Math. Proc. Camb. Phil. Soc. (2022), **173**, 511–523
doi:10.1017/S0305004121000700
First published online 14 December 2021

Non-normality, topological transitivity and expanding families

BY THIERRY MEYRATH
University of Luxembourg, Department of Mathematics, L-4364 Esch-sur-Alzette, Luxembourg.
e-mail: thierry.meyrath@uni.lu

JÜRGEN MÜLLER
Universität Trier, Fachbereich IV – Mathematik, D-54286 Trier, Germany.
e-mail: jmueller@uni-trier.de

(*Received* 19 *October* 2020*; revised* 17 *November* 2021*; accepted* 18 *October* 2021)

Abstract

We investigate the behaviour of families of meromorphic functions in the neighbourhood of points of non-normality and prove certain covering properties that complement Montel's Theorem. In particular, we also obtain characterisations of non-normality in terms of such properties.

2020 Mathematics Subject Classification: 30D45, 30D30 (Primary)

1. *Introduction*

For an open set $\Omega \subset \mathbb{C}$ we denote by $M(\Omega)$ the set of meromorphic functions on Ω, by which we mean all functions whose restriction to a connected component of Ω is either meromorphic or constant infinity. Endowed with the topology of spherically uniform convergence (i.e. uniform convergence with respect to the chordal metric χ) on compact subsets of Ω, the space $M(\Omega)$ becomes a complete metric space (e.g. [12, Chapter VII]). As usual, we say that a family $\mathcal{F} \subset M(\Omega)$ is normal in Ω, if every sequence $(f_n) \subset \mathcal{F}$ contains a subsequence (f_{n_k}) that converges spherically uniformly on compact subsets of Ω to a function $f \in M(\Omega)$. The family $\mathcal{F}$ is called normal at a point $z_0 \in \Omega$, if there exists an open neighbourhood U of z_0, such that $\mathcal{F}$ is normal in U. By $J(\mathcal{F})$ we denote the set of points in Ω, at which the family $\mathcal{F}$ is non-normal. If $z_0 \in J(\mathcal{F})$, the family $\mathcal{F}$ can still have infinite subfamilies $\tilde{\mathcal{F}} \subset \mathcal{F}$ that are normal at z_0, in other words, $z_0 \in J(\mathcal{F})$ does in general not imply $z_0 \in J(\tilde{\mathcal{F}})$. We say that $\mathcal{F}$ is strongly non-normal at a point $z_0 \in \Omega$, if we have $z_0 \in J(\tilde{\mathcal{F}})$ for every infinite subfamily $\tilde{\mathcal{F}} \subset \mathcal{F}$. We further say that $\mathcal{F}$ is strongly non-normal on a relatively closed set $B \subset \Omega$, if $\mathcal{F}$ is strongly non-normal at every $z_0 \in B$, that is if $B \subset J(\tilde{\mathcal{F}})$ for every infinite subfamily $\tilde{\mathcal{F}} \subset \mathcal{F}$. Moreover, we call $\mathcal{F}$ hereditarily non-normal on B, if some infinite subfamily $\tilde{\mathcal{F}} \subset \mathcal{F}$ is strongly non-normal on B. Note that on a single point set, hereditary non-normality is equivalent to non-normality, while this is in general not true for sets containing at least two points.

For a family $\mathcal{F} \subset M(\Omega)$ and an open set $U \subset \Omega$, we write $\limsup \mathcal{F}(U)$ for the intersection of all $\bigcup_{f \in \tilde{\mathcal{F}}} f(U)$, where $\tilde{\mathcal{F}}$ ranges over the cofinite subsets of $\mathcal{F}$. Moreover, for $z_0 \in \Omega$

we denote by $\limsup_{z_0} \mathcal{F}$ the intersection of $\limsup \mathcal{F}(U)$ taken over all open neighbourhoods $U \subset \Omega$ of z_0. Similarly, we write $\liminf \mathcal{F}(U)$ for the union of all $\bigcap_{f \in \tilde{\mathcal{F}}} f(U)$, where $\tilde{\mathcal{F}}$ ranges over the cofinite subsets of $\mathcal{F}$ and $\liminf_{z_0} \mathcal{F}$ for the intersection of $\liminf \mathcal{F}(U)$ taken over all open neighbourhoods $U \subset \Omega$ of z_0. Obviously, we have that $\liminf_{z_0} \mathcal{F} \subset \limsup_{z_0} \mathcal{F}$, furthermore $\liminf_{z_0} \mathcal{F} = \bigcap_{\tilde{\mathcal{F}} \subset \mathcal{F} \text{ infinite}} \limsup_{z_0} \tilde{\mathcal{F}}$. For instance, if $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ with $f_n(z) = nz$ for even integers n and $f_n(z) = z$ for odd n, then $\limsup_0 \mathcal{F} = \mathbb{C}$ and $\liminf_0 \mathcal{F} = \{0\}$.

The classical Montel Theorem suggests that the behaviour of families $\mathcal{F} \subset M(\Omega)$ in neighbourhoods of points $z_0 \in J(\mathcal{F})$ consists in some sense of spreading points, since it asserts that for every $z_0 \in J(\mathcal{F})$, the set $E_{z_0}(\mathcal{F}) := \mathbb{C}_\infty \setminus \limsup_{z_0} \mathcal{F}$ contains at most two points, where $\mathbb{C}_\infty := \mathbb{C} \cup \{\infty\}$. Hence, for every neighbourhood U of z_0, every point $a \in \mathbb{C}_\infty$ is covered by $f(U)$ for infinitely many $f \in \mathcal{F}$, with at most two exceptions. In case that $E_{z_0}(\mathcal{F})$ contains two points and $\mathcal{F}$ is strongly non-normal at z_0, a further consequence of Montel's Theorem is that $\liminf_{z_0} \mathcal{F} = \limsup_{z_0} \mathcal{F}$, so that for every neighbourhood U of z_0, every point $a \in \mathbb{C}_\infty \setminus E_{z_0}(\mathcal{F})$ is covered by $f(U)$ for cofinitely many $f \in \mathcal{F}$. Note, however, that Montel's Theorem does not contain any information about the 'size' of the individual sets $f(U)$, for instance, if U is any neighbourhood of a point $z_0 \in J(\mathcal{F})$, it is in general not clear if for a given set $A \subset \limsup_{z_0} \mathcal{F}$ we have $A \subset f(U)$ for infinitely many $f \in \mathcal{F}$.

In this paper, we will further investigate the behaviour of (strongly) non-normal families near points of non-normality and show certain covering and 'expanding' properties that complement the statement of Montel's Theorem. In particular, we will also derive different characterisations of (strong) non-normality in terms of these properties.

2. *Non-normality and topological transitivity*

In the sequel, for $\lambda > 0$ and $z_0 \in \mathbb{C}$ we set $D_\lambda(z_0) := \{z \in \mathbb{C} : |z - z_0| < \lambda\}$ and denote by $\overline{D}_\lambda(z_0)$ the closure of $D_\lambda(z_0)$ in $\mathbb{C}$. For $w_0 \in \mathbb{C}_\infty$, we further set $D_\lambda^\chi(w_0) := \{w \in \mathbb{C}_\infty : \chi(w, w_0) < \lambda\}$. We say that a family $\mathcal{F} \subset M(\Omega)$ is (topologically) transitive with respect to a point $z_0 \in \Omega$, if for every pair of non-empty open sets $U \subset \Omega$ and $V \subset \mathbb{C}_\infty$ with $z_0 \in U$, there exists $f \in \mathcal{F}$ such that $f(U) \cap V \neq \emptyset$. Note that in this case we have $f(U) \cap V \neq \emptyset$ for infinitely many $f \in \mathcal{F}$. If $f(U) \cap V \neq \emptyset$ holds for cofinitely many $f \in \mathcal{F}$, we say that $\mathcal{F}$ is (topologically) mixing with respect to z_0. Furthermore, if for every non-empty open set $U \subset \Omega$ with $z_0 \in U$ and every pair of non-empty open sets $V_1, V_2 \subset \mathbb{C}_\infty$, there exists $f \in \mathcal{F}$ such that $f(U) \cap V_i \neq \emptyset$ for $i = 1, 2$, we say that $\mathcal{F}$ is weakly mixing with respect to z_0. Finally, we say that $\mathcal{F}$ is transitive (or (weakly) mixing) with respect to a relatively closed set $B \subset \Omega$, if $\mathcal{F}$ is transitive (or (weakly) mixing) with respect to every $z_0 \in B$.

