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BISECTORS IN VECTOR GROUPS OVER INTEGERS

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Abstract

We present an example of an isometric subspace of a metric space that has a greater metric dimension. We also show that the metric spaces of vector groups over the integers, defined by the generating set of unit vectors, cannot be resolved by a finite set. Bisectors in the spaces of vector groups, defined by the generating set consisting of unit vectors, are completely determined.

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1. Vector groups

Let G be a group and $S \subseteq G$ be a generating set for G such that $1 \notin S$ and $S^{-1} = S$. Define the *Cayley graph* $X = X(G, S)$ by the specification of the set of vertices and the set of edges of X:

$$V(X) = G, \quad E(X) = \{gh : g, h \in G, gh^{-1} \in S\}.$$

This definition is classical (see, for example, [4, page 34]). A more general concept of a graph of a semigroup was given in [1, page 56]. It was pointed out in [1] that the class of Cayley graphs and the class of Toeplitz graphs are subclasses of semigroup graphs, while the class of semigroup graphs contains graphs that are neither Cayley nor Toeplitz (for example, Cayley graphs of vector semigroups).

In the definition of a Cayley graph, the condition $S^{-1} = S$ implies that the resulting graph is undirected and the condition $1 \notin S$ implies that the graph has no loops. The condition that S is a generating set of G ensures that X is connected. Connectivity is imposed for the simple reason that our interest in this paper is in metric properties of a special family of Cayley graphs. A graph is a metric space with its intrinsic path metric. We study a basic metric space property of these graphs.

Let G be a group. If for $x \in G$ there exists $n \in \mathbb{N}$ such that $x^n = 1$ then, by the well ordering principle, there exists a smallest positive integer n such that $x^n = 1$. The smallest positive integer n for which $x^n = 1$ is called the *order* of x and x is called a *torsion* element. Note that the identity element is always a torsion element of order 1.

If for every $x \in G$ with $x \neq 1$, there is no $n \in \mathbb{N}$ such that $x^n = 1$, then G is called *torsion-free*. For any finite group G, every element is of finite order. If G is torsion-free then G is necessarily infinite. Examples of torsion-free abelian groups are $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ under addition and vector groups over these groups. Torsion-free abelian groups of finite rank are determined in [7]. As is customary, we use additive notation in an additive abelian group. The identity element for addition is called the *zero* element and is denoted 0. The binary operation is denoted + and the inverse of an element x is its *negative* and is denoted $-x$. The condition imposed on the generating set S now becomes $0 \notin S$ and $-S = S$. We consider locally finite Cayley graphs (in which the degree of every vertex is a finite nonnegative integer). Thus it is assumed that the torsion-free abelian group is finitely generated with S a generating set. The torsion condition in additive notation is: there exists $n \in \mathbb{N}$ such that $nx = \underbrace{x + x + \cdots + x}_{n} = 0$.

The concepts of a metric space, distance and geodesic are basic (see, for example, [3, 8]). The concept of metric dimension was introduced in [3] and has been widely studied for graphs, usually with a motivation from applications in radio and telecommunication technology. A recent application is to global positioning systems (GPS). As in [2], let X be a metric space with distance function $\rho : X \times X \to [0, +\infty)$. Let A be a nonempty subset of X with finite or countably infinite cardinality. Thus we may write $A = \{a_1, a_2, \ldots, a_n, \ldots\}$. If for every $x, y \in X$ with $x \neq y$, there is at least one index i such that $\rho(a_i, x) \neq \rho(a_i, y)$, then A is said to *resolve* X and is called a *resolving set* or briefly a *resolver* for X. A resolving set of minimum cardinality is called a *metric basis* for X. The cardinality of a minimum resolving set is called the *metric dimension* of X and is denoted $\beta(X)$. The condition for A to be a resolver may be written in a logically equivalent form: for all $i \in \{1, 2, \ldots, n, \ldots\}, \rho(a_i, x) = \rho(a_i, y) \Rightarrow x = y$.

