Proceedings of the Royal Society of Edinburgh, **150**, 569–606, *2020*
DOI:10.1017/prm.2018.154

Long-time asymptotic expansions for Navier-Stokes equations with power-decaying forces

Dat Cao and Luan Hoang
Department of Mathematics and Statistics, Texas Tech University, Box 41042, Lubbock, TX 79409-1042, USA (dat.cao@ttu.edu, luan.hoang@ttu.edu)

(MS received 27 March 2018; accepted 4 September 2018)

The Navier-Stokes equations for viscous, incompressible fluids are studied in the three-dimensional periodic domains, with the body force having an asymptotic expansion, when time goes to infinity, in terms of power-decaying functions in a Sobolev-Gevrey space. Any Leray-Hopf weak solution is proved to have an asymptotic expansion of the same type in the same space, which is uniquely determined by the force, and independent of the individual solutions. In case the expansion is convergent, we show that the next asymptotic approximation for the solution must be an exponential decay. Furthermore, the convergence of the expansion and the range of its coefficients, as the force varies are investigated.

Keywords: Navier-Stokes, asymptotic expansion, power-decaying

2010 *Mathematics subject classification:* 35C20, 35B40, 35Q30, 37L99, 76D05

1. Introduction

The Navier-Stokes equations (NSE) are nonlinear partial differential equations that describe the dynamics of viscous, incompressible fluids. The mathematics of NSE has proven to be quite important, intriguing and challenging. In particular, understanding the long-term behaviours of the solutions of NSE would be insightful to many hydrodynamical phenomena. Unfortunately, such a level of mathematical understanding is still not available in general. However, under some circumstances, the mathematics is more accessible and much has been understood. One such case is when the body force is potential, which has many papers devoted to [**3**–**7**, **9**, **11**–**14**, **16**]. (We caution that these works provide a deep understanding of the solutions despite the fact that they go to zero as time becomes large. They are different from those studying the case of large forces, which are more oriented toward the theory of turbulence.) The case when the force is nonpotential and decays exponentially in time has only been studied recently in [**19**]. The current paper follows this direction of research. We aim to understand the long-term behaviour of the solutions in case the force is larger than those considered in [**19**]. More importantly, we hope to find new phenomena due to the different structure of the force, and describe precisely how the asymptotic properties of the force determine the asymptotic behaviour of the solutions.

First, we recall the mathematical formulation of the NSE, and specify the context in which we study them. Let $\mathbf{u}(\mathbf{x},t)\in\mathbb{R}^3$ denote the velocity vector field and $p(\mathbf{x},t)\in\mathbb{R}$ denote the pressure of a viscous, incompressible fluid, where $\mathbf{x}\in\mathbb{R}^3$ is the vector of spatial variables, and $t\in\mathbb{R}$ is the time variable. The (kinematic) viscosity of the fluid is a constant $\nu>0$. The body force acting on the fluid is $\mathbf{f}(\mathbf{x},t)\in\mathbb{R}^3$. The NSE are the following system of partial differential equations

$$\begin{aligned}&\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}-\nu\Delta\mathbf{u}=-\nabla p+\mathbf{f}\quad \text{on } \mathbb{R}^3\times(0,\infty),\\&\text{div }\mathbf{u}=0\quad\text{on } \mathbb{R}^3\times(0,\infty).\end{aligned}\tag{1.1}$$

Above, the first equation is the balance of momentum, while the second one is the incompressibility condition.

The initial condition specified for the velocity is

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}^0(\mathbf{x}),\tag{1.2}$$

where $\mathbf{u}^0(\mathbf{x})$ is a given divergence-free vector field.

In our current study, the force $\mathbf{f}(\mathbf{x},t)$ and solutions $(\mathbf{u}(\mathbf{x},t),p(\mathbf{x},t))$ are considered to belong to the class of L-periodic functions for some $L>0$, that is, the class of functions $g(\mathbf{x})$ that satisfy

$$g(\mathbf{x}+L\mathbf{e}_j)=g(\mathbf{x})\quad\text{for all}\quad \mathbf{x}\in\mathbb{R}^3,\quad j=1,2,3,$$

where $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is the standard basis of $\mathbb{R}^3$. Such a consideration will simplify our mathematical analysis since it avoids the case of unbounded domains, and the no-slip boundary condition usually imposed on bounded domains.

By a Galilean transformation, see for example, [**19**], we can assume that $\mathbf{f}(\mathbf{x},t)$ and $\mathbf{u}(\mathbf{x},t)$, for all $t\geqslant 0$, have zero averages over the domain $\Omega=(-L/2,L/2)^3$, that is, their spatial integrals over Ω are zero.

Thanks to the Leray-Helmholtz decomposition, and for the sake of convenience, we assume further that $\mathbf{f}(\mathbf{x},t)$ is divergence-free for all $t\geqslant 0$.

By rescaling the variables $\mathbf{x}$ and t, we assume throughout, without loss of generality, that $L=2\pi$ and $\nu=1$. With this assumption, the equations in (1.1) are adimensional now.

In studying the dynamics of NSE, the function $\mathbf{u}(\mathbf{x},t)$ of several variables can be viewed as a function of t valued in some functional space. For time-dependent functions of such type, their asymptotic properties, as time goes to infinity, can be understood most precisely if some form of asymptotic expansions is established. We discuss, in this paper, the following two types of expansions. Briefly speaking, one expansion is in terms of exponential decaying functions with polynomial coefficients, and the other of power-decaying ones.

Definition 1.1. Let $(X, \|\cdot\|)$ be a real normed space, g be a function from $(0,\infty)$ to X, and $(\gamma_n)_{n=1}^\infty$ be a strictly increasing, divergent sequence of positive numbers.

(a) The function g is said to have the asymptotic expansion

$$g(t) \overset{\text{exp.}}{\sim} \sum_{n=1}^\infty g_n(t)e^{-\gamma_n t} \text{ in } X, \tag{1.3}$$

where $g_n(t)$'s are X-valued polynomials in t, if for any $N \geqslant 1$, there exists $\beta_N > \gamma_N$ such that

$$\left\| g(t) - \sum_{n=1}^N g_n(t)e^{-\gamma_n t} \right\| = \mathcal{O}(e^{-\beta_N t}) \quad \text{as } t \to \infty.$$

(b) The function g is said to have the asymptotic expansion

$$g(t) \sim \sum_{n=1}^\infty \xi_n t^{-\gamma_n} \text{ in } X, \tag{1.4}$$

where ξ_n's are elements in X, if for any $N \geqslant 1$, there exists $\beta_N > \gamma_N$ such that

$$\left\| g(t) - \sum_{n=1}^N \xi_n t^{-\gamma_n} \right\| = \mathcal{O}(t^{-\beta_N}) \quad \text{as } t \to \infty.$$

Throughout the paper, we will make use of the following notation

$$u(t) = \mathbf{u}(\cdot,t),\ f(t) = \mathbf{f}(\cdot,t), \quad u^0 = \mathbf{u}^0(\cdot).$$

Note that u^0, and each value of $u(t)$, $f(t)$ belong to some functional spaces.

In case the force $\mathbf{f}$ in NSE is a potential function, that is, $\mathbf{f}(\mathbf{x},t) = -\nabla\phi(\mathbf{x},t)$ for some scalar function ϕ, it is well-known that any Leray-Hopf weak solution becomes regular eventually and decays in $H^1(\Omega)$-norm exponentially. The first precise asymptotic behaviour is proved by Foias and Saut [5]. Namely, for any nontrivial, regular solution $u(t)$ in bounded or periodic domains, there exist an eigenvalue λ of the Stokes operator and a corresponding eigenfunction ξ such that

$$\lim_{t\to\infty} e^{\lambda t}u(t) = \xi, \quad \text{where the limit holds in all Sobolev norms.}$$

Moreover, they showed in [7] that any such solution admits an asymptotic expansion

$$u(t) \overset{\text{exp.}}{\sim} \sum_{n=1}^\infty q_n(t)e^{-\mu_n t} \tag{1.5}$$

in Sobolev spaces $H^m(\Omega)^3$ for all $m \geqslant 0$. Here, $\{\mu_n : n \in \mathbb{N}\}$ is the additive semi-group generated by the spectrum of the Stokes operator. It was then improved in [18], for the case of periodic domains, that the expansion holds in any Gevrey spaces $G_{\alpha,\sigma}$, see § 2 for details.

Studying the asymptotic expansion (1.5) leads to theories of associated normalization map and invariant nonlinear manifolds [4–7, 9], Poincaré-Dulac normal form

https://doi.org/10.1017/prm.2018.154 Published online by Cambridge University Press

for NSE [11, 14, 16]; they were applied to the analysis of helicity, statistical solutions of the NSE, and decaying turbulence [12, 13]. It provides fine details for the long-time behaviour of the solutions, and sheds some insights into the nonlinear structure of NSE. See also [20] for a result in $\mathbb{R}^3$, [22] for expansions for dissipative wave equations, and the survey paper [17] for more information on the subject.

Regarding the problem of establishing the expansion (1.5), the simplified approach in [18], for NSE in the periodic domains, turns out to be easily adapted to the case of nonpotential forces [19]. We recall here a result in this direction – theorem 2.2 of [19].

Assume there exists $\sigma \geqslant 0$, such that

$$f(t) \overset{\text{exp.}}{\sim} \sum_{n=1}^{\infty} f_n(t)e^{-nt} \quad \text{in } G_{\alpha,\sigma} \text{ for all } \alpha \geqslant 0. \tag{1.6}$$

Then any Leray-Hopf weak solution $u(t)$ of (1.1) and (1.2) admits an asymptotic expansion

$$u(t) \overset{\text{exp.}}{\sim} \sum_{n=1}^{\infty} q_n(t)e^{-nt} \text{ in } G_{\alpha,\sigma} \text{ for all } \alpha \geqslant 0. \tag{1.7}$$

We note that the expansion (1.6) of f is of type (1.3), and so are the expansions (1.5) and (1.7). A natural question arising is whether one can establish the same results for other types of decaying forces. This paper studies a particular case when f has an asymptotic expansion of type (1.4) instead. More specifically, assume there exist $\alpha \geqslant 1/2$ and $\sigma \geqslant 0$ such that

$$f(t) \sim \sum_{n=1}^{\infty} \psi_n t^{-\gamma_n} \quad \text{in } G_{\alpha,\sigma}. \tag{1.8}$$

We will derive a corresponding expansion for solutions of NSE. First, rewrite (1.8) as

$$f(t) \sim \sum_{n=1}^{\infty} \phi_n t^{-\mu_n} \quad \text{in } G_{\alpha,\sigma},$$

where $(\mu_n)_{n=1}^{\infty}$ is an appropriate sequence of powers generated by γ_n's.

We prove that there exist $\xi_n \in G_{\alpha+1,\sigma}$, for all $n \in \mathbb{N}$, which are explicitly determined by ϕ_n's, such that any Leray-Hopf weak solution $u(t)$ will admit the following expansion

$$u(t) \sim \sum_{n=1}^{\infty} \xi_n t^{-\mu_n} \quad \text{in } G_{\alpha+1-\rho,\sigma}, \text{ for all } \rho \in (0,1). \tag{1.9}$$

The expansion (1.9) has the following new features.

(a) All Leray-Hopf weak solutions have the *same expansion*, depending only on the force, regardless even their uniqueness and global regularity.

(b) The expansion of solution u is established with the force f belonging to a Sobolev-Gevrey space $G_{\alpha,\sigma}$ for *fixed* α and σ. This contrasts with the requirement in (1.6) that $f \in G_{\alpha,\sigma}$ for *all* $\alpha \geqslant 0$.

We also note that the force in (1.8), though decays to zero, is much larger than the one in (1.6), as $t \to \infty$.

Although our proof follows the scheme installed in [**7**] and [**18**, **19**], we take advantage of the new structure of the force f to make significant improvements in estimates, and succeed in quantifying the effects of such structure on that of the solution u.

Since the expansion (1.9) is convergent in many cases, we investigate what may be the next approximation of the solution after this expansion. Specifically, if $f(t) = \sum_{n=1}^\infty \phi_n t^{-n}$ and $\bar{u}(t) \stackrel{\text{def}}{=} \sum_{n=1}^\infty \xi_n t^{-n}$ are uniformly convergent in appropriate spaces for large t, then $u(t) - \bar{u}(t)$ is proved to decay at least at the rate $t^\beta e^{-t}$, as $t \to \infty$, for some number $\beta \geqslant 0$. This result rules out any intermediate approximation of u after $\bar{u}$ that is between the power and exponential decays.

The paper is organized as follows. Section 2 reviews the functional setting for NSE, some basic inequalities for Sobolev and Gevrey norms, and estimates for the bi-linear form $B(u, v)$ in NSE. Lemma 2.2 describes the asymptotic behaviour of certain integrals which will be utilized repeatedly in asymptotic estimates for large time. Particularly, it is used in lemma 2.3 to establish the limit, as $t \to \infty$, and the remainder estimates for solutions of certain linearized NSE. This will be a building block of proving the asymptotic expansion (1.9). In § 3, we establish the power-decay for any Leray-Hopf weak solutions, cf. theorem 3.2. It combines standard energy estimates, when time is large, with theorem 3.1, which proves strong asymptotic bounds for solutions in Gevrey spaces when the initial data and the force are small. In § 4, the asymptotic expansion (1.9) is obtained, either as a finite sum in theorem 4.1, or an infinite sum in theorem 4.3. As mentioned in remarks (a) and (b) above, the same expansion holds for *all* Leray-Hopf weak solutions, and only requires the force f to belong to a *fixed* Sobolev-Gevrey space, namely, $G_{\alpha,\sigma}$. Moreover, the expansion of the solution u holds in even more regular space, $G_{\alpha+1-\rho,\sigma}$, than that of f. This feature is possible because of the higher regularity for the elements ξ_n's in lemma 4.2, and the remainder estimate in lemma 2.3. It is also worth mentioning that the ξ_n's are explicitly determined by the recursive formulas (4.6) and (4.7) without solving any ordinary differential equations (in functional spaces) which was the case for the expansions (1.5) and (1.7). Section 5 deals with the convergence of the expansions, and the range of ξ_n's as the force varies. In case $\gamma_n = \mu_n = n$ for all n, it turns out that the expansion (1.9) can be any finite sum, or an infinite sum with the norms $\|\xi_n\|_{G_{\alpha+1,\sigma}}$ decaying in a certain, but still very general, way, see theorem 5.1, example 5.2 and corollary 5.3. Since the sequence $(\xi_n)_{n=1}^\infty$ completely determines the asymptotic expansion (1.9), it plays a similar role to the normalization map W in [**7**, **9**]. Therefore, a number of comparisons between them are made in remark 5.5. Another topic in this section is to find out what will be the next approximation of the solution u after the expansion (1.9). It is proved in theorem 5.6 that, in case the expansion converges, say, to $\bar{u}$, the remainder $u - \bar{u}$ must decay exponentially.

2. Preliminaries

2.1. Background for NSE

Let $L^2(\Omega) = H^0(\Omega)$ and $H^m(\Omega) = W^{m,2}(\Omega)$, for integers $m \geqslant 0$, denote the standard Lebesgue and Sobolev spaces on Ω. The standard inner product and norm in $L^2(\Omega)^3$ are denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$, respectively. (We warn that this notation $|\cdot|$ also denotes the Euclidean norm in $\mathbb{R}^n$ and $\mathbb{C}^n$, for any $n \in \mathbb{N}$, but its meaning will be clear based on the context.)

Let $\mathcal{V}$ be the set of all 2π-periodic trigonometric polynomial vector fields which are divergence-free and have zero average over Ω. Define

$$H, \text{ resp. } V = \text{ closure of } \mathcal{V} \text{ in } L^2(\Omega)^3, \text{ resp. } H^1(\Omega)^3.$$

Notice that each element of H is divergence-free and has zero average over Ω, and each element of V is 2π-periodic.

We use the following embeddings and identification

$$V \subset H = H' \subset V',$$

where each space is dense in the next one, and the embeddings are compact.

Let $\mathcal{P}$ denote the orthogonal (Leray) projection in $L^2(\Omega)^3$ onto H.

The Stokes operator A is a bounded linear mapping from V to its dual space V' defined by

$$\langle A\mathbf{u}, \mathbf{v} \rangle_{V',V} = \langle\!\langle \mathbf{u}, \mathbf{v} \rangle\!\rangle \stackrel{\text{def}}{=} \sum_{j=1}^{3} \left\langle \frac{\partial \mathbf{u}}{\partial x_j}, \frac{\partial \mathbf{v}}{\partial x_j} \right\rangle \quad \text{for all } \mathbf{u}, \mathbf{v} \in V.$$

As an unbounded operator on H, the operator A has the domain $\mathcal{D}(A) = V \cap H^2(\Omega)^3$, and, under the current consideration of periodicity conditions,

$$A\mathbf{u} = -\mathcal{P}\Delta\mathbf{u} = -\Delta\mathbf{u} \in H \quad \text{for all} \quad \mathbf{u} \in \mathcal{D}(A).$$

The spectrum of A is known to be

$$\sigma(A) = \{|\mathbf{k}|^2 : \ \mathbf{k} \in \mathbb{Z}^3, \mathbf{k} \neq \mathbf{0}\},$$

and each $\lambda \in \sigma(A)$ is an eigenvalue. Note that $\sigma(A) \subset \mathbb{N}$ and $1 \in \sigma(A)$, hence, the additive semigroup generated by $\sigma(A)$ is $\mathbb{N}$.