With these notations, we obtain the following characterisation of (strong) non-normality.

THEOREM 1. *Let $\Omega \subset \mathbb{C}$ be open, $\mathcal{F} \subset M(\Omega)$ a family of meromorphic functions and $z_0 \in \Omega$. Then we have*:

(a) *$\mathcal{F}$ is strongly non-normal at z_0 if and only if $\mathcal{F}$ is mixing with respect to z_0;*

(b) *the following are equivalent*:

(i) *$\mathcal{F}$ is non-normal at z_0;*

(ii) *there exists an infinite subfamily $\tilde{\mathcal{F}} \subset \mathcal{F}$ that is mixing with respect to z_0;*

(iii) *$\mathcal{F}$ is weakly mixing with respect to z_0.*

Proof. (a) Let $\mathcal{F}$ be strongly non-normal at z_0 and suppose that $\mathcal{F}$ is not mixing with respect to z_0. Then there exist non-empty open sets $U \subset \Omega$ and $V \subset \mathbb{C}_\infty$ with $z_0 \in U$, and an infinite subfamily $\tilde{\mathcal{F}} \subset \mathcal{F}$ such that $f(U) \cap V = \emptyset$ for every $f \in \tilde{\mathcal{F}}$. By Montel's Theorem, we obtain that $\tilde{\mathcal{F}}$ is normal on U, hence also at z_0, in contradiction to the strong non-normality of $\mathcal{F}$ at z_0.

On the other hand, suppose that $\mathcal{F}$ is mixing with respect to $z_0 \in \Omega$, but not strongly non-normal at z_0. Then there exists an open neighbourhood U of z_0 and a sequence $(f_n) \subset \mathcal{F}$, such that (f_n) converges spherically uniformly on compact subsets of U to a function $f \in M(U)$. Hence, for $\varepsilon > 0$ sufficiently small, we have that $\overline{D}_\varepsilon(z_0) \subset U$ and there exists $\delta > 0$ and $w_0 \in \mathbb{C}_\infty$ such that $f(\overline{D}_\varepsilon(z_0)) \cap D^\chi_\delta(w_0) = \emptyset$. Since (f_n) is mixing with respect to z_0, we obtain that $f_n(D_\varepsilon(z_0)) \cap D^\chi_{\frac{\delta}{2}}(w_0) \neq \emptyset$ for all n sufficiently large, in contradiction to the spherically uniform convergence of (f_n) to f on $\overline{D}_\varepsilon(z_0)$.

(b) (i) $\Rightarrow$ (ii): Since $\mathcal{F}$ is non-normal at z_0, there exists an infinite subfamily $\tilde{\mathcal{F}} \subset \mathcal{F}$ that is strongly non-normal at z_0. This subfamily is mixing with respect to z_0 according to the first statement of the theorem.

(ii) $\Rightarrow$ (iii): This is clear, since a mixing family is also weakly mixing.

(iii) $\Rightarrow$ (i): Suppose that $\mathcal{F}$ is weakly mixing with respect to z_0. Further consider two non-empty open sets $V_1, V_2 \subset \mathbb{C}_\infty$ such that $\inf_{z\in V_1, w\in V_2} \chi(z,w) > \varepsilon$ for some $\varepsilon > 0$. For $k \in \mathbb{N}$, we set $U_k := D_{\frac{1}{k}}(z_0) \cap \Omega$. By assumption, for every $k \in \mathbb{N}$ there is a function $f_k \in \mathcal{F}$ such that $f_k(U_k) \cap V_1 \neq \emptyset$ and $f_k(U_k) \cap V_2 \neq \emptyset$, and hence points $z_k^{(1)}, z_k^{(2)} \in U_k$ such that $f_k(z_k^{(1)}) \in V_1$ and $f_k(z_k^{(2)}) \in V_2$. Note that $z_k^{(1)}, z_k^{(2)} \in U_k$ implies that $z_k^{(1)} \to z_0$ and $z_k^{(2)} \to z_0$ for $k \to \infty$, furthermore we have that $\chi(f_k(z_k^{(1)}), f_k(z_k^{(2)})) > \varepsilon$ for every $k \in \mathbb{N}$, and hence

$$\chi(f_k(z_0), f_k(z_k^{(1)})) > \frac{\varepsilon}{2} \quad \text{or} \quad \chi(f_k(z_0), f_k(z_k^{(2)})) > \frac{\varepsilon}{2}.$$

Hence, we can find a sequence (z_k) with $z_k \to z_0$ for $k \to \infty$ and $\chi(f_k(z_0), f_k(z_k)) > \frac{\varepsilon}{2}$ for every $k \in \mathbb{N}$, implying that the family $\mathcal{F}$ is not spherically equicontinuous at z_0, and thus also not normal.

By Montel's Theorem, it is clear that $z_0 \in J(\mathcal{F})$ implies that $\mathcal{F}$ is transitive with respect to z_0. On the other hand, it is easily seen that transitivity of a family with respect to some point $z_0 \in \Omega$ is in general not sufficient for non-normality at z_0. For instance, if (z_n) is a sequence that is dense in $\mathbb{C}_\infty$, the family (f_n) of constant functions $f_n \equiv z_n$ is transitive with respect to any $z_0 \in \Omega$, while at the same time we have $J(f_n) = \emptyset$. However, the following proposition shows that this example is in some sense typical:

PROPOSITION 1. *Let $\Omega \subset \mathbb{C}$ be open, $\mathcal{F} \subset M(\Omega)$ a family of meromorphic functions and $z_0 \in \Omega$. Suppose that $\mathcal{F}$ is transitive with respect to z_0 and that $z_0 \notin J(\mathcal{F})$. Then $\{f(z_0) : f \in \mathcal{F}\}$ is dense in $\mathbb{C}_\infty$.*

Proof. Suppose that $\{f(z_0) : f \in \mathcal{F}\}$ is not dense in $\mathbb{C}_\infty$. Then there is $w \in \mathbb{C}_\infty$ and $\varepsilon > 0$, such that $\{f(z_0) : f \in \mathcal{F}\} \cap D^\chi_\varepsilon(w) = \emptyset$. Consider now for $k \in \mathbb{N}$ the sets $U_k := D_{\frac{1}{k}}(z_0) \cap \Omega$. Since $\mathcal{F}$ is transitive with respect to z_0, for every $k \in \mathbb{N}$ there is $f_k \in \mathcal{F}$ such that $f_k(U_k) \cap D^\chi_{\frac{\varepsilon}{2}}(w) \neq \emptyset$. In particular, there is a sequence (z_k) with $z_k \in U_k$, and hence $z_k \to z_0$ for $k \to \infty$, such that $f_k(z_k) \in D^\chi_{\frac{\varepsilon}{2}}(w)$ for $k \in \mathbb{N}$. On the other hand, we have $f_k(z_0) \notin D^\chi_\varepsilon(w)$

https://doi.org/10.1017/S0305004121000700 Published online by Cambridge University Press

for $k \in \mathbb{N}$. Finally, we obtain that

$$\chi(f_k(z_0), f_k(z_k)) > \frac{\varepsilon}{2} \qquad \text{for every } k \in \mathbb{N},$$

so that $\mathcal{F}$ is not spherically equicontinuous at z_0, and thus also not normal, that is $z_0 \in J(\mathcal{F})$.

Example 1.

