Canad. Math. Bull. Vol. **43** (3), 2000 pp. 330–342

Maximal Operators and Cantor Sets

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Abstract. We consider maximal operators in the plane, defined by Cantor sets of directions, and show such operators are not bounded on L^2 if the Cantor set has positive Hausdorff dimension.

1 Introduction

An important problem in harmonic analysis is the differentiability of integrals and the L^p boundedness of the Hardy-Littlewood maximal operator. One formulation of this in $\mathbf{R}^2$ is the following: Consider A a subset of the unit circle in $\mathbf{R}^2$, and view A as a set of directions. Let A_x be the set of all rectangles in $\mathbf{R}^2$ containing x and oriented in one of the directions of A. The associated Hardy-Littlewood maximal operator M_A is defined by

$$M_A f(x) = \sup_{R \in A_x} \frac{1}{|R|} \int_R |f|.$$

When M_A maps L^p to L^p boundedly for some $1 < p < \infty$ then A is called a $\mathrm{Max}(p)$ set. If

$$\lim \left\{ \frac{1}{|R|} \int_R f(y)\, dy : R \in A_x,\ \operatorname{diam} R \to 0 \right\} = f(x) \text{ a.e.}$$

for all characteristic functions f, then A is said to differentiate characteristic functions and is called a density basis.

Not all sets are $\mathrm{Max}(p)$ sets or density bases; indeed, any set which is dense in a subset of the circle of positive measure is neither (see [5, p. 228] or [6]). In the positive direction Sjogren and Sjolin [9], improving upon earlier work of Nagel, Stein and Wainger [8], Cordoba and Fefferman [4] and Stromberg [10], showed that an n-fold sum of lacunary sets of directions is both a $\mathrm{Max}(p)$ set for all $p > 1$ and a density basis. It was conjectured that the Cantor middle third set (which can be thought of as an "infinite" fold sum of the lacunary set $\{3^{-j}\}$) was a $\mathrm{Max}(2)$ set [11], but very recently this was shown to be false by Katz [7]. Using other methods, Arutyunyants [1] proved that any central Cantor set, with ratios of dissection tending to $1/2$, was neither a density basis nor $\mathrm{Max}(p)$ for any finite p.

By generalizing Katz's ideas we show that central Cantor sets of positive Hausdorff dimension, and more generally, subsets of these Cantor sets of positive Cantor measure, are not $\mathrm{Max}(2)$ sets. (Non-central) Cantor sets with ratios of dissection which are both bounded away from zero "on average" and bounded away from one are also shown to fail to be $\mathrm{Max}(2)$. We conjecture that these Cantor sets are neither a density basis nor $\mathrm{Max}(p)$ for any finite p.

Received by the editors November 12, 1997.
This research was partially supported by NSERC.
AMS subject classification: Primary: 42B25; secondary: 43A46.
Keywords: maximal function, Cantor set, lacunary set.

https://doi.org/10.4153/CMB-2000-040-5 Published online by Cambridge University Press

2 Main Result

2.1 Cantor Set Construction

By a Cantor set we mean a compact, totally disconnected, perfect subset of $[0, 1]$. Cantor sets can be constructed in a similar fashion to the classical middle third Cantor set. Beginning with a closed subinterval of $[0, 1]$ we remove from it a non-empty open interval, leaving two closed intervals of positive length (to avoid isolated points) called the Cantor intervals of step one. The relation between the length of these intervals and the initial subinterval are called the ratios of dissection at step one. A similar operation is performed on each Cantor interval of step one, producing the (4) closed Cantor intervals (of positive length) of step two and the ratios of dissection at step two. This construction yields a decreasing sequence of closed sets whose intersection is a Cantor set.

An interesting special example is when the ratios of dissection at step k are all the same; we call this a central Cantor set (as the removed intervals are centred) and refer to this construction as the standard construction. The classical middle third Cantor set is a central Cantor set because the usual construction has all ratios of dissection equal to $1/3$. The central Cantor set with ratio r_k at step k and initial interval $[0, 1]$ can also be described as

$$\Big\{x \in [0, 1] : x = \sum_{k=1}^{\infty} \varepsilon_k r_1 \cdots r_{k-1}(1 - r_k), \varepsilon_k = 0, 1\Big\}$$

and so can be thought of as the "infinite" sum of the lacunary set

$$\{r_1 \cdots r_{k-1}(1 - r_k)\}_{k=1}^{\infty}.$$

To simplify the exposition we will assume the Cantor sets have initial interval $[0, 1]$.