Let X be a metric space with distance function ρ. For $x, y \in X$, the *bisector* of x, y is defined to be $B(x, y) = \{z \in X : \rho(x, z) = \rho(y, z)\}$. Note that $B(y, x) = B(x, y)$. In $\mathbb{R}^n$, the bisector of x, y is $B(x, y) = \{z : |z - x| = |z - y|\}$ in the usual Euclidean metric.

In [2], the metric dimensions of the three classical geometric spaces were determined and the metric dimension of Riemann surfaces was shown to be 3. In [5, 6], metric dimensions of geometric spaces and geometric manifolds were determined. The present paper is on Cayley graphs of vector groups over $\mathbb{Z}$. We focus on spaces of vector groups over the integers. These graphs are metric spaces with their natural path metrics. Consider $G = \mathbb{Z} \times \mathbb{Z}$. Then G is a vector group and a generating set satisfying the conditions $(0, 0) \notin S$ and $-S = S$ is given by $S = \{u \in G : |u| = 1\} = \{(-1, 0), (0, -1), (0, 1), (1, 0)\}$. Let $X = X(G, S)$. If $x, y \in X$ with $x = (x_1, y_1)$ and $y = (x_2, y_2)$, then the distance in $X(G, S)$ is explicitly given by $\rho_1(x, y) = |x_2 - x_1| + |y_2 - y_1|$.

Let X be a metric space with distance function ρ. A subset $A \subseteq X$ is not a resolver of X if and only if there exist $u, v \in X$ such that $u \neq v$ and $\rho(a, u) = \rho(a, v)$ for every $a \in A$. Hence $A \subseteq X$ resolves X if and only if A is not contained in any bisector. This shows that a determination of bisectors is directly relevant to the investigation of metric dimensions.

2. Monotonicity

In this section, we consider monotonicity of metric dimension. An example was presented in [2, page 296] with an intention to show that metric dimension is not monotonic. If X, Y are metric spaces with distance functions ρ_X, ρ_Y, then monotonicity states that $X \subseteq Y \Rightarrow \beta(X) \leq \beta(Y)$. Monotonicity is a natural assumption for a well-defined concept of a dimension.

Let X, Y be metric spaces with distance functions ρ_X and ρ_Y. Let $f : X \to Y$ be a mapping. If $\rho_Y(f(x), f(y)) = \rho_X(x, y)$ for $x, y \in X$, then the mapping f is called an *isometry*. If f is also injective, then we say that f is an *isometric embedding* of X in Y, X may be considered as an *isometric* subspace of Y and the metric in X is a restriction of that in Y, that is, $\rho_X = \rho_Y|_X$.

We can now see the simple reason for the proposed counterexample to monotonicity given in [2]. In that example, a metric space $X \subseteq \mathbb{R}^2$ was given such that $\beta(X)$ is unbounded, while $\beta(\mathbb{R}^2)$ was shown to be 3. However, the metric ρ_X is the intrinsic path metric of X while $\rho_{\mathbb{R}^2}$ is the Euclidean metric which is intrinsic in $\mathbb{R}^2$. The metric ρ_X is not a restriction of the Euclidean metric of $\mathbb{R}^2$. Therefore, X is not an isometric subspace of the metric space $\mathbb{R}^2$.

Let $G = \mathbb{Z} \times \mathbb{Z}$ and $S = \{u \in \mathbb{Z} \times \mathbb{Z} : |u| = 1\}$. Let $X = X(G, S)$. This space is usually called the *grid plane* or the *Gaussian integers*. We now show that $\beta(X)$ is unbounded.