(i) Let f be an entire function that is neither constant nor a polynomial of degree 1, and let $\mathcal{F} := \{f^{\circ n} : n \in \mathbb{N}\}$ be the family of iterates of f. Then $\mathcal{F}$ is strongly non-normal on the Julia set $J = J(\mathcal{F})$, as follows e.g. from the facts that the repelling periodic points are dense in J and that J is the boundary of the escaping set (e.g. [6,14,26,30]). Here we have $\liminf_{z_0} \mathcal{F} \supset \mathbb{C} \setminus E$ for each $z_0 \in J$, where E is the (empty or one-point) set of Fatou exceptional values of f, that is the set of points $w \in \mathbb{C}$ whose backward orbit $O^-(w) := \bigcup_{n\geq 1}\{z : f^{\circ n}(z) = w\}$ is finite. Indeed, consider $z_0 \in J$ and an infinite subfamily $\tilde{\mathcal{F}} = \{f^{\circ n_k} : k \in \mathbb{N}\}$ with $n_k > 2$. It follows from Picard's Theorem that if $a \in \mathbb{C}$ is not Fatou exceptional, there are points $a_1, a_2 \in \mathbb{C}$ with $a_1 \neq a_2$ and $f^{\circ 2}(a_1) = a = f^{\circ 2}(a_2)$. Since $\mathcal{F}$ is strongly non-normal at z_0, Montel's Theorem implies that the set $\mathbb{C} \setminus \limsup_{z_0} \tilde{\mathcal{F}}^-$ contains at most one point, where $\tilde{\mathcal{F}}^- := \{f^{\circ(n_k-2)} : k \in \mathbb{N}\}$. Hence, $\{a_1, a_2\} \cap \limsup_{z_0} \tilde{\mathcal{F}}^- \neq \emptyset$, which implies $a \in \limsup_{z_0} \tilde{\mathcal{F}}$.

(ii) Let M denote the Mandelbrot set and let, with $p_0 := \mathrm{id}_{\mathbb{C}}$, the family (p_n) of polynomials of degree 2^n be recursively defined by $p_n := p_{n-1}^2 + \mathrm{id}_{\mathbb{C}}$. Since $p_n \to \infty$ pointwise on $\mathbb{C} \setminus M$ for $n \to \infty$ and $|p_n| \leq 2$ on M (e.g. [6]), we have $\partial M \subset J(\mathcal{F})$, where $\mathcal{F} := \{p_n : n \in \mathbb{N}_0\}$, and no infinite subfamily of $\mathcal{F}$ can be normal at any point of ∂M. Hence, $\mathcal{F}$ is strongly non-normal and thus mixing on ∂M.

(iii) A function $f \in M(\mathbb{C})$ is called Yosida function, if it has bounded spherical derivative $f^\#$ (e.g. [24,32]). Hence, if f is not a Yosida function, there exists a sequence (z_n) in $\mathbb{C}$ with $z_n \to \infty$ and $f^\#(z_n) \to \infty$ for $n \to \infty$. Marty's Theorem (e.g. [29, p.75]) implies that the family (f_n) with $f_n(z) := f(z + z_n)$ is strongly non-normal at 0, hence by Theorem 1, we obtain that (f_n) is mixing with respect to 0. Note that it is easily seen that if $f \in M(\mathbb{C})$ is a Yosida function, then its order of growth is at most 2, while entire Yosida functions are necessarily of exponential type (e.g. [11,24]).

For a family of meromorphic functions $\mathcal{F} \subset M(\Omega)$ and $N \in \mathbb{N}$, we consider the family $\mathcal{F}^{\times N} := \{f^{\times N} : f \in \mathcal{F}\}$, where $f^{\times N} : \Omega^N \to \mathbb{C}_\infty^N$ with $f^{\times N}(z_1, \ldots, z_N) = (f(z_1), \ldots, f(z_N))$. We say that $\mathcal{F}^{\times N}$ is transitive with respect to $z \in \Omega^N$, if for every pair of non-empty open sets $U \subset \Omega^N$ and $V \subset \mathbb{C}_\infty^N$ with $z \in U$, there exists $f^{\times N} \in \mathcal{F}^{\times N}$ such that $f^{\times N}(U) \cap V \neq \emptyset$. Furthermore, for a relatively closed set $B \subset \Omega$, we say that $\mathcal{F}^{\times N}$ is transitive with respect to B^N, if $\mathcal{F}^{\times N}$ is transitive with respect to every $z \in B^N$. We then have the following characterisation of hereditary non-normality.

PROPOSITION 2. *Let $\Omega \subset \mathbb{C}$ be open, $\mathcal{F} \subset M(\Omega)$ a family of meromorphic functions and $B \subset \Omega$ closed in Ω. Then the following are equivalent*:

(i) *$\mathcal{F}$ is hereditarily non-normal on B;*

(ii) *there exists an infinite subfamily $\tilde{\mathcal{F}} \subset \mathcal{F}$ that is mixing with respect to B;*

(iii) *for all $N \in \mathbb{N}$ the family $\mathcal{F}^{\times N}$ is transitive with respect to B^N.*

https://doi.org/10.1017/S0305004121000700 Published online by Cambridge University Press

Proof. The equivalence of (i) and (ii) follows from Theorem 1.

(ii) $\Rightarrow$ (iii): Without loss of generality consider $\tilde{\mathcal{F}}$ to be countable, $\tilde{\mathcal{F}} = \{f_n : n \in \mathbb{N}\}$ say. Let $N \in \mathbb{N}$ and consider non-empty open sets $U \subset \Omega^N$ and $V \subset \mathbb{C}_\infty^N$ with $B^N \cap U \neq \emptyset$. Then there exist non-empty open sets $U_1, \ldots, U_N$ with $U_1 \times \cdots \times U_N \subset U$ and $B \cap U_i \neq \emptyset$ for $i = 1, \ldots, N$, and non-empty open sets $V_1, \ldots, V_N \subset \mathbb{C}_\infty$ with $V_1 \times \cdots \times V_N \subset V$. According to the assumption, $\{f_n : n > m\}$ is transitive with respect to B, for all $m \in \mathbb{N}$. Inductively, we can find a strictly increasing sequence (n_k) in $\mathbb{N}$ with $f_{n_k}(U_1) \cap V_1 \neq \emptyset$ for all $k \in \mathbb{N}$. By assumption, the family $\{f_{n_k} : k \in \mathbb{N}\}$ is transitive with respect to B. Thus, the same argument as above yields the existence of a subsequence $(n_k^{(2)})$ of $(n_k^{(1)}) := (n_k)$ with $f_{n_k^{(2)}}(U_2) \cap V_2 \neq \emptyset$ for all $k \in \mathbb{N}$. Proceeding in the same way, for any j with $2 \leq j \leq N$ we find subsequences $(n_k^{(j)})$ of $(n_k^{(j-1)})$ with $f_{n_k^{(j)}}(U_j) \cap V_j \neq \emptyset$ for all $k \in \mathbb{N}$. In particular, for $n := n_1^{(N)}$, we obtain that

$$(f_n(U_1) \times \cdots \times f_n(U_N)) \cap (V_1 \times \cdots \times V_N) \neq \emptyset,$$

hence also $f_n^{\times N}(U) \cap V \neq \emptyset$, implying that $\mathcal{F}^{\times N}$ is transitive with respect to B^N.

(iii) $\Rightarrow$ (ii) The proof follows along the same lines as the proof of the corresponding part of the Bès–Peris Theorem (e.g. [21, pp.76]).

Remark 1.

(i) Let $\mathcal{K}(A)$ denote the hyperspace of $A \subset \mathbb{C}$, that is the space of all non-empty compact subsets of A endowed with the Hausdorff metric, and suppose that B as in Proposition 2 has non-empty interior. Then [2, corollay 1·2] shows that, under the conditions of Proposition 2, for each $\mathbb{C}$-closed set $A \subset B$ which coincides with the closure of its interior, the family $\mathcal{F}|_E$ is dense in $C(E, \mathbb{C}_\infty)$ for generically many sets $E \in \mathcal{K}(A)$.

(ii) We mention that Proposition 2 is an extension of Theorem 3·7 from the recent paper [4].

Example 2.

