On Mellin's inversion formula. By Mr J. C. Burkill, Trinity College.

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1. The extension of Mellin's inversion formula expressed by the equations

$$f(s)=\int_0^\infty y^{-s}\,d\phi(y),$$

$$\phi(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} f(s)\,x^s\frac{ds}{s}$$

has been considered by Fowler* who shows that some form of Stieltjes integral is essential to Poincaré's proof of the necessity of the quantum hypothesis. Fowler confines his discussion to a restricted type of function $\phi(y)$ which is sufficient for the physical problem. It will be proved here that the formulae hold with a general Stieltjes integral in the first equation.

I have remarked elsewhere† that inversion formulae in Stieltjes integrals arise from integrals of the "discontinuous factor" type. The discontinuous integral in the present instance is

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} k^s\frac{ds}{s} \qquad (k>0),$$

which is equal to 1, $\frac{1}{2}$ or 0 according as $k >, =$ or < 1, the integral being interpreted as the principal value

$$\lim_{T\to\infty} I(k, T),$$

where
$$I(k, T)=\frac{1}{2\pi i}\int_{a-iT}^{a+iT} k^s\frac{ds}{s}.$$

2. We need two lemmas on the behaviour of $I(k, T)$.

Lemma 1. *If* $a>0$, $T>0$, *then*

$$|I(k, T)-1|\leqslant\frac{1}{\pi T}\frac{k^a}{\log k} \qquad (k>1),$$

$$|I(k, T)|\leqslant\frac{1}{\pi T}\frac{k^a}{\log(1/k)} \qquad (k<1),$$

$$|I(k, T)-\tfrac{1}{2}|\leqslant\frac{a}{\pi T} \qquad (k=1).$$

* *Proc. Royal Soc.* (A), vol. 99 (1921), pp. 462–471. An account of Mellin's formula in the ordinary form (without Stieltjes integrals) is given by Hardy, *Messenger of Math.* vol. 47 (1918), pp. 178–184 and vol. 50 (1921), pp. 165–171.

† "The expression in Stieltjes integrals of the inversion formulae of Fourier and Hankel," *Proc. London Math. Soc.* (unpublished).

Cor. $I(k, T)$ *tends to its limit uniformly in the intervals* $0 < \delta \leqslant k \leqslant 1 - \delta$ *and* $1 + \delta \leqslant k \leqslant k_0$.

For proofs, see Landau, *Primzahlen*, 342.

Lemma 2. *As* $T \to \infty$,

$$I(k, T) = O(1)$$

uniformly in k *for* $0 < k_0 \leqslant k \leqslant k_1$.

Writing $m = \log k$—so that m lies between fixed bounds—we have

$$\begin{aligned} I(k, T) &= \frac{k^a}{2\pi} \int_{-T}^{T} \frac{\cos mt + i \sin mt}{a + it} dt \\ &= \frac{k^a}{2\pi} \int_{-T}^{T} \frac{a \cos mt + t \sin mt}{a^2 + t^2} dt \\ &= O(1) + O \left| \int_0^{mT} \frac{u \sin u}{m^2 a^2 + u^2} du \right|. \end{aligned}$$

Since $u^2/(m^2 a^2 + u^2)$ is an increasing function of u, the second mean value theorem shows that the integral in the second term is

$$\frac{T^2}{a^2 + T^2} \int_{\alpha}^{mT} \frac{\sin u}{u} du,$$

where α is between 0 and mT, which is $O(1)$, and the lemma is proved.

3. We can now prove the main theorem.

Theorem A. *If*

(1) $\phi(y)$ *has bounded variation in* $0 \leqslant y \leqslant k$ *for every* $k > 0$,

(2) $\phi(y) = \frac{1}{2}\{\phi(y + 0) + \phi(y - 0)\}$,

(3) $\phi(+0) = 0$,

(4) $\int_0 y^{-\alpha} |d\phi(y)|$ *converges where* $\alpha > 0$

and $\int^{\infty} y^{-\beta} |d\phi(y)|$ *converges where* $\beta < \alpha$,

and if

(3·1) $$f(s) = \int_0^{\infty} y^{-s} d\phi(y) \qquad (\beta < \sigma < \alpha),$$

then will

(3·2) $$\phi(x) = \frac{1}{2\pi i} \int_{a - i\infty}^{a + i\infty} f(s) x^s \frac{ds}{s} \quad (\beta < a < \alpha,\ a > 0),$$

where the integrals (3·1), (3·2) *are respectively* $\lim \int_{\epsilon}^{\infty}$, $\lim \int_{a-iT}^{a+iT}$

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From (4), $\int_0^\infty y^{-s} d\phi(y)$ is uniformly convergent in any rectangle

$$\alpha < \alpha' \leqslant \sigma \leqslant \beta' < \beta, \quad -T \leqslant t \leqslant T$$

and $f(s)$ is regular in $\alpha < \sigma < \beta$.