For $n \in \sigma(A)$, we denote by R_n the orthogonal projection in H on the eigenspace of A corresponding to n, and set

$$P_n = \sum_{j \in \sigma(A), j \leqslant n} R_j.$$

Note that each vector space $P_n H$ is finite dimensional.

For $\alpha, \sigma \in \mathbb{R}$ and $\mathbf{u} = \sum_{\mathbf{k}\neq\mathbf{0}} \widehat{\mathbf{u}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}$, define

$$A^\alpha \mathbf{u} = \sum_{\mathbf{k}\neq\mathbf{0}} |\mathbf{k}|^{2\alpha}\widehat{\mathbf{u}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}, \quad e^{\sigma A^{1/2}}\mathbf{u} = \sum_{\mathbf{k}\neq\mathbf{0}} e^{\sigma|\mathbf{k}|}\widehat{\mathbf{u}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}},$$

and, hence,

$$A^\alpha e^{\sigma A^{1/2}}\mathbf{u} = e^{\sigma A^{1/2}}A^\alpha\mathbf{u} = \sum_{\mathbf{k}\neq\mathbf{0}} |\mathbf{k}|^{2\alpha}e^{\sigma|\mathbf{k}|}\widehat{\mathbf{u}}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}.$$

For $\alpha \in \mathbb{R}$, $\sigma > 0$ or $\alpha \geqslant 0$, $\sigma = 0$, the Gevrey spaces are defined by

$$G_{\alpha,\sigma} = \mathcal{D}(A^\alpha e^{\sigma A^{1/2}}) \overset{\text{def}}{=} \{\mathbf{u} \in H : |\mathbf{u}|_{\alpha,\sigma} \overset{\text{def}}{=} |A^\alpha e^{\sigma A^{1/2}}\mathbf{u}| < \infty\}.$$

In particular, when $\sigma = 0$, the domain of the fractional operator A^α is

$$\mathcal{D}(A^\alpha) = G_{\alpha,0} = \{\mathbf{u} \in H : |A^\alpha\mathbf{u}| = |\mathbf{u}|_{\alpha,0} < \infty\} \quad \text{for } \alpha \geqslant 0.$$

Clearly, each space $G_{\alpha,\sigma}$ with the norm $|\cdot|_{\alpha,\sigma}$ is a Banach space.

Thanks to the zero-average condition, the norm $|A^{m/2}\mathbf{u}|$ is equivalent to $\|\mathbf{u}\|_{H^m(\Omega)^3}$ on the space $\mathcal{D}(A^{m/2})$ for $m = 0, 1, 2, \ldots$

Note that $\mathcal{D}(A^0) = H$, $\mathcal{D}(A^{1/2}) = V$, and $\|\mathbf{u}\| \overset{\text{def}}{=} |\nabla\mathbf{u}|$ is equal to $|A^{1/2}\mathbf{u}|$ for $\mathbf{u} \in V$. Also, the norms $|\cdot|_{\alpha,\sigma}$ are increasing in α, σ, hence, the spaces $G_{\alpha,\sigma}$ are decreasing in α, σ.

Regarding the nonlinear term in the NSE, a bounded linear map $B : V \times V \to V'$ is defined by

$$\langle B(\mathbf{u}, \mathbf{v}), \mathbf{w}\rangle_{V',V} = b(\mathbf{u}, \mathbf{v}, \mathbf{w}) \overset{\text{def}}{=} \int_\Omega ((\mathbf{u}\cdot\nabla)\mathbf{v})\cdot\mathbf{w}\,\mathrm{d}\mathbf{x}, \quad \text{for all} \quad \mathbf{u}, \mathbf{v}, \mathbf{w} \in V.$$

In particular,

$$B(\mathbf{u}, \mathbf{v}) = \mathcal{P}((\mathbf{u}\cdot\nabla)\mathbf{v}), \quad \text{for all} \quad \mathbf{u}, \mathbf{v} \in \mathcal{D}(A).$$

The problems (1.1) and (1.2) can now be rewritten in the functional form as

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} + Au(t) + B(u(t), u(t)) = f(t) \quad \text{in } V' \text{ on } (0, \infty), \tag{2.1}$$

$$u(0) = u^0 \in H. \tag{2.2}$$

(We refer the reader to the books [**1**, **21**, **23**, **25**] for more details.)

The next definition makes precise the meaning of weak solutions of (2.1).

Definition 2.1. Let $f \in L^2_{\mathrm{loc}}([0,\infty),H)$. A *Leray-Hopf weak solution* $u(t)$ of (2.1) is a mapping from $[0,\infty)$ to H such that

$$u \in C([0,\infty),H_{\mathrm{w}}) \cap L^2_{\mathrm{loc}}([0,\infty),V), \quad u' \in L^{4/3}_{\mathrm{loc}}([0,\infty),V'),$$

and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u(t),v\rangle + \langle\!\langle u(t),v\rangle\!\rangle + b(u(t),u(t),v) = \langle f(t),v\rangle \tag{2.3}$$

in the distribution sense in $(0,\infty)$, for all $v \in V$, and the energy inequality

$$\frac{1}{2}|u(t)|^2 + \int_{t_0}^t \|u(\tau)\|^2\,\mathrm{d}\tau \leqslant \frac{1}{2}|u(t_0)|^2 + \int_{t_0}^t \langle f(\tau),u(\tau)\rangle\,\mathrm{d}\tau \tag{2.4}$$

holds for $t_0 = 0$ and almost all $t_0 \in (0,\infty)$, and all $t \geqslant t_0$. Here, H_{w} denotes the topological vector space H with the weak topology.

If a Leray-Hopf weak solution belongs to $C([0,\infty),V)$, it is called a regular solution.

If $T \geqslant 0$ and $t \mapsto u(T+t)$ is a regular solution, then we say u is a regular solution on $[T,\infty)$.

We denote by $\mathcal{T}$ the set of $t_0 \geqslant 0$ such that (2.4) holds for all $t \geqslant t_0$. Then $\mathbb{R}\setminus\mathcal{T}$ has zero measure.

We assume throughout the paper that

(A) *The function f belongs to $L^\infty_{\mathrm{loc}}([0,\infty),H)$.*

Under assumption (A), for any $u^0 \in H$, there exists a Leray-Hopf weak solution $u(t)$ of (2.1) and (2.2), see for example, [**10**]. The large-time behaviour of $u(t)$ is the focus of our study. More specific conditions on f will be imposed later.

We note that, thanks to remark 1(e) of [**15**], the Leray-Hopf weak solutions in definition 2.1 are the same as the weak solutions used in [**10**, Chapter II, § 7], even though they have slightly different formulations. Hence, according to inequality (A.39) in [**10**, Chapter II], we have for any Leray-Hopf weak solution $u(t)$ (in definition 2.1) that

$$|u(t)|^2 \leqslant e^{-t}|u(0)|^2 + \int_0^t e^{-(t-\tau)}|f(\tau)|^2\,\mathrm{d}\tau \quad \forall t > 0. \tag{2.5}$$

2.2. Basic inequalities

Below are some inequalities that will be needed in later estimates. First, for any $\sigma,\alpha > 0$, one has

$$\max_{x\geqslant 0}(x^\alpha e^{-\sigma x}) = d_0(\alpha,\sigma) \stackrel{\mathrm{def}}{=} \left(\frac{\alpha}{e\sigma}\right)^\alpha, \tag{2.6}$$

and, hence,

$$e^{-\sigma x} = e^{-\sigma(x+1)}e^\sigma \leqslant d_0(\alpha,\sigma)e^\sigma(1+x)^{-\alpha} \quad \forall x \geqslant 0. \tag{2.7}$$

https://doi.org/10.1017/prm.2018.154 Published online by Cambridge University Press

Thanks to (2.6), one can verify, for all $\alpha, \sigma > 0$, that

$$|A^\alpha e^{-\sigma A} v| \leqslant d_0(\alpha, \sigma)|v| \quad \forall v \in H, \tag{2.8}$$

$$|A^\alpha e^{-\sigma A^{1/2}} v| \leqslant d_0(2\alpha, \sigma)|v| \quad \forall v \in H,$$

and, consequently,

$$|A^\alpha v| = |(A^\alpha e^{-\sigma A^{1/2}}) e^{\sigma A^{1/2}} v| \leqslant d_0(2\alpha, \sigma)|e^{\sigma A^{1/2}} v| \quad \forall v \in G_{0,\sigma}. \tag{2.9}$$

For the bi-linear mapping $B(u, v)$, it follows from its boundedness that there exists a constant $K_* > 0$ such that

$$\|B(u, v)\|_{V'} \leqslant K_* \|u\| \|v\| \quad \forall u, v \in V. \tag{2.10}$$

For stronger norms of $B(u, v)$, we recall from [**18**, lemma 2.1] a convenient inequality. (See Foias-Temam paper [**8**] for the original version.)

There exists a constant $K > 1$ such that if $\sigma \geqslant 0$ and $\alpha \geqslant 1/2$, then

$$|B(u, v)|_{\alpha,\sigma} \leqslant K^\alpha |u|_{\alpha+1/2,\sigma} |v|_{\alpha+1/2,\sigma} \quad \forall u, v \in G_{\alpha+1/2,\sigma}. \tag{2.11}$$

Lemma 2.2. *Let $\sigma, \lambda > 0$. One has, for all $t \geqslant 0$, that*

$$\int_0^t \frac{e^{-\sigma(t-\tau)}}{(1+\tau)^\lambda} \, \mathrm{d}\tau \leqslant \frac{d_1(\lambda, \sigma)}{(1+t)^\lambda}, \tag{2.12}$$

where

$$d_1(\lambda, \sigma) \stackrel{\text{def}}{=} 2^\lambda (d_0(\lambda + 1, \sigma) e^\sigma + \sigma^{-1}) = 2^\lambda \left[\left(\frac{\lambda + 1}{e\sigma} \right)^{\lambda+1} e^\sigma + \frac{1}{\sigma} \right].$$

Proof. First, we have

$$\begin{aligned} I &\stackrel{\text{def}}{=} \int_0^t \frac{e^{-\sigma(t-\tau)}}{(1+\tau)^\lambda} \, \mathrm{d}\tau = \int_0^{t/2} \frac{e^{-\sigma(t-\tau)}}{(1+\tau)^\lambda} \, \mathrm{d}\tau + \int_{t/2}^t \frac{e^{-\sigma(t-\tau)}}{(1+\tau)^\lambda} \, \mathrm{d}\tau \\ &\leqslant \int_0^{t/2} e^{-\sigma t/2} \, \mathrm{d}\tau + \frac{1}{(1+t/2)^\lambda} \int_{t/2}^t e^{-\sigma(t-\tau)} \, \mathrm{d}\tau \leqslant \frac{t}{2} e^{-\sigma t/2} + \frac{1}{(1+t/2)^\lambda} \cdot \frac{1}{\sigma}. \end{aligned}$$

Note, by (2.7), that

$$e^{-\sigma t/2} \leqslant \frac{d_0(\lambda + 1, \sigma) e^\sigma}{(1 + t/2)^{\lambda+1}},$$

which implies

$$\frac{t}{2} e^{-\sigma t/2} \leqslant \frac{C}{(1+t/2)^\lambda}, \quad \text{where } C = d_0(\lambda + 1, \sigma) e^\sigma.$$

Thus, we obtain

$$I \leqslant \frac{1}{(1+t/2)^\lambda} \left(C + \frac{1}{\sigma} \right) \leqslant \frac{2^\lambda}{(1+t)^\lambda} \left(C + \frac{1}{\sigma} \right) = \frac{d_1(\sigma, \lambda)}{(1+t)^\lambda},$$

which proves inequality (2.12). □

2.3. Large-time behaviour of solutions of the linearized NSE

We discuss the asymptotic behaviour, in Sobolev-Gevrey spaces, of weak solutions of the linearized NSE.

LEMMA 2.3. *Let $\alpha, \sigma \geqslant 0$, $\xi \in G_{\alpha,\sigma}$, and f be a function from $(0,\infty)$ to $G_{\alpha,\sigma}$ that satisfies*

$$|f(t)|_{\alpha,\sigma} \leqslant M(1+t)^{-\lambda} \quad \text{a.e. in } (0,\infty) \text{ for some } M>0. \tag{2.13}$$

Suppose

$$w \in C([0,\infty), H_{\rm w}) \cap L^1_{\rm loc}([0,\infty), V), \quad \text{with } w' \in L^1_{\rm loc}([0,\infty), V'), \tag{2.14}$$

is a weak solution of

$$w' = -Aw + \xi + f \text{ in } V' \text{ on } (0,\infty), \tag{2.15}$$

that is, it holds, for all $v \in V$, that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle w, v\rangle = -\langle\langle w, v\rangle\rangle + \langle \xi + f, v\rangle \quad \text{in the distribution sense on } (0,\infty). \tag{2.16}$$

Assume $w(0) = w_0 \in G_{\alpha,\sigma}$. Then, for any $\varepsilon \in (0,1)$, there exists $C>0$ depending on ε, λ, M, $|\xi|_{\alpha,\sigma}$ and $|w_0|_{\alpha,\sigma}$ such that

$$|w(t) - A^{-1}\xi|_{\alpha+1-\varepsilon,\sigma} \leqslant C(1+t)^{-\lambda} \quad \forall t \geqslant 1. \tag{2.17}$$

Proof. Let $N \in \sigma(A)$, and set $A_N = A|_{P_N H}$ which is an invertible linear map from $P_N H$ onto itself. By taking $v \in P_N H$ in equation (2.16), we deduce that $P_N w$ solves, in the $P_N H$-valued distribution sense on $(0,\infty)$, the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}(P_N w) = -A_N(P_N w) + P_N\xi + P_N f \quad \text{in } P_N H \text{ on } (0,\infty). \tag{2.18}$$

Since $P_N H$ is a finite-dimensional Euclidean space, one has $P_N w \in C([0,\infty), P_N H)$ and $(P_N w)' \in L^1_{\rm loc}([0,\infty), P_N H)$. Then the variation of constants formula still holds true for the solution $P_N w$ of (2.18). (See, e.g., the arguments in [**19**, lemma 4.2].) We have, for any $t>0$,

$$\begin{aligned} P_N w(t) &= e^{-tA_N} P_N w_0 + \int_0^t e^{-(t-\tau)A_N}(P_N\xi + P_N f(\tau))\,\mathrm{d}\tau \\ &= e^{-tA_N} P_N w_0 + A_N^{-1}(P_N\xi - e^{-tA_N}P_N\xi) + \int_0^t e^{-(t-\tau)A_N} P_N f(\tau)\,\mathrm{d}\tau \\ &= e^{-tA} P_N w_0 + A^{-1}(P_N\xi - e^{-tA}P_N\xi) + \int_0^t e^{-(t-\tau)A} P_N f(\tau)\,\mathrm{d}\tau, \end{aligned}$$

which yields

$$P_N\big(w(t) - A^{-1}\xi\big) = e^{-tA}P_N w_0 - A^{-1}e^{-tA}P_N\xi + \int_0^t e^{-(t-\tau)A}P_N f(\tau)\,\mathrm{d}\tau. \tag{2.19}$$

Let $\varepsilon \in (0,1)$. Applying $A^{1-\varepsilon}$ to both sides of (2.19), and estimating the $|\cdot|_{\alpha,\sigma}$ norm of the resulting quantities, we obtain, for all $t > 0$,

$$|P_N(w(t) - A^{-1}\xi)|_{\alpha+1-\varepsilon,\sigma} \leqslant |A^{1-\varepsilon}e^{-tA}w_0|_{\alpha,\sigma} + |A^{-\varepsilon}e^{-tA}\xi|_{\alpha,\sigma} + \int_0^t |e^{-(t-\tau)A}A^{1-\varepsilon}f(\tau)|_{\alpha,\sigma}\,\mathrm{d}\tau. \tag{2.20}$$

We find bounds for each term on the right-hand side of the preceding inequality.