Theorem 2.1 *Let C be a Cantor set with ratios of dissection bounded away from one. Let p_k be the minimum of the lengths of the Cantor intervals of step k. If $\delta \equiv \inf(p_k)^{1/k} > 0$ then C is not a* $\mathrm{Max}(2)$ *set. Indeed, the maximal operator is not even of weak type* $(2, 2)$.

The condition that $\inf(p_k)^{1/k} > 0$ is what was meant in the introduction by the phrase 'ratios of dissection ... bounded away from zero "on average"'. The results mentioned in the introduction about (subsets of) Cantor sets of positive Hausdorff dimension will be derived from this theorem (and its proof) in Section 3.

Before beginning the proof we mention one easy consequence of the theorem.

Corollary 2.2 *A Cantor set with ratios of dissection bounded away from* 0 *is not a* $\mathrm{Max}(2)$ *set.*

Proof There are two step k Cantor intervals contained in each step $k - 1$ interval, thus if all ratios of dissection are at least c, then all are at most $1 - c$, and hence are bounded away from one. Furthermore, in this case $p_k \geq c^k$, so the second condition is satisfied as well. ■

2.2 Proof of the Main Result

We begin by introducing notation which will be used throughout the remainder of the paper. Let W be the set of binary words of finite length,

$$W = \{w_1 w_2 \cdots w_n : w_i \in \{0, 1\}, n \in \mathbf{N}\} \cup \{e\}$$

where e is the empty word. If $w, w' \in W$ then, as usual, ww' will denote the concatenation of w and w', and the length of the word w will be denoted by $|w|$. It is convenient to use W to label the elements of the construction of the Cantor set.

We will set $I_e = [0, 1]$, and let I_0 and I_1 be the left and right closed intervals remaining after removing the open interval from $[0, 1]$ in the first step of the construction. I_0 and I_1 are the Cantor intervals of step one. We set $C_1 = I_0 \cup I_1$. In general, for each $w \in W$, $|w| = k$, we denote by I_{w0} and I_{w1} the left and right Cantor intervals of step $k + 1$ obtained after removing the open interval from I_w, a Cantor interval of step k. We set $C_k = \bigcup\{I_w : w \in W, |w| = k\}$. The Cantor set C is $\bigcap C_k$.

For $w \in W$ let $l_w = |I_w|$, let $r_{w0} = \frac{l_{w0}}{l_w}$ and $r_{w1} = \frac{l_{w1}}{l_w}$. The numbers r_w for $|w| = k$ are the ratios of dissection at step k. By passing to a Cantor subset of C, if necessary, we can assume $r_w \leq 1/3$ for all w.

The first step in the argument is to inductively define a partitioning of $[0, 1)$ and a function $s\colon [0, 1) \to C$ which have the property that if x, y belong to the same interval of the partition at step k, then $s(x)$ and $s(y)$ belong to the same Cantor interval of step k; and the interval of the partition and the Cantor interval are of comparable size. To do this we begin by dividing the interval $[0, 1/2)$ into disjoint, subintervals $[a, b)$ of lengths between l_0 and $2l_0$. This can be done since $0 < l_0 \leq 1/3$. Similarly, divide $[1/2, 1)$ into subintervals of lengths between l_1 and $2l_1$. These subintervals will be referred to as the (word) 0, or respectively 1, intervals, or more generally, the step 1 intervals (in the partition). Thus $[0, 1)$ is the union of the step 1 intervals.

Now assume inductively that $[0, 1)$ is the union of the step $n-1$ intervals, *i.e.*, w-intervals for $|w| = n - 1$, and that these are of lengths between l_w and $2l_w$. Fix one such w-interval and partition the left half of the interval into disjoint subintervals of lengths between l_{w0} and $2l_{w0}$, and the right half into subintervals of lengths between l_{w1} and $2l_{w1}$. Since l_{wi}/l_w, $i = 0, 1$ are ratios of dissection, and hence bounded by $1/3$, this is possible. Making such a choice for each w-interval, $|w| = n - 1$, gives a partition of $[0, 1)$ into a union of step n intervals, the left half subintervals of a w-interval being called $w0$-intervals, and the right half subintervals, $w1$-intervals. Notice that the construction ensures that if x belongs to a w-interval for $w = \alpha\beta$, then x belongs to an α-interval as well (from an earlier step). Also, notice that if x, y belong to the same step n interval, then

$$|x - y| \leq 2 \max\{l_w : |w| = n\} \leq 2/3^n.$$

Given $x \in C$ we may address x as $\{w_1, w_2, \dots\}$ where $w_i \in \{0, 1\}$ and if $w = w_1 \cdots w_n$ then $x \in I_w$. Define a map $s\colon [0, 1) \to C$ by the rule that if x belongs to one of the $w_1 \cdots w_n$-intervals for each n then $s(x)$ is the element in the Cantor set with address $\{w_i\}$. An important property of this map is that if x, y belong to the same w-interval, then $s(x)$ and $s(y)$ belong to I_w, an interval of comparable length.

