the corresponding $\hat{f}_\mathfrak{d}$ is in $\Bbbk_q[\widehat{P_\varphi}]$.

By Proposition 4.4, the Hamiltonian $\hat{H}_\mathfrak{d}$ attached to an outgoing ray $\mathfrak{d}$ of $S(\hat{\mathfrak{D}}_\mathfrak{m}) - \hat{\mathfrak{D}}_\mathfrak{m}$ is given by

$$\hat{H}_\mathfrak{d} = \left(\frac{i}{\hbar}\right) \sum_{p \in P_{m_\mathfrak{d}}} \left(\sum_{g \geqslant 0} N^{Y_\mathfrak{m}/\partial Y_\mathfrak{m}}_{g,\beta_p} \hbar^{2g}\right) \hat{z}^{\beta_p - \bar\varphi(\ell_\beta m_\mathfrak{d})}.$$

According to [Bou18, Theorem 33], for every $p \in P_{m_\mathfrak{d}}$, there exists

$$\Omega^{Y_\mathfrak{m}}_p(q^{1/2}) = \sum_{j \in \mathbb{Z}} \Omega^{Y_\mathfrak{m}}_{p,j} q^{j/2} \in \mathbb{Z}[q^{\pm 1/2}]$$

such that

$$\left(\frac{i}{\hbar}\right)\left(\sum_{g \geqslant 0} N^{Y_\mathfrak{m}}_{g,p} \hbar^{2g-1}\right) = -(-1)^{\beta_p.\partial Y_\mathfrak{m}+1} \sum_{p=\ell p'} \frac{1}{\ell} \frac{1}{q^{\ell/2} - q^{-\ell/2}} \Omega^{Y_\mathfrak{m}}_{p'}(q^{\ell/2}),$$

which can be rewritten

$$\left(\frac{i}{\hbar}\right)\left(\sum_{g \geqslant 0} N^{Y_\mathfrak{m}}_{g,p} \hbar^{2g-1}\right) = \sum_{j \in \mathbb{Z}} \sum_{p=\ell p'} \frac{1}{\ell} \frac{1}{q^{\ell/2} - q^{-\ell/2}} (-1)^{\ell\beta_{p'}.\partial Y_\mathfrak{m}} \Omega^{Y_\mathfrak{m}}_{p',j} q^{j\ell/2}.$$

Using Lemma 3.2, we get that

$$\hat{f}_\mathfrak{d} = \prod_{p \in P_{m_\mathfrak{d}}} \prod_{j \in \mathbb{Z}} (1 + q^{(j-1)/2} \hat{z}^{\beta_p - \bar\varphi(\ell_\beta m_\mathfrak{d})})^{\Omega^{Y_\mathfrak{m}}_{p,j}},$$

which concludes the proof. $\qquad\square$

Remark. As the initial rays of the scattering diagram $\hat{\mathfrak{D}}_{\mathfrak{m}}$ depend on $\hbar$ through rational functions of $q^{1/2}$, it follows directly from the Kontsevich–Soibelman algorithm producing $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ that all the dependence on $\hbar$ in $S(\hat{\mathfrak{D}}_{\mathfrak{m}})$ is through rational functions of $q^{1/2}$. But we do not know of an elementary way to see directly that the functions $\hat{f}_{\mathfrak{d}}$ have coefficients which are Laurent polynomials in $q^{1/2}$ and not general rational functions in $q^{1/2}$. In the above proof of Proposition 5.7, we use [Bou18, Theorem 33], which relies on some quite deep results of [KS11].

6. Example: degree 5 del Pezzo surfaces

Let Y be a del Pezzo surface of degree 5, that is, a blow-up of $\mathbb{P}^2$ at four points in general position, and let D be an anticanonical cycle of five (-1)-curves on Y. Then (Y, D) is a positive Looijenga pair. The Looijenga pair (Y, D) is studied in [GHK15a, Examples 1.9, 3.7 and 6.12]. Remark that the interior $U = Y - D$ has topological Euler characteristic $e(U) = 2$.