THEOREM 2.1. *Let $G = \mathbb{Z}^n$, $S = \{u \in \mathbb{Z}^n : |u| = 1\}$ and $X = X(G, S)$. If $A \subseteq G$ is any finite set, then there exist $x, y \in G$ such that $A \subseteq B(x, y)$.*

PROOF. For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, the distance in $X(G, S)$ is explicitly given by

$$\rho(x, y) = |y_1 - x_1| + |y_2 - x_2| + \cdots + |y_n - x_n|.$$

Let $A \subseteq G$ be any finite set. Then there exists $u = (a_1, a_2, \ldots, a_n) \in G$ such that if $x = (x_1, x_2, \ldots, x_n) \in A$ then $x_i \leq a_i$ for $1 \leq i \leq n$ and, subject to this condition, each a_i is minimal. Consider $p = (a_1 + 1, a_2, \ldots, a_n)$ and $q = (a_1, a_2 + 1, \ldots, a_n)$. Then $p, q \in G$ and $pu, qu \in E(X)$, and so $\rho(p, u) = 1 = \rho(q, u)$. For each $c \in A$,

$$\begin{aligned}
\rho(c, p) &= (a_1 + 1 - c_1) + (a_2 - c_2) + \cdots + (a_n - c_n)\\
\rho(c, q) &= (a_1 - c_1) + (a_2 + 1 - c_2) + \cdots + (a_n - c_n)\\
\rho(c, u) &= (a_1 - c_1) + (a_2 - c_2) + \cdots + (a_n - c_n).
\end{aligned}$$

Hence, $\rho(c, p) = \rho(c, u) + \rho(u, p)$ and $\rho(c, q) = \rho(c, u) + \rho(u, q)$. Each geodesic connecting c and p passes through u and each geodesic connecting c and q passes through u. This shows that

$$\rho(c, p) = \rho(c, u) + 1 = \rho(c, q).$$

Thus $c \in B(p, q)$ for each $c \in A$, that is, $A \subseteq B(p, q)$. □

Theorem 2.1 may be restated in the following way.

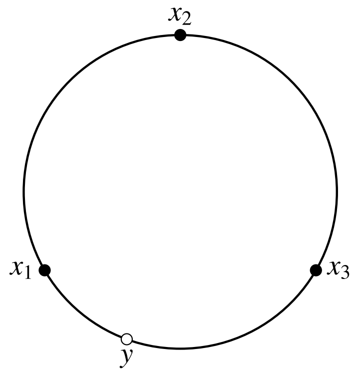

FIGURE 1. An isometric subspace with a higher dimension.

COROLLARY 2.2. *Let $G = \mathbb{Z}^n$, $S = \{u \in \mathbb{Z}^n : |u| = 1\}$ and $X = X(G, S)$. If $A \subseteq G$ is a finite set then A does not resolve X.*

The metric of X is an isometric restriction of the metric

$$\rho_1(x, y) = \sum_{i=1}^{n} |y_i - x_i|$$

of $\mathbb{R}^n$. However, under this metric the metric dimension of $\mathbb{R}^n$ is also unbounded. The metric space X is an isometric subspace of $\mathbb{R}^n$ under the metric ρ_1. Moreover, $\beta(X)$ and $\beta(\mathbb{R}^n)$ under ρ_1 are both unbounded. Therefore, the monotonicity holds and this X does not form a counterexample to the monotonicity of dimension.

We now properly address the question of whether for $X \subseteq Y$ with $\rho_X = \rho_Y|_X$ and X an isometric subspace of Y, we have $\beta(X) \leq \beta(Y)$. We provide an example to show that this is not true. It is this example that shows that monotonicity fails to hold in general for the metric dimension.