Fix n (large) and let $q_n = [1/p_n]$. Notice that there exists an integer N such that the length of any step $Nn+1$ interval is strictly less than p_n. ($N \geq |\log\delta|/\log 3$ works.)

For $j = 0, 1, \ldots, q_n - 1$ set $s(j) = s(jp_n)$. We will say integers $j, k \in$ step m if jp_n and kp_n belong to the same step m interval, but not the same step $m+1$ interval. If $j \neq k$ then $|j-k|p_n \geq p_n$ and it follows from our previous comment that j, k are not in the same step m interval for any $m > Nn$.

We work with rectangles defined in a similar way to those used by Katz: For each $j = 0, 1, \ldots, q_n - 1$ let Q_j be the cube $[0, p_n] \times [jp_n, (j+1)p_n]$, and let R_j be the rectangle of dimension $40 \times p_n/4$, centred at the centre of Q_j and oriented in the direction $s(j)$. Denote by f the function

$$f = \sum_{k=0}^{q_n-1} \chi_{R_k}.$$

The maximal operator M_n, associated with the set of directions $\{s(j)\}$, is given by

$$M_n g(x) = \sup_{x \in R_j} \frac{1}{|R_j|} \int_{R_j} |g|.$$

This operator is clearly dominated by the maximal operator of the Cantor set since in the latter case the supremum is taken over all rectangles containing x, having direction in the Cantor set.

Motivated by the proof of Theorem 2 in [7] we next prove several preliminary lemmas. The notation above and the assumptions of our theorem apply in each. The reader should note that in the calculations which follow the constant B may vary from one occurrence to another.

Lemma 2.3 *There is a constant $B > 0$ (independent of n) such that $\int f^2 \leq Bn$.*

Proof As in Katz's proof (which follows from [3]) one easily sees that $\int f^2 \leq B \log q_n$. But the assumption that $\inf(p_k)^{1/k} > 0$ ensures that $\log q_n = O(n)$.

Lemma 2.4 *There is a constant $B > 0$ so that for all $0 \leq j < q_n$,*

$$\frac{1}{|R_j|} \int_{R_j} |f| \geq Bn.$$

Proof Fix j. By definition

$$\frac{1}{|R_j|} \int_{R_j} |f| = Bq_n \sum_k |R_j \cap R_k|.$$

Elementary geometry shows that if $j \neq k$ and

$$|s(j) - s(k)| \geq \frac{|j-k|p_n}{6}$$

https://doi.org/10.4153/CMB-2000-040-5 Published online by Cambridge University Press

then

$$|R_j \cap R_k| \geq \frac{Bp_n^2}{|s(j) - s(k)|}.$$

Suppose $j, k \in$ step m, say to a w-interval for a word w of length m. Then $s(j)$ and $s(k)$ belong to the Cantor interval I_w, so $|s(j) - s(k)| \leq l_w$, while $|j - k|p_n \leq 2l_w$. Suppose, however, $jp_n \in w0$-interval and $kp_n \in w1$-interval, or vice versa. Then $s(j)$ and $s(k)$ do not belong to the same Cantor interval of step $m + 1$ and therefore $|s(j) - s(k)|$ is at least the length of the gap of I_w, hence at least $l_w/3$. Thus for all such k

$$|s(j) - s(k)| \geq \frac{|j - k|p_n}{6},$$

and there are at least $[l_w q_n/2]$ integers k with this property. It follows that

$$Bq_n \sum_k |R_j \cap R_k| \geq Bq_n \sum_{m=0}^{n} \frac{[l_w q_n/2]p_n^2}{l_w} \geq Bn.$$

■

Lemma 2.5 *For $\lambda > 0$ let $R_j^\lambda = \{(x, y) \in R_j : 1/\lambda \leq |x| \leq 2/\lambda\}$. There is a constant $B > 0$ such that*

$$\sum_{j \neq k} |R_j^\lambda \cap R_k^\lambda| \leq \frac{Bn}{\lambda^2}$$

(provided λ is sufficiently large).

