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Abstract

We prove that, for any small $\varepsilon > 0$, the number of irrationals among the following odd zeta values: $\zeta(3), \zeta(5), \zeta(7), \ldots, \zeta(s)$ is at least $(c_0 - \varepsilon)(s^{1/2}/(\log s)^{1/2})$, provided s is a sufficiently large odd integer with respect to ε. The constant $c_0 = 1.192507\ldots$ can be expressed in closed form. Our work improves the lower bound $2^{(1-\varepsilon)(\log s/\log\log s)}$ of the previous work of Fischler, Sprang and Zudilin. We follow the same strategy of Fischler, Sprang and Zudilin. The main new ingredient is an asymptotically optimal design for the zeros of the auxiliary rational functions, which relates to the inverse totient problem.

1. Introduction

The Riemann zeta function $\zeta(s)$ is one of the most fascinating objects in mathematics. Due to the work of Euler and Lindemann, it is well known that for any positive integer k, the Riemann zeta value $\zeta(2k)$ is a (non-zero) rational multiple of π^{2k}; therefore, is transcendental. One may want to further investigate the odd zeta values, i.e., the numbers $\zeta(2k+1)$. It is conjectured that $\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}$, but very little is known.

We mention a few works on this subject. In 1978, Apéry proved that $\zeta(3)$ is irrational [Apé79]. (See also van der Poorten's report [Poo79] and Beukers' alternative proof [Beu79]. For a survey, see [Fis04].) In 2000, Ball and Rivoal [BR01] (see also Rivoal [Riv00]) showed that for all odd integers $s \geqslant 3$, we have the following asymptotics as $s \to +\infty$:

$$\dim_{\mathbb{Q}}\left(\operatorname{Span}_{\mathbb{Q}}(1, \zeta(3), \zeta(5), \zeta(7), \ldots, \zeta(s))\right) \geqslant \frac{1+o(1)}{1+\log 2}\log s.$$

The proof of Ball and Rivoal makes use of Nesterenko's linear independence criterion [Nes85] and the following auxiliary rational functions:

$$R_n^{(\mathrm{BR})}(t) = n!^{s-2r}\frac{\prod_{j=0}^{(2r+1)n}(t-rn+j)}{\prod_{j=0}^{n}(t+j)^{s+1}}.$$

As a corollary, there are infinitely many irrational numbers among odd zeta values. In 2018, Zudilin [Zud18] studied the following rational functions (with $s = 25$):

$$R_n^{(\mathrm{Z})}(t) = 2^{6n} n!^{s-5}\frac{\prod_{j=0}^{6n}(t-n+j/2)}{\prod_{j=0}^{n}(t+j)^{s+1}}$$

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and proved that both series $\sum_{t=1}^{\infty} R_n^{(\mathrm{Z})}(t)$ and $\sum_{t=1}^{\infty} R_n^{(\mathrm{Z})}(t+\frac{1}{2})$ are $\mathbb{Q}$-linear combinations of odd zeta values with related coefficients; it provides a new elimination procedure. Zudilin's new idea inspired many works afterwards (see Sprang [Spr18] and Fischler [Fis20]). Based on further developments in [Zud18] and an important arithmetic observation of Sprang [Spr18, Lemma 1.4], in 2018, Fischler, Sprang and Zudilin [FSZ19] proved for all $\varepsilon > 0$, for any odd integer s which is sufficiently large with respect to ε, with the help of the following rational functions:

$$R_n^{(\mathrm{FSZ})}(t) = D^{3Dn} n!^{s+1-3D} \frac{\prod_{j=0}^{3Dn}(t-n+j/D)}{\prod_{j=0}^{n}(t+j)^{s+1}},$$

that the number of irrationals in the set $\{\zeta(3), \zeta(5), \zeta(7), \ldots, \zeta(s)\}$ is at least $2^{(1-\varepsilon)(\log s/\log\log s)}$. (See also [FSZ18].)

In the current work, we prove the following result.

Theorem 1.1. *For any small $\varepsilon > 0$ and for all odd integers s sufficiently large with respect to ε, there are at least*

$$(c_0 - \varepsilon)\, \frac{s^{1/2}}{\log^{1/2} s}$$

irrational numbers among $\{\zeta(3), \zeta(5), \zeta(7), \ldots, \zeta(s)\}$, where the constant

$$c_0 = \sqrt{\frac{4\zeta(2)\zeta(3)}{\zeta(6)}\left(1 - \log\frac{\sqrt{4e^2+1}-1}{2}\right)} = 1.192507\ldots.$$

The proof is a natural extension to the original ideas of Zudilin [Zud18] and Sprang [Spr18]. Our main strategies are exactly the same as [FSZ19], though an amount of small technical modifications is involved. The major new ingredient of our work is an asymptotically optimal design for the rational zeros of the auxiliary rational functions. In an earlier version of this note,[1] we made use of the rational functions

$$\left(\prod_{d=1}^{D} d^{3dn+1}\right) n!^{s+D-(3/2)D(D+1)} \frac{\prod_{d=1}^{D}\prod_{j=0}^{3dn}(t-n+j/d)}{\prod_{j=0}^{n}(t+j)^{s+D}}$$

and proved that the number of irrationals in the set $\{\zeta(3), \zeta(5), \ldots, \zeta(s)\}$ is at least $\frac{1}{10}s^{1/2}/\log^{1/2} s$ for any odd integer $s \geqslant 10^4$. In order to improve the constant $\frac{1}{10}$, we need to consider more technical modifications; see Definition 2.3. Since our constructions inherit the spirit of $R_n^{(\mathrm{FSZ})}(t)$, we refer to them (and a class of similar constructions) as FSZ constructions. It turns out that the optimal design for the zeros of FSZ constructions is in connection with the inverse totient problem.

The structure of this note is as follows: in § 2, we introduce the auxiliary rational functions $R_n(t)$ and related linear forms. In § 3, we study the arithmetic of the denominators appearing in the linear forms. In § 4, we bound the growth of the linear forms. In § 5, we prove Theorem 1.1. Finally, in § 6, we show that under certain constraints, the FSZ auxiliary functions constructed in this note are the most economical ones (up to an $o(1)$ term).

[1] https://arxiv.org/abs/1911.08458v1.

2. Auxiliary functions and linear forms

Let $r = \operatorname{num}(r)/\operatorname{den}(r)$ be a positive rational number, where $\operatorname{num}(r)$ and $\operatorname{den}(r)$ are the numerator and denominator of r in reduced form, respectively. We refer to r as the Ball–Rivoal length parameter [BR01]. Eventually we will take the rational number r arbitrarily close to $r_0 = (\sqrt{4e^2+1}-1)/2 \approx 2.26388$ in order to maximize a certain quantity.

Let s be a positive odd integer and B be a positive real number. We will always assume that:

(1) both s and B are larger than some absolute constant;
(2) $s \geqslant 10(2r+1)B^2$.

Eventually we will take $B = cs^{1/2}/\log^{1/2} s$ for some constant c.

DEFINITION 2.1. We define the following two sets which depend only on B:

(1) the *denominator set*

$$\Psi_B = \{b \in \mathbb{N} \mid \varphi(b) \leqslant B\},$$

where $\varphi(\cdot)$ is the Euler totient function;

(2) the *zero set*

$$\mathcal{F}_B = \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; b \in \Psi_B, 1 \leqslant a \leqslant b \text{ and } \gcd(a,b) = 1 \right\}.$$

The zero set $\mathcal{F}_B$ consists of the zeros in the interval $(0,1]$ of our auxiliary rational functions, and the denominator set Ψ_B consists of different denominators of the zeros. We collect some properties of these two sets. The first property is known in the topic about the inverse totient problem.