- Firstly, for $t \geqslant 1$, rewriting the first term on the right-hand side of (2.20) and applying (2.8) yield

$$|A^{1-\varepsilon}e^{-tA}w_0|_{\alpha,\sigma} = |A^{1-\varepsilon}e^{-tA/2}e^{-tA/2}w_0|_{\alpha,\sigma} \leqslant \left[\frac{(1-\varepsilon)}{et/2}\right]^{1-\varepsilon} |e^{-tA/2}w_0|_{\alpha,\sigma} \leqslant \left[\frac{2(1-\varepsilon)}{e}\right]^{1-\varepsilon} e^{-t/2}|w_0|_{\alpha,\sigma}.$$

  To compare $e^{-t/2}$ and $(1+t)^{-\lambda}$, we apply (2.7) to obtain

$$|A^{1-\varepsilon}e^{-tA}w_0|_{\alpha,\sigma} \leqslant \left[\frac{2(1-\varepsilon)}{e}\right]^{1-\varepsilon} \frac{d_0(\lambda,1/2)e^{1/2}}{(1+t)^\lambda}|w_0|_{\alpha,\sigma}. \tag{2.21}$$

- Secondly, the second term on the right-hand side of (2.20) can be easily estimated by

$$|A^{-\varepsilon}e^{-tA}\xi|_{\alpha,\sigma} \leqslant |e^{-tA}\xi|_{\alpha,\sigma} \leqslant e^{-t}|\xi|_{\alpha,\sigma} \leqslant \frac{d_0(\lambda,1)e}{(1+t)^\lambda}|\xi|_{\alpha,\sigma}. \tag{2.22}$$

- Thirdly, dealing with the last integral in (2.20), we split it into two integrals

$$\int_0^t |e^{-(t-\tau)A}A^{1-\varepsilon}f(\tau)|_{\alpha,\sigma}\,\mathrm{d}\tau = I_1 + I_2, \tag{2.23}$$

  where

$$I_1 = \int_0^{t/2} |e^{-(t-\tau)A}A^{1-\varepsilon}f(\tau)|_{\alpha,\sigma}\,\mathrm{d}\tau, \quad I_2 = \int_{t/2}^t |e^{-(t-\tau)A}A^{1-\varepsilon}f(\tau)|_{\alpha,\sigma}\,\mathrm{d}\tau.$$

For I_1, we have for $t \geqslant 1$ that

$$I_1 = \int_0^{t/2} \left|e^{-(t-\tau)A/2}\Big(e^{-(t-\tau)A/2}A^{1-\varepsilon}f(\tau)\Big)\right|_{\alpha,\sigma}\mathrm{d}\tau \leqslant \int_0^{t/2} \left|e^{-(t/4)A}A^{1-\varepsilon}e^{-(t-\tau)A/2}f(\tau)\right|_{\alpha,\sigma}\mathrm{d}\tau.$$

https://doi.org/10.1017/prm.2018.154 Published online by Cambridge University Press

Utilizing (2.8) and then using hypothesis (2.13), we obtain

$$\begin{aligned}I_1 &\leqslant \int_0^{t/2} \left[\frac{1-\varepsilon}{et/4}\right]^{1-\varepsilon} |e^{-(t-\tau)A/2} f(\tau)|_{\alpha,\sigma}\,\mathrm{d}\tau \\ &\leqslant \left[\frac{1-\varepsilon}{et/4}\right]^{1-\varepsilon} \int_0^{t/2} e^{-(t-\tau)/2} M(1+\tau)^{-\lambda}\,\mathrm{d}\tau \\ &= M\left[\frac{1-\varepsilon}{et/4}\right]^{1-\varepsilon} e^{-t/4} \int_0^{t/2} e^{-(t/2-\tau)/2}(1+\tau)^{-\lambda}\,\mathrm{d}\tau.\end{aligned}$$

Then by lemma 2.2

$$I_1 \leqslant M\left[\frac{4(1-\varepsilon)}{et}\right]^{1-\varepsilon} e^{-t/4}\frac{d_1(\lambda,1/2)}{(1+t/2)^{\lambda}} \leqslant M\left[\frac{4(1-\varepsilon)}{e}\right]^{1-\varepsilon} \frac{e^{-t/4}}{t^{1-\varepsilon}}\frac{2^{\lambda} d_1(\lambda,1/2)}{(1+t)^{\lambda}}.$$

Thus, for $t \geqslant 1$

$$I_1 \leqslant M\left[\frac{4(1-\varepsilon)}{e}\right]^{1-\varepsilon} e^{-1/4} 2^{\lambda} d_1(\lambda,1/2)\frac{1}{(1+t)^{\lambda}}. \tag{2.24}$$

For I_2, we apply (2.8) and use (2.13) to estimates its integrand, for $t/2 < \tau < t$, by

$$\begin{aligned}&|e^{-(t-\tau)A}A^{1-\varepsilon} f(\tau)|_{\alpha,\sigma} \leqslant e^{-(t-\tau)/2}|e^{-(t-\tau)A/2}A^{1-\varepsilon}f(\tau)|_{\alpha,\sigma} \\ &\quad\leqslant e^{-(t-\tau)/2}\left[\frac{1-\varepsilon}{e(t-\tau)/2}\right]^{1-\varepsilon} |f(\tau)|_{\alpha,\sigma} \leqslant \left[\frac{2(1-\varepsilon)}{e}\right]^{1-\varepsilon} \frac{e^{-(t-\tau)/2}}{(t-\tau)^{1-\varepsilon}}\cdot\frac{M}{(1+\tau)^{\lambda}} \\ &\quad\leqslant \left[\frac{2(1-\varepsilon)}{e}\right]^{1-\varepsilon} \frac{M}{(1+t/2)^{\lambda}}\cdot\frac{e^{-(t-\tau)/2}}{(t-\tau)^{1-\varepsilon}}.\end{aligned}$$

Hence,

$$\begin{aligned}I_2 &\leqslant \left[\frac{2(1-\varepsilon)}{e}\right]^{1-\varepsilon} \frac{M}{(1+t/2)^{\lambda}} \int_{t/2}^{t} \frac{e^{-(t-\tau)/2}}{(t-\tau)^{1-\varepsilon}}\,\mathrm{d}\tau \\ &\leqslant \left[\frac{2(1-\varepsilon)}{e}\right]^{1-\varepsilon} \frac{2^{\lambda}M}{(1+t)^{\lambda}} \int_0^{t/2} \frac{e^{-z/2}}{z^{1-\varepsilon}}\,\mathrm{d}z.\end{aligned}$$

We estimate the last integral by

$$\begin{aligned}\int_0^{t/2} \frac{e^{-z/2}}{z^{1-\varepsilon}}\,\mathrm{d}z &= \int_0^{1/2} \frac{e^{-z/2}}{z^{1-\varepsilon}}\,\mathrm{d}z + \int_{1/2}^{t/2} \frac{e^{-z/2}}{z^{1-\varepsilon}}\,\mathrm{d}z \\ &\leqslant \int_0^{1/2} \frac{1}{z^{1-\varepsilon}}\,\mathrm{d}z + 2^{1-\varepsilon}\int_{1/2}^{t/2} e^{-z/2}\,\mathrm{d}z \\ &\leqslant 2^{-\varepsilon}\varepsilon^{-1} + 2^{2-\varepsilon}e^{-1/4} = 2^{-\varepsilon}(\varepsilon^{-1} + 4e^{-1/4}).\end{aligned}$$

Therefore,

$$I_2 \leqslant M\left[\frac{1-\varepsilon}{e}\right]^{1-\varepsilon} 2^{1-2\varepsilon+\lambda}(\varepsilon^{-1}+4e^{-1/4})\cdot\frac{1}{(1+t)^{\lambda}}. \tag{2.25}$$

Combining (2.20)–(2.25), we obtain

$$|P_N\big(w(t)-A^{-1}\xi\big)|_{\alpha+1-\varepsilon,\sigma} \leqslant C(1+t)^{-\lambda} \quad \forall t\geqslant 1, \tag{2.26}$$

with constant C independent of N. Since $A^{-1}\xi$ belongs to $G_{\alpha+1-\varepsilon,\sigma}$, this bound shows that $w(t)$ also belongs to $G_{\alpha+1-\varepsilon,\sigma}$. By passing $N\to\infty$ in (2.26), we obtain (2.17). The proof is complete. □

The particular case $\xi=0$ has a special meaning, and we state the result separately here.

LEMMA 2.4. *Let $\alpha,\sigma\geqslant 0$, and suppose f is a function in $L^\infty_{\rm loc}([0,\infty),G_{\alpha,\sigma})$. Let w satisfy* (2.14) *and be a weak solution of*

$$w' = -Aw + f \ \textit{in}\ V' \ \textit{on}\ (0,\infty).$$

(i) *Then $w(t)\in G_{\alpha+1-\varepsilon,\sigma}$ for all $\varepsilon\in(0,1)$ and $t>0$.*

(ii) *If, in addition, f satisfies* (2.13)*, then, for any $\varepsilon\in(0,1)$, there exists $C>0$ depending on ε, λ, M and $|w(0)|_{\alpha,\sigma}$ such that*

$$|w(t)|_{\alpha+1-\varepsilon,\sigma}\leqslant C(1+t)^{-\lambda}\quad \forall t\geqslant 1. \tag{2.27}$$

Proof. We set $\xi=0$ in (2.15) and follow the proof of lemma 2.3. In this case, (2.20) reads, for all $t>0$, as

$$|P_Nw(t)|_{\alpha+1-\varepsilon,\sigma}\leqslant |A^{1-\varepsilon}e^{-tA}w_0|_{\alpha,\sigma}+\int_0^t |e^{-(t-\tau)A}A^{1-\varepsilon}f(\tau)|_{\alpha,\sigma}\,\mathrm{d}\tau. \tag{2.28}$$

(i) Let $T>0$. There is $M_0>0$ such that $|f(t)|_{\alpha,\sigma}\leqslant M_0$ a.e. in $(0,T)$. For $t\in(0,T)$, we use (2.8) to estimate

$$|A^{1-\varepsilon}e^{-tA}w_0|_{\alpha,\sigma}\leqslant\left[\frac{1-\varepsilon}{et}\right]^{1-\varepsilon}|w_0|_{\alpha,\sigma},$$

$$\begin{aligned}\int_0^t |e^{-(t-\tau)A}A^{1-\varepsilon}f(\tau)|_{\alpha,\sigma}\,\mathrm{d}\tau &\leqslant \left[\frac{1-\varepsilon}{e}\right]^{1-\varepsilon}\int_0^t\frac{|f(\tau)|_{\alpha,\sigma}}{(t-\tau)^{1-\varepsilon}}\,\mathrm{d}\tau\\ &\leqslant \left[\frac{1-\varepsilon}{e}\right]^{1-\varepsilon}\int_0^t\frac{M_0}{(t-\tau)^{1-\varepsilon}}\,\mathrm{d}\tau = \left[\frac{1-\varepsilon}{e}\right]^{1-\varepsilon}M_0\varepsilon^{-1}t^{\varepsilon}.\end{aligned}$$

https://doi.org/10.1017/prm.2018.154 Published online by Cambridge University Press

Utilizing these estimates, we can pass $N \to \infty$ in (2.28), and obtain

$$|w(t)|_{\alpha+1-\varepsilon,\sigma} \leqslant \left[\frac{1-\varepsilon}{e}\right]^{1-\varepsilon} \Big(t^{\varepsilon-1}|w_0|_{\alpha,\sigma} + M_0\varepsilon^{-1}t^{\varepsilon}\Big),$$

thus, $w(t) \in G_{\alpha+1-\varepsilon,\sigma}$.

(ii) This part is the same as lemma 2.3, and (2.27) follows (2.17). □

3. Asymptotic estimates for the Leray-Hopf weak solutions

The goal of this section is to establish the power-decay for any Leray-Hopf weak solutions whenever the force is power-decaying. The first theorem concerns the Gevrey estimates for the solutions for positive time when the initial data is small in a Sobolev norm, and the force is small in a Gevrey norm.

THEOREM 3.1. *Let $\lambda > 0$, $\sigma \geqslant 0$, and $\alpha \geqslant 1/2$ be given numbers. Suppose*

$$|A^{\alpha}u^0| \leqslant c_0, \tag{3.1}$$

$$|f(t)|_{\alpha-1/2,\sigma} \leqslant c_1(1+t)^{-\lambda} \quad \text{a.e. in } (0,\infty), \tag{3.2}$$

where

$$c_0 = c_0(\alpha,\lambda) \stackrel{\text{def}}{=} \frac{c_*}{\max\{1,\sqrt{M_1}\}} \quad \text{and} \quad c_1 = c_1(\alpha,\lambda) \stackrel{\text{def}}{=} \frac{c_*}{\sqrt{3M_2}}, \tag{3.3}$$

with

$$c_* = c_{*,\alpha} \stackrel{\text{def}}{=} \frac{1}{12K^{\alpha}}, \quad M_1 = M_{1,\lambda} \stackrel{\text{def}}{=} d_0(2\lambda,1)e, \quad M_2 = M_{2,\lambda} \stackrel{\text{def}}{=} d_1(2\lambda,1).$$

Then there exists a unique regular solution $u(t)$ of (2.1) and (2.2), which, furthermore, satisfies $u \in C([0,\infty),\mathcal{D}(A^{\alpha}))$ and

$$|u(t)|_{\alpha,\sigma} \leqslant \sqrt{2}c_*(1+t)^{-\lambda} \quad \forall t \geqslant t_*, \tag{3.4}$$

$$\int_t^{t+1} |u(\tau)|^2_{\alpha+1/2,\sigma}\, \mathrm{d}\tau \leqslant 2c_*^2\left(1+\frac{1}{2M_2}\right)(1+t)^{-2\lambda} \quad \forall t \geqslant t_*, \tag{3.5}$$

where $t_* = 12\sigma$.

Proof. We will perform formal calculations below. They can be made rigorous by applying to solutions of the Galerkin approximations and then pass to the limit.

(a) *Case $\sigma > 0$.* We denote by φ a C^{∞}-function on $\mathbb{R}$ that satisfies $\varphi(t) = 0$ on $(-\infty,0]$, $\varphi(t) = \sigma$ on $[t_*,\infty)$, and $0 < \varphi'(t) < 2\sigma/t_* = 1/6$ on $(0,t_*)$.

We derive from (2.1) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(A^\alpha e^{\varphi(t)A^{1/2}}u) = \varphi'(t)A^{1/2}A^\alpha e^{\varphi(t)A^{1/2}}u + A^\alpha e^{\varphi(t)A^{1/2}}\frac{\mathrm{d}u}{\mathrm{d}t}$$
$$= \varphi'(t)A^{\alpha+1/2}e^{\varphi(t)A^{1/2}}u + A^\alpha e^{\varphi(t)A^{1/2}}(-Au - B(u,u) + f). \tag{3.6}$$

By taking the inner product in H of (3.6) with $A^\alpha e^{\varphi(t)A^{1/2}}u(t)$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2_{\alpha,\varphi(t)} + |A^{1/2}u|^2_{\alpha,\varphi(t)} = \varphi'(t)\langle A^{\alpha+1/2}e^{\varphi(t)A^{1/2}}u, A^\alpha e^{\varphi(t)A^{1/2}}u\rangle$$
$$- \langle A^\alpha e^{\varphi(t)A^{1/2}}B(u,u), A^\alpha e^{\varphi(t)A^{1/2}}u\rangle$$
$$+ \langle A^{\alpha-1/2}e^{\varphi(t)A^{1/2}}f, A^{\alpha+1/2}e^{\varphi(t)A^{1/2}}u\rangle.$$

Using the Cauchy-Schwarz inequality, and estimating the second term on the right-hand side by (2.11), we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2_{\alpha,\varphi(t)} + |A^{1/2}u|^2_{\alpha,\varphi(t)} \leqslant \varphi'(t)|A^{1/2}u|^2_{\alpha,\varphi(t)}$$
$$+ K^\alpha|A^{1/2}u|^2_{\alpha,\varphi(t)}|u|_{\alpha,\varphi(t)}$$
$$+ |f(t)|_{\alpha-1/2,\varphi(t)}|A^{1/2}u|_{\alpha,\varphi(t)}. \tag{3.7}$$

Using the bound of $\varphi'(t)$ and applying Cauchy's inequality to the last term gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2_{\alpha,\varphi(t)} + |A^{1/2}u|^2_{\alpha,\varphi(t)} \leqslant \frac{1}{6}|A^{1/2}u|^2_{\alpha,\varphi(t)}$$
$$+ K^\alpha|u|_{\alpha,\varphi(t)}|A^{1/2}u|^2_{\alpha,\varphi(t)}$$
$$+ \frac{1}{6}|A^{1/2}u|^2_{\alpha,\varphi(t)} + \frac{3}{2}|f(t)|^2_{\alpha-1/2,\varphi(t)},$$

which, together with the fact $\varphi(t) \leqslant \sigma$, implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2_{\alpha,\varphi(t)} + \left(1 - \frac{1}{3} - K^\alpha|u|_{\alpha,\varphi(t)}\right)|A^{1/2}u|^2_{\alpha,\varphi(t)} \leqslant \frac{3}{2}|f(t)|^2_{\alpha-1/2,\sigma}. \tag{3.8}$$