Proof Again elementary geometry shows that $R_j^\lambda \cap R_k^\lambda \neq \phi$ only if

$$|j - k|p_n \leq \frac{C}{\lambda}|s(j) - s(k)|$$

for an appropriate choice of constant C. Moreover,

$$|R_j^\lambda \cap R_k^\lambda| \leq \frac{Bp_n^2}{|s(j) - s(k)|}.$$

Temporarily fix m, and first we will consider the number of pairs $j, k \in$ step m, $j \neq k$, (recall that this implies that $m \leq Nn$) with $R_j^\lambda \cap R_k^\lambda \neq \phi$. Assume that jp_n, kp_n belong to a w-interval, $|w| = m$. Consequently $|j - k|p_n \leq 2l_w$ while $|s(j) - s(k)| \leq l_w$.

Case 1: $s(j)_{m+1} \neq s(k)_{m+1}$, *i.e.*, $jp_n \in w0$-interval and $kp_n \in w1$-interval, or vice versa (a more geometric way to say this is that they belong to opposite halves of the w-interval).

Since

$$|j - k|p_n \leq \frac{C}{\lambda}|s(j) - s(k)| \leq \frac{C}{\lambda} l_w,$$

we must have jp_n and kp_n each lying within $\frac{C}{\lambda} l_w$ of the midpoint of the w-interval, and hence there are at most $(\frac{C}{\lambda} l_w q_n)^2$ pairs of integers of this type with $R_j^\lambda \cap R_k^\lambda \neq \phi$. Furthermore, in this case $s(j)$ and $s(k)$ belong to different Cantor intervals of step $m + 1$, thus $|s(j) - s(k)| \geq l_w/3$ and therefore $|R_j^\lambda \cap R_k^\lambda| \leq Bp_n^2/l_w$.

Case 2: $s(j)_{m+1} = s(k)_{m+1}$. Then either jp_n and kp_n both belong to $w0$-subintervals, or both belong to $w1$-subintervals. Let's assume first that both are in $w0$-intervals (but necessarily different ones since $j, k \notin$ step $m+1$). In this case $|s(j) - s(k)| \le l_{w0}$, hence in order for $R_j^\lambda \cap R_k^\lambda$ to be non-empty we must have

$$|j-k|p_n \le \frac{C}{\lambda} l_{w0}.$$

This means jp_n and kp_n must each lie within $\frac{C}{\lambda} l_{w0}$ of an endpoint of some $w0$-interval. There are at most l_w/l_{w0} such intervals and therefore at most $(\frac{C}{\lambda} l_{w0} q_n)^2 l_w/l_{w0}$ pairs of this type with $R_j^\lambda \cap R_k^\lambda \ne \phi$.

If, in addition, $s(j)_{m+2} = s(k)_{m+2}$ (say, without loss of generality, jp_n and kp_n both belong to $w00$-subintervals) then since jp_n and kp_n belong to different step $m+1$ intervals, and both are in the left half of those step $m+1$ intervals, we must have $|j-k|p_n \ge l_{w0}/2$. But we also know that the additional assumption means $|s(j) - s(k)| \le l_{w00}$, and if we assume that $\lambda \ge C$ then it is impossible to satisfy

$$|j-k|p_n \le \frac{C}{\lambda} |s(j) - s(k)|.$$

Hence $R_j^\lambda \cap R_k^\lambda = \phi$.

So it must be the case that $s(j)_{m+2} \ne s(k)_{m+2}$. But then $s(j)$ and $s(k)$ differ by at least the size of the gap of I_{w0}, and hence $|s(j) - s(k)| \ge l_{w0}/3$.

To summarize, there are at most $(\frac{C}{\lambda} l_{w0} q_n)^2 l_w/l_{w0}$ pairs of this type with $R_j^\lambda \cap R_k^\lambda \ne \phi$, and for these j, k we have the bound

$$|R_j^\lambda \cap R_k^\lambda| \le \frac{Bp_n^2}{l_{w0}}.$$

If, instead, jp_n and kp_n belong to (different) $w1$-intervals, then similarly there are at most $(\frac{C}{\lambda} l_{w1} q_n)^2 l_w/l_{w1}$ pairs of integers j, k of this type with $R_j^\lambda \cap R_k^\lambda \ne \phi$, and for these j, k we have the bound