PROPOSITION 2.2.

(1) *The size of the set* Ψ_B *is*

$$|\Psi_B| = \left(\frac{\zeta(2)\zeta(3)}{\zeta(6)} + o_{B\to+\infty}(1) \right) B.$$

(2) *For any* $b \in \Psi_B$, *we have*

$$\left\{ \frac{1}{b}, \frac{2}{b}, \ldots, \frac{b}{b} \right\} \subset \mathcal{F}_B.$$

(3) *If* B *is larger than some absolute constant, then*

$$|\mathcal{F}_B| \leqslant B^2.$$

Proof. For the first proposition, we refer the reader to [Dre70] or [Bat72]. Since

$$|\mathcal{F}_B| = \sum_{b\in\Psi_B} \varphi(b) = \sum_{m=1}^{\lfloor B \rfloor} m\left(|\Psi_m| - |\Psi_{m-1}|\right) = \lfloor B \rfloor \left|\Psi_{\lfloor B \rfloor}\right| - \sum_{m=1}^{\lfloor B \rfloor -1} |\Psi_m|,$$

the first proposition implies that $|\mathcal{F}_B| = (\frac{1}{2}(\zeta(2)\zeta(3)/\zeta(6)) + o(1))B^2$. Now, $\zeta(2)\zeta(3)/\zeta(6) = 1.94\ldots < 2$ and the third proposition follows. For the second proposition, note that if $b \in \Psi_B$ and b' is any divisor of b, then $\varphi(b') \leqslant \varphi(b) \leqslant B$, so $b' \in \Psi_B$. Therefore, for any $k \in \{1,2,\ldots,b\}$, we have $k/b = (k/\gcd(k,b))/(b/\gcd(k,b)) \in \mathcal{F}_B$. □

We define the integer

$$P_{B,\mathrm{den}(r)} = 2\,\mathrm{den}(r)\cdot \mathrm{LCM}_{\substack{b\in\Psi_B\\ p\mid b}}\{p-1\}, \tag{2.1}$$

where LCM means taking the least common multiple. As a convention, the letter p always denotes prime numbers.

For given r, s and B, we define the following auxiliary rational functions.

Definition 2.3 (FSZ constructions). For any positive integer n which is a multiple of $P_{B,\mathrm{den}(r)}$, we define the rational function

$$R_n(t) = A_1(B)^n A_2(B)^n \frac{n!^{s+1}}{(n/\mathrm{den}(r))!^{\;\mathrm{den}(r)(2r+1)|\mathcal{F}_B|}} \frac{(t-rn)\prod_{\theta\in\mathcal{F}_B}\prod_{j=0}^{(2r+1)n-1}(t-rn+j+\theta)}{\prod_{j=0}^{n}(t+j)^{s+1}},$$

where

$$A_1(B) = \prod_{b\in\Psi_B} b^{(2r+1)\varphi(b)},$$

we refer to $A_1(B)^n$ the *major arithmetic (wasting) factor*, and

$$A_2(B) = \prod_{b\in\Psi_B}\prod_{p\mid b} p^{(2r+1)\varphi(b)/(p-1)},$$

we refer to $A_2(B)^n$ the *minor arithmetic (wasting) factor*.

Notice that by (2.1), both $A_1(B)^n$ and $A_2(B)^n$ are integers; also, $n/\mathrm{den}(r)$, rn and $(2r+1)n$ are integers. The factors $A_1(B)^n$ and $A_2(B)^n$ are technical: approximately speaking, they are designed to remedy the arithmetic loss from the denominators of rational zeros (it will be clear later in § 3). In the following lemma we estimate $A_1(B)$ and $A_2(B)$.

Lemma 2.4. *We have*

$$A_1(B) = \exp\left(\left(\frac{1}{2}\frac{\zeta(2)\zeta(3)}{\zeta(6)} + o_{B\to+\infty}(1)\right)(2r+1)\,B^2\log B\right)$$

and, for any B larger than some absolute constant,

$$A_2(B) \leqslant \exp\left(10(2r+1)B^2(\log\log B)^2\right).$$

Proof. We start by $\log A_1(B) = (2r+1)\sum_{b\in\Psi_B}\varphi(b)\log b$. Firstly,

$$\begin{aligned}\log A_1(B) &\geqslant (2r+1)\sum_{b\in\Psi_B}\varphi(b)\log\varphi(b)\\ &= (2r+1)\int_{1^-}^{B} x\log x\, d|\Psi_x|;\end{aligned}$$

an integration by parts argument with the fact that $|\Psi_x| = ((\zeta(2)\zeta(3)/\zeta(6)) + o_{x\to+\infty}(1))x$ (see Proposition 2.2(1)) gives $\log A_1(B) \geqslant (2r+1)(\frac{1}{2}(\zeta(2)\zeta(3)/\zeta(6)) + o_{B\to+\infty}(1))B^2\log B$. On the other hand, it is well known (see, for instance, [MV06, Theorem 2.9]) that

$$\varphi(m) \geqslant (e^{-\gamma} + o_{m\to+\infty}(1))\frac{m}{\log\log m},$$

where $\gamma = 0.577\ldots$ is Euler's constant. For any $b \in \Psi_B$, since $\varphi(b) \leqslant B$, we derive that

$$b \leqslant (e^{\gamma} + o_{B\to+\infty}(1))B\log\log B; \tag{2.2}$$

thus, $\log A_1(B) \leqslant (2r+1)(1+o_{B\to+\infty}(1))\log B\sum_{b\in\Psi_B}\varphi(b)$ and a summation by parts argument as above gives $\log A_1(B) \leqslant (2r+1)(\frac{1}{2}(\zeta(2)\zeta(3)/\zeta(6)) + o_{B\to+\infty}(1))B^2\log B$. Combining the two parts, we obtain the estimate for $A_1(B)$.

Now, for $A_2(B)$, by (2.2) and $e^{\gamma} = 1.78\ldots < 2$, when B is larger than some absolute constant, we have

$$\begin{aligned}\log A_2(B) &\leqslant (2r+1)\sum_{b\leqslant 2B\log\log B}\varphi(b)\sum_{p\mid b}\frac{\log p}{p-1}\\ &= (2r+1)\sum_{p\leqslant 2B\log\log B}\frac{\log p}{p-1}\sum_{\substack{b\leqslant 2B\log\log B\\ p\mid b}}\varphi(b).\end{aligned}$$

Since $\sum_{\substack{b\leqslant 2B\log\log B\\ p\mid b}}\varphi(b) \leqslant \sum_{\substack{b\leqslant 2B\log\log B\\ p\mid b}} b \leqslant 4B^2(\log\log B)^2/p$, it implies that

$$\log A_2(B) \leqslant 4(2r+1)B^2(\log\log B)^2\sum_p\frac{\log p}{p(p-1)}.$$

Because $\sum_p((\log p)/p(p-1)) \leqslant \sum_{p\leqslant 5}((\log p)/p(p-1)) + \sum_{p\geqslant 7}(1/p^{1.5}) \leqslant 2$, the estimate for $A_2(B)$ follows. □

We proceed to construct linear forms in Hurwitz zeta values. Since the numerator and denominator of $R_n(t)$ have a common factor $\prod_{j=0}^{n}(t+j)$, it can be rewritten as $R_n(t) = Q_n(t)/\prod_{j=0}^{n}(t+j)^s$, where $Q_n(t)$ is a polynomial in t with rational coefficients. Since $\deg R_n < 0$ (see below), we know that $R_n(t)$ has a (unique) partial fraction expansion

$$R_n(t) = \sum_{i=1}^{s}\sum_{k=0}^{n}\frac{a_{i,k}}{(t+k)^i} \tag{2.3}$$

with coefficients $a_{i,k} \in \mathbb{Q}$. Note that these coefficients $a_{i,k}$ also depend on n, r, s and B.