(b) *Case* $\sigma = 0$. Let $\varphi(t) = 0$ for all $t \in \mathbb{R}$. Then the first term on the right-hand side of (3.7) vanishes. Applying Cauchy's inequality to the last term of (3.7), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|A^\alpha u|^2 + \left(1 - \frac{1}{3} - K^\alpha|A^\alpha u|\right)|A^{\alpha+1/2}u|^2 \leqslant \frac{3}{4}|A^{\alpha-1/2}f|^2 \leqslant \frac{3}{2}|A^{\alpha-1/2}f|^2. \tag{3.9}$$

Hence, we have the same inequality as (3.8).

https://doi.org/10.1017/prm.2018.154 Published online by Cambridge University Press

(c) For both cases $\sigma > 0$ and $\sigma = 0$, let $T \in (0, \infty)$. Note that

$$|u(0)|_{\alpha,\varphi(0)} = |A^\alpha u^0| < 2c_0 \leqslant 2c_*.$$

Assume that

$$|u(t)|_{\alpha,\varphi(t)} \leqslant 2c_* \quad \forall t \in [0, T). \tag{3.10}$$

This and the definition of c_* give

$$K^\alpha |u(t)|_{\alpha,\varphi(t)} \leqslant 2c_* K^\alpha = 1/6 \quad \forall t \in [0, T). \tag{3.11}$$

For $t \in (0, T)$, we have from (3.8), (3.9), and (3.11) that

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2_{\alpha,\varphi(t)} + |A^{1/2}u|^2_{\alpha,\varphi(t)} \leqslant 3|f(t)|^2_{\alpha-1/2,\sigma}. \tag{3.12}$$

Applying Gronwall's inequality in (3.12) yields for all $t \in (0, T)$ that

$$\begin{aligned} |u(t)|^2_{\alpha,\varphi(t)} &\leqslant e^{-t}|u^0|^2_{\alpha,0} + 3\int_0^t e^{-(t-\tau)}|f(\tau)|^2_{\alpha-1/2,\sigma}\,\mathrm{d}\tau \\ \text{(by (3.1) and (3.2))} \quad &\leqslant e^{-t}c_0^2 + 3c_1^2\int_0^t \frac{e^{-(t-\tau)}}{(1+\tau)^{2\lambda}}\,\mathrm{d}\tau. \end{aligned}$$

Using (2.7) to compare e^{-t} with $(1+t)^{-2\lambda}$, and estimating the last integral by (2.12) yield

$$|u(t)|^2_{\alpha,\varphi(t)} \leqslant \frac{M_1 c_0^2}{(1+t)^{2\lambda}} + \frac{3c_1^2 M_2}{(1+t)^{2\lambda}} \leqslant \frac{2c_*^2}{(1+t)^{2\lambda}}.$$

This implies

$$|u(t)|_{\alpha,\varphi(t)} \leqslant \sqrt{2}c_*(1+t)^{-\lambda} \quad \forall t \in [0, T). \tag{3.13}$$

Letting $t \to T^-$ in (3.13) gives

$$\lim_{t\to T^-} |u(t)|_{\alpha,\varphi(t)} \leqslant \sqrt{2}c_*(1+T)^{-\lambda} < 2c_*. \tag{3.14}$$

Comparing (3.14) with (3.10), and by the standard contradiction argument, we deduce that the inequalities (3.10) and (3.13) hold for $T = \infty$. Then, thanks to $\varphi(t) = \sigma$ for all $t \geqslant t_*$, inequality (3.13) implies (3.4).

(d) For $t \geqslant t_*$, by integrating (3.12) from t to $t+1$, and using estimates (3.4), (3.2), we obtain

$$\begin{aligned} \int_t^{t+1} |A^{1/2}u(\tau)|^2_{\alpha,\sigma}\,\mathrm{d}\tau &\leqslant |u(t)|^2_{\alpha,\sigma} + 3c_1^2\int_t^{t+1}(1+\tau)^{-2\lambda}\,\mathrm{d}\tau \\ &\leqslant 2c_*^2(1+t)^{-2\lambda} + 3c_1^2(1+t)^{-2\lambda} \\ &= \left(2c_*^2 + \frac{c_*^2}{M_2}\right)(1+t)^{-2\lambda}. \end{aligned}$$

Then inequality (3.5) follows. The proof is complete. □

https://doi.org/10.1017/prm.2018.154 Published online by Cambridge University Press

In the next theorem, we establish the power decay, as $t \to \infty$, for any Leray-Hopf weak solutions. Its proof combines the energy inequalities (2.4) and (2.5) with successive use of theorem 3.1.

THEOREM 3.2. *Assume that there are* $\sigma \geqslant 0$, $\alpha \geqslant 1/2$ *and* $\mu_1 > 0$ *such that*

$$|f(t)|_{\alpha,\sigma} = \mathcal{O}(t^{-\mu_1}) \quad \text{as } t \to \infty. \tag{3.15}$$

Let $u(t)$ *be a Leray-Hopf weak solution of* (2.1). *Then there exists* $T_* > 0$ *such that* $u(t)$ *is a regular solution of* (2.1) *on* $[T_*, \infty)$, *and for any* $\varepsilon \in (0,1)$, *there exists* $C > 0$ *such that*

$$|u(T_* + t)|_{\alpha+1-\varepsilon,\sigma} \leqslant C(1+t)^{-\mu_1}, \tag{3.16}$$

$$|B(u(T_* + t), u(T_* + t))|_{\alpha+1/2-\varepsilon,\sigma} \leqslant C(1+t)^{-2\mu_1}, \tag{3.17}$$

for all $t \geqslant 0$.

Proof. The proof is divided into two parts.

Part A. We prove the following weaker version of the statements.

For any $\lambda \in (0, \mu_1)$, *there exists* $T_* > 0$ *such that* $u(t)$ *is a regular solution of* (2.1) *on* $[T_*, \infty)$, *and one has for all* $t \geqslant 0$ *that*

$$|u(T_* + t)|_{\alpha+1/2,\sigma} \leqslant K^{-\alpha-1/2}(1+t)^{-\lambda}, \tag{3.18}$$

$$|B(u(T_* + t), u(T_* + t))|_{\alpha,\sigma} \leqslant K^{-\alpha-1}(1+t)^{-2\lambda}, \tag{3.19}$$

where K *is the constant in inequality* (2.11).

The proof of this part consists several steps.

Step 1. By assumption (A) and (3.15), there exists $M > 0$ such that

$$|f(t)| \leqslant M(1+t)^{-\mu_1} \quad \text{a.e. in } (0,\infty). \tag{3.20}$$

It follows (2.5) and (3.20) that, for all $t > 0$,

$$\begin{aligned} |u(t)|^2 &\leqslant e^{-t}|u_0|^2 + M^2 \int_0^t \frac{e^{-(t-\tau)}}{(1+\tau)^{2\mu_1}}\,\mathrm{d}\tau \\ (\text{by (2.7) and (2.12)}) &\leqslant C_1(1+t)^{-2\mu_1}|u_0|^2 + M^2 C_2 (1+t)^{-2\mu_1}, \end{aligned}$$

where $C_1 = d_0(2\mu_1, 1)e$ and $C_2 = d_1(2\mu_1, 1)$. Thus,

$$|u(t)|^2 \leqslant (|u_0|^2 C_1 + M^2 C_2)(1+t)^{-2\mu_1} \quad \forall t \geqslant 0. \tag{3.21}$$

In (2.4), we estimate

$$|\langle f(\tau), u(\tau)\rangle| \leqslant \frac{1}{2}|u(\tau)|^2 + \frac{1}{2}|f(\tau)|^2 \leqslant \frac{1}{2}\|u(\tau)\|^2 + \frac{1}{2}|f(\tau)|^2.$$

Hence, we obtain

$$|u(t)|^2 + \int_{t_0}^t \|u(\tau)\|^2\,\mathrm{d}\tau \leqslant |u(t_0)|^2 + \int_{t_0}^t |f(\tau)|^2\ \mathrm{d}\tau, \tag{3.22}$$

for all $t_0 \in \mathcal{T}$ and all $t \geqslant t_0$.

Letting $t = t_0 + 1$ in (3.22), using (3.21) to estimate $|u(t_0)|^2$, and (3.20) to estimate $|f(\tau)|$, we derive

$$\begin{aligned}\int_{t_0}^{t_0+1} \|u(\tau)\|^2\,\mathrm{d}\tau &\leqslant (|u_0|^2 C_1 + M^2 C_2)(1+t_0)^{-2\mu_1} + M^2(1+t_0)^{-2\mu_1}\\ &= (|u_0|^2 C_1 + M^2 C_2 + M^2)(1+t_0)^{-2\mu_1}. \qquad (3.23)\end{aligned}$$

To establish (3.23) for any t_0, we use the following approximation. Let $t \geqslant 0$ be arbitrary. There exists a sequence $\{t_n\}_{n=1}^{\infty} \subset \mathcal{T} \cap (0,\infty)$ such that $\lim_{n\to\infty} t_n = t$. By (3.23) with $t_0 := t_n$, we have

$$\int_{t_n}^{t_n+1} \|u(\tau)\|^2\,\mathrm{d}\tau \leqslant \Big(|u_0|^2 C_1 + M^2 C_2 + M^2\Big)(1+t_n)^{-2\mu_1}.$$

Then letting $n \to \infty$ gives

$$\int_{t}^{t+1} \|u(\tau)\|^2\,\mathrm{d}\tau \leqslant \Big(|u_0|^2 C_1 + M^2 C_2 + M^2\Big)(1+t)^{-2\mu_1} \quad \forall t \geqslant 0. \qquad (3.24)$$

Step 2. We prove that there exists $T \in \mathcal{T} \cap (0,\infty)$ so that

$$|A^{\alpha+1/2} u(T)| \leqslant c_0(\alpha + 1/2, \lambda), \qquad (3.25)$$

$$|f(T+t)|_{\alpha,\sigma} \leqslant c_1(\alpha+1/2, \lambda)(1+t)^{-\lambda} \quad \forall t \geqslant 0. \qquad (3.26)$$

(a) *Case* $\sigma > 0$.

Set $\lambda' = (\lambda + \mu_1)/2$, which is a number in the interval (λ, μ_1). Thanks to the decay in (3.24) and (3.15), there exists $t_0 \in \mathcal{T} \cap (0,\infty)$ such that

$$|A^{1/2} u(t_0)| < c_0(1/2, \lambda'),$$

$$|f(t_0 + t)|_{0,\sigma} \leqslant c_1(1/2, \lambda')(1+t)^{-\lambda'} \quad \forall t \geqslant 0,$$

where $c_0(\cdot,\cdot)$ and $c_1(\cdot,\cdot)$ are defined in (3.3).

Applying theorem 3.1 to the solution $t \mapsto u(t_0+t)$, force $t \mapsto f(t_0+t)$ with parameters $\alpha = 1/2$ and $\lambda = \lambda'$, we obtain from (3.4) that

$$|u(t_0+t)|_{1/2,\sigma} \leqslant \sqrt{2} c_{*,1/2}(1+t)^{-\lambda'} \leqslant K^{-1/2}(1+t)^{-\lambda'} \quad \forall t \geqslant t_* \overset{\text{def}}{=} 12\sigma. \quad (3.27)$$

Then by (2.9), we have for all $t \geqslant t_*$ that

$$|A^{\alpha+1/2} u(t_0+t)| \leqslant d_0(2\alpha+1,\sigma)|e^{\sigma A^{1/2}} u(t_0+t)| \leqslant d_0(2\alpha+1,\sigma)|u(t_0+t)|_{1/2,\sigma},$$

and, thanks to (3.27),

$$|A^{\alpha+1/2} u(t_0+t)| \leqslant d_0(2\alpha+1,\sigma) K^{-1/2}(1+t)^{-\lambda'}. \qquad (3.28)$$

Since $\lambda' > \lambda$ it follows (3.28) and (3.15) that there is a sufficiently large $T \in \mathcal{T}$ and $T > t_0 + t_*$ so that (3.25) and (3.26) hold.

(b) *Case* $\sigma = 0$. First, we claim that

Claim: If $j \in \mathbb{N}$ such that $j \leqslant 2\alpha + 1$ and

$$\lim_{t\to\infty} \int_t^{t+1} |A^{j/2}u(\tau)|^2 \, \mathrm{d}\tau = 0, \tag{3.29}$$

then

$$\lim_{t\to\infty} \int_t^{t+1} |A^{(j+1)/2}u(\tau)|^2 \, \mathrm{d}\tau = 0. \tag{3.30}$$

Proof of Claim. Note that $(j-1)/2 \leqslant \alpha$, and thanks to (3.15), we have

$$|A^{(j-1)/2}f(t)| = \mathcal{O}(t^{-\mu_1}) \quad \text{as } t \to \infty. \tag{3.31}$$

By (3.29) and (3.31), there is $T \in \mathcal{T} \cap (0, \infty)$ so that

$$|A^{j/2}u(T)| \leqslant c_0(j/2, \lambda),$$

$$|A^{j/2-1/2}f(T+t)| \leqslant c_1(j/2, \lambda)(1+t)^{-\lambda} \quad \forall t \geqslant 0.$$

Applying theorem 3.1 to $u(T+\cdot)$, $f(T+\cdot)$, $\alpha := j/2$, $\sigma := 0$, we obtain

$$\int_t^{t+1} |A^{(j+1)/2}u(\tau)|^2 \, \mathrm{d}\tau = \mathcal{O}(t^{-2\lambda}) \quad \text{as } t \to \infty,$$

which proves (3.30).

Now, let m be a nonnegative integer such that $2\alpha \leqslant m < 2\alpha + 1$.

Note that $m \geqslant 1$, and, because of (3.24), condition (3.29) holds true for $j = 1$. Hence we obtain (3.30) with $j = 1$, which is (3.29) for $j = 2$. This way, we are able to apply the *Claim* recursively for $j = 1, 2, \ldots, m$, and obtain, when $j = m$, from (3.30) that

$$\lim_{t\to\infty} \int_t^{t+1} |A^{(m+1)/2}u(\tau)|^2 \, \mathrm{d}\tau = 0.$$

Since $\alpha \leqslant m/2$, it follows that

$$\lim_{t\to\infty} \int_t^{t+1} |A^{\alpha+1/2}u(\tau)|^2 \, \mathrm{d}\tau = 0. \tag{3.32}$$

By (3.32) and (3.15), we assert that there is $T \in \mathcal{T} \cap (0, \infty)$ so that (3.25) and (3.26) hold.

Step 3. With $T > 0$ in Step 2, we apply theorem 3.1 to $u(T+\cdot)$, $f(T+\cdot)$, $\alpha := \alpha + 1/2$, and obtain that there is $T_* > T + t_*$ such that

$$|u(T_*+t)|_{\alpha+1/2,\sigma} \leqslant \sqrt{2}c_{*,\alpha+1/2}(1+t)^{-\lambda} \leqslant \frac{1}{K^{\alpha+1/2}}(1+t)^{-\lambda} \quad \forall t \geqslant 0.$$

https://doi.org/10.1017/prm.2018.154 Published online by Cambridge University Press

This proves (3.18). Then applying inequality (2.11) with the use of estimate (3.18) yields (3.19).

Part B. We now prove (3.16) and (3.17). We write equation (2.1) as

$$u_t + Au = F(t) \stackrel{\text{def}}{=} -B(u(t), u(t)) + f(t). \tag{3.33}$$

By Part A, we set $\lambda = \mu_1/2$ in (3.19) to obtain

$$|B(u(t), u(t))|_{\alpha,\sigma} = \mathcal{O}(t^{-\mu_1}) \text{ as } t \to \infty. \tag{3.34}$$

From this and (3.15), we have

$$|F(t)|_{\alpha,\sigma} = \mathcal{O}(t^{-\mu_1}) \text{ as } t \to \infty.$$

Applying part (ii) of lemma 2.4 to (3.33) on (T, ∞) for some sufficiently large T and any $\varepsilon \in (0, 1)$, we obtain the first inequality (3.16). Then the second inequality (3.17) follows (2.11) and (3.16). (In these arguments, the values of T_* and C were adjusted appropriately.) The proof is complete. □

Remark 3.3. The estimates (3.18) and (3.19) are similar to those in [**19**, proposition 3.4]. The improved estimates (3.16) and (3.17) with the stronger norms come from the better regularity result in lemma 2.4.

4. Asymptotic expansions

This section consists of the first main results of this paper. Briefly speaking, when the force has a finite or infinite expansion in terms of power-decaying functions, then any Leray-Hopf weak solution will have an asymptotic expansion of the same type.

4.1. Finite expansions

We start with the following consideration for the force $f(t)$.