$$|R_j^\lambda \cap R_k^\lambda| \le \frac{Bp_n^2}{l_{w1}}.$$

Putting these cases together we obtain

$$\begin{aligned}
\sum_{j \ne k} |R_j^\lambda \cap R_k^\lambda| &\le \sum_{m=0}^{Nn} \sum_{j,k \in \text{step } m} |R_j^\lambda \cap R_k^\lambda| \\
&\le Bp_n^2 \sum_{m=0}^{Nn} \sum_{|w|=m} \Big[\Big(\frac{(\frac{C}{\lambda} l_w q_n)^2}{l_w} \Big) + \Big(\frac{(\frac{C}{\lambda} l_{w0} q_n)^2 l_w/l_{w0}}{l_{w0}} \Big) + \Big(\frac{(\frac{C}{\lambda} l_{w1} q_n)^2 l_w/l_{w1}}{l_{w1}} \Big) \Big] \\
&\le \frac{B}{\lambda^2} \sum_{m=0}^{Nn} \sum_{|w|=m} l_w
\end{aligned}$$

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where the notation $\sum_{|w|=m}$ means to sum over all w-intervals with $|w| = m$. Since the union of the w-intervals is $[0,1)$, and they are disjoint intervals of length at least l_w, we must have $\sum_{|w|=m} l_w \le 1$. Therefore

$$\sum_{j\neq k} |R_j^\lambda \cap R_k^\lambda| \le \frac{Bn}{\lambda^2}$$

as claimed. ■

We need one other preliminary lemma in order to prove the theorem, however, we will actually prove a more general result which will be helpful for later.

Lemma 2.6 *Suppose $\sigma > 0$ and let X_n be any subset of $\{0,1,\dots,q_n-1\}$ of cardinality at least σq_n. There is a constant $B(\sigma) > 0$ such that*

$$\Big|\bigcup_{j\in X_n} R_j\Big| \ge \frac{B(\sigma)\log n}{n}.$$

Proof We continue to use the notation introduced in Lemma 5. We let

$$t_j(x) = \frac{\chi_{R_j}(x)}{\sum_{k\in X_n}\chi_{R_k}(x)} \quad \text{for } x \in \bigcup_{k\in X_n} R_k.$$

Clearly $\sum |R_j^\lambda| \le B/\lambda$ and

$$\sum_{j\in X_n}\int_{R_j^\lambda} t_j^{-1}(x)\,dx \le \sum_{j,k}|R_j^\lambda \cap R_k^\lambda|.$$

Combined with Lemma 5 this means that if we restrict to $\lambda \le n$, then

$$\sum_{j\in X_n}\int_{R_j^\lambda} t_j^{-1}(x)\,dx \le B\Big(\frac{1}{\lambda}+\frac{n}{\lambda^2}\Big) \le \frac{Bn}{\lambda^2}.$$

By Tchebycheff's inequality, for at least $[|X_n|/2]$ choices of $j \in X_n$ (and a new constant B as usual) we must have

$$\int_{R_j^\lambda} t_j^{-1}(x)\,dx \le \frac{Bn}{\lambda^2|X_n|}.$$

Since $|X_n| \ge \sigma q_n$ and $|R_j^\lambda| \ge p_n/4\lambda$, applying Holder's inequality gives the lower bound

$$\frac{1}{|R_j^\lambda|}\int_{R_j^\lambda} t_j(x)\,dx \ge \frac{B(\sigma)\lambda}{n}$$

for these j's. Summing over all $j \in X_n$ we obtain

$$\sum_{j\in X_n}\int_{R_j^\lambda} t_j(x)\,dx \ge \frac{B(\sigma)}{n}.$$

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Recall that Lemma 5 is only valid if λ is sufficiently large, say $\lambda \geq C$. We further restrict our attention to

$$\lambda \in \Lambda \equiv \left\{2^k C : k = 0, 1, \dots, \left[\frac{\log n/C}{\log 2}\right]\right\}.$$

For any fixed j and $\lambda_1, \lambda_2 \in \Lambda$, $\lambda_1 \neq \lambda_2$, the sets $R_j^{\lambda_1}$ and $R_j^{\lambda_2}$ are disjoint, thus

$$\Big|\bigcup_{j \in X_n} R_j\Big| = \sum_{j \in X_n} \int_{R_j} t_j(x)\,dx \geq \sum_{j \in X_n} \sum_{\lambda \in \Lambda} \int_{R_j^\lambda} t_j(x)\,dx$$
$$\geq \frac{B(\sigma)\log n}{n}.$$