We list two properties of $R_n(t)$ and $a_{i,k}$ which will be used later in Lemma 2.5.

(1) As a rational function, the degree of $R_n(t)$ is

$$\deg R_n = 1 + (2r+1)|\mathcal{F}_B|n - (s+1)(n+1) \leqslant -2.$$

This is due to $|\mathcal{F}_B| \leqslant B^2$ and $s \geqslant 10(2r+1)B^2$.

(2) The auxiliary function $R_n(t)$ has the following symmetry:

$$R_n(-t-n) = -R_n(t).$$

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In fact, if we substitute t by $-t-n$ in the definition of $R_n(t)$ (see Definition 2.3), the factor

$$\frac{(t-rn)\prod_{\theta\in\mathcal{F}_B}\prod_{j=0}^{(2r+1)n-1}(t-rn+j+\theta)}{\prod_{j=0}^{n}(t+j)^{s+1}}$$

becomes

$$\begin{aligned}
&\frac{(-t-(r+1)n)\prod_{\theta\in\mathcal{F}_B}\prod_{j=0}^{(2r+1)n-1}(-t-(r+1)n+j+\theta)}{\prod_{j=0}^{n}(-t-n+j)^{s+1}}\\
&\quad=(-1)^{1+(2r+1)n|\mathcal{F}_B|-(s+1)(n+1)}\times\frac{(t+(r+1)n)\prod_{\theta\in\mathcal{F}_B}\prod_{j=0}^{(2r+1)n-1}(t+(r+1)n-j-\theta)}{\prod_{j=0}^{n}(t+n-j)^{s+1}}\\
&\quad=(-1)\times\prod_{j=-rn}^{(r+1)n}(t+j)\times\frac{\prod_{\theta\in\mathcal{F}_B\setminus\{1\}}\prod_{j=0}^{(2r+1)n-1}(t-rn+j+1-\theta)}{\prod_{j=0}^{n}(t+j)^{s+1}}.
\end{aligned}$$

We have used the facts that $(2r+1)n$ is even and s is odd in the above computation. By observing that

$$\{1-\theta\mid\theta\in\mathcal{F}_B\setminus\{1\}\}=\mathcal{F}_B\setminus\{1\},$$

we obtain $R_n(-t-n)=-R_n(t)$. In particular, since the partial fraction expansion for $R_n(t)$ is unique, we derive that

$$(-1)^i a_{i,k}=-a_{i,n-k}$$

for all $1\leqslant i\leqslant s$, $0\leqslant k\leqslant n$.

For all $\theta\in\mathcal{F}_B$, we define the following quantities:

$$r_{n,\theta}=\sum_{m=1}^{\infty}R_n(m+\theta). \tag{2.4}$$

The notation $r_{n,\theta}$ is adopted to keep pace with that in [FSZ19]. There is no risk for it to be confused with the Ball–Rivoal length parameter r.

We recall the definition of the Hurwitz zeta values:

$$\zeta(i,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^i},$$

where $i\geqslant 2$ is an integer and α is a positive real number.

The following lemma is the same as [FSZ19, Lemma 1] (for a proof, see therein).

Lemma 2.5 (Linear forms). *For all $\theta\in\mathcal{F}_B$, we have*

$$r_{n,\theta}=\rho_{0,\theta}+\sum_{\substack{3\leqslant i\leqslant s\\ i\ \text{odd}}}\rho_i\zeta(i,\theta),$$

where the rational coefficient

$$\rho_i=\sum_{k=0}^{n}a_{i,k}\quad\text{for } 3\leqslant i\leqslant s, i\text{ odd},$$

does not depend on $\theta \in \mathcal{F}_B$, *and*

$$\rho_{0,\theta} = -\sum_{k=0}^{n}\sum_{\ell=0}^{k}\sum_{i=1}^{s}\frac{a_{i,k}}{(\ell+\theta)^i}.$$

3. Arithmetic lemma

The following proposition is elementary; we omit the proof.

PROPOSITION 3.1. *Let* $L \in \mathbb{N} \cup \{0\}$. *Suppose that* $x_1, x_2, \ldots, x_L$ *are any* L *consecutive terms in an integer arithmetic progression with common difference* $b \in \mathbb{N}$; *then, for any prime* $q \nmid b$, *we have*

$$v_q(x_1x_2\ldots x_L) \geqslant \sum_{i=1}^{\infty}\left\lfloor \frac{L}{q^i}\right\rfloor.$$

In the degenerate case of $L = 0$, *we view* $x_1x_2\ldots x_L = 1$.

For any $a/b \in \mathcal{F}_B$ with $\gcd(a,b) = 1$, we define the following polynomials:

$$\begin{aligned} F_{b,a}(t) &= \frac{\prod_{p\mid b} p^{(2r+1)n/(p-1)}}{(n/\mathrm{den}(r))!^{\,\mathrm{den}(r)(2r+1)}}\cdot b^{(2r+1)n}\prod_{j=0}^{(2r+1)n-1}\left(t - rn + j + \frac{a}{b}\right) \\ &= \frac{\prod_{p\mid b} p^{(2r+1)n/(p-1)}}{(n/\mathrm{den}(r))!^{\,\mathrm{den}(r)(2r+1)}}\prod_{j=0}^{(2r+1)n-1}(bt - brn + a + bj)\,. \end{aligned} \tag{3.1}$$

Then we define

$$\widetilde{F}_{b,a}(t) = \begin{cases} F_{b,a}(t) & \text{if } a/b \neq 1, \\ (t - rn)F_{1,1}(t) & \text{if } a/b = 1. \end{cases}$$

Notice that since $n \in P_{B,\mathrm{den}(r)}\mathbb{N}$, by (2.1), all of $(2r+1)n/(p-1)$, $n/\mathrm{den}(r)$, rn and $(2r+1)n$ are integers. By Definition 2.3, we have

$$R_n(t) = n!^{s+1}\frac{\prod_{a/b\in\mathcal{F}_B}\widetilde{F}_{b,a}(t)}{\prod_{j=0}^{n}(t+j)^{s+1}}. \tag{3.2}$$

For a formal series $U(t) = \sum_{\ell=0}^{\infty} u_\ell t^\ell \in \mathbb{Q}[[t]]$, we denote by $[t^\ell](U(t))$ the ℓth coefficient of $U(t)$, i.e., $[t^\ell](U(t)) = u_\ell$.

As usual, we denote by $d_n = \mathrm{LCM}\{1, 2, \ldots, n\}$ the least common multiple of the first n positive integers. By the prime number theorem, we have $\lim_{n\to+\infty} d_n^{1/n} = e$. We first establish the following arithmetic property of $\widetilde{F}_{b,a}(t)$.