(i) Consider a function $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ that is holomorphic on the unit disk $\mathbb{D}$. Suppose that f has at least one singularity on $\partial\mathbb{D}$ and denote by $D \subset \partial\mathbb{D}$ the set of all singularities. Then, denoting by $s_n(z) := (s_n f)(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ the nth partial sum of f, the family (s_n) is non-normal on $\partial\mathbb{D}$ and strongly non-normal on D. Moreover, in case $D \neq \partial\mathbb{D}$, Vitali's Theorem implies that a subsequence of (s_n) forms a normal family at a point $z_0 \in \partial\mathbb{D} \setminus D$ if and only if it converges to an analytic continuation of f in some neighbourhood of z_0. From refined versions of Ostrowski's results on overconvergence ([16, Theorems 3 and 4]), it follows that a subsequence (s_{n_k}) is strongly non-normal at $z_0 \in \partial\mathbb{D} \setminus D$ if and only if (s_n) has no Hadamard–Ostrowski gaps relative to (n_k), that is, if and only if there is a sequence (δ_k) of positive numbers tending to 0 with

$$\sup_{(1-\delta_k)n_k \leq \nu \leq n_k} |a_\nu|^{1/\nu} \to 1$$

as $k \to \infty$. In this case, the sequence (s_{n_k}) is already strongly non-normal at all $z \in \partial\mathbb{D}$. Since the non-normality of (s_n) on $\partial\mathbb{D}$ implies that, given $z_0 \in \partial\mathbb{D} \setminus D$, some subsequence of (s_n) is strongly non-normal at z_0, we finally obtain that the family (s_n) is always hereditarily non-normal on $\partial\mathbb{D}$. According to a result of Gardiner ([15, corollary 3]), for each f that is holomorphic on $\mathbb{D}$ and analytically continuable to some domain U such that $\mathbb{C}\setminus U$ is thin at some $z_0 \in \partial\mathbb{D}$ but not continuable to the point z_0, the sequence (s_n) has no Hadamard–Ostrowski gaps with respect to any (n_k), hence (s_n) is strongly non-normal on $\partial\mathbb{D}$. In particular, this holds for each f that has an isolated singularity at some point $z_0 \in \partial\mathbb{D}$.

(ii) We write H_0 for the space of functions holomorphic on $\mathbb{C}\setminus\{1\}$ that vanish at ∞. For $f(z) = 1/(1-z)$, the sequence $(s_n f)$ is the geometric series which tends to ∞ spherically uniformly on compact subsets of $\mathbb{C}\setminus\overline{\mathbb{D}}$. From [3, theorem 1·1] it can be deduced that generically many functions $f \in H_0$ enjoy the property that some subsequence of the sequence $((f - s_n f)(z)/z^n)$ converges to $1/(1-z)$ spherically uniformly on compact subsets of $\mathbb{C}_\infty\setminus\{1\}$. This implies that the corresponding subsequence of $(s_n f)$ converges to ∞ spherically uniformly on compact subsets of $\mathbb{C}\setminus\overline{\mathbb{D}}$ and thus forms a normal family on $\mathbb{C}\setminus\overline{\mathbb{D}}$. In particular, $(s_n f)$ is not strongly non-normal at any point $z_0 \in \mathbb{C}\setminus\overline{\mathbb{D}}$. On the other hand, if A is a countable and dense subset of $\mathbb{C}\setminus\mathbb{D}$, from [23, theorem 2] it follows that for generically many functions $f \in H_0$, a subsequence $(s_{n_k} f)$ of $(s_n f)$ converges to 0 pointwise on A. Since a result from [22] implies that for $f \in H_0$, normality of a subsequence of $(s_n f)$ at a point $z_0 \in \mathbb{C}\setminus\overline{\mathbb{D}}$ forces the subsequence to tend to ∞ spherically uniformly on compact subsets of some neighbourhood of z_0, it follows that no subsequence of $(s_{n_k} f)$ can form a normal family at any point of $\mathbb{C}\setminus\overline{\mathbb{D}}$. By the previous example, $(s_n f)$ is strongly non-normal on $\partial\mathbb{D}$ for $f \in H_0$, thus we obtain that for generically many $f \in H_0$, the family $(s_n f)$ is hereditarily non-normal on $\mathbb{C}\setminus\mathbb{D}$. By Remark 1, for generically many $f \in H_0$, the sequence $(s_n f|_E)$ is dense in $C(E, \mathbb{C}_\infty)$ for generically many $E \in \mathcal{K}(\mathbb{C}\setminus\mathbb{D})$ (see also [1, theorem 2]).

3. *Non-normality and expanding families*

We define the following 'expanding' property of families $\mathcal{F} \subset M(\Omega)$.

Definition 1. Let $\Omega \subset \mathbb{C}$ be open, $\mathcal{F} \subset M(\Omega)$ a family of meromorphic functions and $z_0 \in \Omega$. Consider further a set $A \subset \mathbb{C}_\infty$. We say that $\mathcal{F}$ is expanding at z_0 with respect to A, if for every open neighbourhood $U \subset \Omega$ of z_0 and every compact set $K \subset A$ we have $K \subset f(U)$ for infinitely many $f \in \mathcal{F}$. If $K \subset f(U)$ holds for cofinitely many $f \in \mathcal{F}$, we say that $\mathcal{F}$ is strongly expanding at z_0 with respect to A. Finally, we say that $\mathcal{F}$ is (strongly) expanding on a set $B \subset \Omega$ with respect to A, if $\mathcal{F}$ is (strongly) expanding with respect to A at every $z_0 \in B$.

Note that if $\mathcal{F}$ is expanding at z_0 with respect to A, there exists an infinite subfamily $\tilde{\mathcal{F}} \subset \mathcal{F}$ that is strongly expanding at z_0 with respect to A. Moreover, in this case we have that A is contained in $\limsup_{z_0} \mathcal{F}$. Also note that $\mathcal{F}$ is strongly expanding at z_0 with respect to A if and only if every infinite subfamily $\tilde{\mathcal{F}} \subset \mathcal{F}$ is expanding at z_0 with respect to A, and in this case, A is contained in $\liminf_{z_0} \mathcal{F}$. On the other hand, we remark that $A \subset \liminf_{z_0} \mathcal{F}$ does in general not imply that $\mathcal{F}$ is (strongly) expanding at z_0 with respect to A. This can for instance be seen by considering the family $\mathcal{F} := \{e^{nz} + (1 - 1/n) : n \in \mathbb{N}\}$, for which we have $\liminf_0 \mathcal{F} = \mathbb{C}$ (note that for each neighbourhood U of 0 and each $w \in \mathbb{C}$, there is some N such that $w - 1 + 1/n \in \exp(nU)$ for all $n \geq N$), but $\mathcal{F}$ is not expanding at 0 with respect to any set $A \subset \mathbb{C}$ with $1 \in A^\circ$.

Our next result establishes a relationship between strong non-normality and the expanding property. Here and in the following, we denote by $|E| \in \mathbb{N}_0 \cup \{\infty\}$ the number of elements of a set $E \subset \mathbb{C}_\infty$.

THEOREM 2. *Let $\Omega \subset \mathbb{C}$ be open, $\mathcal{F} \subset M(\Omega)$ a family of meromorphic functions and $z_0 \in \Omega$. Then we have*:

(i) *if $\mathcal{F}$ is strongly non-normal at z_0, then for each infinite subfamily $\tilde{\mathcal{F}} \subset \mathcal{F}$ there exists $E \subset \mathbb{C}_\infty$ with $|E| \leq 2$, such that $\tilde{\mathcal{F}}$ is expanding at z_0 with respect to $\mathbb{C}_\infty \setminus E$. Moreover, $\mathcal{F}$ is strongly expanding at z_0 with respect to $\mathbb{C}_\infty \setminus \mathcal{E}$, where $\mathcal{E} := \bigcup_{\tilde{\mathcal{F}} \subset \mathcal{F} \text{ infinite}} E_{\tilde{\mathcal{F}}}$ with $E_{\tilde{\mathcal{F}}} \subset \mathbb{C}_\infty$ being some set such that $\tilde{\mathcal{F}}$ is expanding at z_0 with respect to $\mathbb{C}_\infty \setminus E_{\tilde{\mathcal{F}}}$;*

(ii) *if $|\liminf_{z_0} \mathcal{F}| \geq 2$, then $\mathcal{F}$ is strongly non-normal at z_0. In particular, this holds if $\mathcal{F}$ is strongly expanding at z_0 with respect to some $A \subset \mathbb{C}_\infty$ with $|A| \geq 2$.*