(B1) *Suppose there exist numbers $\sigma \geqslant 0$, $\alpha \geqslant 1/2$, an integer $N_0 \geqslant 1$, strictly increasing, positive numbers γ_n and functions $\psi_n \in G_{\alpha,\sigma}$ for $1 \leqslant n \leqslant N_0$, and a number $\delta > 0$ such that*

$$\left| f(t) - \sum_{n=1}^{N_0} \psi_n t^{-\gamma_n} \right|_{\alpha,\sigma} = \mathcal{O}(t^{-\gamma_{N_0}-\delta}) \quad \text{as } t \to \infty. \tag{4.1}$$

Note from (4.1) that $f(t)$ belongs to $G_{\alpha,\sigma}$ for all t sufficiently large.

Although the force $f(t)$ has an expansion in terms of $t^{-\gamma_n}$'s, the solution $u(t)$ of the NSE may not. In fact, due to the system's nonlinearity and time derivative, $u(t)$ may be expanded in terms of functions of different powers, which we describe now.

https://doi.org/10.1017/prm.2018.154 Published online by Cambridge University Press

We define the following set of powers generated by γ_n's and 1:

$$S_* = \left\{ \left(\sum_{j=1}^{p} \gamma_{n_j} \right) + k : \text{for some } p \geqslant 1, \quad 1 \leqslant n_1, n_2, \ldots, n_p \leqslant N_0, \right.$$
$$\left. \text{and some integer } k \geqslant 0 \right\}.$$

Note that S_* contains γ_n for all $1 \leqslant n \leqslant N_0$, is an infinite subset of $(0, \infty)$, and possesses the property

$$\forall x, y \in S_* : \quad x + 1, x + y \in S_*. \tag{4.2}$$

By ordering this set, one has

$$S_* = \{\mu_n : n \in \mathbb{N}\}, \quad \text{where } \mu_n\text{'s are strictly increasing.} \tag{4.3}$$

The set of powers that will be used for the expansion of $u(t)$ is

$$S = S_* \cap [\gamma_1, \gamma_{N_0}].$$

This set S is finite, and

$$S = \{\mu_n : 1 \leqslant n \leqslant N_*\} \quad \text{for some } N_* \geqslant N_0. \tag{4.4}$$

Note that $\mu_1 = \gamma_1$ and $\mu_{N_*} = \gamma_{N_0}$. Then we rewrite (4.1) as

$$\left| f(t) - \sum_{n=1}^{N_*} \phi_n t^{-\mu_n} \right|_{\alpha,\sigma} = \mathcal{O}(t^{-\mu_{N_*} - \delta}) \quad \text{as } t \to \infty, \tag{4.5}$$

where $\phi_n \in G_{\alpha,\sigma}$ for $1 \leqslant n \leqslant N_*$, which can be defined explicitly as follows. If there exists $k \in [1, N_0]$ such that $\mu_n = \gamma_k$, then, with such k, $\phi_n = \psi_k$; otherwise, $\phi_n = 0$.

Our first result on the expansion of Leray-Hopf weak solutions is the following.

Theorem 4.1. *Assume* (B1). *Let μ_n's be as in* (4.3), *and let the corresponding equation* (4.5) *hold true. Define ξ_1, ξ_2, …, ξ_{N_*} recursively by*

$$\xi_1 = A^{-1}\phi_1, \tag{4.6}$$

$$\xi_n = A^{-1} \left(\phi_n + \chi_n - \sum_{\substack{1 \leqslant k, m \leqslant n-1, \\ \mu_k + \mu_m = \mu_n}} B(\xi_k, \xi_m) \right) \quad \textit{for } 2 \leqslant n \leqslant N_*, \tag{4.7}$$

where

$$\chi_n = \begin{cases} \mu_p \xi_p, & \textit{if there exists an integer } p \in [1, n-1] \textit{ such that } \mu_p + 1 = \mu_n, \\ 0, & \textit{otherwise.} \end{cases} \tag{4.8}$$

https://doi.org/10.1017/prm.2018.154 Published online by Cambridge University Press

Let $u(t)$ be a Leray-Hopf weak solution of (2.1) and (2.2). Then it holds for all $\rho \in (0,1)$ that

$$\left| u(t) - \sum_{n=1}^{N_*} \xi_n t^{-\mu_n} \right|_{\alpha+1-\rho,\sigma} = \mathcal{O}(t^{-\mu_{N_*}-\varepsilon_*}) \quad \text{as } t \to \infty, \tag{4.9}$$

where

$$\varepsilon_* = \begin{cases} \min\{\delta, \mu_1, 1\} & \text{if } N_* = 1, \\ \min\{\delta, \mu_{N_*} - \mu_{N_*-1}, \mu_{N_*+1} - \mu_{N_*}\} & \text{if } N_* \geqslant 2. \end{cases} \tag{4.10}$$

We make a couple of notes on the formulas of ξ_n's.

(a) In case $n = 1$, we set $\chi_1 = 0$, and use the convention that the last term on the right-hand side of (4.7) vanishes, then formula (4.7) agrees with (4.6), and hence holds also for $n = 1$.

(b) The relation between ϕ_n's and ξ_n's is one-to-one. Indeed, the ϕ_n's can be solved recursively from (4.6) and (4.7) by

$$\begin{cases} \phi_1 = A\xi_1, \\ \phi_n = A\xi_n - \chi_n + \displaystyle\sum_{\substack{k,m\geqslant 1,\\ \mu_k+\mu_m=\mu_n}} B(\xi_k, \xi_m), \quad n \geqslant 2. \end{cases} \tag{4.11}$$

where the χ_n's are still defined by (4.8).

The fact that we can have α fixed instead of requiring (4.5) to hold for all $\alpha > 0$ comes from the following regularity property of ξ_n's.

Lemma 4.2. *Let ϕ_n and ξ_n, for $1 \leqslant n \leqslant N_*$, be as in theorem 4.1. Then*

$$\xi_n \in G_{\alpha+1,\sigma} \quad \forall n = 1, 2, \dots N_*. \tag{4.12}$$

Proof. We prove (4.12) by induction.

When $n = 1$, since $\phi_1 \in G_{\alpha,\sigma}$, we have $\xi_1 \in A^{-1}\phi_1 \in G_{\alpha+1,\sigma}$.

Let $1 \leqslant n < N_*$, and assume that all $\xi_1, \dots, \xi_n \in G_{\alpha+1,\sigma}$. It implies that

$$\chi_{n+1} \in G_{\alpha+1,\sigma} \subset G_{\alpha,\sigma}.$$

This and the fact $\phi_{n+1} \in G_{\alpha,\sigma}$ yield $A^{-1}(\phi_{n+1} + \chi_{n+1}) \in G_{\alpha+1,\sigma}$.

For $1 \leqslant k, m \leqslant n$, we have from the induction hypothesis that $\xi_k, \xi_m \in G_{\alpha+1,\sigma}$. Then, by (2.11),

$$B(\xi_k, \xi_m) \in G_{\alpha+1-1/2,\sigma} = G_{\alpha+1/2,\sigma},$$

which yields

$$A^{-1}B(\xi_k, \xi_j) \in G_{\alpha+3/2,\sigma} \subset G_{\alpha+1,\sigma}.$$

Therefore,

$$\xi_{n+1} = A^{-1}(\phi_{n+1} + \chi_{n+1}) - \sum_{\substack{1\leqslant k,m\leqslant n,\\ \mu_k+\mu_m=\mu_{n+1}}} A^{-1}B(\xi_k, \xi_m) \in G_{\alpha+1,\sigma}.$$

By the induction principle, $\xi_n \in G_{\alpha+1,\sigma}$ for all $1 \leqslant n \leqslant N_*$. □

Before proceeding with the proof of theorem 4.1, we observe from (4.5) that if $1 \leqslant N < N_*$ then

$$\begin{aligned} \left| f(t) - \sum_{n=1}^{N} \phi_n t^{-\mu_n} \right|_{\alpha,\sigma} &\leqslant \left| f(t) - \sum_{n=1}^{N_*} \phi_n t^{-\mu_n} \right|_{\alpha,\sigma} + \left| \sum_{n=N+1}^{N_*} \phi_n t^{-\mu_n} \right| \\ &= \mathcal{O}(t^{-\mu_{N_*}-\delta}) + \mathcal{O}(t^{-\mu_{N+1}}) \text{ as } t \to \infty, \end{aligned}$$

hence,

$$\left| f(t) - \sum_{n=1}^{N} \phi_n t^{-\mu_n} \right|_{\alpha,\sigma} = \mathcal{O}(t^{-\mu_{N+1}}). \tag{4.13}$$

Thus, one has, for $1 \leqslant N \leqslant N_*$, that

$$\left| f(t) - \sum_{n=1}^{N} \phi_n t^{-\mu_n} \right|_{\alpha,\sigma} = \mathcal{O}(t^{-\mu_N-\delta_N}), \tag{4.14}$$

where

$$\delta_N = \begin{cases} \mu_{N+1} - \mu_N & \text{for } 1 \leqslant N < N_*, \\ \delta & \text{for } N = N_*. \end{cases} \tag{4.15}$$

Proof of theorem 4.1. (i) We first prove that if N is any integer in $[1, N_*]$, then there exists a number $\varepsilon_N > 0$ such that for all $\rho \in (0, 1)$

$$\left| u(t) - \sum_{n=1}^{N} \xi_n t^{-\mu_n} \right|_{\alpha+1-\rho,\sigma} = \mathcal{O}(t^{-\mu_N-\varepsilon_N}) \quad \text{as } t \to \infty. \tag{4.16}$$

Proof of (4.16). We use the following notation. For an integer $n \in [1, N_*]$, define

$$F_n(t) = \phi_n t^{-\mu_n}, \quad \bar{F}_n(t) = \sum_{j=1}^{n} F_j(t), \quad \text{and} \quad \tilde{F}_n(t) = f(t) - \bar{F}_n(t),$$

$$u_n(t) = \xi_n t^{-\mu_n}, \quad \bar{u}_n(t) = \sum_{j=1}^{n} u_j(t), \quad \text{and} \quad v_n = u(t) - \bar{u}_n(t).$$

In calculations below, all differential equations hold in V'-valued distribution sense on (T, ∞) for any $T > 0$, which is similar to (2.3). One can easily verify them by using (2.10), and the facts $u \in L^2_{\rm loc}([0, \infty), V)$ and $u' \in L^1_{\rm loc}([0, \infty), V')$ in definition 2.1.

https://doi.org/10.1017/prm.2018.154 Published online by Cambridge University Press

We prove (4.16) by induction in N.

First step: $N = 1$. Let $w_1(t) = t^{\mu_1} u(t)$. We have, for $t > 0$, that

$$w_1'(t) + Aw_1(t) = \phi_1 + H_1(t), \tag{4.17}$$

where

$$H_1(t) = t^{\mu_1}[\tilde{F}_1(t) - B(u(t), u(t))] + \mu_1 t^{\mu_1 - 1} u(t).$$

(a) We estimate $|H_1(t)|_{\alpha,\sigma}$. Equation (4.14), for $N = 1$, particularly reads

$$|f(t) - \phi_1 t^{-\mu_1}|_{\alpha,\sigma} = \mathcal{O}(t^{-\mu_1 - \delta_1}). \tag{4.18}$$

It follows that

$$|f(t)|_{\alpha,\sigma} = \mathcal{O}(t^{-\mu_1}). \tag{4.19}$$

Thanks to (4.19), we can apply theorem 3.2 with $\varepsilon = 1/2$ and obtain from inequalities (3.16) and (3.17) that, as $t \to \infty$,

$$|u(t)|_{\alpha+1/2,\sigma} = \mathcal{O}(t^{-\mu_1}), \tag{4.20}$$

$$|B(u(t), u(t))|_{\alpha,\sigma} = \mathcal{O}(t^{-2\mu_1}). \tag{4.21}$$

Combining estimates (4.18), (4.20) and (4.21), we deduce that there exist $T_0 > 0$ and $D_0 > 0$ such that, for $t \geqslant 0$,

$$(T_0 + t)^{\mu_1} |\tilde{F}_1(T_0 + t)|_{\alpha,\sigma} \leqslant D_0(1 + t)^{-\delta_1},$$

$$(T_0 + t)^{\mu_1 - 1} |u(T_0 + t)|_{\alpha+1/2,\sigma} \leqslant D_0(1 + t)^{-1},$$

$$(T_0 + t)^{\mu_1} |B(u(T_0 + t), u(T_0 + t))|_{\alpha,\sigma} \leqslant D_0(1 + t)^{-\mu_1}.$$

Thus, we have

$$|H_1(T_0 + t)|_{\alpha,\sigma} \leqslant 3D_0(1 + t)^{-\varepsilon_1} \quad \forall t \geqslant 0,$$

where

$$\varepsilon_1 = \min\{\delta_1, \mu_1, 1\}. \tag{4.22}$$

(b) Applying lemma 2.3 to equation (4.17) in $G_{\alpha,\sigma}$ with solution $w_1(T_0 + t)$, for $t \in [0, \infty)$, yields

$$|w_1(T_0 + t) - A^{-1}\phi_1|_{\alpha+1-\rho,\sigma} = \mathcal{O}(t^{-\varepsilon_1})$$

for any $\rho \in (0, 1)$, and consequently,

$$|w_1(t) - A^{-1}\phi_1|_{\alpha+1-\rho,\sigma} = \mathcal{O}(t^{-\varepsilon_1}).$$

Multiplying this equation by $t^{-\mu_1}$ yields

$$|u(t) - \xi_1 t^{-\mu_1}|_{\alpha+1-\rho,\sigma} = \mathcal{O}(t^{-\mu_1 - \varepsilon_1}) \quad \forall \rho \in (0, 1).$$

https://doi.org/10.1017/prm.2018.154 Published online by Cambridge University Press

This proves that

$$\text{the statement (4.16) holds true for } N = 1 \text{ with } \varepsilon_1 \text{ defined by (4.22).} \tag{4.23}$$

Induction step. Let N be an integer with $1 \leqslant N < N_*$, and assume there exists $\varepsilon_N > 0$ such that

$$|v_N(t)|_{\alpha+1-\rho,\sigma} = \mathcal{O}(t^{-\mu_N-\varepsilon_N}) \quad \forall \rho \in (0,1). \tag{4.24}$$

(a) Equation for v_N. Since $v_N' = u' - \bar{u}_N'$, we calculate u' and $\bar{u}_N'$ in terms of the quantities that are more appropriate for the analysis of v_N.