THEOREM 2.3. *There exist metric spaces X and Y such that X is an isometric subspace of Y and $\beta(X) > \beta(Y)$.*

PROOF. Consider the circle shown in Figure 1. Let $t > 0$ and $X = \{x_1, x_2, x_3\}$ with

$$\rho_X(x_1, x_2) = \rho_X(x_1, x_3) = \rho_X(x_2, x_3) = 3t,$$

and $Y = X \cup \{y\}$ with

$$\rho_Y(x_1, x_2) = \rho_Y(x_1, x_3) = \rho_Y(x_2, x_3) = 3t,$$

and

$$\rho_Y(x_1, y) = t, \quad \rho_Y(x_2, y) = 4t, \quad \rho_Y(x_3, y) = 2t.$$

Then X is isometrically embedded in Y, $\{y\}$ is a metric basis for Y, no set with one element resolves X and $\{x_1, x_2\}$ resolves X.

Hence we have shown that $\beta(X) = 2$ and $\beta(Y) = 1$. □

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3. Bisectors

Let $G = \mathbb{Z} \times \mathbb{Z}$ and $S = \{u \in \mathbb{Z} \times \mathbb{Z} : |u| = 1\}$. In this section we obtain a complete characterisation of bisectors in $X = X(G, S)$. First, we give an algebraic verification of a basic geometric fact.

Lemma 3.1. *Suppose $x = (x_1, x_2)$ and $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. Let*

$$f_1(x) = (x_2, -x_1), \quad f_2(x) = (-x_1, x_2), \quad f_{(a,b)}(x) = (x_1 + a, x_2 + b).$$

Then $f_1, f_2, f_{(a,b)}$ are isometries of X.

Proof. Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. By the definition of the three mappings,

$$\begin{aligned}
\rho(f_1(x), f_1(y)) &= \rho((x_2, -x_1), (y_2, -y_1)) \\
&= |x_2 - y_2| + |-x_1 - y_1| = |x_1 - y_1| + |y_1 - y_2| \\
&= \rho((x_1, x_2), (y_1, y_2)) = \rho(x, y); \\
\rho(f_2(x), f_2(y)) &= \rho((-x_1, x_2), (-y_1, y_2)) \\
&= | - x_1 - y_1| + |x_2 - y_2| = |x_1 - y_1| + |y_1 - y_2| \\
&= \rho((x_1, x_2), (y_1, y_2)) = \rho(x, y); \\
\rho(f_{(a,b)}(x), f_{(a,b)}(y)) &= \rho((x_1 + a, x_2 + b), (y_1 + a, y_2 + b)) \\
&= |x_1 + a - y_1 - a| + |x_2 + b - y_2 - b| = |x_1 - y_1| + |y_1 - y_2| \\
&= \rho((x_1, x_2), (y_1, y_2)) = \rho(x, y).
\end{aligned}$$

Hence $f_1, f_2, f_{(a,b)}$ are isometries of X. □

By the definition of isometry, if X, Y are any metric spaces, $f : X \to Y$ any isometry and $A \subseteq X$, then A and $f(A)$ possess the same metric properties. This observation gives the following lemma.

Lemma 3.2. *If $x, y \in \mathbb{Z} \times \mathbb{Z}$ and f is an isometry of X, then $f(B(x, y)) = B(f(x), f(y))$.*

By Lemmas 3.1 and 3.2, it suffices to determine bisectors of $O = (0, 0)$ and any point $p = (p_1, p_2) \in \mathbb{Z} \times \mathbb{Z}$. Since $f_{(a,b)}$ is an isometry, one of the two points may be chosen to be the origin. Since f_1, f_2 are isometries, the other point is chosen arbitrarily in the first quadrant not above the line $y = x$. In the proof of the main result of this section, we distinguish two cases according as $p_1 \equiv p_2 \pmod 2$ or not. Choosing the point in the first quadrant not above $y = x$ amounts to the condition $p_1 \geq p_2 \geq 0$.