Proof. (i) Suppose that $\mathcal{F}$ is strongly non-normal at z_0 and consider an infinite subfamily $\tilde{\mathcal{F}} \subset \mathcal{F}$. Then $\tilde{\mathcal{F}}$ is strongly non-normal at z_0 and assuming that $\tilde{\mathcal{F}}$ is not expanding at z_0 with respect to $\mathbb{C}_\infty \setminus E$ for any $E \subset \mathbb{C}_\infty$ with $|E| \leq 2$, we obtain that for every $E \subset \mathbb{C}_\infty$ with $|E| \leq 2$, there is an open neighbourhood U of z_0 and a compact set $K \subset \mathbb{C}_\infty \setminus E$, such that $K \setminus f(U) \neq \emptyset$ for cofinitely many $f \in \tilde{\mathcal{F}}$. In particular, if $\tilde{\mathcal{F}}$ is not expanding at z_0 with respect to $\mathbb{C}_\infty$, we can find an open neighbourhood U_1 of z_0, a sequence (f_n) in $\tilde{\mathcal{F}}$, and a sequence (a_n) in $\mathbb{C}_\infty$ with $a_n \to a \in \mathbb{C}_\infty$ for $n \to \infty$, such that $a_n \notin f_n(U_1)$ for every $n \in \mathbb{N}$. By assumption, $\tilde{\mathcal{F}}$ is not expanding at z_0 with respect to $\mathbb{C}_\infty \setminus \{a\}$, hence, there is an open neighbourhood U_2 of z_0 and a compact set $K_2 \subset \mathbb{C}_\infty \setminus \{a\}$, such that $K_2 \setminus f(U_2) \neq \emptyset$ for cofinitely many $f \in \tilde{\mathcal{F}}$. In particular, there is a subsequence (f_{n_k}) of (f_n), and a sequence (b_k) in K_2 with $b_k \to b \in K_2$ for $k \to \infty$, such that $b_k \notin f_{n_k}(U_2)$ for every $k \in \mathbb{N}$. Since $\tilde{\mathcal{F}}$ is not expanding at z_0 with respect to $\mathbb{C}_\infty \setminus \{a, b\}$, a similar argumentation leads to an open neighbourhood U_3 of z_0, a compact set $K_3 \subset \mathbb{C}_\infty \setminus \{a, b\}$, a subsequence $(f_{n_{k_l}})$ of (f_{n_k}) and a sequence (c_l) in K_3 with $c_l \to c \in K_3$ for $l \to \infty$, such that $c_l \notin f_{n_{k_l}}(U_3)$ for every $l \in \mathbb{N}$.

Finally, setting $U = U_1 \cap U_2 \cap U_3$ we obtain that

$$\{a_{n_{k_l}}, b_{k_l}, c_l\} \cap f_{n_{k_l}}(U) = \emptyset \quad \text{for every } l \in \mathbb{N}.$$

Furthermore, since a,b,c are pairwise distinct, there exists $\varepsilon > 0$ such that

$$\chi(a_{n_{k_l}}, b_{k_l})\, \chi(b_{k_l}, c_l)\, \chi(a_{n_{k_l}}, c_l) > \varepsilon,$$

for $l \in \mathbb{N}$ sufficiently large, so that Carathéodory's extension of Montel's Theorem (e.g. [29, p.104]) implies that $(f_{n_{k_l}}) \subset \tilde{\mathcal{F}}$ is normal in U, hence also at z_0, in contradiction to the strong non-normality of $\tilde{\mathcal{F}}$ at z_0.

To prove the second statement, suppose that $\mathcal{F}$ is not strongly expanding at z_0 with respect to $\mathbb{C}_\infty \setminus \mathcal{E}$. Then there is an infinite subfamily $\tilde{\mathcal{F}} \subset \mathcal{F}$ that is not expanding at z_0 with respect to $\mathbb{C}_\infty \setminus \mathcal{E}$, contradicting the fact that $\tilde{\mathcal{F}}$ is expanding at z_0 with respect to $\mathbb{C}_\infty \setminus E_{\tilde{\mathcal{F}}}$ for some set $E_{\tilde{\mathcal{F}}} \subset \mathbb{C}_\infty$ with $E_{\tilde{\mathcal{F}}} \subset \mathcal{E}$.

(ii) Suppose that for some infinite subfamily $\tilde{\mathcal{F}} = \{f_n : n \in \mathbb{N}\}$ of $\mathcal{F}$ the sequence (f_n) is spherically uniformly convergent on compact subsets of a neighbourhood of z_0. Then

https://doi.org/10.1017/S0305004121000700 Published online by Cambridge University Press

$\limsup_{z_0} \tilde{\mathcal{F}}$ is a one-point set, and hence $|\liminf_{z_0} \mathcal{F}| \leq 1$. The second statement follows from the fact that in this case we have $A \subset \liminf_{z_0} \mathcal{F}$.

Remark 2. Note that if $\mathcal{F}$ is strongly non-normal at z_0, $\mathcal{F}$ does not need to be strongly expanding at z_0 with respect to any open set $A \subset \mathbb{C}_\infty$. Indeed, let (q_n) be an enumeration of the Gaussian rational numbers with $q_n^2/n \to 0$ as $n \to \infty$ and consider the family (f_n) with $f_n(z) := e^{nz} + q_n$ for $z \in \mathbb{C}$. From Marty's Theorem, it is easily seen that (f_n) is strongly non-normal on the imaginary axis $i\mathbb{R}$, but for a point $z_0 \in i\mathbb{R}$ and an open neighbourhood U of z_0, there is no $N \in \mathbb{N}$ such that $K \subset f_n(U)$ holds for all $n \geq N$ for any compact set $K \subset \mathbb{C}$ with $K^\circ \neq \emptyset$.

From Theorem 2 we easily obtain the following characterisation of non-normality in terms of the expanding property, which in some sense complements the statement of Montel's Theorem:

COROLLARY 1. *Let $\Omega \subset \mathbb{C}$ be open, $\mathcal{F} \subset M(\Omega)$ a family of meromorphic functions and $z_0 \in \Omega$. Then the following are equivalent*:

(i) *there exists $A \subset \mathbb{C}_\infty$ with $|A| \geq 2$ such that $\mathcal{F}$ is expanding at z_0 with respect to A;*

(ii) *$\mathcal{F}$ is non-normal at z_0;*

(iii) *there exists $E \subset \mathbb{C}_\infty$ with $|E| \leq 2$ such that $\mathcal{F}$ is expanding at z_0 with respect to $\mathbb{C}_\infty \setminus E$.*

Proof. (i) $\Rightarrow$ (ii) Suppose that $\mathcal{F}$ is expanding at z_0 with respect to some $A \subset \mathbb{C}_\infty$ with $|A| \geq 2$. Then there exists an infinite subfamily $\tilde{\mathcal{F}} \subset \mathcal{F}$ that is strongly expanding at z_0 with respect to A. By Theorem 2, the family $\tilde{\mathcal{F}}$ is strongly non-normal at z_0, hence $\mathcal{F}$ is non-normal at z_0.

(ii) $\Rightarrow$ (iii) If $\mathcal{F}$ is non-normal at z_0, there exists an infinite subfamily $\tilde{\mathcal{F}} \subset \mathcal{F}$ that is strongly non-normal at z_0. By Theorem 2, there then exists $E \subset \mathbb{C}_\infty$ with $|E| \leq 2$ such that $\tilde{\mathcal{F}}$ is expanding at z_0 with respect to $\mathbb{C}_\infty \setminus E$. The same then holds for the family $\mathcal{F}$.

(iii) $\Rightarrow$ (i) is obvious.

Let $\mathcal{F} \subset M(\Omega)$ be a family that is non-normal at a point $z_0 \in \Omega$ and consider the set $E_{z_0}(\mathcal{F}) = \mathbb{C}_\infty \setminus \limsup_{z_0} \mathcal{F}$. If $\mathcal{F}$ is expanding at z_0 with respect to $\mathbb{C}_\infty \setminus E$ for some set $E \subset \mathbb{C}_\infty$, we obviously have $E_{z_0}(\mathcal{F}) \subset E$. If $\mathcal{F}$ is a family of holomorphic functions on Ω that is (strongly) non-normal at z_0, we have $\infty \in E_{z_0}(\mathcal{F})$, so that in this case we obtain that the expanding property of $\mathcal{F}$ at z_0 in Theorem 2 and Corollary 1 holds with respect to $\mathbb{C} \setminus E$ for some set $E \subset \mathbb{C}$ with $|E| \leq 1$.

Example 3.