Let j be an index modulo 5. We denote by D_j the components of D and ρ_j the corresponding one-dimensional cones in the tropicalization (B, Σ) of (Y, D). Let v_j be the primitive generator of ρ_j and E_j be the unique (-1)-curve in Y which is not contained in D and meets D_j transversally at one point.

The only curve classes contributing to the canonical quantum scattering diagram $\hat{\mathfrak{D}}^{\mathrm{can}}$ are multiples of some $[E_j]$, and so $\hat{\mathfrak{D}}^{\mathrm{can}}$ consists of five rays $(\rho_j, \hat{H}_{\rho_j})$. By [Bou18, Lemma 23] we have

$$\hat{H}_{\rho_j} = i \sum_{\ell \geqslant 1} \frac{1}{\ell} \frac{(-1)^{\ell-1}}{2 \sin(\ell \hbar/2)} \hat{z}^{\ell \eta([E_j]) - \ell \varphi_{\rho_j}(v_j)},$$

and so, by Lemma 3.2, the corresponding $\hat{f}_{\rho_j}$ are given by

$$\hat{f}_{\rho_j} = 1 + q^{-1/2} \hat{z}^{E_j - \varphi_{\rho_j}(v_j)}.$$

PROPOSITION 6.1. *The $\Bbbk[NE(Y)]$-algebra defined by the product formula of Theorem 3.2 is generated by the quantum theta functions $\hat{\vartheta}_{v_j}$, satisfying the relations*

$$\hat{\vartheta}_{v_{j-1}} \hat{\vartheta}_{v_{j+1}} = \hat{z}^{[D_j]}(\hat{z}^{[E_j]} + q^{1/2} \hat{\vartheta}_{v_j}),$$
$$\hat{\vartheta}_{v_{j+1}} \hat{\vartheta}_{v_{j-1}} = \hat{z}^{[D_j]}(\hat{z}^{[E_j]} + q^{-1/2} \hat{\vartheta}_{v_j}).$$

Proof. The description of quantum broken lines is identical to the description of classical broken lines given in [GHK15a, Example 3.7].

The term $\hat{z}^{[D_j]} \hat{z}^{[E_j]}$ is the coefficient of $\hat{\vartheta}_0 = 1$. The final directions of the broken lines γ_1 and γ_2 satisfy $s(\gamma_1) + s(\gamma_2) = 0$, so $\langle s(\gamma_1), s(\gamma_2) \rangle = 0$ and the quantum result is identical to the classical one.

The term $\hat{z}^{[D_j]} \hat{\vartheta}_{v_j}$ corresponds to two straight broken lines for v_{j-1} and v_{j+1}, with endpoint the point v_j of ρ_j. The corresponding extra power of q in Theorem 3.2 is $q^{\pm(1/2)\langle v_{j-1}, v_{j+1} \rangle} = q^{\pm 1/2}$. □

Remark. Setting $[E_j] = [D_j] = 0$, we recover a well-known description of the A_2 quantum $\mathcal{X}$-cluster algebra; see formula (60) in [FG09a, § 3.3].

7. Higher-genus mirror symmetry and string theory

7.1 From higher genus to quantization via Chern–Simons theory

In [Bou18, §9], we compared our enumerative interpretation of the q-refined two-dimensional Kontsevich–Soibelman scattering diagrams in terms of higher-genus log Gromov–Witten invariants of log Calabi–Yau surfaces with the physical derivation of the refined wall-crossing formula from topological string given by Cecotti and Vafa [CV09].

A parallel discussion shows that the main result of the present paper, the connection between higher-genus log Gromov–Witten invariants of log Calabi–Yau surfaces and quantization of the mirror geometry, also fits naturally into this story.

Let (Y, D) be a Looijenga pair. The complement $U := Y - D$ is a non-compact holomorphic symplectic surface admitting a Lagrangian torus fibration [Sym03]. According to the SYZ picture of mirror symmetry, the mirror of U should be obtained by taking the dual Lagrangian torus fibration, corrected by counts of holomorphic discs in U with boundary on the torus fibers. In some cases, U admits a hyperkähler metric, such that the original complex structure of U is the compatible complex structure J, and such that the SYZ fibration becomes I-holomorphic Lagrangian. Typical examples include two-dimensional Hitchin moduli spaces; see [Boa12] for a nice review. From now on, we assume that we are in such a case, and so we should be able to consider the kind of twistorial construction considered by Cecotti and Vafa.