PROPOSITION 3.2. *For any non-negative integers* ℓ *and* k, *we have*

$$d_n^\ell \cdot [t^\ell](\widetilde{F}_{b,a}(t-k)) \in \mathbb{Z}.$$

Proof. Note that we only need to prove the proposition with $\widetilde{F}_{b,a}$ replaced by $F_{b,a}$. If $\ell > \deg F_{b,a} = (2r+1)n$, the proposition trivially holds. In the rest of the proof, we assume that $\ell \leqslant \deg F_{b,a}$.

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For a prime $q \mid b$, the q-adic order of the factor $\prod_{p|b} p^{(2r+1)n/(p-1)}/(n/\mathrm{den}(r))!^{\mathrm{den}(r)(2r+1)}$ is non-negative (recall that $v_q(m!) \leqslant m/(q-1)$). So, by (3.1), the q-adic order of every coefficient of $F_{b,a}(t-k)$ is non-negative. Therefore, for any prime $q \mid b$, we have $v_q(d_n^{\ell} \cdot [t^{\ell}](F_{b,a}(t-k))) \geqslant 0$.

Now consider a prime $q \nmid b$. Notice that $[t^{\ell}](\prod_{j=0}^{(2r+1)n-1}(b(t-k)-brn+a+bj))$ is a sum of finitely many terms all of the form

$$b^{\ell} \prod_{i=1}^{\ell+1} \prod_{j \in J_i} (-bk - brn + a + bj)\,, \tag{3.3}$$

where J_i is a set consisting of $L_i \in \mathbb{N} \cup \{0\}$ consecutive integers such that $L_1 + L_2 + \cdots + L_{\ell+1} = (2r+1)n - \ell$. By Proposition 3.1, we derive that the q-adic order of the expression (3.3) is

$$v_q(\text{Equation (3.3)}) \geqslant \sum_{i=1}^{\infty} \sum_{j=1}^{\ell+1} \left\lfloor \frac{L_j}{q^i} \right\rfloor.$$

For a fixed $i \geqslant 1$, we have

$$\sum_{j=1}^{\ell+1} \left\lfloor \frac{L_j}{q^i} \right\rfloor \geqslant \sum_{j=1}^{\ell+1} \frac{L_j - (q^i - 1)}{q^i} = \frac{(2r+1)n+1}{q^i} - \ell - 1 > \left\lfloor \frac{(2r+1)n}{q^i} \right\rfloor - \ell - 1,$$

but the left-hand side is a non-negative integer, so we obtain that $\sum_{j=1}^{\ell+1} \lfloor L_j/q^i \rfloor \geqslant \max(0, \lfloor (2r+1)n/q^i \rfloor - \ell)$. Therefore,

$$\begin{aligned} v_q(\text{Equation (3.3)}) &\geqslant \sum_{i=1}^{\lfloor \log_q n \rfloor} \left(\left\lfloor \frac{(2r+1)n}{q^i} \right\rfloor - \ell \right) \\ &\geqslant \sum_{i=1}^{\lfloor \log_q n \rfloor} \left(\mathrm{den}(r)(2r+1) \left\lfloor \frac{n/\mathrm{den}(r)}{q^i} \right\rfloor - \ell \right) \\ &= v_q\left(\left(\frac{n}{\mathrm{den}(r)} \right)!^{\,\mathrm{den}(r)(2r+1)} \right) - \ell v_q(d_n). \end{aligned} \tag{3.4}$$

(The non-trivial part is for cases $q \leqslant n$; for $q > n$, the above derivation is also valid but degenerates to trivial results.) In conclusion, for any prime $q \nmid b$, by (3.1) and inequality (3.4), we find that $d_n^{\ell} \cdot [t^{\ell}](F_{b,a}(t-k))$ is a sum of finitely many terms, each of these terms having non-negative q-adic order; this completes the proof of Proposition 3.2. □

We prove the following arithmetic lemma, which corresponds to [FSZ19, Lemma 2]. In our situation, the Ball–Rivoal length parameter r is just a rational number (not necessarily an integer or a half integer), so we have to modify the proof for $d_n^{s+1-i} a_{i,k} \in \mathbb{Z}$, but the rest of proof is the same as [FSZ19, Lemma 2].

LEMMA 3.3 (Arithmetic lemma). *We have*

$$d_n^{s+1-i} \rho_i \in \mathbb{Z}$$

for all odd integers i with $3 \leqslant i \leqslant s$, and we have

$$d_{n+1}^{s+1} \rho_{0,\theta} \in \mathbb{Z}$$

for all $\theta \in \mathcal{F}_B$.

Proof. For any $k \in \{0,1,\ldots,n\}$ and any $i \in \{1,2,\ldots,s\}$, by comparing (3.2) with the partial fraction expansion (2.3) of $R_n(t)$, and by viewing $t^{s+1}R_n(t-k) \in \mathbb{Q}[[t]]$ as a formal series, we have

$$\begin{aligned} a_{i,k} &= [t^{s+1-i}]\big(t^{s+1}R_n(t-k)\big) \\ &= (-1)^{(s+1)k}\frac{n!^{s+1}}{k!^{s+1}(n-k)!^{s+1}}[t^{s+1-i}]\Bigg(\prod_{a/b\in\mathcal{F}_B}\widetilde{F}_{b,a}(t-k)\prod_{\substack{0\leqslant j\leqslant n\\ j\neq k}}(1+\frac{t}{j-k})^{-s-1}\Bigg) \\ &= \binom{n}{k}^{s+1}\sum_{\substack{\underline{\ell} \\ \mathrm{sum}(\underline{\ell})=s+1-i}}\prod_{a/b\in\mathcal{F}_B}[t^{\ell_{b,a}}](\widetilde{F}_{b,a}(t-k))\prod_{\substack{0\leqslant j\leqslant n\\ j\neq k}}\frac{(-1)^{\ell_j}\binom{s+\ell_j}{\ell_j}}{(j-k)^{\ell_j}}, \end{aligned}$$

where the sum is taken for all tuples $\underline{\ell}$ consisting of non-negative integers $\ell_{b,a}$ and ℓ_j such that

$$\mathrm{sum}(\underline{\ell}) = \sum_{a/b\in\mathcal{F}_B}\ell_{b,a} + \sum_{\substack{0\leqslant j\leqslant n\\ j\neq k}}\ell_j = s+1-i.$$

By Proposition 3.2 and the fact that $d_n^{\ell_j}(1/(j-k)^{\ell_j}) \in \mathbb{Z}$, we derive that

$$d_n^{s+1-i}a_{i,k} \in \mathbb{Z}.$$

Once $d_n^{s+1-i}a_{i,k} \in \mathbb{Z}$ is established, the rest of the arguments are the same as [FSZ19, Lemma 2]. We only outline the proof as follows: $d_n^{s+1-i}\rho_i \in \mathbb{Z}$ follows immediately from the expression of ρ_i. The proof for $d_{n+1}^{s+1}\rho_{0,\theta} \in \mathbb{Z}$ is more involved; it is proved by showing that

$$\sum_{i=1}^{s}\frac{d_{n+1}^{s+1}a_{i,k}}{(\ell+\theta)^i}$$

is an integer for any $0 \leqslant \ell \leqslant k \leqslant n$ and $\theta \in \mathcal{F}_B$. In order to do this, we will use the fact that $R_n(t)$ has zeros $-n+\theta, -n+1+\theta, -n+2+\theta, \ldots, \theta$ for $\theta \in \mathcal{F}_B \setminus \{1\}$. □

4. Analysis lemma

Under our assumptions that $s \geqslant 10(2r+1)B^2$ and B is larger than some absolute constant, we have the following result.