(i) Consider a compact set $K \subset \mathbb{C}$ with connected complement and let f be a function that is continuous on K and holomorphic in K°. Further assume that f has at least one singularity on ∂K and denote by $D \subset \partial K$ the set of all singularities. Let (p_n) be a sequence of polynomials converging uniformly on K to f (such a sequence exists by Mergelian's Theorem). Then, (p_n) is strongly non-normal on D, hence also expanding at every point $z_0 \in D$ with respect to $\mathbb{C} \setminus E$ for some set $E \subset \mathbb{C}$ with $|E| \leq 1$. Indeed, since otherwise there exists a point $z_0 \in D$, an open neighbourhood U of z_0, and a subsequence (p_{n_k}) of (p_n) that converges uniformly on compact subsets of U to a function holomorphic in U, contradicting that f does not have an analytic continuation across $z_0 \in D$.

(ii) Consider the function $f(z) = |z|$ on the interval $[-1, 1]$ and denote by $(p_n^\star)$ the sequence of polynomials of best uniform approximation to f on $[-1, 1]$. Then, according to the previous example, $(p_n^\star)$ is strongly non-normal at the point 0. However, since $p_n^\star(z) \to \infty$ for $n \to \infty$ spherically uniformly on compact subsets of $\mathbb{C} \setminus [-1, 1]$ (e.g. [28]), the family $(p_n^\star)$ is strongly non-normal on $[-1, 1]$, hence expanding at every point $z_0 \in [-1, 1]$ with respect to $\mathbb{C} \setminus E$ for some set $E \subset \mathbb{C}$ with $|E| \leq 1$. (Note that the strong non-normality on $[-1, 1]$ also holds for several specific ray sequences of best uniform rational approximants to f on $[-1, 1]$ ([28, corollary 1·3]).) In fact, [5, corollary 2] implies that $(p_n^\star)$ is expanding on $[-1, 1]$ with respect to $\mathbb{C}$, as it shows the existence of a subsequence $(p_{n_k}^\star)$ of $(p_n^\star)$ that is strongly expanding on $[-1, 1]$ with respect to $\mathbb{C}$.

(iii) Consider again a function $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ that is holomorphic on $\mathbb{D}$ and has at least one singularity on $\partial\mathbb{D}$. Then the family of partial sums (s_n) is non-normal on $\partial\mathbb{D}$, hence, (s_n) is expanding at every $z_0 \in \partial\mathbb{D}$ with respect to $\mathbb{C} \setminus E$ for some set $E \subset \mathbb{C}$ with $|E| \leq 1$. In fact, (s_n) is expanding on $\partial\mathbb{D}$ with respect to $\mathbb{C}$, as results in [5,13] show that if (a_{n_k}) is a sequence such that $\lim_{k\to\infty} \left|a_{n_k}\right|^{\frac{1}{n_k}} = 1$, the subfamily (s_{n_k}) is strongly expanding on $\partial\mathbb{D}$ with respect to $\mathbb{C}$.

A further consequence of Theorem 2 and the fact that we have $E_{z_0}(\mathcal{F}) \subset E$ if $\mathcal{F} \subset M(\Omega)$ is expanding at $z_0 \in \Omega$ with respect to $\mathbb{C}_\infty \setminus E$ is the following statement for the case $\left|E_{z_0}(\mathcal{F})\right| = 2$.

COROLLARY 2. *Let $\Omega \subset \mathbb{C}$ be open and $\mathcal{F} \subset M(\Omega)$ be a family of meromorphic functions. Consider $z_0 \in \Omega$ and suppose that $\mathcal{F}$ is (strongly) non-normal at z_0 with $\left|E_{z_0}(\mathcal{F})\right| = 2$. Then $\mathcal{F}$ is (strongly) expanding at z_0 with respect to $\mathbb{C}_\infty \setminus E_{z_0}(\mathcal{F})$.*

Proof. Suppose that $\mathcal{F}$ is non-normal at z_0. By Corollary 1, there then exists $E \subset \mathbb{C}_\infty$ with $|E| \leq 2$ such that $\mathcal{F}$ is expanding at z_0 with respect to $\mathbb{C}_\infty \setminus E$. Since $E_{z_0}(\mathcal{F}) \subset E$, we obtain $E_{z_0}(\mathcal{F}) = E$. If $\mathcal{F}$ is strongly non-normal at z_0, every infinite subfamily $\tilde{\mathcal{F}} \subset \mathcal{F}$ is non-normal at z_0 with $E_{z_0}(\tilde{\mathcal{F}}) = E_{z_0}(\mathcal{F})$, hence expanding at z_0 with respect to $\mathbb{C}_\infty \setminus E_{z_0}(\mathcal{F})$.

Example 4.

(i) Consider again the family $\mathcal{F} := \{e^{nz} + (1 - 1/n) : n \in \mathbb{N}\}$, which is strongly non-normal at the point 0. It is easily seen that $\mathcal{F}$ is strongly expanding at 0 with respect to $\mathbb{C}_\infty \setminus \{1, \infty\}$, but since $E_0(\mathcal{F}) = \{\infty\}$, this can not be derived from Corollary 2. On the other hand, the family $\mathcal{F} := \{e^{nz} + (1 - 1/n!) : n \in \mathbb{N}\}$ is strongly non-normal at the point 0 with $E_0(\mathcal{F}) = \{1, \infty\}$ (note that for each neighbourhood U of 0 and each $1 \neq w \in \mathbb{C}$, there is some N with $w - 1 + 1/n! \in \exp(nU)$ for all $n \geq N$, but $1/n! \notin \exp(n\mathbb{D})$ for sufficiently large n). So, in this case Corollary 2 can be applied.

(ii) Consider again a power series $f(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu$ with radius of convergence 1 and denote by (s_n) its partial sums. As mentioned in Example 3, the family $\mathcal{F} = \{s_n : n \in \mathbb{N}\}$ is expanding on $\partial\mathbb{D}$ with respect to $\mathbb{C}$, so that for every $z_0 \in \partial\mathbb{D}$ we

https://doi.org/10.1017/S0305004121000700 Published online by Cambridge University Press

have $E_{z_0}(\mathcal{F}) = \{\infty\}$ (note that this is also easily derived from the classical Jentzsch Theorem ([19]) stating that for every $a \in \mathbb{C}$, every $z_0 \in \partial\mathbb{D}$ is a limit point of a-points of the partial sums). However, a further result of Jentzsch ([20]) states that there exist power series with radius of convergence 1, such that the zeros of some subsequence (s_{n_k}) of the partial sums do not have a finite limit point. Hence, in this case Corollary 2 shows that the family $\tilde{\mathcal{F}} = \{s_{n_k} : k \in \mathbb{N}\}$ is strongly expanding with respect to $\mathbb{C} \setminus \{0\}$ at every point $z_0 \in \partial\mathbb{D}$ at which the function does not admit an analytic continuation (there must be at least one such point), since $\tilde{\mathcal{F}}$ is strongly non-normal at such z_0 with $E_{z_0}(\tilde{\mathcal{F}}) = \{0, \infty\}$. In a similar vein, it was shown in [18, theorem 1] that there exists a function f holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ with at least one singularity on $\partial\mathbb{D}$, for which the zeros of some subsequence $(p^\star_{n_k})$ of the sequence $(p^\star_n)$ of polynomials of best uniform approximation do not have a finite limit point. Hence, as before, Corollary 2 can be applied to the family $\mathcal{F} = \{p^\star_{n_k} : k \in \mathbb{N}\}$ at every singular point $z_0 \in \partial\mathbb{D}$ of f, since $\mathcal{F}$ is strongly non-normal at z_0 (see Example 3 (i)) and we have $E_{z_0}(\mathcal{F}) = \{0, \infty\}$. Moreover, [18, theorem 2] shows the existence of a function f that is holomorphic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ with at least one singularity on $\partial\mathbb{D}$, for which there is a sequence (q_n) of polynomials of near-best uniform approximation that has no finite limit point of zeros. Hence, in this case Corollary 2 implies that the family $\mathcal{F} = \{q_n : n \in \mathbb{N}\}$ is strongly expanding with respect to $\mathbb{C} \setminus \{0\}$ at every singular point $z_0 \in \partial\mathbb{D}$ of f.