- Calculating u'. By NSE,

$$\begin{aligned} u' &= -Au - B(u,u) + f(t) \\ &= -Av_N - A\bar{u}_N - B(\bar{u}_N + v_N, \bar{u}_N + v_N) + \bar{F}_N + F_{N+1} + \tilde{F}_{N+1}, \end{aligned}$$

hence,

$$u' = -Av_N - A\bar{u}_N + \bar{F}_N - B(\bar{u}_N, \bar{u}_N) + \phi_{N+1} t^{-\mu_{N+1}} + h_{N+1,1}, \tag{4.25}$$

where

$$h_{N+1,1} = -B(\bar{u}_N, v_N) - B(v_N, \bar{u}_N) - B(v_N, v_N) + \tilde{F}_{N+1}.$$

Firstly, note that

$$-A\bar{u}_N + \bar{F}_N = -\sum_{n=1}^{N} \frac{1}{t^{\mu_n}} \Big(A\xi_n - \phi_n \Big).$$

Secondly, we write

$$\begin{aligned} B(\bar{u}_N, \bar{u}_N) &= \sum_{m,j=1}^{N} t^{-\mu_m-\mu_j} B(\xi_m, \xi_j) \\ &= \sum_{n=1}^{N} \frac{1}{t^{\mu_n}} \left(\sum_{\substack{1\leqslant m,j\leqslant N, \\ \mu_m+\mu_j=\mu_n}} B(\xi_m, \xi_j) \right) \\ &\quad + \frac{1}{t^{\mu_{N+1}}} \sum_{\substack{1\leqslant m,j\leqslant N, \\ \mu_m+\mu_j=\mu_{N+1}}} B(\xi_m, \xi_j) + h_{N+1,2}, \end{aligned}$$

where

$$h_{N+1,2} = \sum_{\substack{1\leqslant m,j\leqslant N, \\ \mu_m+\mu_j\geqslant\mu_{N+2}}} t^{-\mu_m-\mu_j} B(\xi_m, \xi_j).$$

- Calculating $\bar{u}_N'$. Note that $\mu_N + 1 \geqslant \mu_{N+1}$ and

$$\{\mu_p + 1 : 1 \leqslant p \leqslant N\} \cap [\mu_1, \mu_{N+1}] \subset \{\mu_k : 1 \leqslant k \leqslant N+1\}.$$

Thus,

$$-\bar{u}_N' = \sum_{p=1}^{N} \frac{\mu_p \xi_p}{t^{\mu_p+1}} = \sum_{n=1}^{N} \frac{\chi_n}{t^{\mu_n}} + \frac{\chi_{N+1}}{t^{\mu_{N+1}}} + h_{N+1,3}, \tag{4.26}$$

where

$$h_{N+1,3} = \sum_{\substack{1\leqslant p\leqslant N,\\ \mu_p+1\geqslant \mu_{N+2}}} \frac{\mu_p \xi_p}{t^{\mu_p+1}}.$$

- Combining the above equations (4.25)–(4.26) yields

$$v_N' = -Av_N + \frac{1}{t^{\mu_{N+1}}}\Bigg(-\sum_{\substack{1\leqslant m,j\leqslant N,\\ \mu_m+\mu_j=\mu_{N+1}}} B(\xi_m,\xi_j) + \phi_{N+1} + \chi_{N+1}\Bigg)$$
$$-\sum_{n=1}^{N}\frac{1}{t^{\mu_n}}\Bigg(A\xi_n + \sum_{\substack{1\leqslant m,j\leqslant N,\\ \mu_m+\mu_j=\mu_n}} B(\xi_m,\xi_j) - \phi_n - \chi_n\Bigg) + h_{N+1,4},$$

where

$$h_{N+1,4} = h_{N+1,1} - h_{N+1,2} + h_{N+1,3}. \tag{4.27}$$

Note, for $1 \leqslant n \leqslant N+1$, that

$$\sum_{\substack{1\leqslant m,j\leqslant N,\\ \mu_m+\mu_j=\mu_n}} B(\xi_m,\xi_j) = \sum_{\substack{1\leqslant m,j\leqslant n-1,\\ \mu_m+\mu_j=\mu_n}} B(\xi_m,\xi_j).$$

Therefore, one has, for $1 \leqslant n \leqslant N$,

$$A\xi_n + \sum_{\substack{1\leqslant m,j\leqslant N,\\ \mu_m+\mu_j=\mu_n}} B(\xi_m,\xi_j) - \phi_n - \chi_n = 0,$$

and

$$-\sum_{\substack{1\leqslant m,j\leqslant N,\\ \mu_m+\mu_j=\mu_{N+1}}} B(\xi_m,\xi_j) + \phi_{N+1} + \chi_{N+1} = A\xi_{N+1}.$$

These yield

$$v_N' = -Av_N + t^{-\mu_{N+1}} A\xi_{N+1} + h_{N+1,4}(t). \tag{4.28}$$

(b) Define $w_{N+1}(t) = t^{\mu_{N+1}} v_N(t)$ for $t > 0$. We have

$$w_{N+1}' = t^{\mu_{N+1}} v_N' + \mu_{N+1} t^{\mu_{N+1}-1} v_N,$$

which, thanks to (4.28), yields

$$w'_{N+1} = -Aw_{N+1} + A\xi_{N+1} + H_{N+1}(t), \tag{4.29}$$

where

$$H_{N+1}(t) = t^{\mu_{N+1}} h_{N+1,4}(t) + \mu_{N+1} t^{\mu_{N+1}-1} v_N(t). \tag{4.30}$$

Let ρ be any number in the interval $(0,1)$.
(c) We estimate $H_{N+1}(t)$ now. From (4.24) and the fact $\mu_N + 1 \geqslant \mu_{N+1}$, it follows

$$t^{\mu_{N+1}-1}|v_N(t)|_{\alpha+1-\rho,\sigma} = \mathcal{O}(t^{\mu_{N+1}-1}t^{-\mu_N-\varepsilon_N}) = \mathcal{O}(t^{-\varepsilon_N}). \tag{4.31}$$

By (4.14), we have

$$t^{\mu_{N+1}}|\tilde{F}_{N+1}(t)|_{\alpha,\sigma} = \mathcal{O}(t^{-\delta_{N+1}}).$$

Clearly,

$$|\bar{u}_N(t)|_{\alpha+1,\sigma} = \mathcal{O}(t^{-\mu_1}). \tag{4.32}$$

By inequality (2.11), estimate (4.24) for $\rho = 1/2$, and (4.32), it follows that

$$\begin{aligned} &t^{\mu_{N+1}}|B(v_N(t),\bar{u}_N(t))|_{\alpha,\sigma},\\ &t^{\mu_{N+1}}|B(\bar{u}_N(t),v_N(t))|_{\alpha,\sigma} = \mathcal{O}(t^{\mu_{N+1}}t^{-\mu_N-\varepsilon_N}t^{-\mu_1}) = \mathcal{O}(t^{-\varepsilon_N}),\\ &t^{\mu_{N+1}}|B(v_N(t),v_N(t))|_{\alpha,\sigma} = \mathcal{O}(t^{\mu_{N+1}}t^{-\mu_N-\varepsilon_N}t^{-\mu_N-\varepsilon_N}) = \mathcal{O}(t^{-\varepsilon_N}). \end{aligned}$$

Above, we used the fact $2\mu_N \geqslant \mu_N + \mu_1 \geqslant \mu_{N+1}$. Hence,

$$t^{\mu_{N+1}}|h_{N+1,1}(t)|_{\alpha,\sigma} = \mathcal{O}(t^{-\delta_{N+1}}) + \mathcal{O}(t^{-\varepsilon_N}).$$

It is obvious that

$$t^{\mu_{N+1}} \sum_{\substack{1\leqslant m,j\leqslant N,\\ \mu_m+\mu_j\geqslant\mu_{N+2}}} t^{-\mu_m}t^{-\mu_j}|B(\xi_m,\xi_j)|_{\alpha,\sigma} = \mathcal{O}(t^{-(\mu_{N+2}-\mu_{N+1})}),$$

and thus,

$$t^{\mu_{N+1}}|h_{N+1,2}(t)|_{\alpha,\sigma} = \mathcal{O}(t^{-(\mu_{N+2}-\mu_{N+1})}).$$

It is also clear that

$$t^{\mu_{N+1}}|h_{N+1,3}(t)|_{\alpha,\sigma} = \mathcal{O}(t^{-(\mu_{N+2}-\mu_{N+1})}).$$

Combining these estimates of $t^{\mu_{N+1}}h_{N+1,j}(t)$ for $j = 1,2,3$, with (4.30), (4.27) and (4.31) gives

$$|H_{N+1}(t)|_{\alpha,\sigma} = \mathcal{O}(t^{-\delta_{N+1}}) + \mathcal{O}(t^{-\varepsilon_N}) + \mathcal{O}(t^{-(\mu_{N+2}-\mu_{N+1})}) = \mathcal{O}(t^{-\varepsilon_{N+1}}), \tag{4.33}$$

where

$$\varepsilon_{N+1} = \min\{\delta_{N+1}, \varepsilon_N, \mu_{N+2} - \mu_{N+1}\}. \tag{4.34}$$

(d) Note that from lemma 4.2 that $A\xi_{N+1} \in G_{\alpha,\sigma}$. By applying lemma 2.3 to equation (4.29) and solution $w_{N+1}(T_1 + t)$ for some sufficiently large $T_1 > 0$

with the use of (4.33), we obtain

$$|w_{N+1}(T_1+t) - A^{-1}(A\xi_{N+1})|_{\alpha+1-\rho,\sigma} \leqslant C(1+t)^{-\varepsilon_{N+1}} \quad \forall t \geqslant 1.$$

Thus,

$$|w_{N+1}(t) - \xi_{N+1}|_{\alpha+1-\rho,\sigma} = \mathcal{O}(t^{-\varepsilon_{N+1}}).$$

Multiplying this equation by $t^{-\mu_{N+1}}$ yields

$$|v_N(t) - \xi_{N+1}t^{-\mu_{N+1}}|_{\alpha+1-\rho,\sigma} = \mathcal{O}(t^{-\mu_{N+1}-\varepsilon_{N+1}}),$$

that is,

$$|v_{N+1}(t)|_{\alpha+1-\rho,\sigma} = \mathcal{O}(t^{-\mu_{N+1}-\varepsilon_{N+1}}).$$

This proves

$$\text{the statement (4.16) holds for } N := N+1 \text{ with } \varepsilon_{N+1} \text{ defined by (4.34).} \tag{4.35}$$

By the induction principle, we have (4.16) holds true for all $N = 1, 2, \ldots, N_*$. This completes the proof of (4.16).

(ii) We now prove (4.9).

Case $N_* = 1$. We have in this case, thanks to (4.15), $\delta_1 = \delta$ and ε_1 in (4.22) equals ε_* in (4.10). Thus, the statement (4.9) just follows (4.23).

Case $N_* \geqslant 2$. Similar to (4.13), one has from (4.16), for $N = N_*$, that

$$|v_{N_*-1}(t)|_{\alpha+1-\rho,\sigma} = \mathcal{O}(t^{-\mu_{N_*}}). \tag{4.36}$$

We repeat the induction step in part (i) for $N = N_* - 1$, but now with

$$\delta_{N+1} = \delta_{N_*} = \delta \text{ and, thanks to (4.36), } \varepsilon_N = \varepsilon_{N_*-1} = \mu_{N_*} - \mu_{N_*-1}.$$

Then one obtains from (4.35) that

$$|v_{N_*}(t)|_{\alpha+1-\rho,\sigma} = \mathcal{O}(t^{-\mu_{N_*}-\varepsilon_{N_*}}),$$

where, according to formula (4.34), $\varepsilon_{N_*} = \varepsilon_{N+1} = \min\{\delta, \mu_{N_*} - \mu_{N_*-1}, \mu_{N_*+1} - \mu_{N_*}\}$ which exactly is ε_*. This completes our proof. □

4.2. Infinite expansions

We focus, in this subsection, the case when the force has an infinite expansion, and obtain an infinite expansion for any Leray-Hopf weak solution of the NSE. The force's expansion considered will be of the following type.

(B2) *Suppose there exist real numbers $\sigma \geqslant 0$, $\alpha \geqslant 1/2$, a strictly increasing, divergent sequence of positive numbers $(\gamma_n)_{n=1}^\infty$ and a sequence of functions $(\psi_n)_{n=1}^\infty$ in $G_{\alpha,\sigma}$ such that, in the sense of definition 1.1,*

$$f(t) \sim \sum_{n=1}^{\infty} \psi_n t^{-\gamma_n} \text{ in } G_{\alpha,\sigma}. \tag{4.37}$$

https://doi.org/10.1017/prm.2018.154 Published online by Cambridge University Press

Similar to the previous subsection, the appropriate set of powers generated by γ_n's and 1 is

$$S_\infty = \left\{ \left(\sum_{j=1}^{p} \gamma_{n_j} \right) + k : \text{ for some } p \geqslant 1,\ n_1, n_2, \ldots, n_p \geqslant 1, \right.$$
$$\left. \text{and some integer } k \geqslant 0 \right\}.$$

Then $S_\infty \subset (0, \infty)$, and property (4.2) still holds with S_∞ replacing S_*. Since $\gamma_n \to \infty$ as $n \to \infty$, we can order S_∞, and denote

$$S_\infty = \{\mu_n : n \in \mathbb{N}\} \text{ with } \mu_n\text{'s being strictly increasing.} \tag{4.38}$$

(This is possible by ordering finitely many elements in each set $S_\infty \cap (n-1, n]$, for all $n \in \mathbb{N}$.) Note that $\mu_n \to \infty$ as $n \to \infty$.

Then rewrite (4.37) as

$$f(t) \sim \sum_{n=1}^{\infty} \phi_n t^{-\mu_n} \quad \text{in } G_{\alpha,\sigma} \quad \text{as } t \to \infty, \tag{4.39}$$

where the sequence $(\phi_n)_{n=1}^\infty$ in $G_{\alpha,\sigma}$ is defined by $\phi_n = \psi_k$ if there exists $k \geqslant 1$ such that $\mu_n = \gamma_k$, and $\phi_n = 0$ otherwise.

By the same arguments as in § 4.1, the estimate (4.13) now holds for all $N \geqslant 1$.

THEOREM 4.3. *Assume* (B2) *and the corresponding expansion* (4.39). *Then any Leray-Hopf weak solution $u(t)$ of* (2.1) *and* (2.2) *has the asymptotic expansion*

$$u(t) \sim \sum_{n=1}^{\infty} \xi_n t^{-\mu_n} \quad \textit{in } G_{\alpha+1-\rho,\sigma}, \quad \forall \rho \in (0,1), \tag{4.40}$$

where ξ_n is defined by (4.6) *for $n = 1$, and by* (4.7) *for $n \geqslant 2$. More precisely, one has for any $N \geqslant 1$ that*

$$\left| u(t) - \sum_{n=1}^{N} \xi_n t^{-\mu_n} \right|_{\alpha+1-\rho,\sigma} = \mathcal{O}(t^{-\mu_{N+1}}) \quad \textit{as } t \to \infty, \quad \forall \rho \in (0,1). \tag{4.41}$$

Proof. Clearly, f satisfies condition (B1) for any $N_0 \geqslant 1$, and (4.5) for any $N_* \geqslant 1$. Hence, applying theorem 4.1 for each $N_* \geqslant 1$, we obtain the expansion (4.40) for $u(t)$. Then similar to (4.13), we obtain from (4.40) that (4.41) holds for all $N \geqslant 1$. □

5. Properties of the expansions

According to theorem 4.3, for each force f satisfying the required conditions, there exists a unique sequence $(\xi_n)_{n=1}^\infty$ such that the expansion (4.40) holds for all Leray-Hopf weak solution $u(t)$. The first part of this section investigates the range of $(\xi_n)_{n=1}^\infty$ when the force f varies.

Below, we focus on the infinite expansions in § 4.2, with $\gamma_n = n$ for all $n \in \mathbb{N}$ in Assumption (B2), which implies that $\mu_n = n$ for all $n \in \mathbb{N}$. In this case, (4.39) and (4.40) read as

$$f(t) \sim \sum_{n=1}^{\infty} \phi_n t^{-n} \quad \text{in } G_{\alpha,\sigma}, \tag{5.1}$$

$$u(t) \sim \sum_{n=1}^{\infty} \xi_n t^{-n} \quad \text{in } G_{\alpha+1-\rho,\sigma}, \quad \forall \rho \in (0,1), \tag{5.2}$$

where ϕ_n's and ξ_n's, referring to (4.6) and (4.7), are related by

$$\begin{cases} \xi_1 = A^{-1}\phi_1, \\ \xi_n = A^{-1}\left[\phi_n + (n-1)\xi_{n-1} - \displaystyle\sum_{k=1}^{n-1} B(\xi_k, \xi_{n-k})\right], \quad n \geqslant 2. \end{cases} \tag{5.3}$$

Above, we used the fact that χ_n given by (4.8) now is $(n-1)\xi_{n-1}$ for all $n \geqslant 2$.

We recall note (b) after theorem 4.1 that the relation between $(\phi_n)_{n=1}^{\infty}$ and $(\xi_n)_{n=1}^{\infty}$ in (5.3) is one-to-one, and (4.11) now reads as

$$\begin{cases} \phi_1 = A\xi_1, \\ \phi_n = A\xi_n - (n-1)\xi_{n-1} + \displaystyle\sum_{k=1}^{n-1} B(\xi_k, \xi_{n-k}), \quad n \geqslant 2. \end{cases} \tag{5.4}$$

The following proposition gives a sufficient condition for $(\xi_n)_{n=1}^{\infty}$ so that $\sum_{n=1}^{\infty} \xi_n t^{-n}$ is an expansion of a Leray-Hopf weak solution with some force f as in (5.1).

Theorem 5.1. *Let $(c_n)_{n=1}^{\infty}$ be a sequence of nonnegative numbers such that*

$$\sum_{n=2}^{\infty} n d_n < \infty, \ \textit{where } d_n = \max\{c_k c_{n-k} : 1 \leqslant k \leqslant n-1\}. \tag{5.5}$$

Given $\alpha \geqslant 1/2$ and $\sigma \geqslant 0$. Suppose $(\zeta_n)_{n=1}^{\infty}$ is a sequence in $G_{\alpha+1,\sigma}$ such that

$$|\zeta_n|_{\alpha+1,\sigma} \leqslant c_n \quad \forall n \in \mathbb{N}. \tag{5.6}$$

Then there exists a forcing function $f(t)$ such that any Leray-Hopf weak solution $u(t)$ of (2.1) and (2.2) satisfies

$$u(t) \sim \sum_{n=1}^{\infty} \zeta_n t^{-n} \quad \textit{in } G_{\alpha+1-\rho,\sigma}, \quad \forall \rho \in (0,1). \tag{5.7}$$

Moreover, the series of the expansion, $\sum_{n=1}^{\infty} \zeta_n t^{-n}$, converges in $G_{\alpha+1,\sigma}$ absolutely and uniformly on $[1,\infty)$.