Take $p_1, p_2 \in \mathbb{Z}$ with $p_1 \geq p_2 \geq 0$ and $p_1 \equiv p_2 \pmod 2$. Then $\frac{1}{2}(p_2 \pm p_1) \in \mathbb{Z}$. For $p_1 > p_2$, define

$$\begin{aligned}
L_1 &= \left\{\left(\frac{p_1 - p_2}{2} + i, p_2 - i\right) : 0 \leq i \leq p_2, i \in \mathbb{Z}\right\} \\
L_2 &= \left\{\left(\frac{p_1 - p_2}{2}, p_2 + i\right) : i > 0, i \in \mathbb{Z}\right\} \\
L_3 &= \left\{\left(\frac{p_1 + p_2}{2} + i, -i\right) : i > 0, i \in \mathbb{Z}\right\},
\end{aligned}$$

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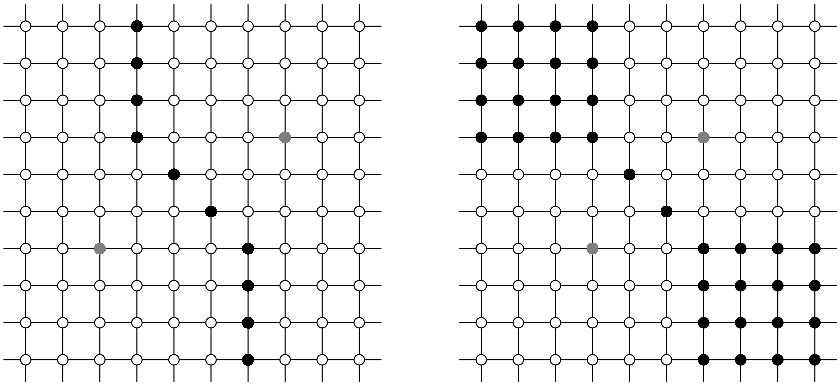

FIGURE 2. An illustration of bisectors $B((2,3),(7,6))$ and $B((3,3),(6,6))$.

and for $p_1 = p_2$ define

$$\begin{aligned} M &= \{(p_1 - i, i) : 0 < i < p_1, i \in \mathbb{Z}\} \\ Q_1 &= \{(-i, p_1 + j) : i \geq 0, j \geq 0, i, j \in \mathbb{Z}\} \\ Q_2 &= \{(p_1 + i, -j) : i \geq 0, j \geq 0, i, j \in \mathbb{Z}\}. \end{aligned}$$

For two pairs of points, the bisectors $L_1 \cup L_2 \cup L_3$ and $M \cup Q_1 \cup Q_2$ are illustrated in Figure 2. The pairs of points are shown in grey and the points in their bisectors are shown as solid points. With a slight abuse of notation, the point $(0,0)$ may be denoted by 0, while we use the former if clarity is required.

LEMMA 3.3. *$B(0, p) = \emptyset$ if and only if $p_1 \not\equiv p_2 \pmod 2$.*

PROOF. Suppose that $p_1 \not\equiv p_2 \pmod 2$. Let $x = (x_1, x_2) \in G$. Then

$$\rho((x_1, x_2), (0,0)) = |x_1| + |x_2|, \quad \rho((x_1, x_2), (p_1, p_2)) = |x_1 - p_1| + |x_2 - p_2|.$$

If $p_1 \equiv 0 \pmod 2$ and $p_2 \equiv 1 \pmod 2$, then $|x_1| \equiv |x_1 - p_1| \pmod 2$ and $|x_2| \not\equiv |x_2 - p_2| \pmod 2$. It follows that $\rho((x_1, x_2), (0,0)) \neq \rho((x_1, x_2), (p_1, p_2))$ and $(x_1, x_2) \notin B(0, p)$. The proof is almost verbatim if $p_1 \equiv 1 \pmod 2$ and $p_2 \equiv 0 \pmod 2$.