4. *Expanding families of derivatives*

In the following, we show that under certain conditions, (strong) non-normality of a family $\mathcal{F} \subset M(\Omega)$ at a point $z_0 \in \Omega$ implies that the family of derivatives is (strongly) expanding at z_0 with respect to $\mathbb{C} \setminus \{0\}$, hence in particular (strongly) non-normal at z_0. Throughout this section, we denote by $\mathcal{F}^{(k)}$ the family of kth derivatives of the functions in $\mathcal{F}$, that is $\mathcal{F}^{(k)} = \{f^{(k)} : f \in \mathcal{F}\}$, where k is some natural number.

THEOREM 3. *Let $\Omega \subset \mathbb{C}$ be open and $\mathcal{F} \subset M(\Omega)$ be a family of meromorphic functions. Consider $z_0 \in \Omega$ and suppose that $\mathcal{F}$ is (strongly) non-normal at z_0. Further assume that $\mathcal{F}$ is not expanding at z_0 with respect to $\mathbb{C}$. Then, for every $k \in \mathbb{N}$, the family $\mathcal{F}^{(k)}$ is (strongly) expanding at z_0 with respect to $\mathbb{C} \setminus \{0\}$.*

Proof. We first assume that $\mathcal{F}$ is strongly non-normal at z_0. By assumption, $\mathcal{F}$ is not expanding at z_0 with respect to $\mathbb{C}$, hence there exists an open neighbourhood U_1 of z_0 and a compact set $K_1 \subset \mathbb{C}$ such that $K_1 \setminus f(U_1) \neq \emptyset$ holds for cofinitely many $f \in \mathcal{F}$.

Now assume that there exists $k \in \mathbb{N}$, such that $\mathcal{F}^{(k)}$ is not strongly expanding at z_0 with respect to $\mathbb{C} \setminus \{0\}$. Then there exists an open neighbourhood U_2 of z_0 and a compact set $K_2 \subset \mathbb{C} \setminus \{0\}$ such that $K_2 \setminus f^{(k)}(U_2) \neq \emptyset$ holds for infinitely many $f \in \mathcal{F}$.

In particular, we can find a sequence (f_n) in $\mathcal{F}$, and sequences $(c_n^{(1)})$ in K_1 and $(c_n^{(2)})$ in K_2, such that the equations $f_n(z) = c_n^{(1)}$ and $f_n^{(k)}(z) = c_n^{(2)}$ have no roots in $U := U_1 \cap U_2$ for every $n \in \mathbb{N}$. From [10, theorem 3·17], which is an extension of Gu's famous normality criterion (e.g. [17,29]), we obtain that (f_n) is normal in U, hence also at z_0, in contradiction to the strong non-normality of $\mathcal{F}$ at z_0.

If $\mathcal{F}$ is non-normal at z_0, there exists an infinite subfamily $\tilde{\mathcal{F}} \subset \mathcal{F}$ that is strongly non-normal at z_0. By assumption, $\mathcal{F}$ is not expanding at z_0 with respect to $\mathbb{C}$, hence the same holds for $\tilde{\mathcal{F}}$, so that by the above argumentation, $\tilde{\mathcal{F}}^{(k)}$ is strongly expanding at z_0 with respect

https://doi.org/10.1017/S0305004121000700 Published online by Cambridge University Press

to $\mathbb{C}\setminus\{0\}$ for every $k\in\mathbb{N}$. Hence, $\mathcal{F}^{(k)}$ is expanding at z_0 with respect to $\mathbb{C}\setminus\{0\}$ for every $k\in\mathbb{N}$.

Remark 3. Note that the assumption that $\mathcal{F}$ is not expanding at z_0 with respect to $\mathbb{C}$ turns out to be necessary, as is seen e.g. by considering the sequence of polynomials $f_n(z)=nz$ and $z_0=0$. Moreover, it is easily seen that a similar argumentation as in the proof of the theorem leads to the following result: Let $\Omega\subset\mathbb{C}$ be open and $\mathcal{F}\subset M(\Omega)$ be a family of meromorphic functions. Consider $z_0\in\Omega$ and suppose that $\mathcal{F}$ is (strongly) non-normal at z_0. Further assume that for some $k\in\mathbb{N}$, the family $\mathcal{F}^{(k)}$ is not expanding at z_0 with respect to $\mathbb{C}\setminus\{0\}$. Then, the family $\mathcal{F}$ is (strongly) expanding at z_0 with respect to $\mathbb{C}$.

COROLLARY 3. *Let $\Omega\subset\mathbb{C}$ be open and $\mathcal{F}\subset M(\Omega)$ be a family of meromorphic functions. Consider $z_0\in\Omega$ and suppose that $\mathcal{F}$ is (strongly) non-normal at z_0. Suppose further that there exists an open neighbourhood U of z_0 and a number $M>0$, such that for cofinitely many $f\in\mathcal{F}$ there is a point $a_f\in\mathbb{C}$ with $|a_f|<M$ and $a_f\notin f(U)$. Then, for every $k\in\mathbb{N}$, the family $\mathcal{F}^{(k)}$ is (strongly) expanding at z_0 with respect to $\mathbb{C}\setminus\{0\}$.*

Proof. Since it follows from the assumptions that $\mathcal{F}$ is not expanding at z_0 with respect to $\mathbb{C}$, the statement follows from Theorem 3.

Note that the assumptions of Corollary 3 are fulfilled if $\mathcal{F}\subset M(\Omega)$ is (strongly) non-normal at $z_0\in\Omega$ and for some $a\in\mathbb{C}$ we have $a\in E_{z_0}(\mathcal{F})$, hence in particular if $|E_{z_0}(\mathcal{F})|=2$.

Example 5.

(i) In Example 4 (ii) we considered strongly non-normal families $\mathcal{F}$ of polynomials for which $E_{z_0}(\mathcal{F})=\{0,\infty\}$, hence we obtain that the corresponding families of derivatives $\mathcal{F}^{(k)}$ are strongly expanding at z_0 with respect to $\mathbb{C}\setminus\{0\}$ for every $k\in\mathbb{N}$.

(ii) Consider the family (f_n) with $f_n:=\exp^{\circ n}$, the nth iterate of e^z. Then $J(f_n)$ coincides with the Julia set of e^z, which is known to equal $\mathbb{C}$ ([25]). According to Example 1 (i), we thus have that (f_n) is strongly non-normal on $\mathbb{C}$. Furthermore, we obviously have $0\in E_{z_0}(f_n)$ for every $z_0\in\mathbb{C}$, so that Corollary 3 implies that for every $k\in\mathbb{N}$, the family $(f_n^{(k)})$ is strongly expanding on $\mathbb{C}$ with respect to $\mathbb{C}\setminus\{0\}$.

We mention that the statement of Corollary 3 remains valid to some extent, if instead of omitting a value a_f in some neighbourhood of z_0, cofinitely many functions $f\in\mathcal{F}$ have a value a_f that they take with sufficiently high multiplicity in that neighbourhood.

PROPOSITION 3. *Let $\Omega\subset\mathbb{C}$ be open and $\mathcal{F}\subset M(\Omega)$ be a family of meromorphic functions. Consider $z_0\in\Omega$ and suppose that $\mathcal{F}$ is (strongly) non-normal at z_0. Suppose further that there exists an open neighbourhood U of z_0, a number $M>0$ and some $k\in\mathbb{N}$, such that for cofinitely many $f\in\mathcal{F}$ there is a point $a_f\in\mathbb{C}$ with $|a_f|<M$, such that the a_f-points of f in U have multiplicity at least $k+2$. Then the family $\mathcal{F}^{(k)}$ is (strongly) expanding at z_0 with respect to $\mathbb{C}\setminus\{0\}$.*

Proof. Again, we first consider the case that $\mathcal{F}$ is strongly non-normal at z_0. Assuming that $\mathcal{F}^{(k)}$ is not strongly expanding at z_0 with respect to $\mathbb{C}\setminus\{0\}$, there exists an open

https://doi.org/10.1017/S0305004121000700 Published online by Cambridge University Press

neighbourhood U_1 of z_0 and a compact set $K \subset \mathbb{C}\setminus\{0\}$ such that $K \setminus f^{(k)}(U_1) \neq \emptyset$ for infinitely many $f \in \mathcal{F}$. In particular, we can find a sequence (c_n) in K with $c_n \to c$ for some $c \neq 0$, and a sequence (f_n) in $\mathcal{F}$ such that $c_n \notin f_n^{(k)}(U_1)$ for every $n \in \mathbb{N}$. For $n \in \mathbb{N}$ sufficiently large, say $n > N$, there is a point $a_{f_n} \in \mathbb{C}$ with $|a_{f_n}| < M$, such that the a_{f_n}-points of f_n in U have multiplicity at least $k+2$. Setting $g_n(z) = f_n(z) - a_{f_n}$ for $n > N$, we obtain that the functions g_n only have zeros of multiplicity at least $k+2$ in $U' := U \cap U_1$. Furthermore, since $c_n \notin g_n^{(k)}(U')$ for every $n > N$, it follows from [9, lemma 2·7] that the family $\{g_n : n > N\}$ is normal in U', and as $|a_{f_n}| < M$ for every $n > N$, the same holds for the family $\{f_n : n > N\}$. This is in contradiction to the strong non-normality of $\mathcal{F}$ at z_0.