Completion of the Proof of the Theorem From Lemma 4, $M_n f(x) \geq Bn$ for all $x \in \bigcup R_j$, and hence by Lemma 6 (applied with $X_n = \{0, 1, \dots, q_n - 1\}$, $\sigma = 1$) $M_n f(x) \geq Bn$ on a set of measure at least $B \log n/n$. Thus

$$\int (M_n f)^2 \geq Bn \log n$$

and since $\|f\|_2 \leq O(\sqrt{n})$ (Lemma 3)

$$\frac{\|M_n(f)\|_2}{\|f\|_2} \geq B\sqrt{\log n}.$$

It follows immediately from this that the maximal operator is not of strong type $(2, 2)$, *i.e.*, the Cantor set is not a $\mathrm{Max}(2)$ set. Moreover, since

$$n^2 m\{x : M_C f(x) \geq Bn\} \geq O(n \log n)$$

and $\|f\|_2^2 = O(n)$ the maximal operator is not even of weak type $(2, 2)$. ■

Remark 2.1 Since $M_C f \geq \sup_n M_n f$ it suffices to have the operator norm of M_n at least $B \log n$ for infinitely many n, and this will be true if we only assume the weaker condition that $p_n \geq \delta^n$ infinitely often.

3 Applications

This result has a number of other corollaries. One obvious one is:

Corollary 3.1 *If C is a Cantor set as in the theorem, then C is not a* $\mathrm{Max}(p)$ *set for any* $p < 2$.

Proof If the maximal operator M_C was of strong type (p, p) for some $p < 2$ (*i.e.*, C was a $\mathrm{Max}(p)$ set) or even of weak type (p, p), then since M_C is always a bounded operator on L^∞, by interpolation theory it would be of strong type $(2, 2)$ which we have shown it is not. ■

An important corollary is the result highlighted in the introduction.

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Corollary 3.2 *If C is a central Cantor set of positive Hausdorff dimension, then C is not a* $\mathrm{Max}(p)$ *set for any $p \le 2$.*

Proof Assume C is the central Cantor set with ratios of dissection, in the standard construction, equal to r_k $(< 1/2)$ at step k. Then all Cantor intervals at step k are of length $r_1 \cdots r_k$. It is known [2] that the Hausdorff dimension of this Cantor set is positive if and only if

$$\inf\Big(\prod_{j=1}^{k} r_j\Big)^{1/k} > 0,$$

hence both hypotheses of the theorem are clearly met. ∎

In fact, the assumption of positive Hausdorff dimension is more than what is necessary.

Corollary 3.3 *Suppose C is a central Cantor set with ratios of dissection r_k satisfying $\prod_{k=1}^{n} r_k \ge n^{-ng(n)}$ for some function $g(n)$ tending to zero. Then C is not a* $\mathrm{Max}(2)$ *set.*

Proof To prove this we need to look at the proof of the theorem. We work with the standard construction and assume (as before) that the ratios of dissection are at most $1/3$. Thus $p_n = \prod_{k=1}^{n} r_k$ and from Lemma 3 we obtain

$$\|f\|_2 \le B\sqrt{\log q_n} \le B\sqrt{ng(n)\log n}.$$

Any step $n+1$ interval in the partition has length less than p_n, so if $j, k \in$ step m, $j \ne k$, then $m \le n$. Thus Lemmas 4–6 can be proved as before. Using the new bound for the 2-norm of f in the final calculation of the $(2,2)$ operator norm of M_n we again are able to conclude that maximal operator is not bounded on L^2. ∎

By making an appropriate choice for the partition we can strengthen the theorem in another way.

Proposition 3.4 *Let C be a central Cantor set of positive Hausdorff dimension, and denote by E_n the set of left hand endpoints in the n'-th step of the standard construction. Then the set of left endpoints, $\bigcup E_n$, is not a* $\mathrm{Max}(2)$ *set.*

Proof An important observation to make here is that the function s, as defined in the theorem, maps left endpoints of the intervals in step n of the given partition, to left endpoints of Cantor intervals in step n in the Cantor set construction. Our strategy will be to show that we can define a partition in such a way that jp_n is an endpoint of step n in the partition for "enough" integers j so that one may conclude that the operator norm of the maximal function associated with E_n is at least $O(\log n)$.

We work with the standard construction so p_n is the length of each Cantor interval of step n. By passing to a Cantor subset, if necessary, we may assume all ratios of dissection are at most $1/6$.