Proof. In case $c_n = 0$ for all n, then $\zeta_n = 0$ for all n. We simply take $f = 0$, which gives $\phi_n = 0$ for all n. Then we have expansion (5.2), where the ξ_n's are given by (5.3), which obviously yields $\xi_n = 0 = \zeta_n$ for all n. Hence (5.7) follows (5.2).

We now consider the case that there exists $n_0 \in \mathbb{N}$ such that $c_{n_0} > 0$. Clearly, from the definition of d_n one has $c_{n_0} c_n \leqslant d_{n+n_0}$ for $n \geqslant 1$. Using this and (5.5) yield

$$\sum_{n=1}^{\infty} c_n \leqslant \sum_{n=1}^{\infty} n c_n \leqslant \sum_{n=1}^{\infty} n \frac{d_{n+n_0}}{c_{n_0}} \leqslant \frac{1}{c_{n_0}} \sum_{n=1}^{\infty} (n+n_0) d_{n+n_0} < \infty. \tag{5.8}$$

Define

$$\begin{cases} \phi_1 = A\zeta_1, \\ \phi_n = A\zeta_n - (n-1)\zeta_{n-1} + \displaystyle\sum_{k=1}^{n-1} B(\zeta_k, \zeta_{n-k}), \quad n \geqslant 2. \end{cases} \tag{5.9}$$

(This, in fact, is the construction of ϕ_n's in (5.4) with ξ_n's being replaced with ζ_n's.)

We estimate, for $n = 1$,

$$|\phi_1|_{\alpha,\sigma} = |A\zeta_1|_{\alpha,\sigma} = |\zeta_1|_{\alpha+1,\sigma} \leqslant c_1.$$

For $n \geqslant 2$, we have from (5.9) and (2.11) that

$$\begin{aligned} |\phi_n|_{\alpha,\sigma} &\leqslant |A\zeta_n|_{\alpha,\sigma} + (n-1)|\zeta_{n-1}|_{\alpha,\sigma} + \sum_{k=1}^{n-1} |B(\zeta_k, \zeta_{n-k})|_{\alpha,\sigma} \\ &\leqslant |\zeta_n|_{\alpha+1,\sigma} + (n-1)|\zeta_{n-1}|_{\alpha+1,\sigma} + K^{\alpha} \sum_{k=1}^{n-1} |\zeta_k|_{\alpha+1/2,\sigma} |\zeta_{n-k}|_{\alpha+1/2,\sigma}. \end{aligned}$$

Then, by (5.6),

$$\begin{aligned} |\phi_n|_{\alpha,\sigma} &\leqslant c_n + (n-1)c_{n-1} + K^{\alpha} \sum_{k=1}^{n-1} c_k c_{n-k} \\ &\leqslant c_n + (n-1)c_{n-1} + K^{\alpha}(n-1)d_n < \infty. \end{aligned}$$

Therefore, by (5.8) and (5.5),

$$\sum_{n=1}^{\infty} |\phi_n|_{\alpha,\sigma} \leqslant \sum_{n=1}^{\infty} c_n + \sum_{n=1}^{\infty} n c_n + K^{\alpha} \sum_{n=2}^{\infty} (n-1) d_n < \infty.$$

It follows that $\sum_{n=1}^{\infty} \phi_n t^{-n}$ converges in $G_{\alpha,\sigma}$ absolutely and uniformly on $[1,\infty)$. Thus, we can define $f(t)$ for $t \geqslant 0$ as following:

$$f(t) = \begin{cases} \sum_{n=1}^{\infty} \phi_n & \text{if } 0 \leqslant t < 1, \\ \sum_{n=1}^{\infty} \phi_n t^{-n} & \text{if } t \geqslant 1. \end{cases} \tag{5.10}$$

Clearly, f satisfies (A), and $f(t) \sim \sum_{n=1}^{\infty} \phi_n t^{-n}$ in $G_{\alpha,\sigma}$, hence f satisfies (B2) too.

https://doi.org/10.1017/prm.2018.154 Published online by Cambridge University Press

Let u be a Leray-Hopf weak solution of (2.1) and (2.2). Applying theorem 4.3 gives the expansion (5.2) for $u(t)$, where the ξ_n's are given by (5.3).

Solving for ζ_n's from (5.9) gives

$$\begin{cases} \zeta_1 = A^{-1}\phi_1, \\ \zeta_n = A^{-1}\left[\phi_n + (n-1)\zeta_{n-1} - \sum_{k=1}^{n-1} B(\zeta_k, \zeta_{n-k})\right], \quad n \geqslant 2. \end{cases} \tag{5.11}$$

Comparing (5.3) and (5.11) shows $\xi_n = \zeta_n$ for all $n \in \mathbb{N}$. Therefore, (5.7) follows (5.2).

By (5.6), we have, for all $t \geqslant 1$ and $n \geqslant 1$, that

$$|\zeta_n t^{-n}|_{\alpha+1,\sigma} \leqslant |\zeta_n|_{\alpha+1,\sigma} \leqslant c_n. \tag{5.12}$$

This estimate and (5.8) imply that $\sum_{n=1}^{\infty} \zeta_n t^{-n}$ converges in $G_{\alpha+1,\sigma}$ absolutely and uniformly on $[1, \infty)$. The proof is complete. □

Example 5.2. In theorem 5.1, assume there exist $M > 0$, $\lambda > 2$ and $N_0 \geqslant 1$ such that $c_n \leqslant M n^{-\lambda}$ for all $n \geqslant N_0$. Then the sequence $(c_n)_{n=1}^{\infty}$ is bounded by, say, a number $c_* > 0$. Let $n \geqslant 2N_0$. If $1 \leqslant k \leqslant n/2$, then $n - k \geqslant n/2 \geqslant N_0$ and

$$c_k c_{n-k} \leqslant c_* c_{n-k} \leqslant c_* M (n-k)^{-\lambda} \leqslant c_* M (n/2)^{-\lambda}.$$

If $n/2 < k < n$, then $k > N_0$ and

$$c_k c_{n-k} \leqslant c_k c_* \leqslant c_* M k^{-\lambda} \leqslant c_* M (n/2)^{-\lambda}.$$

Hence, $d_n \leqslant 2^{\lambda} M n^{-\lambda}$ for all $n \geqslant 2N_0$. Therefore, condition (5.5) is satisfied.

As a special case of theorem 5.1, the next corollary shows that the expansion of $u(t)$, essentially, can be any finite sum in $G_{\alpha+1,\sigma}$ (of course, of the same type.)

Corollary 5.3. *Let $\alpha \geqslant 1/2$, $\sigma \geqslant 0$ be given numbers, and $\zeta_1, \ldots, \zeta_N$ be given elements in $G_{\alpha+1,\sigma}$, for some $N \geqslant 1$. Then there exists a forcing function $f(t)$ such that any Leray-Hopf weak solution $u(t)$ of (2.1) and (2.2) has the expansion $u(t) \sim \sum_{n=1}^{\infty} \xi_n t^{-n}$ in $G_{\alpha+1-\rho,\sigma}$ for all $\rho \in (0, 1)$, where $\xi_n = \zeta_n$ for $1 \leqslant n \leqslant N$ and $\xi_n = 0$ for all $n > N$.*

Proof. For $n > N$, set $\zeta_n = 0$. Define $c_n = |\zeta_n|_{\alpha+1,\sigma}$ for $1 \leqslant n \leqslant N$, and $c_n = 0$ for $n > N$. Let d_n be defined as in (5.5). One can verify that $d_n = 0$ for all $n > 2N$. Hence, the condition (5.5) is satisfied. Then the conclusion of this corollary follows theorem 5.1. (In fact, following the construction of $f(t)$, we have $\phi_n = 0$ for $n > 2N$, and $f(t)$, for $t \geqslant 1$, is simply the finite sum $\sum_{n=1}^{2N} \phi_n t^{-n}$.) □

Example 5.4 (Divergent expansions). For a given and fixed force f, the expansion (4.40) may not converge. We give here a simple example. Let $\phi_1 \neq 0$ be a function in $R_1 H$ such that $B(\phi_1, \phi_1) = 0$, and let $\phi_n = 0$ for all $n \geqslant 2$. (For e.g., $\phi_1(\mathbf{x}) = \varepsilon \mathbf{e}_2 [e^{i\mathbf{e}_1 \cdot \mathbf{x}} + e^{-i\mathbf{e}_1 \cdot \mathbf{x}}]$ for any $\varepsilon > 0$.) From (5.3), one can easily verify that $\xi_n =$

$(n-1)!\phi_1$ for all $n \in \mathbb{N}$. Hence, the expansion $\sum_{n=1}^{\infty} \xi_n t^{-n}$ is divergent in H for all $t > 0$.

REMARK 5.5. We recall the normalization map for NSE, in case the force is potential, defined by Foias and Saut [**7**, **9**]. First, rewrite $\sigma(A) = \{\lambda_n : n \in \mathbb{N}\}$, where λ_n is strictly increasing. For any $u^0 \in V$ such that the solution $u(t)$ of (2.1) and (2.2) is regular on $[0, \infty)$, there exists $\xi = (\xi_n)_{n=1}^{\infty}$, with $\xi_n \in R_{\lambda_n} H$ for all $n \in \mathbb{N}$, such that the expansion (1.5) of $u(t)$ can be reconstructed explicitly based on ξ only, that is, $q_n(t) = q_n(t, \xi)$ – a polynomial in both t and ξ. The normalization map is defined as $W(u^0) = \xi = (\xi_n)_{n=1}^{\infty}$. Thus, regarding the reconstructions of the asymptotic expansions for solutions, the sequence $(\xi_n)_{n=1}^{\infty}$ in (5.2) and $W(u^0)$ have similar roles, because they totally determine the expansions (5.2) and (1.5), respectively. We now briefly compare (a) their ranges, and (b) the convergence of their generated expansions.

Regarding (a), it is known that given *small* elements $\zeta_n \in R_{\lambda_n} H$, for $n = 1, 2, \ldots, N$, there exists u^0 such that $W(u^0) = (\xi_n)_{n=1}^{\infty}$ with $\xi_n = \zeta_n$ for $1 \leqslant n \leqslant N$. However, it is not known what ξ_n's might be for $n > N$. For the expansion (5.2), theorem 5.1, example 5.2 and corollary 5.3 give specific and simple characteristics of the possible values of $(\xi_n)_{n=1}^{\infty}$.

Regarding (b), it is not known what might be a general, nontrivial $W(u^0)$ such that the expansion (1.5), when generated by $W(u^0)$, is convergent. (See [**11**, **14**] for more information about this topic.) In contrast, the expansion (5.2) is just a power series having ξ_n's as its coefficients. Hence, a simple condition such as $\limsup_{n\to\infty} |\xi_n|_{\alpha,\sigma}^{1/n} < \infty$ is enough to conclude that the expansion (5.2) converges in $G_{\alpha,\sigma}$ for sufficiently large t.

The second part of this section investigates the possible type of decay for the Leray-Hopf weak solutions after all the power-decaying terms. More specifically, in theorem 5.1, $u(t) \sim \sum_{n=1}^{\infty} \xi_n t^{-n}$ in $G_{\alpha,\sigma}$ with $\sum_{n=1}^{\infty} \xi_n t^{-n}$ converging to $\bar{u}(t)$ in $G_{\alpha+1,\sigma}$ for all $t \geqslant 1$. Hence, by this expansion and the theory of power series,

$$|u(t) - \bar{u}(t)|_{\alpha,\sigma} = \mathcal{O}(t^{-\mu}) \quad \forall \mu > 0. \tag{5.13}$$

The next theorem states that, the remainder in (5.13), decays faster and, in fact, it decays exponentially.

THEOREM 5.6. *Given $\alpha \geqslant 1/2$ and $\sigma \geqslant 0$. Suppose that*

$$f(t) = \sum_{n=1}^{\infty} \phi_n t^{-n} \quad \text{in } G_{\alpha,\sigma}, \quad \forall t \geqslant T_0, \text{ for some } T_0 > 0, \tag{5.14}$$

where $(\phi_n)_{n=1}^{\infty}$ is a sequence in $G_{\alpha,\sigma}$.

Let $(\xi_n)_{n=1}^{\infty}$ be defined by (5.3), and assume

$$\limsup_{n\to\infty} \left(|\xi_n|_{\alpha+1,\sigma}^{1/n} \right) < \infty. \tag{5.15}$$

https://doi.org/10.1017/prm.2018.154 Published online by Cambridge University Press

Then

(i) *The series* $\sum_{n=1}^{\infty} \xi_n t^{-n}$ *converges in* $G_{\alpha+1,\sigma}$ *to a function* $\bar{u}(t)$ *for sufficiently large* t.

(ii) *If* $u(t)$ *is a Leray-Hopf weak solution of* (2.1) *and* (2.2), *then one has for all* $\rho \in (0, 1/2]$ *that, as* $t \to \infty$,

$$|u(t) - \bar{u}(t)|_{\alpha+\frac{1}{2}-\rho,\sigma} = \begin{cases} \mathcal{O}(e^{-t}), & \text{if } \xi_1 = 0, \\ \mathcal{O}(t^{\beta} e^{-t}) \text{ for some } \beta > 0, & \text{if } \xi_1 \neq 0. \end{cases} \tag{5.16}$$

Proof.

(i) This part is a straight consequence of power series theory in Banach spaces, see for example, [**2**, Chapter IX]. Indeed, (5.15) implies that there is $R > 0$ such that $\sum_{n=1}^{\infty} \xi_n z^n$ converges in $G_{\alpha+1,\sigma}$ absolutely and uniformly for $z \in [-R, R]$. Hence, denoting $T_1 = 1/R$, we have

$$\bar{u}(t) = \sum_{n=1}^{\infty} \xi_n t^{-n} \text{ converges in } G_{\alpha+1,\sigma} \text{ absolutely and uniformly for all } t \geqslant T_1. \tag{5.17}$$

(ii) One can verify from (5.14) that f satisfies (A) and (B2). Let $T_2 = \max\{T_0, T_1\}$.

(a) First, we claim that, for all $t > T_2$,

$$\bar{u}'(t) + A\bar{u}(t) + B(\bar{u}(t), \bar{u}(t)) = f(t) \quad \text{in } G_{\alpha,\sigma}. \tag{5.18}$$

Indeed, if $t > T_2$ then

$$\bar{u}'(t) = \sum_{n=1}^{\infty} u_n'(t) = \sum_{n=2}^{\infty} \frac{-(n-1)\xi_{n-1}}{t^n} \quad \text{in } G_{\alpha+1,\sigma},$$

$$A\bar{u}(t) = \sum_{n=1}^{\infty} \frac{A\xi_n}{t^n} \quad \text{in } G_{\alpha,\sigma}.$$

By (2.11), (5.17), and Cauchy's product, we infer that

$$B(\bar{u}(t), \bar{u}(t)) = \sum_{n=2}^{\infty} \frac{1}{t^n} \left[\sum_{k=1}^{n-1} B(\xi_k, \xi_{n-k}) \right] \quad \text{in } G_{\alpha+1/2,\sigma} \text{ for } t > T_2. \tag{5.19}$$

Thus, we have for $t > T_2$ that the following identities hold in $G_{\alpha,\sigma}$

$$\begin{aligned} &\bar{u}'(t) + A\bar{u}(t) + B(\bar{u}(t), \bar{u}(t)) \\ &\quad = \frac{A\xi_1}{t} + \sum_{n=2}^{\infty} \frac{1}{t^n} \left\{ -(n-1)\xi_{n-1} + A\xi_n + \sum_{k=1}^{n-1} B(\xi_k, \xi_{n-k}) \right\} \\ &\quad = \sum_{n=1}^{\infty} \phi_n t^{-n} = f(t). \end{aligned}$$

This proves (5.18).