For the converse, suppose that $p_1 \equiv p_2 \pmod 2$. Then $\frac{1}{2}(p_1 + p_2) \in \mathbb{Z}$. Consider $(0, \frac{1}{2}(p_1 + p_2)) \in \mathbb{Z} \times \mathbb{Z}$. Then

$$\rho\Big(\Big(0, \frac{p_1 + p_2}{2}\Big), (0,0)\Big) = \frac{p_1 + p_2}{2} = \rho\Big(\Big(0, \frac{p_1 + p_2}{2}\Big), (p_1, p_2)\Big),$$

that is $(0, \frac{1}{2}(p_1 + p_2)) \in B(0, p)$ so that the bisector is nonempty. □

LEMMA 3.4. *If $p = (p_1, p_2)$ and $p_1 > p_2 \geq 0$, then $B(0, p) = L_1 \cup L_2 \cup L_3$.*

PROOF. We first show that $L_1 \cup L_2 \cup L_3 \subseteq B(0, p)$.

By the definition of the set L_1, for any $x = (x_1, x_2) \in L_1$, we have $0 \leq x_1 \leq p_1$ and $0 \leq x_2 \leq p_2$. Hence there is $i \in \mathbb{Z}$ with $0 \leq i \leq p_2$ such that $x_1 = \frac{1}{2}(p_1 - p_2) + i$ and $x_2 = p_2 - i$. It follows that

$$\rho(x, 0) = \left|\frac{p_1 - p_2}{2} + i - 0\right| + |p_2 - i - 0| = \frac{p_1 - p_2}{2} + i + p_2 - i = \frac{p_1 + p_2}{2}$$

and

$$\begin{aligned}\rho(x, p) &= \left|\frac{p_1 - p_2}{2} + i - p_1\right| + |p_2 - i - p_2| \\ &= p_1 - \frac{p_1 - p_2}{2} - i + p_2 - p_2 + i = \frac{p_1 + p_2}{2} = \rho(x, 0).\end{aligned}$$

Hence $L_1 \subseteq B(0, p)$.

Let $x = (x_1, x_2) \in L_2$. By the definition of L_2, $0 \leq x_1 \leq p_1$ and $x_2 \geq p_2 \geq 0$. Hence there exists a positive integer i such that $x_1 = \frac{1}{2}(p_1 - p_2)$ and $x_2 = p_2 + i$. Then

$$\rho(x, 0) = \left|\frac{p_1 - p_2}{2}\right| + |p_2 + i| = \frac{p_1 - p_2}{2} + i + p_2 = \frac{p_1 + p_2}{2} + i$$

and

$$\begin{aligned}\rho(x, p) &= \left|\frac{p_1 - p_2}{2} - p_1\right| + |p_2 + i - p_2| \\ &= p_1 - \frac{p_1 - p_2}{2} + i + p_2 - p_2 = \frac{p_1 + p_2}{2} + i = \rho(x, 0).\end{aligned}$$

Hence $L_2 \subseteq B(0, p)$.

Let $x = (x_1, x_2) \in L_3$. By the definition of L_3, $0 \leq x_1 \leq p_1$ and $x_2 \leq 0 \leq p_2$. Then there exists a positive integer i such that $x_1 = \frac{1}{2}(p_1 + p_2)$ and $x_2 = -i$. Then

$$\rho(x, 0) = \left|\frac{p_1 + p_2}{2}\right| + |-i| = \frac{p_1 + p_2}{2} + i$$

and

$$\rho(x, p) = \left|\frac{p_1 + p_2}{2} - p_1\right| + |-i - p_2| = p_1 - \frac{p_1 + p_2}{2} + i + p_2 = \frac{p_1 + p_2}{2} + i = \rho(x, 0).$$

Hence $L_3 \subseteq B(0, p)$. Therefore, we have shown that $L_1 \cup L_2 \cup L_3 \subseteq B(0, p)$.

We now show that $B(0, p) \subseteq L_1 \cup L_2 \cup L_3$. Let $x = (x_1, x_2) \in B(0, p)$. Suppose that $x \notin L_1 \cup L_2 \cup L_3$. Denote $a = (0, p_2)$ and $b = (p_1, 0)$.