If $\mathcal{F}$ is non-normal at z_0, the statement follows as before from the fact that $\mathcal{F}$ contains a strongly non-normal subfamily.

In general, the number $k+2$ can not be replaced by $k+1$ in Proposition 3. Indeed, for fixed $k \in \mathbb{N}$, the family (f_n) with

$$f_n(z) = \frac{1}{k!}\,\frac{z^{k+1}}{(z - \frac{1}{n})}$$

is strongly non-normal at the point 0 and has only zeros of multiplicity $k+1$ (see also [31]). But as $f_n^{(k)}(z) \neq 1$ for every $n \in \mathbb{N}$ and every $z \in \mathbb{C}$, the familiy $(f_n^{(k)})$ is obviously not expanding at 0 with respect to $\mathbb{C}\setminus\{0\}$. Nevertheless, under certain additional conditions, $k+2$ can be replaced by $k+1$:

PROPOSITION 4. *Under each of the following additional conditions, the statement of Proposition 3 remains valid if $k+2$ is replaced by $k+1$:*

(i) *the functions $f \in \mathcal{F}$ are holomorphic in Ω;*

(ii) *the functions $f \in \mathcal{F}$ only have multiple poles;*

(iii) *there exists a sequence (z_n) in Ω with $z_n \to z_0$ and $\mathcal{F}$ is strongly non-normal at z_n for every $n \in \mathbb{N}$.*

Proof. Using [7, lemma 4] and [27, lemma 6], respectively, the proofs of (i) and (ii) are similar to the proof of Proposition 3. In order to prove the third statement, we note that using [8, lemma 2·9], a similar argumentation as in the proof of Proposition 3 implies that the family (g_n) with $g_n(z) = f_n(z) - a_{f_n}$ is quasinormal in some neighbourhood U of z_0. Since $|a_{f_n}| < M$ for every $n \in \mathbb{N}$, the same then holds for the family (f_n) ([10, lemma 5·2]). This contradicts the assumption that the set $\{z : \mathcal{F}$ is strongly non-normal at $z\}$ has an accumulation point in U.

Acknowledgements. The authors would like to thank the anonymous reviewer for his careful reading of the manuscript and his valuable comments and suggestions.

REFERENCES

[1] H.-P. BEISE, T. MEYRATH and J. MÜLLER. Universality properties of Taylor series inside the domain of holomorphy. *J. Math. Anal. Appl.* **383** (2011), 234–238.

[2] H.-P. BEISE, T. MEYRATH and J. MÜLLER. Limit functions of discrete dynamical systems. *Conform. Geom. Dyn.* **18** (2014), 56–64.

[3] H.-P. BEISE, T. MEYRATH and J. MÜLLER. Mixing Taylor shifts and universal Taylor series. *Bull. London Math. Soc.* **47** (2015), 136–142.

https://doi.org/10.1017/S0305004121000700 Published online by Cambridge University Press

[4] L. BERNAL-GONZÁLEZ, A. JUNG and J. MÜLLER. Universality vs. non-normality of families of meromorphic functions. *Proc. Amer. Math. Soc.* **149** (2021), 761–771.
[5] H. P. BLATT, S. BLATT and W. LUH. On a generalization of Jentzsch's theorem. *J. Approx. Theory* **159** (2009), 26–38.
[6] L. CARLESON and T. W. GAMELIN. Complex Dynamics (Springer, New York, 1993).
[7] J.-F. CHEN. Exceptional functions and normal families of holomorphic functions with multiple zeros. *Georgian Math. J.* **18** (2011), 31–38.
[8] Q. CHEN, X. PANG and P. YANG. A new Picard type theorem concerning elliptic functions. *Ann. Acad. Sci. Fenn. Math.* **40** (2015), 17–30.
[9] C. CHENG and Y. XU. Normality concerning exceptional functions. *Rocky Mt. J. Math.* **45** (2015), 157–168.
[10] C.-T. CHUANG. *Normal families of meromorphic functions*, (World Scientific, 1993).
[11] J. CLUNIE and W. K. HAYMAN. The spherical derivative of integral and meromorphic functions. *Comment. Math. Helv.* **40** (1966), 117–148.
[12] J. B. CONWAY. *Functions of One Complex Variable*, I 2nd ed., (Springer, New York, 1978).
[13] A. DVORETZKY. On sections of power series. *Ann. of Math.* **51** (1950), 643–696.
[14] P. FATOU. Sur l'itération des fonctions transcendantes entières. *Acta Math.* **47** (1926), 337–360.
[15] S. GARDINER. Existence of universal Taylor series for nonsimply connected domains. *Constr. Approx.* **35** (2012), 245–257.
[16] W. GEHLEN. Overconvergent power series and conformal maps. *J. Math. Anal. Appl.* **198** (1996), 490–505.
[17] Y. X. GU. A normal criterion of meromorphic families. *Sci. Sinica* **1** (1979), 267–274.
[18] K. G. IVANOV, E. B. SAFF and V. TOTIK. On the behavior of zeros of polynomials of best and near-best approximation. *Canad. J. Math.* **43** (1991), 1010–1021.
[19] R. JENTZSCH. Untersuchungen zur Theorie der Folgen analytischer Funktionen. *Acta. Math.* **41** (1918) 219–251.
[20] R. JENTZSCH. Fortgesetzte Untersuchungen über die Abschnitte von Potenzreihen. *Acta. Math.* **41** (1918), 253–270.
[21] K.-G. GROSSE-ERDMANN and A. PERIS. *Linear Chaos*, (Springer, London, 2011).
[22] T. KALMES, J. MÜLLER and M. NIESS. On the behaviour of power series in the absence of Hadamard–Ostrowski gaps. *C. R. Math. Acad. Sci. Paris* **351** (2013), 255–259.
[23] A. MELAS. Universal functions on nonsimply connected domains. *Ann. Inst. Fourier (Grenoble)* **51** (2001), 1539–1551.
[24] D. MINDA. *Yosida functions, Lectures on Complex Analysis (Xian, 1987),* C.-T. CHUANG (ed.) (World Scientific, Singapore, 1988), 197–213.
[25] M. MISIUREWICZ. On iterates of e^z. *Ergod. Theory Dynam. Systems.* **1** (1981), 103–106.
[26] S. MOROSAWA, Y. NISHIMURA, M. TANIGUCHI and T. UEDA. *Holomorphic Dynamics.* (Cambridge University Press, Cambridge, 2000).
[27] S. NEVO, X. PANG and L. ZALCMAN. Quasinormality and meromorphic functions with multiple zeros. *J. Anal. Math.* **101** (2007), 1–23.
[28] E. B. SAFF and H. STAHL. Ray sequences of best rational approximants for $|x|^\alpha$. *Canad. J. Math.* **49** (1997), 1034–1065.
[29] J. L. SCHIFF. *Normal Families*, Springer, (New York, Berlin, Heidelberg, 1993).
[30] D. SCHLEICHER. *Dynamics of entire functions.* In: *Holomorphic Dynamical Systems.* Lecture Notes in Math. vol. 1998 (Springer, Berlin, 2010), 295–339.
[31] Y. WANG and M. FANG. Picard values and normal families of meromorphic functions with multiple zeros. *Acta Math. Sinica (N.S.)* **14** (1998), 17–26.
[32] K. YOSIDA. On a class of meromorphic functions. *Proc. Phys.-Math. Soc. Japan* **16** (1934), 227–235.