(b) Let $\rho \in (0, 1/2]$. Let $w = u - \bar{u}$. Suppose there exists $T_* \geqslant T_2$ such that

$$K^{\alpha+(1/2)-\rho}(|u(t)|_{\alpha+1-\rho,\sigma} + |\bar{u}(t)|_{\alpha+1-\rho,\sigma}) < 1 \quad \forall t > T_*. \tag{5.20}$$

We claim that, for $t \geqslant T_*$,

$$|w(t)|_{\alpha+1/2-\rho,\sigma} \leqslant |w(T_*)|_{\alpha+1/2-\rho,\sigma} e^{-t+T_*} \times e^{K^{\alpha+(1/2)-\rho} \int_{T_*}^t (|u(\tau)|_{\alpha+1-\rho,\sigma} + |\bar{u}(\tau)|_{\alpha+1-\rho,\sigma})\,\mathrm{d}\tau}. \tag{5.21}$$

Proof of this claim. Subtracting (5.18) from the NSE (2.1) yields

$$w' + Aw + B(w, u) + B(\bar{u}, w) = 0 \text{ in } V' \text{ on } (T_2, \infty). \tag{5.22}$$

Let $N \in \sigma(A)$, taking P_N of (5.22) gives

$$(P_N w)' + A(P_N w) + P_N(B(w, u) + B(\bar{u}, w)) = 0 \text{ on } (T_2, \infty), \tag{5.23}$$

in the $P_N H$-valued distribution sense. Denote

$$A_N = AP_N \text{ and } \tilde{w}_N = A^{\alpha+(1/2)-\rho} e^{\sigma A^{1/2}} P_N w = A_N^{\alpha+(1/2)-\rho} e^{\sigma A_N^{1/2}} P_N w.$$

Then it follows (5.23) that

$$\tilde{w}_N' = -A\tilde{w}_N - A^{\alpha+(1/2)-\rho} e^{\sigma A^{1/2}} P_N(B(w, u) + B(\bar{u}, w)). \tag{5.24}$$

In the finite dimensional space $P_N H$, we have for $t > T_*$,

$$|A\tilde{w}_N| \leqslant \sqrt{N}|A^{1/2}\tilde{w}_N| \leqslant \sqrt{N}(|u(t)|_{\alpha+1-\rho,\sigma} + |\bar{u}(t)|_{\alpha+1-\rho,\sigma}) < \sqrt{N} K^{-(\alpha+1/2-\rho)}, \tag{5.25}$$

and, by using inequality (2.11),

$$\begin{aligned}
&|A^{\alpha+(1/2)-\rho} e^{\sigma A^{1/2}} P_N(B(w, u) + B(\bar{u}, w))| \\
&\quad \leqslant |B(w, u)|_{\alpha+(1/2)-\rho,\sigma} + |B(\bar{u}, w)|_{\alpha+(1/2)-\rho,\sigma} \\
&\quad \leqslant K^{\alpha+1/2-\rho} |w|_{\alpha+1-\rho,\sigma}(|u|_{\alpha+1-\rho,\sigma} + |\bar{u}|_{\alpha+1-\rho,\sigma}) \\
&\quad \leqslant |w|_{\alpha+1-\rho,\sigma} \leqslant |u|_{\alpha+1-\rho,\sigma} + |\bar{u}|_{\alpha+1-\rho,\sigma} \leqslant K^{-(\alpha+1/2-\rho)}.
\end{aligned}$$

Hence, both $\tilde{w}_N$ and $\tilde{w}_N'$ belong to $L^\infty(T_*, \infty; P_N H)$. Thus, see for example, [**24**, Chapter II, lemma 3.2], equation (5.24) implies that, in the distribution sense on (T_*, ∞),

$$\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{w}_N|^2 = 2\langle \tilde{w}_N', \tilde{w}_N\rangle = -2\langle A\tilde{w}_N, \tilde{w}_N\rangle - 2\langle A^{\alpha+1/2-\rho} e^{\sigma A^{1/2}} P_N(B(w, u) + B(\bar{u}, w)), \tilde{w}_N\rangle.$$

For $t > T_*$, applying Cauchy-Schwarz inequality and (2.11) yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{w}_N|^2 &\leqslant -2|A^{1/2}\tilde{w}_N|^2 + 2(|B(w, u)|_{\alpha+1/2-\rho,\sigma} + |B(\bar{u}, w)|_{\alpha+1/2-\rho,\sigma})|\tilde{w}_N| \\
&\leqslant -2|A^{1/2}\tilde{w}_N|^2 + 2K^{\alpha+1/2-\rho}|w|_{\alpha+1-\rho,\sigma}(|u|_{\alpha+1-\rho,\sigma} + |\bar{u}|_{\alpha+1-\rho,\sigma})|A^{1/2}\tilde{w}_N|.
\end{aligned}$$

In the last term,

$$|w|_{\alpha+1-\rho,\sigma} \leqslant |P_N w|_{\alpha+1-\rho,\sigma} + |(\mathrm{Id} - P_N)w|_{\alpha+1-\rho,\sigma}$$
$$= |A^{1/2}\tilde{w}_N| + |(\mathrm{Id} - P_N)w|_{\alpha+1-\rho,\sigma}.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{w}_N|^2 \leqslant -2\Big[1 - K^{\alpha+(1/2)-\rho}(|u|_{\alpha+1-\rho,\sigma} + |\bar{u}|_{\alpha+1-\rho,\sigma})\Big]|A^{1/2}\tilde{w}_N|^2$$
$$+ 2K^{\alpha+(1/2)-\rho}(|u|_{\alpha+1-\rho,\sigma} + |\bar{u}|_{\alpha+1-\rho,\sigma})|(\mathrm{Id} - P_N)w|_{\alpha+1-\rho,\sigma}|A^{1/2}\tilde{w}_N|.$$

Then using condition (5.20) and the relation $|A^{1/2}\tilde{w}_N| \geqslant |\tilde{w}_N|$ for the first term on the right-hand side, we derive for $t > T_*$,

$$\frac{\mathrm{d}}{\mathrm{d}t}|\tilde{w}_N|^2 \leqslant -2\Big[1 - K^{\alpha+(1/2)-\rho}(|u|_{\alpha+1-\rho,\sigma} + |\bar{u}|_{\alpha+1-\rho,\sigma})\Big]|\tilde{w}_N|^2$$
$$+ 2|(\mathrm{Id} - P_N)w|_{\alpha+1-\rho,\sigma}|A^{1/2}\tilde{w}_N|. \tag{5.26}$$

By using the integrating factor, even for weak derivatives, one still obtains from (5.26) the following elementary inequality

$$|P_N w(t)|^2_{\alpha+1/2-\rho,\sigma}$$
$$\leqslant |P_N w(T_*)|^2_{\alpha+1/2-\rho,\sigma} e^{-2(t-T_*)} e^{2K^{\alpha+(1/2)-\rho}\int_{T_*}^t (|u(\tau)|_{\alpha+1-\rho,\sigma} + |\bar{u}(\tau)|_{\alpha+1-\rho,\sigma})\,\mathrm{d}\tau}$$
$$+ 2\int_{T_*}^t e^{-2\int_s^t \left[1 - K^{\alpha+(1/2)-\rho}(|u(\tau)|_{\alpha+1-\rho,\sigma} + |\bar{u}(\tau)|_{\alpha+1-\rho,\sigma})\right]\mathrm{d}\tau}$$
$$\times |(\mathrm{Id} - P_N)w(s)|_{\alpha+1-\rho,\sigma}|A^{1/2}\tilde{w}_N(s)|\mathrm{d}s.$$

Utilizing (5.20) in the second summand on the right-hand side of the preceding inequality yields

$$|P_N w(t)|^2_{\alpha+1/2-\rho,\sigma} \leqslant |P_N w(T_*)|^2_{\alpha,\sigma} e^{-2(t-T_*)}$$
$$\times e^{2K^{\alpha+(1/2)-\rho}\int_{T_*}^t (|u(\tau)|_{\alpha+1-\rho,\sigma} + |\bar{u}(\tau)|_{\alpha+1-\rho,\sigma})\,\mathrm{d}\tau}$$
$$+ 2\int_{T_*}^t |(\mathrm{Id} - P_N)w(s)|_{\alpha+1-\rho,\sigma}|A^{1/2}\tilde{w}_N(s)|\mathrm{d}s. \tag{5.27}$$

Observe, for all $s > T_*$, that

$$|(\mathrm{Id} - P_N)w(s)|_{\alpha+1-\rho,\sigma}|A^{1/2}\tilde{w}_N(s)|$$
$$\leqslant |w(s)|^2_{\alpha+1-\rho,\sigma}$$
$$\leqslant (|u(s)|_{\alpha+1-\rho,\sigma} + |\bar{u}(s)|_{\alpha+1-\rho,\sigma})^2 < 4K^{-2(\alpha+(1/2)-\rho)}.$$

We pass $N \to \infty$ in (5.27), noticing, by Lebesgue's dominated convergence theorem, that the last integral goes to zero, and obtain

$$|w(t)|^2_{\alpha+1/2-\rho,\sigma}$$
$$\leqslant |w(T_*)|^2_{\alpha+1/2-\rho,\sigma} e^{-2(t-T_*)} e^{2K^{\alpha+(1/2)-\rho}\int_{T_*}^t (|u(\tau)|_{\alpha+1-\rho,\sigma} + |\bar{u}(\tau)|_{\alpha+1-\rho,\sigma})\,\mathrm{d}\tau}.$$

https://doi.org/10.1017/prm.2018.154 Published online by Cambridge University Press

Inequality (5.21) then follows.

(c) We consider the two specified cases in (5.16). First, we note, thanks to the expansion (4.40) and (5.17), that

$$|u(t) - \xi_1 t^{-1} - \xi_2 t^{-2}|_{\alpha+1-\rho,\sigma},\ |\bar{u}(t) - \xi_1 t^{-1} - \xi_2 t^{-2}|_{\alpha+1-\rho,\sigma} = \mathcal{O}(t^{-3}). \tag{5.28}$$

Case $\xi_1 = 0$. Thanks to (5.28), there are $T_* > 0$ and $D_0 = |\xi_2|_{\alpha+1-\rho,\sigma} + 1$ such that

$$|u(t)|_{\alpha+1-\rho,\sigma},\ |\bar{u}(t)|_{\alpha+1-\rho,\sigma} \leqslant D_0/t^2 < \frac{K^{-(\alpha+(1/2)-\rho)}}{2} \quad \forall t \geqslant T_*. \tag{5.29}$$

Hence, condition (5.20) is met. By (5.21) and (5.29), one has for $t \geqslant T_*$ that

$$|w(t)|_{\alpha+1/2-\rho,\sigma} \leqslant |w(T_*)|_{\alpha+1/2-\rho,\sigma} e^{-t+T_*} e^{K^{\alpha+(1/2)-\rho} \int_{T_*}^t 2D_0/\tau^2\,\mathrm{d}\tau} \leqslant M_1 e^{-t},$$

where $M_1 = |w(T_*)|_{\alpha+1/2-\rho,\sigma} e^{T_*+2K^{\alpha+(1/2)-\rho} D_0/T_*}$. This proves the first relation in (5.16).

Case $\xi_1 \neq 0$. Again, thanks to (5.28), there are $T_* \geqslant 1$ and $D_0 = 2|\xi_1|_{\alpha+1-\rho,\sigma}$ such that

$$|u(t)|_{\alpha+1-\rho,\sigma},\ |\bar{u}(t)|_{\alpha+1-\rho,\sigma} \leqslant D_0/t < \frac{K^{-(\alpha+(1/2)-\rho)}}{2} \quad \forall t \geqslant T_*. \tag{5.30}$$

Again, condition (5.20) is satisfied, and (5.21), together with (5.30), implies, for $t \geqslant T_*$, that

$$\begin{aligned} |w(t)|_{\alpha+1/2-\rho,\sigma} &\leqslant |w(T_*)|_{\alpha+1/2-\rho,\sigma} e^{-t+T_*} e^{K^{\alpha+(1/2)-\rho} \int_{T_*}^t 2D_0/\tau\,\mathrm{d}\tau} \\ &\leqslant |w(T_*)|_{\alpha+1/2-\rho,\sigma} e^{-t+T_*} e^{2K^{\alpha+(1/2)-\rho} D_0 \ln t} = M_2 t^{\beta} e^{-t}, \end{aligned}$$

where $M_2 = |w(T_*)|_{\alpha+1/2-\rho,\sigma} e^{T_*}$ and $\beta = 2K^{\alpha+(1/2)-\rho} D_0$. This proves the second relation in (5.16). □

Remark 5.7.

(a) An equivalent condition to (5.15) is that the series $\sum_{n=1}^{\infty} \xi_n t^{-n}$ of expansion (5.2) converges in $G_{\alpha+1,\sigma}$ at least at one point $t = t_0 \in (0, \infty)$.

(b) According to part (ii) of theorem 5.6, the remainder $u(t) - \bar{u}(t)$ cannot have any intermediate decay between the power and exponential ones. For example, it cannot be approximated by any $e^{-\mu\sqrt{t}}$ for $\mu \in (0, \infty)$.

Acknowledgements

L.H. gratefully acknowledges the support for his research by the NSF grant DMS-1412796.

References

1 P. Constantin and C. Foias. *Navier-Stokes equations*. Chicago Lectures in Mathematics. (Chicago, IL: University of Chicago Press, 1988).

2 J. Dieudonné. *Foundations of modern analysis* (New York-London: Academic Press, 1969). Enlarged and corrected printing, Pure and Applied Mathematics, Vol. 10-I.

3 R. H. Dyer and D. E. Edmunds. Lower bounds for solutions of the Navier-Stokes equations. *Proc. London Math. Soc. (3)* **18** (1968), 169–178.

4 C. Foias, and J.-C. Saut. Asymptotic behavior, as $t \to \infty$, of solutions of the Navier-Stokes equations. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. IV (Paris, 1981/1982)*, pp. 74–86, vol. 84 of *Res. Notes in Math* (Boston, MA: Pitman, 1983).

5 C. Foias and J.-C. Saut. Asymptotic behavior, as $t \to +\infty$, of solutions of Navier-Stokes equations and nonlinear spectral manifolds. *Indiana Univ. Math. J.* **33** (1984a), 459–477.

6 C. Foias and J.-C. Saut. On the smoothness of the nonlinear spectral manifolds associated to the Navier-Stokes equations. *Indiana Univ. Math. J.* **33** (1984b), 911–926.

7 C. Foias and J.-C. Saut. Linearization and normal form of the Navier-Stokes equations with potential forces. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **4** (1987), 1–47.

8 C. Foias and R. Temam. Gevrey class regularity for the solutions of the Navier-Stokes equations. *J. Funct. Anal.* **87** (1989), 359–369.

9 C. Foias and J.-C. Saut. Asymptotic integration of Navier-Stokes equations with potential forces. I. *Indiana Univ. Math. J.* **40** (1991), 305–320.

10 C. Foias, O. Manley, R. Rosa and R. Temam. *Navier-Stokes equations and turbulence*, vol. 83 of *Encyclopedia of Mathematics and its Applications* (Cambridge: Cambridge University Press, 2001).

11 C. Foias, L. Hoang, E. Olson and M. Ziane. On the solutions to the normal form of the Navier-Stokes equations. *Indiana Univ. Math. J.* **55** (2006), 631–686.

12 C. Foias, L. Hoang and B. Nicolaenko. On the helicity in 3D-periodic Navier-Stokes equations. I. The non-statistical case. *Proc. Lond. Math. Soc. (3)* **94** (2007), 53–90.

13 C. Foias, L. Hoang and B. Nicolaenko. On the helicity in 3D-periodic Navier-Stokes equations. II. The statistical case. *Comm. Math. Phys.* **290** (2009a), 679–717.

14 C. Foias, L. Hoang, E. Olson and M. Ziane. The normal form of the Navier-Stokes equations in suitable normed spaces. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26** (2009b), 1635–1673.

15 C. Foias, R. Rosa and R. Temam. Topological properties of the weak global attractor of the three-dimensional Navier-Stokes equations. *Discrete Contin. Dyn. Syst.* **27** (2010), 1611–1631.

16 C. Foias, L. Hoang and J.-C. Saut. Asymptotic integration of Navier-Stokes equations with potential forces. II. An explicit Poincaré-Dulac normal form. *J. Funct. Anal.* **260** (2011), 3007–3035.

17 C. Foias, L. Hoang and J.-C. Saut. Navier and Stokes meet Poincaré and Dulac. *J. Appl. Anal. Comput.* **8** (2018), 727–763.

18 L. T. Hoang and V. R. Martinez. Asymptotic expansion in Gevrey spaces for solutions of Navier-Stokes equations. *Asymptot. Anal.* **104** (2017), 167–190.

19 L. T. Hoang and V. R. Martinez. Asymptotic expansion for solutions of the Navier-Stokes equations with non-potential body forces. *J. Math. Anal. Appl.* **462** (2018), 84–113.

20 I. Kukavica and E. Reis. Asymptotic expansion for solutions of the Navier-Stokes equations with potential forces. *J. Diff. Equ.* **250** (2011), 607–622.

21 O.A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow*, 2nd English edn, revised and enlarged. Translated from the Russian by Richard A. Silverman and John Chu. Mathematics and its Applications, Vol. 2 (New York: Gordon and Breach Science Publishers, 1969).

22 Y. Shi. A Foias-Saut type of expansion for dissipative wave equations. *Comm. Partial Diff. Equ.* **25** (2000), 2287–2331.

23 R. Temam. *Navier-Stokes equations and nonlinear functional analysis*, 2nd edn, vol. 66 of *CBMS-NSF Regional Conference Series in Applied Mathematics* (Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), 1995).

24 R. Temam. *Infinite-dimensional dynamical systems in mechanics and physics*, 2nd edn, vol. 68 of *Applied Mathematical Sciences* (New York: Springer-Verlag, 1997).

25 R. Temam. *Navier-Stokes equations* (Providence, RI: AMS Chelsea Publishing, 2001). Theory and numerical analysis, Reprint of the 1984 edition.