If $x_1 \leq 0$, then

$$\rho(x, 0) \leq \rho(x, a) + \rho(0, a) < \rho(x, a) + \rho(p, a) = \rho(x, p).$$

If $x_1 \geq p_1$, then

$$\rho(x, p) \leq \rho(x, b) + \rho(p, b) < \rho(x, b) + \rho(0, b) = \rho(x, 0).$$

Now let $0 < x_1 < p_1$. Since the set $L_1 \cup L_2 \cup L_3$ is the set of integral points on the union of two vertical half lines and a line segment connecting their end points and since $y = x_2$ is a horizontal line, the intersection of $y = x_2$ and $L_1 \cup L_2 \cup L_3$ is a single integral

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point. Let this point be denoted $z = (z_1, z_2)$. Then $z_2 = x_2$ and $z_1 \neq x_1$. If $z_1 > x_1$, then $\rho(x, 0) < \rho(z, 0) = \rho(z, p) < \rho(x, p)$. If $z_1 < x_1$, then $\rho(x, 0) > \rho(z, 0) = \rho(z, p) > \rho(x, p)$. Hence $\rho(x, 0) \neq \rho(x, p)$. That is, $x \notin B(0, p)$. This contradicts the assumption that $x \in B(0, p)$. Hence $B(0, p) \subseteq L_1 \cup L_2 \cup L_3$.

Therefore, $B(0, p) = L_1 \cup L_2 \cup L_3$. □

Lemma 3.5. *If $p_1 = p_2 > 0$ then $B(0, p) = M \cup Q_1 \cup Q_2$.*

Proof. We first show that $M \cup Q_1 \cup Q_2 \subseteq B(0, p)$. By the definition of M, if $x = (x_1, x_2) \in M$ then $0 < x_1 < p_1$. Then there exists $i \in \mathbb{Z}$ with $0 < i < p_1$, such that $x_1 = p_1 - i$ and $x_2 = i$. Consequently,

$$\rho(x, 0) = |p_1 - i| + |i| = p_1 - i + i = p_1$$

and

$$\rho(x, p) = |p_1 - i - p_1| + |i - p_1| = i + p_1 - i = p_1 = \rho(x, 0).$$

Hence $M \subseteq B(0, p)$.

By the definition of Q_1, if $x = (x_1, x_2) \in Q_1$, then $x_1 \leq 0 < p_1$ and $x_2 \geq p_1 > 0$. Hence there exist positive integers i, j such that $x_1 = -i$ and $x_2 = p_1 + j$. Then

$$\rho(x, 0) = |-i| + |p_1 + j| = p_1 + i + j$$

and

$$\rho(x, p) = |-i - p_1| + |p_1 + j - p_1| = p_1 + i + j = \rho(x, 0).$$

Hence $Q_1 \subseteq B(0, p)$.

By the definition of Q_2, if $x = (x_1, x_2) \in Q_2$ then $x_1 \geq p_1 > 0$ and $x_2 \leq 0 \leq p_1$. Hence there exist positive integers i, j such that $x_1 = p_1 + i$ and $x_2 = -j$. Then

$$\rho(x, 0) = |p_1 + i| + |-j| = p_1 + i + j$$

and

$$\rho(x, p) = |-i - p_1| + |p_1 + j - p_1| = p_1 + i + j = \rho(x, 0).$$

Hence $Q_2 \subseteq B(0, p)$. We have shown that $B(0, p) \supseteq M \cup Q_1 \cup Q_2$.