LEMMA 4.1 (Analysis lemma). *We have*

$$\lim_{n\to+\infty}(r_{n,1})^{1/n} = g(x_0),$$

where

$$g(X) = A_1(B)A_2(B)\mathrm{den}(r)^{(2r+1)|\mathcal{F}_B|}(X+2r+1)^{(2r+1)|\mathcal{F}_B|}\left(\frac{(X+r)^r}{(X+r+1)^{r+1}}\right)^{s+1}$$

and x_0 is the unique positive real solution of the equation

$$f(X) = \left(\frac{X+2r+1}{X}\right)^{|\mathcal{F}_B|}\left(\frac{X+r}{X+r+1}\right)^{s+1} = 1.$$

Moreover, for any $\theta \in \mathcal{F}_B$*, we have*

$$\lim_{n\to+\infty} \frac{r_{n,1}}{r_{n,\theta}} = 1.$$

Before proving the analysis lemma, we first collect some properties of the functions f and g. Note that these two functions depend only on r, s and B.

Proposition 4.2. *Let* $f(x)$ *and* $g(x)$ *be the functions in Lemma 4.1* (*defined on* $x \in (0, +\infty)$). *Then:*

(1) *there exists a unique* $x_0 \in (0, +\infty)$ *such that* $f(x_0) = 1$, $f(x) > 1$ *on* $(0, x_0)$ *and* $f(x) < 1$ *on* $(x_0, +\infty)$*; moreover,*

$$x_0 < \frac{r(r+1)|\mathcal{F}_B|}{s+1-(2r+1)|\mathcal{F}_B|};$$

(2) *if we fix* $r \in \mathbb{Q}_+$ *and assume in addition that* $B = cs^{1/2}/\log^{1/2} s$ *for some positive constant* c*, when* $s \to +\infty$*, we have*

$$\lim_{\substack{s\to+\infty \\ B=cs^{1/2}/\log^{1/2} s}} g(x_0)^{1/(s+1)} = \exp\left(\frac{\zeta(2)\zeta(3)}{4\zeta(6)}(2r+1)c^2\right)\frac{r^r}{(r+1)^{r+1}}.$$

Proof. For the first proposition, by calculating $f'(x)/f(x)$, we find that $f'(x) = 0$ has a unique positive solution x_1 which satisfies

$$(s+1-(2r+1)|\mathcal{F}_B|)x_1^2 + (2r+1)(s+1-(2r+1)|\mathcal{F}_B|)x_1 - r(r+1)(2r+1)|\mathcal{F}_B| = 0 \quad (4.1)$$

and f is decreasing on $(0, x_1)$, increasing on $(x_1, +\infty)$. Since $f(0^+) = +\infty$ and $f(+\infty) = 1$, there exists a unique x_0 satisfying all the requirements. The last (very weak) bound for x_0 comes from $x_0 < x_1$ and (4.1).

The second proposition follows from the estimates for $A_1(B)$, $A_2(B)$, $|\mathcal{F}_B|$ (see Proposition 2.2 and Lemma 2.4) and $x_0 \to 0$. □

Now we prove Lemma 4.1. We claim that it can be proved by the same strategy in [FSZ19, Lemma 3], but we give a slightly modified proof.

Proof of Lemma 4.1. For any $\theta \in \mathcal{F}_B$, since $R_n(m+\theta) = 0$ for $m = 1, 2, \ldots, rn-1$, we define the shift version of the auxiliary rational functions:

$$\widehat{R}_n(t) = R_n(t + rn);$$

then, by (2.4), we have

$$r_{n,\theta} = \sum_{k=0}^{\infty} \widehat{R}_n(k+\theta). \quad (4.2)$$

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We have the following two expressions for $\widehat{R}_n(t)$:

$$\widehat{R}_n(t) = A_1(B)^n A_2(B)^n \frac{n!^{s+1}}{(n/\mathrm{den}(r))!^{\,\mathrm{den}(r)(2r+1)|\mathcal{F}_B|}} \frac{t\prod_{\theta'\in\mathcal{F}_B}\prod_{j=0}^{(2r+1)n-1}(t+j+\theta')}{\prod_{j=0}^{n}(t+rn+j)^{s+1}} \tag{4.3}$$

$$= A_1(B)^n A_2(B)^n \frac{n!^{s+1}}{(n/\mathrm{den}(r))!^{\,\mathrm{den}(r)(2r+1)|\mathcal{F}_B|}} \times t\left(\prod_{\theta'\in\mathcal{F}_B}\frac{\Gamma(t+(2r+1)n+\theta')}{\Gamma(t+\theta')}\right)\left(\frac{\Gamma(t+rn)}{\Gamma(t+(r+1)n+1)}\right)^{s+1}. \tag{4.4}$$

We define $c_1 = \min(\frac{1}{2}e^{-10s/r}, x_0/2)$, which is independent of n. To estimate the series (4.2) for $r_{n,\theta}$, we divide it into three parts:

$$r_{n,\theta} = \left(\sum_{0\leqslant k<c_1 n} + \sum_{c_1 n\leqslant k\leqslant n^{10}} + \sum_{k>n^{10}}\right)(\widehat{R}_n(k+\theta)).$$

For the first part, by (4.3), for all $t\in(0,2c_1 n]$, we have

$$\begin{aligned}\frac{\widehat{R}'_n(t)}{\widehat{R}_n(t)} &> \sum_{j=0}^{(2r+1)n}\frac{1}{t+j} - (s+1)\sum_{j=0}^{n}\frac{1}{t+rn+j}\\ &> \log\left(\frac{t+(2r+1)n}{t}\right) - (s+1)\frac{n+1}{rn}\\ &> \log\left(\frac{1}{2c_1}\right) - 4s/r\\ &> 0.\end{aligned}$$

So, $\widehat{R}_n(t)$ is increasing on $t\in(0,2c_1 n]$; we derive that (when $n > c_1^{-1}$)

$$\sum_{0\leqslant k<c_1 n}\widehat{R}_n(k+\theta) < (c_1 n+1)\,\widehat{R}_n([c_1 n]+\theta). \tag{4.5}$$

To deal with the middle part, for all $c_1 n\leqslant k\leqslant n^{10}$, we denote $\kappa=\kappa(k,n)=k/n\in[c_1,+\infty)$. By applying Stirling's formula in the weak form

$$\Gamma(x) = x^{O_{x\to+\infty}(1)}\left(\frac{x}{e}\right)^x$$

for the equation (4.4), a calculation shows that as $n\to+\infty$,

$$\widehat{R}_n(k+\theta) = n^{O(1)}\cdot A_1(B)^n A_2(B)^n \frac{(n/e)^{(s+1)n}}{((n/\mathrm{den}(r))/e)^{(2r+1)|\mathcal{F}_B|n}} \times\left(\frac{((k+(2r+1)n)/e)^{k+(2r+1)n}}{(k/e)^k}\right)^{|\mathcal{F}_B|}\frac{((k+rn)/e)^{(s+1)(k+rn)}}{((k+(r+1)n)/e)^{(s+1)(k+(r+1)n)}}$$