Let (I, J, K) be a quaternionic triple of compatible complex structure, $(\omega_I, \omega_J, \omega_K)$ be the corresponding triple of real symplectic forms and $(\Omega_I, \Omega_J, \Omega_K)$ be the corresponding triple of holomorphic symplectic forms. Let $\Sigma \subset U$ be a fiber of the original SYZ fibration. It is a I-holomorphic Lagrangian subvariety of U, that is, a submanifold such that $\Omega_I|_\Sigma = 0$. It is an example of a (B, A, A)-brane in U in the sense of [KW07], that is, a complex subvariety for the complex structure I and a Lagrangian for any of the real symplectic forms $(\cos \theta) \omega_J + (\sin \theta) \omega_K$, $\theta \in \mathbb{R}$. There is in fact a twistor sphere J_ζ, $\zeta \in \mathbb{P}^1$, of compatible complex structures, such that $I = J_0$, $J = J_1$ and $K = J_i$. Let X be the non-compact Calabi–Yau 3-fold, of underlying smooth manifold $U \times \mathbb{C}^*$ and equipped with a complex structure twisted in a twistorial way, that is, such that the fiber over $\zeta \in \mathbb{C}^*$ is the complex variety (U, J_ζ). Consider $S^1 \subset \mathbb{C}^*$ and $L := \Sigma \times S^1 \subset X$.

We consider the open topological string A-model on (X, L), that is, the count of holomorphic maps $(C, \partial C) \to (X, L)$ from an open Riemann surface C to X with boundary ∂C mapping to L.[13] We restrict ourselves to open Riemann surfaces with only one boundary component. Given a class $\beta \in H_2(X, L)$, let $N_{g,\beta} \in \mathbb{Q}$ be the 'count' of holomorphic maps $\varphi : (C, \partial C) \to (X, L)$ with C a genus-g Riemann surface with one boundary component and $[\varphi(C, \partial C)] = \beta$. A holomorphic map $\varphi : (C, \partial C) \to (X, L)$ of class $\beta \in H_2(X, L)$ is a $J_{e^{i\theta}}$-holomorphic map to U, at a constant value $e^{i\theta} \in S^1$, where θ is the argument of $\int_\beta \Omega_I$.

The log Gromov–Witten invariants with insertion of a top lambda class $N_{g,\beta}$, introduced in §4, should be viewed as a rigorous definition of the open Gromov–Witten invariants in the twistorial geometry X, with boundary on a torus fiber Σ 'near infinity'. We refer to [MPT10, Lemma 7] for comparison, in the compact analogue given by K3 surfaces, between Gromov–Witten invariants of a holomorphic symplectic surface with insertion of a top lambda class and Gromov–Witten invariants of a corresponding three-dimensional twistorial geometry. The key

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[13] Usually, A-branes, that is, boundary conditions for the A-model, have to be Lagrangian submanifolds. In fact, L is not Lagrangian in X but only totally real. Combined with specific aspects of the twistorial geometry, it is probably enough to have well-defined worldsheet instanton contributions. As suggested in [CV09], it would be interesting to clarify this point.

point is that the lambda class comes from the comparison of the deformation theories of stable maps mapping to the surface or to the 3-fold.

According to Witten [Wit95], in the absence of non-constant worldsheet instantons, the effective spacetime theory of the A-model on the A-brane L is Chern–Simons theory of gauge group $U(1)$. The non-constant worldsheet instantons deform this result; see [Wit95, §4.4]. The effective spacetime theory on the A-brane L is still a $U(1)$ gauge theory but the Chern–Simons action is deformed by additional terms involving the worldsheet instantons. The genus-0 worldsheet instantons correct the classical action, whereas higher-genus worldsheet instantons give higher quantum corrections.