The partition of $[0,1]$ will be defined inductively as follows, with step 0 being the interval $[0,1]$. At step k the intervals in the partition will have lengths at least p_k and have as

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endpoints: the endpoints of the intervals of step $k-1$; the midpoints of the intervals from step $k-1$; and as many integer multiples of p_k as possible.

Since the ratios of dissection are at most $1/6$ and the intervals from step $k-1$ have lengths at least p_{k-1}, there are at least 3 multiples of p_k in each "half" interval from step $k-1$. It follows that the partition we have defined will have the property that the length of each interval at step k is less than $2p_k$. As we observed in a previous corollary, the positive Hausdorff dimension condition ensures that $\inf(p_k)^{1/k} > 0$. Thus we have an appropriate partition to use our previous work, and so if we fix n, and define rectangles R_j for $j = 0, 1, \ldots, q_n - 1$ and the function $f = \sum_{k=0}^{q_n-1} \chi_{R_k}$ as before, Lemmas 3–5 hold.

Let X_n be the set of integers $0 \le j < q_n - 1$ such that jp_n is an endpoint of an interval at step n in the partition. (Thus if $j \in X_n$ then rectangle R_j has direction $s(jp_n) \in E_n$.) All the multiples of p_n in each "half" interval from step $n-1$, except perhaps the smallest and largest, must be endpoints of the partition at step n by construction, and since there are at most $2/p_{n-1}$ half intervals at step $n-1$ it follows that

$$|X_n| \ge q_n - \frac{4}{p_{n-1}} \ge \frac{q_n}{4}$$

(for sufficiently large n). Thus Lemma 6 applies, with $\sigma = 1/4$, to show that for all n

$$\Big|\bigcup_{j\in X_n} R_j\Big| \ge \frac{B\log n}{n}.$$

The maximal operator associated with E_n obviously dominates the operator M_n' defined by

$$M_n' g(x) \equiv \sup_{x\in R_j,\, j\in X_n} \frac{1}{|R_j|}\int_{R_j} |g|.$$

For the function f defined above we clearly have $M_n' f \ge Bn$ on $\bigcup_{j\in X_n} R_j$, and the proof can be completed essentially as before. ■

We commented earlier that a central Cantor set can be interpreted as a lacunary set added to itself "infinitely often". There is a converse to this as well, for suppose $F = \{n_j\}$ is a lacunary set in $[0,1]$ with $\inf \frac{n_j}{n_{j+1}} > 2$. Construct a central Cantor set C in the interval $[0, \sum_{j=1}^\infty n_j]$, in the usual way, with the intervals at step k having length $\sum_{j=k+1}^\infty n_j$. (The condition that $\inf \frac{n_j}{n_{j+1}} > 2$ ensures that the ratios in the Cantor set construction are strictly less than $1/2$, so this is a Cantor set of measure zero.) It is a routine exercise to verify that

$$C = \Big\{\sum_{j=1}^\infty \varepsilon_j n_j : \varepsilon_j = 0, 1\Big\}$$

and that the left endpoints of the intervals at step k in the construction are the 2^k points $\{\sum_{j=1}^k \varepsilon_j n_j : \varepsilon_j = 0, 1\}$. The set of left endpoints can also be described as $\bigcup_N F^{(N)}$ where

$$F^{(N)} = \Big\{\sum_{j=1}^\infty \varepsilon_j n_j : \varepsilon_j = 0, 1, \sum \varepsilon_j = N\Big\}.$$

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Each set $F^{(N)}$ has been shown to be $\mathrm{Max}(p)$ for all $p > 1$ in [9]. In contrast the next corollary shows that this fact is not in general true for the union.

Corollary 3.5 *Let* $F = \{n_j\}$ *be a lacunary set in* $[0,1]$ *with* $\inf(\sum_{j=k+1}^{\infty} n_j)^{1/k} > 0$. *Then*

$$E \equiv \Big\{ \sum_{j=1}^{\infty} \varepsilon_j n_j : \varepsilon_j = 0, 1, \sum \varepsilon_j < \infty \Big\}$$

is not a $\mathrm{Max}(2)$ *set.*

Proof Consider a subsequence $\{n_{jl}\}$ for appropriate l so that $\inf \frac{n_{jl}}{n_{(j+1)l}} > 2$. The Cantor set constructed from $\{n_{jl}\}$, as outlined above, has intervals at step k of length

$$\sum_{j=k+1}^{\infty} n_{jl} \geq \frac{1}{l} \sum_{j=(k+1)l}^{\infty} n_j,$$

hence this Cantor set has positive Hausdorff dimension. By Proposition 10 even the set of left endpoints of the Cantor set constructed from $\{n_{jl}\}$, which is a subset of E, is not a $\mathrm{Max}(2)$ set. ■

Remark 3.1 The condition that $\inf(\sum_{j=k+1}^{\infty} n_j)^{1/k} > 0$ is satisfied if $\sup \frac{n_j}{n_{j+1}} < \infty$.