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$$
\begin{aligned}
&= n^{O(1)} \cdot A_1(B)^n A_2(B)^n \mathrm{den}(r)^{(2r+1)|\mathcal{F}_B| n} \\
&\quad \times \left(\frac{(\kappa+2r+1)^{\kappa+2r+1}}{\kappa^\kappa}\right)^{|\mathcal{F}_B|n} \left(\frac{(\kappa+r)^{\kappa+r}}{(\kappa+r+1)^{\kappa+r+1}}\right)^{(s+1)n} \\
&= n^{O(1)} \cdot (f(\kappa)^\kappa g(\kappa))^n \\
&= n^{O(1)} \cdot h(\kappa)^n
\end{aligned} \tag{4.6}
$$

uniformly for any $k \in [c_1 n, n^{10}]$ and any $\theta \in \mathcal{F}_B$ (the absolute bound for $O(1)$ depends only on s, B, r and $\mathrm{den}(r)$); here the function $h(x)$ is defined for $x > 0$ as $h(x) = f(x)^x g(x)$ and a direct computation shows that $h'(x)/h(x) = \log f(x)$. Hence, $h(x)$ achieves its maximum only at $x = x_0$ with maximal value $h(x_0) = g(x_0)$.

In particular, we have the following bound for each $k \in [c_1 n, n^{10}]$:

$$
\widehat{R}_n(k+\theta) \leqslant n^{O(1)} \cdot g(x_0)^n. \tag{4.7}
$$

Finally, we treat the tail part. For any $k > n^{10}$, when n is sufficiently large in terms of r, s and B (more precisely, when $n \geqslant \max(10(2r+1), 10A_1(B)A_2(B), 10/g(x_0)))$, by (4.3) and our assumption that $s \geqslant 10(2r+1)B^2$, we have

$$
\begin{aligned}
\widehat{R}_n(k+\theta) &< A_1(B)^n A_2(B)^n n^{(s+1)n} \cdot \frac{(2k)\prod_{\theta' \in \mathcal{F}_B} \prod_{j=0}^{(2r+1)n-1} (2k)}{\prod_{j=0}^{n} (k)^{s+1}} \\
&< \frac{(2A_1(B)A_2(B)n)^{(s+1)n}}{k^{(9/10)(s+1)n+2}} \\
&< \left(\frac{2A_1(B)A_2(B)}{n^8}\right)^{(s+1)n} \frac{1}{k^2} \\
&< \left(\frac{g(x_0)}{2}\right)^n \frac{1}{k^2}.
\end{aligned}
$$

As a conclusion, we obtain the following bound for the tail part for all sufficiently large n:

$$
\sum_{k > n^{10}} \widehat{R}_n(k+\theta) \leqslant \left(\frac{g(x_0)}{2}\right)^n. \tag{4.8}
$$

Now, in view of the estimates (4.5), (4.7) and (4.8), we have $r_{n,1} \leqslant n^{O(1)} g(x_0)^n$. On the other hand, (4.6) implies that $r_{n,1} \geqslant \widehat{R}_n(\lfloor x_0 n \rfloor) = n^{O(1)} h(x_0 + o(1))^n$. Therefore,

$$
\lim_{n \to +\infty} (r_{n,1})^{1/n} = g(x_0).
$$

To prove the last statement in the lemma, we first fix an arbitrary (sufficiently) small $\varepsilon_0 > 0$. For all $\theta \in \mathcal{F}_B$, we have

$$
r_{n,\theta} \geqslant \sum_{(x_0 - \varepsilon_0)n \leqslant k \leqslant (x_0+\varepsilon_0)n} \widehat{R}_n(k+\theta). \tag{4.9}
$$

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In view of the estimates (4.5), (4.6) and (4.8), we also have

$$\begin{aligned} r_{n,\theta} &\leqslant n^{O(1)} \max(h(x_0-\varepsilon_0), h(x_0+\varepsilon_0))^n \quad + \sum_{(x_0-\varepsilon_0)n\leqslant k\leqslant(x_0+\varepsilon_0)n} \widehat{R}_n(k+\theta) \\ &< (1+\varepsilon_0) \sum_{(x_0-\varepsilon_0)n\leqslant k\leqslant(x_0+\varepsilon_0)n} \widehat{R}_n(k+\theta), \end{aligned} \tag{4.10}$$

provided n is sufficiently large with respect to ε_0, i.e., $n \geqslant n_0(\varepsilon_0)$. For all k with $(x_0-\varepsilon_0)n \leqslant k \leqslant (x_0+\varepsilon_0)n$, let $\kappa = \kappa(n,k) = k/n$ as before. We now use the fact that, for any fixed real number τ,

$$\frac{\Gamma(x+\tau)}{\Gamma(x)} = (1+o_{x\to+\infty}(1))\, x^{\tau}. \tag{4.11}$$

Applying (4.11) to (4.4), we derive that

$$\begin{aligned} \frac{\widehat{R}_n(k+1)}{\widehat{R}_n(k+\theta)} &= (1+o(1)) \left(\frac{\kappa+2r+1}{\kappa}\right)^{|\mathcal{F}_B|(1-\theta)} \left(\frac{\kappa+r}{\kappa+r+1}\right)^{(s+1)(1-\theta)} \\ &= (1+o(1))\, f(\kappa)^{1-\theta} \end{aligned} \tag{4.12}$$

uniformly for $k \in [(x_0-\varepsilon_0)n, (x_0+\varepsilon_0)n]$ as $n \to +\infty$. By (4.9), (4.10) and (4.12), we find that

$$(1+o(1)) \frac{1}{1+\varepsilon_0} f(x_0+\varepsilon_0)^{1-\theta} \leqslant \frac{r_{n,1}}{r_{n,\theta}} \leqslant (1+o(1))\,(1+\varepsilon_0) f(x_0-\varepsilon_0)^{1-\theta};$$

thus,

$$\frac{1}{1+\varepsilon_0} f(x_0+\varepsilon_0)^{1-\theta} \leqslant \liminf_{n\to+\infty} \frac{r_{n,1}}{r_{n,\theta}} \leqslant \limsup_{n\to+\infty} \frac{r_{n,1}}{r_{n,\theta}} \leqslant (1+\varepsilon_0) f(x_0-\varepsilon_0)^{1-\theta}.$$

It is true for all sufficiently small $\varepsilon_0 > 0$. Letting $\varepsilon_0 \to 0^+$, we deduce that

$$\lim_{n\to+\infty} \frac{r_{n,1}}{r_{n,\theta}} = 1.$$

This completes the proof of Lemma 4.1. □

5. Elimination procedure and proof of the theorem

We prove Theorem 1.1 in this section. We will use the same strategy as [FSZ19, § 5], namely, an elimination procedure. So, we only give an outline of this elimination procedure.