We now arrive at the key point: the relation between the SYZ mirror construction in terms of dual tori and the Chern–Simons story, whose quantization is supposed to be naturally related to higher-genus curves. As $L = \Sigma \times S^1$, we can adopt a Hamiltonian description where S^1 plays the role of the time direction. The key point is that the classical phase space of $U(1)$ Chern–Simons theory on $L = \Sigma \times S^1$ is the space of $U(1)$ flat connections on Σ, that is, it is exactly the dual torus of Σ used in the construction of the SYZ mirror. The genus-0 worldsheet instanton corrections to $U(1)$ Chern–Simons theory then translate into the genus-0 worldsheet instanton corrections in the SYZ construction of the mirror.

The Poisson structure on the mirror comes from the natural Poisson structure on the classical phase space of Chern–Simons theory. It is then natural to think that a quantization of the mirror should be obtained from quantization of Chern–Simons theory. Quantization of the torus of flat connections gives a quantum torus and higher-genus worldsheet instanton corrections to quantum Chern–Simons theory imply that these quantum tori should be glued together in a non-trivial way. We recover the main construction of the present paper: gluing quantum tori together using higher-genus curve counts in the gluing functions. The fact that we have been able to give a rigorous version of this construction should be viewed as a highly non-trivial mathematical check of the above string-theoretic expectations.

7.2 Quantization and higher-genus mirror symmetry

In the previous section we explained how to understand the connection between higher-genus log Gromov–Witten invariants and deformation quantization using Chern–Simons theory as an intermediate step. In this explanation, a key role is played by the non-compact Calabi–Yau 3-fold X, a partial twistor family of U.

In the present section we adopt a slightly different point of view, and we also consider a similar non-compact Calabi–Yau 3-fold on the mirror side: $Y = V \times \mathbb{C}^*$. It is natural to expect that the mirror symmetry relation between U and V lifts to a mirror symmetry relation between the Calabi–Yau 3-folds X and Y.

As explained in the previous section, the higher-genus log Gromov–Witten invariants considered in the present paper should be viewed as part of an algebraic version of the open higher-genus A-model on X. The open higher-genus A-model should be a mirror to the open higher-genus B-model on Y. We briefly explain below why the open higher-genus B-model on $Y = V \times \mathbb{C}^*$ has something to do with quantization of the holomorphic symplectic variety V.

The string field theory of the open higher-genus B-model for a single B-brane wrapping Y is the holomorphic Chern–Simons theory, of field a $(0,1)$-connection A and of action

$$S(A) = \int_Y \Omega_Y \wedge A \wedge \bar{\partial} A,$$

407

where Ω_Y is the holomorphic volume form of Y. We will be rather interested in a single B-brane wrapping a curve $\mathbb{C}_v^* := \{v\} \times \mathbb{C}^* \subset Y$, where v is a point in V. The dimensional reduction of holomorphic Chern–Simons theory to describe a B-brane wrapping a curve was first studied by Aganagic and Vafa [AV00, § 4]. Writing locally

$$\Omega_Y = dx \wedge dp \wedge \frac{dz}{z},$$

where (x, p) are local holomorphic Darboux coordinates on V near v and z is a linear coordinate along $\mathbb{C}^*$, the fields of the reduced theory on $\mathbb{C}_v^*$ are functions $(x(z, \bar{z}), p(z, \bar{z}))$, and the action is

$$S(x, p) = \int_{\mathbb{C}_v^*} \frac{dz}{z} \wedge p \wedge \bar{\partial} x.$$

A further dimensional reduction from the cylinder $\mathbb{C}_v^*$ to a real line $\mathbb{R}_t$ leads to a theory of a particle moving on V, of position $(x(t), z(t))$, of action

$$S(x, p) = \int_{\mathbb{R}_t} p(t) \, dx(t).$$

In particular, $p(t)$ and $x(t)$ are canonically conjugate variables, and in the corresponding quantum theory, obtained as dimensional reduction of the higher-genus B-model, they should become operators satisfying the canonical commutation relations $[x, p] = \hbar$. We conclude that the higher-genus B-model of the B-branes $\mathbb{C}_v^*$ should lead to a quantization of the holomorphic symplectic surface V. The same relation between the higher-genus B-model and quantization appears in [ADK$^+$06] and follow-ups.

We conclude that our main result, Theorem 1.1, should be viewed as an example of a higher-genus mirror symmetry relation.

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410

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