As a final application we will extend the theorem to subsets of Cantor sets.

Proposition 3.6 *Suppose C is a central Cantor set with positive Hausdorff dimension and ratios of dissection $r_k \leq 1/3$. If A is a subset of C with positive Cantor measure then A is not a* $\mathrm{Max}(2)$ *set.*

Proof To prove this we need to take advantage of the symmetry in the standard construction of a central Cantor set.

We construct our partition of $[0,1)$ in a similar fashion to before, but require, in addition, that the intervals in step k be equal in length, say of length L_k, where $L_k \in [r_1 \cdots r_k, 2r_1 \cdots r_k)$. (Recall that $r_1 \cdots r_k$ is the length of any Cantor interval at step k in the standard construction.) Fix n and let q_n be the integer $1/L_n$. We define $s\colon [0,1) \to C$ in the same manner as previously.

The symmetry of both the partition and the central Cantor set ensures that each interval in C_n (where $C = \bigcap C_n$ is the standard construction) is the image under s of precisely $q_n/2^n$ intervals of the form $\big[jL_n, (j+1)L_n\big)$ for $j = 0, 1, \ldots, q_n - 1$. If we assume the Cantor measure of A is at least $\varepsilon > 0$, then the number of intervals from C_n which intersect A is at least $\varepsilon 2^n$. Thus, for at least εq_n choices of j,

$$s\big[jL_n, (j+1)L_n\big) \cap A \neq \phi.$$

Without loss of generality we may assume that for at least $\varepsilon q_n/2$ choices of $j \in \{0, 1, \ldots, q_n/2 - 1\}$,

$$s\big[2jL_n, (2j+1)L_n\big) \cap A \neq \phi.$$

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For each such j choose
$$x_j \in \big[2jL_n, (2j+1)L_n\big)$$
such that $s(x_j) \in A$. For other $j \in \{0, 1, \dots, q_n/2 - 1\}$ set $x_j = 2jL_n$. An important fact here is that $L_n < x_{j+1} - x_j < 3L_n$. We work with rectangles R_j, $j = 0, 1, \dots, q_n/2 - 1$, with dimension $40 \times L_n/4$, direction $s(x_j)$ and centred at $(0, x_j)$. The purpose of this modification is to ensure that at least $\varepsilon q_n/2$ of the rectangles have direction in A.

Let $f = \sum_{j=0}^{q_n/2-1} \chi_{R_j}$. The fact that $L_n < x_{j+1} - x_j < 3L_n$ allows one to bound, from both above and below, the number of integers k such that x_k belongs to a given subinterval of $[0, 1)$, it implies that if x_j, x_k belong to step m then $m \le n$, and it ensures that the same elementary geometry arguments apply. Thus Lemmas 3–5 can be proved essentially as before. If $X_n = \{j : s(x_j) \in A\}$ then $|X_n| \ge \varepsilon q_n/2$, so by Lemma 6 we derive that
$$\Big|\bigcup_{j \in X_n} R_j\Big| \ge \frac{B \log n}{n}$$
for some constant B depending only on the Cantor measure of A.

To finish the proof of the proposition just note that the maximal operator associated with A dominates the operators M_n^A given by
$$M_n^A g(x) = \sup_{x \in R_j,\ j \in X_n} \frac{1}{|R_j|} \int_{R_j} |g|,$$
and the usual arguments prove that the $(2, 2)$ operator norm of M_n^A is at least $O(\log n)$. ■

Remark 3.2 The same result could be obtained for central Cantor sets having ratio of dissection bounded away from $1/2$. Just define the rectangles R_j to have dimension $B \times L_n/4$ where B is a suitably large constant.

Our result could also be further weakened to apply to even "smaller" subsets A of the types of Cantor sets described in the theorem: It suffices to have the number of intervals of C_n intersecting A at least $g(n)2^n$ for some function g satisfying $g(n)^2 \log n \to \infty$. This involves a straightforward modification of the proof.

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