We denote $I_s = \{3, 5, 7, \dots, s\}$. For any subset $J \subset I_s$ with $|J| = |\Psi_B| - 1$, since the following general Vandermonde matrix (see, for instance, [GK02, pp. 76–77]):

$$\left[b^j\right]_{b\in\Psi_B,\ j\in\{1\}\cup J}$$

is invertible, there exist integers $w_b \in \mathbb{Z}$ for all $b \in \Psi_B$ such that $\sum_{b\in\Psi_B} w_b b^j = 0$ for any $j \in J$ and $\sum_{b\in\Psi_B} w_b b \neq 0$. (Note that these w_b depend only on J and Ψ_B.) Since

$$\sum_{k=1}^{b} \zeta\left(i, \frac{k}{b}\right) = \sum_{k=1}^{b} \sum_{m=0}^{\infty} \frac{b^i}{(mb+k)^i} = b^i \zeta(i), \tag{5.1}$$

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we derive that (recall Proposition 2.2(2), $k/b \in \mathcal{F}_B$)

$$\widehat{r}_{n,b} \coloneqq \sum_{k=1}^{b} r_{n,k/b} = \sum_{k=1}^{b} \rho_{0,k/b} + \sum_{i \in I_s} \rho_i b^i \zeta(i)$$

is a linear combination of 1 and odd zeta values. By Lemma 4.1, we have $\widehat{r}_{n,b} = (b + o(1)) r_{n,1}$ as $n \to +\infty$. Let

$$\widetilde{r}_n \coloneqq \sum_{b \in \Psi_B} w_b \widehat{r}_{n,b};$$

then

$$\widetilde{r}_n = \sum_{b \in \Psi_B} w_b \sum_{k=1}^{b} \rho_{0,k/b} + \sum_{i \in I_s \setminus J} \Bigg(\sum_{b \in \Psi_B} w_b b^i \Bigg) \rho_i \zeta(i) \tag{5.2}$$

and, as $n \to +\infty$,

$$\widetilde{r}_n = \Bigg(\sum_{b \in \Psi_B} w_b b + o(1) \Bigg) r_{n,1} \quad \text{with} \quad \sum_{b \in \Psi_B} w_b b \neq 0. \tag{5.3}$$

Equation (5.2) shows that we can eliminate any $|\Psi_B| - 1$ odd zeta values.

PROPOSITION 5.1. *If $g(x_0) < e^{-(s+1)}$, then the number of irrationals in the odd zeta values set $\{\zeta(i)\}_{i \in I_s}$ is at least $|\Psi_B|$.*

Proof. We argue by contradiction. Suppose that the number of irrationals in $\{\zeta(i)\}_{i \in I_s}$ is less than $|\Psi_B|$; then we can take a subset $J \subset I_s$ with $|J| = |\Psi_B| - 1$ such that $\zeta(i) \in \mathbb{Q}$ for all $I_s \setminus J$; let A be the common denominator of these rational zeta values. Define $\widetilde{r}_n$ as above for this J; then, by (5.2) and Lemma 3.3, for all $n \in P_{B,\mathrm{den}(r)}\mathbb{N}$, we derive that

$$A d_{n+1}^{s+1} \widetilde{r}_n \in \mathbb{Z}.$$

But by (5.3), Lemma 4.1 and the hypothesis $g(x_0) < e^{-(s+1)}$, we have

$$0 < \lim_{n \to +\infty} \left| A d_{n+1}^{s+1} \widetilde{r}_n \right|^{1/n} = e^{s+1} g(x_0) < 1.$$

This is a contradiction. □

So, we seek parameters r, s and B to meet the requirement that $g(x_0) < e^{-(s+1)}$, and at the same time to make $|\Psi_B| \sim (\zeta(2)\zeta(3)/\zeta(6))B$ as large as possible. By Proposition 4.2(2), for a fixed r (such that $r^r/(r+1)^{r+1} < e^{-1}$), if we take $B = cs^{1/2}/\log^{1/2} s$ for some constant c, then $\lim_{s \to +\infty} g(x_0)^{1/(s+1)} < e^{-1}$ if and only if

$$c < \sqrt{\frac{4\zeta(6)}{\zeta(2)\zeta(3)} \frac{(r+1)\log(r+1) - r\log(r) - 1}{2r+1}}.$$

The maximum point of the function $r \mapsto ((r+1)\log(r+1) - r\log(r) - 1)/(2r+1)$ is

$$r_0 = \frac{\sqrt{4e^2+1} - 1}{2} \approx 2.26388,$$

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with maximal value $1-\log r_0$. The constant c_0 in Theorem 1.1 is designed by

$$c_0 = \sqrt{\frac{4\zeta(2)\zeta(3)}{\zeta(6)}(1-\log r_0)}.$$

This leads to the following proof.

Proof of Theorem 1.1. Given any small $\varepsilon > 0$, we first fix a rational number $r = r(\varepsilon)$ sufficiently close to r_0 such that

$$\frac{c_0 - \varepsilon/10}{\zeta(2)\zeta(3)/\zeta(6)} < \sqrt{\frac{4\zeta(6)}{\zeta(2)\zeta(3)}\frac{(r+1)\log(r+1) - r\log(r) - 1}{2r+1}}.$$

Take $B = cs^{1/2}/\log^{1/2} s$ with constant $c = (c_0 - \varepsilon/10)/(\zeta(2)\zeta(3)/\zeta(6))$; by Propositions 4.2(2) and 2.2(1), there exists $s_0(r,\varepsilon)$ such that for any odd integer $s \geqslant s_0(r,\varepsilon)$, we have $g(x_0) < e^{-(s+1)}$ and $|\Psi_B| > (\zeta(2)\zeta(3)/\zeta(6) - \varepsilon/10)B$. Therefore, by Proposition 5.1, the number of irrationals among $\zeta(3), \zeta(5), \ldots, \zeta(s)$ is at least

$$|\Psi_B| > (c_0 - \varepsilon)\frac{s^{1/2}}{\log^{1/2} s}. \qquad \square$$

6. Remarks on FSZ constructions

If we choose a general finite set $\mathcal{F} \subset (0,1]$ of rational numbers to be the *zero set* of the auxiliary function $R(t)$, like [FSZ19] and this note, we design the factor

$$A_1(\mathcal{F})^n = \prod_{\theta \in \mathcal{F}} \mathrm{den}(\theta)^{(2r+1)n} \tag{6.1}$$

to remedy the arithmetic loss from the denominators of rational zeros. Suppose that our goal is to prove that there exist D irrational numbers among $\zeta(3), \zeta(5), \ldots, \zeta(s)$. In order to eliminate $D-1$ zeta values, in view of (5.1), we assume that there exist D pairwise different positive integers $b_1, b_2, \ldots, b_D$ such that

$$\mathcal{F} \supset \left\{\frac{1}{b_i}, \frac{2}{b_i}, \ldots, \frac{b_i}{b_i}\right\} \tag{6.2}$$

for any $i = 1, 2, \ldots, D$. Then $\mathcal{F}$ contains the following disjoint union:

$$\mathcal{F} \supset \bigcup_{i=1}^{D} \left\{ \frac{a}{b_i} \,\middle|\, 1 \leqslant a \leqslant b_i, \gcd(a, b_i) = 1 \right\}.$$

Hence, we have

$$A_1(\mathcal{F}) \geqslant \prod_{i=1}^{D} b_i^{(2r+1)\varphi(b_i)}.$$

Now we consider the minimal magnitude of $A_1(\mathcal{F})$.

Proposition 6.1. *We have*

$$\min \prod_{i=1}^{D} b_i^{\varphi(b_i)} = \exp\left(\left(\frac{1}{2}\frac{\zeta(6)}{\zeta(2)\zeta(3)} + o_{D\to+\infty}(1)\right) D^2 \log D\right),$$

where the minimum is taken for all D-tuples $(b_1, \ldots, b_D)$ such that $b_1, b_2, \ldots, b_D$ are D pairwise distinct positive integers.