Substituting the value of $f(s)$ from (3·1) in the right-hand side of (3·2), we have

$$\lim_{T\to\infty} \frac{1}{2\pi i}\int_{a-iT}^{a+iT} \frac{x^s}{s}\, ds \int_0^\infty y^{-s}\, d\phi(y).$$

Since the inner integral is uniformly convergent for the values of s in question, we may invert the order of integration and obtain

$$\lim_{T\to\infty} \int_0^\infty I\left(\frac{x}{y}, T\right) d\phi(y).$$

Split up this integral into

$$\int_0^\eta + \int_\eta^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^k + \int_k^\infty.$$

By Lemma 1, the integrand in the first and last integrals is $O(y^{-\alpha})$ and in the second and fourth it converges uniformly to its limit as $T\to\infty$; and in the third integral the integrand is bounded by Lemma 2.

The proof is completed by choice of η, k, δ in exactly the same way as that of the main theorem in my Fourier-Stieltjes paper referred to.

4. If in Theorem A we replace y by e^y we obtain the exponential form of Fourier's inversion formula.

Theorem B. *If*

(1) $\phi(y)$ *has bounded variation in* $0 \leqslant y \leqslant k$ *for every* $k > 0$,

(2) $\phi(y) = \frac{1}{2}\{\phi(y+0) + \phi(y-0)\}$,

(3) $\phi(0) = 0$,

(4) $\int^\infty e^{-\beta y} |d\phi(y)|$ *converges*,

and if

$$f(s) = \int_0^\infty e^{-sy}\, d\phi(y) \qquad (\beta < \sigma), \tag{4·1}$$

then will

$$\phi(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} f(s)\, e^{sx} \frac{ds}{s} \qquad (\beta < a,\ a > 0), \tag{4·2}$$

where the integral (4·2) is $\lim \int_{a-iT}^{a+iT}$

If in particular $\phi(y)$ is a step-function, this reduces to the Cahen-Hadamard-Perron formula for the sum of the first n coefficients of a Dirichlet series.

5. We may extend Theorem B to give an inversion formula involving Bessel functions of the third kind instead of exponentials by starting with an appropriate discontinuous factor. Such a factor is

$$(5\cdot1) \qquad \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{\pi}{2}\, H_{\mu}^{(1)}(isy)\, y^{-\mu}\, H_{\mu+1}^{(2)}(isx)\, x^{\mu+1}\, ds,$$

where $x > 0$, $y > 0$, $a > 0$, which has the value 1, $\frac{1}{2}$ or 0 according as $y <, =$ or $> x$. This integral has been applied by Oppenheim* to the discussion of lattice-point problems.

We need lemmas corresponding to (1) and (2) of § 4. These are easily established by writing the integral (5·1) as

$$\frac{1}{2\pi i}\left\{\int_{a-iT}^{a+iT} + \int_{a-i\infty}^{a-iT} + \int_{a+iT}^{a+i\infty}\right\},$$

substituting in the last two integrals the asymptotic value of the integrand †

$$\frac{e^{s(x-y)}}{s}\left(\frac{x}{y}\right)^{\mu+\frac{1}{2}}\left\{1 + O\left(\frac{1}{t}\right)\right\}$$

and then applying the results of lemmas (1) and (2) above with e^{x-y} in place of k.

In generalising from exponentials to Bessel functions an extra condition—(4) below—has to be introduced to secure the convergence of the Stieltjes integral at the origin. The precise statement is as follows.

Theorem C. *If*

(1) $\phi(y)$ *has bounded variation in* $0 \leqslant y \leqslant k$ *for every* $k > 0$,

(2) $\phi(y) = \frac{1}{2}\{\phi(y+0) + \phi(y-0)\}$,

(3) $\phi(+0) = 0$,

(4) $\int_0 \lambda(y)\,|\,d\phi(y)$ *converges,*

where $\lambda(y)$ *is* $y^{-2\mu}$, $\log y$ *or* 1

according as $\mu >, =$ *or* < 0

and $\int^{\infty} e^{-\beta y}\,|\,d\phi(y)\,|$ *converges,*

* A. Oppenheim, "Some identities in the theory of numbers," *Proc. London Math. Soc.* (*Records*), vol. 24 (1925), p. xxiii. I am indebted to Mr Oppenheim for sending me in MS. the part of his work dealing with discontinuous factors.

† Watson, *Bessel Functions*, p. 198.

and if

$$f(s)=\int_0^\infty \left(\frac{\pi}{2}\right)^{\frac{1}{2}} H_\mu^{(1)}(isy)\, y^{-\mu}\, d\phi(y) \qquad (\beta<\sigma), \tag{5·2}$$

then will

$$\phi(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} H_{\mu+1}^{(2)}(isx)\, x^{\mu+1} f(s)\, ds, \tag{5·3}$$

where $\beta<a$, $a>0$ *and the integral is the principal value.*

From a different discontinuous factor we obtain the pair of formulae

$$f(s)=\int_0^\infty \left(\frac{\pi}{2}\right)^{\frac{1}{2}} H_\mu^{(1)}(isy)\, y^{\mu}\, d\phi(y), \tag{5·4}$$

$$\phi(x)=-\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} H_{\mu-1}^{(2)}(isx)\, x^{-\mu+1} f(s)\, ds, \tag{5·5}$$

provided that we now define $\lambda(y)$ in condition (4) to be

$$1,\ \log y \text{ or } y^{-2\mu}$$

according as

$$\mu >, = \text{ or } < 0.$$

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