We next show that $B(0, p) \subseteq M \cup Q_1 \cup Q_2$. Let $x = (x_1, x_2) \in B(0, p)$. Suppose that $x \notin M \cup Q_1 \cup Q_2$. Denote $a = (0, p_1)$ and $b = (p_1, 0)$. If $x_1 \leq 0$, then $x_2 < p_1$ by the definition of Q_1. Denote $z = (0, x_2)$. Then

$$\begin{aligned}
\rho(x, 0) &= \rho(x, z) + \rho(z, 0), \\
\rho(x, p) &= \rho(x, z) + \rho(z, a) + \rho(a, p), \\
\rho(x, p) - \rho(x, 0) &= \rho(z, a) - \rho(z, 0) + p_1 = (p_1 - x_2) - |x_2| + p_1 \\
&= 2p_1 - x_2 - |x_2| > 0.
\end{aligned}$$

If $x_1 \geq p_1$, then by the definition of Q_2, $x_2 > 0$. Let $u = (p_1, x_2)$. Hence

$$\begin{aligned}
\rho(x, 0) &= \rho(x, u) + \rho(u, b) + \rho(b, 0) \\
\rho(x, p) &= \rho(x, u) + \rho(u, p) \\
\rho(x, 0) - \rho(x, p) &= \rho(u, b) - \rho(u, p) + p_1 = x_2 - |x_2 - p_1| + p_1 \\
&= x_2 + p_1 - |x_2 - p_1| > 0.
\end{aligned}$$

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If $0 < x_1 < p_1$ then $p_1 \neq 1$ since $x_1 \in \mathbb{Z}$. The set of integral points on the vertical line $x = x_1$ intersects M in exactly one point. Let this point be $z = (z_1, z_2)$. Then $z_2 = x_2$ and $z_1 \neq x_1$. If $z_1 > x_1$, then $\rho(x, 0) < \rho(z, 0) = \rho(z, p) < \rho(x, p)$. If $z_1 < x_1$, then $\rho(x, 0) > \rho(z, 0) = \rho(z, p) > \rho(x, p)$. Hence $x \notin B(0, p)$. This contradicts the assumption that $x \in B(0, p)$. Hence $B(0, p) \subseteq M \cup Q_1 \cup Q_2$.

In summary, we have $B(0, p) = M \cup Q_1 \cup Q_2$. □

The conjunction of Lemmas 3.3–3.5 gives the main result of this section.

THEOREM 3.6. *Let* $p = (p_1, p_2) \in \mathbb{Z} \times \mathbb{Z}$ *with* $0 \leq p_2 \leq p_1$.

(1) $B(0, p) = \emptyset$ *if and only if* $p_1 \not\equiv p_2 \pmod 2$.
(2) *If* $p_1 \equiv p_2 \pmod 2$ *and* $p_2 < p_1$, *then* $B(0, p) = L_1 \cup L_2 \cup L_3$.
(3) *If* $p_1 \equiv p_2 \pmod 2$ *and* $p_1 = p_2$ *then* $B(0, p) = M \cup Q_1 \cup Q_2$.

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References

[1] S. Bau, 'A generalization of the concept of Toeplitz graphs', *Mong. Math. J.* **15** (2011), 54–61.
[2] S. Bau and A. Beardon, 'The metric dimension of metric spaces', *Comput. Methods Funct. Theory* **13** (2013), 295–305.
[3] L. M. Blumenthal, *Theory and Applications of Distance Geometry* (Clarendon Press, Oxford, 1953).
[4] C. Godsil and G. F. Royle, *Algebraic Graph Theory* (Springer, New York, 2001).
[5] M. Heydarpour and S. Maghsoudi, 'The metric dimension of geometric spaces', *Topology Appl.* **178** (2014), 230–235.
[6] M. Heydarpour and S. Maghsoudi, 'The metric dimension of metric manifolds', *Bull. Aust. Math. Soc.* **91** (2015), 508–513.
[7] A. G. Kurosch, 'Primitive torsionfreie abelsche Gruppen von endlichen Range', *Math. Ann.* **38** (1937), 175–203.
[8] J. G. Ratcliffe, *Foundations of Hyperbolic Manifolds* (Springer, New York, 1964).

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