Proof. We have

$$\begin{aligned}\log \prod_{i=1}^{D} b_i^{\varphi(b_i)} &\geqslant \sum_{i=1}^{D} \varphi(b_i) \log \varphi(b_i) \\ &\geqslant \sum_{i=1}^{D} \varphi(b_i') \log \varphi(b_i'),\end{aligned}$$

where $b_1', b_2', \ldots, b_D'$ are the D smallest positive integers in the linear order $\prec$ defined by

$$m_1 \prec m_2 \Leftrightarrow (\varphi(m_1) < \varphi(m_2) \quad \text{or} \quad (\varphi(m_1) = \varphi(m_2) \text{ and } m_1 < m_2)).$$

Recall that for any positive real number x, we define $\Psi_x = \{b \in \mathbb{N} \mid \varphi(b) \leqslant x\}$. Then there exists an integer B such that $\Psi_{B-1} \subset \{b_1', \ldots, b_D'\} \subset \Psi_B$. By Proposition 2.2(1), we have the asymptotical relation $B = (\zeta(6)/\zeta(2)\zeta(3) + o_{D\to+\infty}(1))D$. Following the first paragraph in the proof of Lemma 2.4, we derive that

$$\begin{aligned}\sum_{i=1}^{D} \varphi(b_i') \log \varphi(b_i') &\geqslant \sum_{b\in\Psi_{B-1}} \varphi(b) \log \varphi(b) \\ &\geqslant \left(\frac{1}{2}\frac{\zeta(2)\zeta(3)}{\zeta(6)} + o_{B\to+\infty}(1)\right) B^2 \log B \\ &= \left(\frac{1}{2}\frac{\zeta(6)}{\zeta(2)\zeta(3)} + o_{D\to+\infty}(1)\right) D^2 \log D.\end{aligned}$$

This shows that

$$\prod_{i=1}^{D} b_i^{\varphi(b_i)} \geqslant \exp\left(\left(\frac{1}{2}\frac{\zeta(6)}{\zeta(2)\zeta(3)} + o_{D\to+\infty}(1)\right) D^2 \log D\right)$$

for any pairwise distinct positive integers $b_1, b_2, \ldots, b_D$. On the other hand, if we take $(b_1', \ldots, b_D')$ as defined above, then Lemma 2.4 also implies that

$$\begin{aligned}\prod_{i=1}^{D} b_i'^{\varphi(b_i')} &\leqslant \prod_{b\in\Psi_B} b^{\varphi(b)} \\ &= \exp\left(\left(\frac{1}{2}\frac{\zeta(2)\zeta(3)}{\zeta(6)} + o_{B\to+\infty}(1)\right) B^2 \log B\right) \\ &= \exp\left(\left(\frac{1}{2}\frac{\zeta(6)}{\zeta(2)\zeta(3)} + o_{D\to+\infty}(1)\right) D^2 \log D\right).\end{aligned}$$

We complete the proof of Proposition 6.1. □

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Proposition 6.1 shows that, under the constraints (6.1) and (6.2), the choice $\mathcal{F} = \mathcal{F}_B$ in Definition 2.1 is optimal up to an $o(1)$ term for minimizing the major arithmetic wasting factor $A_1(\mathcal{F})$. We do not need to consider the minor arithmetic wasting factor $A_2(\mathcal{F})$ for general $\mathcal{F}$, since $A_2(\mathcal{F}_B)$ is asymptotically negligible with respect to $A_1(\mathcal{F}_B)$ (see Lemma 2.4); we already have that

$$A_1(\mathcal{F}_B)A_2(\mathcal{F}_B) = \exp\left(\left(\frac{1}{2}\frac{\zeta(2)\zeta(3)}{\zeta(6)} + o_{B\to+\infty}(1)\right)(2r+1)B^2\log B\right).$$

For the factorial factor

$$\frac{n!^{s+1}}{(n/\mathrm{den}(r))!^{\ \mathrm{den}(r)(2r+1)|\mathcal{F}|}},$$

comparing to the corresponding factor $n!^{s+1-(2r+1)|\mathcal{F}|}$ in [BR01] or [FSZ19], we have an extra waste of

$$\binom{n}{\underbrace{\frac{n}{\mathrm{den}(r)},\ldots,\frac{n}{\mathrm{den}(r)}}_{\mathrm{den}(r)\ \text{in number}}}^{(2r+1)|\mathcal{F}|} \leqslant \mathrm{den}(r)^{(2r+1)|\mathcal{F}|n},$$

which is asymptotically negligible with respect to $A_1(\mathcal{F})^n$ (at least for $\mathcal{F} = \mathcal{F}_B$). Usually the Ball–Rivoal length parameter r is taken to be an integer in the literature, since in the case that r is integral, the corresponding arithmetic lemmas (see [BR01, Lemme 5] and [FSZ19, Lemma 2]) can be proved in a simpler way. We need to take non-integral rational r in order to let it be arbitrarily close to r_0. The idea of taking non-integral r is not new: it has appeared in the literature (for example, [Zud01]), but in a different form.

There exist some arithmetic saving factors known as Φ_n factors. They are certain products over primes in the range $C_{B,r}\sqrt{n} \leqslant p \leqslant n$ (here we can take $C_{B,r} = \sqrt{2(r+1)B\log\log B}$). We mention that the saving from Φ_n^{-1} plays an important role in small cases for the odd zeta problem; see, for instance, [Zud01, RZ20], [Zud02, § 4] or [KR07, Chapitre 11]. However, like [FSZ19, Remark 2], the known types of Φ_n factors have no effect on asymptotics. The reason is that, by Definition 2.3, (2.3) and (3.2), for any $k \in \{0, 1, \ldots, n\}$,

$$\begin{aligned} a_{s,k} &= ((t+k)^s R_n(t))\,|_{t=-k} \\ &= (-1)^k \binom{n}{k}^s \frac{n!(k+1)_{rn}(-k+n+1)_{rn}}{(n/\mathrm{den}(r))!^{\,\mathrm{den}(r)(2r+1)}} \prod_{a/b\in\mathcal{F}_B\setminus\{1\}} F_{b,a}(-k), \end{aligned}$$

where $(x)_m := x(x+1)\cdots(x+m-1)$. For any prime p with $C_{B,r}\sqrt{n} \leqslant p \leqslant n$, the p-adic order of $a_{s,0}$ is relatively small. If we define

$$\widetilde{\Phi}_n = \prod_{C_{B,r}\sqrt{n}\leqslant p\leqslant n} p^{v_p(\gcd\{a_{s,k}\}_{k=0}^n)},$$

then we can show that $\widetilde{\Phi}_n \leqslant A_2(B)^n \cdot d_n^{(\mathrm{den}(r)(2r+1)+1)|\mathcal{F}_B|}$, which is asymptotically negligible with respect to $A_1(\mathcal{F}_B)^n$. One may want to directly save the common divisor of $d_{n+1}^{s+1}\rho_{0,\theta}$ and $d_{n+1}^{s+1}\rho_i$, but it is out of current research. The small cases are more difficult to study; up to now, besides

Apéry's theorem [Apé79] that $\zeta(3)$ is irrational, the most remarkable result is that at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational, due to Zudilin [Zud01] in 2001. The question whether $\zeta(5)$ is irrational remains open.

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