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A NOTE ON SPIRALLIKE FUNCTIONS

Y. J. SIM and D. K. THOMAS

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Abstract

Let f be analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ be the subclass of normalised univalent functions with $f(0) = 0$ and $f'(0) = 1$, given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Let F be the inverse function of f, given by $F(\omega) = \omega + \sum_{n=2}^{\infty} A_n \omega^n$ for $|\omega| \le r_0(f)$. Denote by $\mathcal{S}_p^*(\alpha)$ the subset of $\mathcal{S}$ consisting of the spirallike functions of order α in $\mathbb{D}$, that is, functions satisfying

$$\text{Re}\left\{e^{-i\gamma}\frac{zf'(z)}{f(z)}\right\} > \alpha\cos\gamma,$$

for $z \in \mathbb{D}$, $0 \le \alpha < 1$ and $\gamma \in (-\pi/2, \pi/2)$. We give sharp upper and lower bounds for both $|a_3| - |a_2|$ and $|A_3| - |A_2|$ when $f \in \mathcal{S}_p^*(\alpha)$, thus solving an open problem and presenting some new inequalities for coefficient differences.

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions f in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ normalised by $f(0) = 0 = f'(0) - 1$. For $z \in \mathbb{D}$, a function $f \in \mathcal{A}$ has the representation

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let $\mathcal{S}$ denote the subclass of all univalent (that is, one-to-one) functions in $\mathcal{A}$.

For $f \in \mathcal{S}$ denote by F, the inverse of f, given by

$$F(\omega) = \omega + \sum_{n=2}^{\infty} A_n \omega^n, \tag{1.2}$$

valid on some disk $|\omega| \le r_0(f)$.

In 1985, de Branges [2] solved the famous Bieberbach conjecture by showing that if $f \in \mathcal{S}$, then $|a_n| \le n$ for $n \ge 2$, with equality when $f(z) = k(z) := z/(1-z)^2$, or a rotation of it. It was therefore natural to ask if for $f \in \mathcal{S}$, the inequality $||a_{n+1}| - |a_n|| \le 1$ is true when $n \ge 2$. This was shown not to be the case even when $n = 2$ [3], and the following sharp bounds hold:

$$-1 \le |a_3| - |a_2| \le \tfrac{3}{4} + e^{-\lambda_0}(2e^{-\lambda_0} - 1) = 1.029\cdots,$$

where λ_0 is the unique value of λ in $0 < \lambda < 1$, satisfying the equation $4\lambda = e^{\lambda}$.

Hayman [5] showed that if $f \in \mathcal{S}$, then $||a_{n+1}| - |a_n|| \le C$, where C is an absolute constant. The exact value of C is unknown, the best estimate to date being $C = 3.61\cdots$ [4], which because of the sharp estimate above when $n = 2$, cannot be reduced to 1.

Hayman's seminal result $||a_{n+1}| - |a_n|| \le C$ for $n \ge 2$, was proved in 1963, using his distinctive method developed to study areally mean p-valent functions. A different proof was provided by Milin, using the now well-known Lebedev–Milin inequalities, and an excellent account of this result can be found in Duren's book [3]. Little progress has been made estimating the value of C. It was shown by Ilina [6] in 1968 that $C < 4.26\cdots$. Using a modification of Milin's method, Grispan [4] improved this bound in 1976 to show that for $n \ge 2$,

$$-2.97\cdots < |a_{n+1}| - |a_n| < 3.61\cdots$$

No other advances appear to have been made in this direction during the intervening years, until a recent result of Obradović *et al.* [9] who, using the Grunsky inequalities, have shown that the upper bound for C can be improved when $n = 3$ to $|a_4| - |a_3| \le 2.1033\cdots$.

In [10], the present authors gave sharp upper and lower bounds for $|a_3| - |a_2|$, when f belongs to some important subclasses of starlike and convex functions in $\mathbb{D}$. Also, in [11], the same authors gave corresponding sharp bounds for $|A_3| - |A_2|$ for $f \in \mathcal{S}$ and a variety of subclasses.

2. Definitions

The classes $\mathcal{S}^*(\alpha)$ and $\mathcal{S}_p^*(\alpha)$ of starlike and spirallike functions of order α are defined as follows.

DEFINITION 2.1. Let $f \in \mathcal{A}$ be given by (1.1). For $0 \le \alpha < 1$, denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{S}$ consisting of starlike functions of order α, so that $f \in \mathcal{S}^*(\alpha)$ if and only if, for $z \in \mathbb{D}$,

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha.$$

Although similar in the form of definition, the class of spirallike functions defined below is significantly more difficult to deal with, with relatively little precise information known.

DEFINITION 2.2. Let $f \in \mathcal{A}$ be given by (1.1). For $0 \le \alpha < 1$, denote by $\mathcal{S}_p^*(\alpha)$ the subclass of $\mathcal{S}$ consisting of spirallike functions of order α, so that $f \in \mathcal{S}_p^*(\alpha)$ if and only if, for $z \in \mathbb{D}$ and $\gamma \in (-\pi/2, \pi/2)$,

$$\mathrm{Re}\left\{e^{-i\gamma}\frac{zf'(z)}{f(z)}\right\} > \alpha\cos\gamma. \tag{2.1}$$

We note that $\mathcal{S}^*(0)$ is the class of starlike functions $\mathcal{S}^*$, and that Leung [7] established the sharp inequalities $||a_{n+1}| - |a_n|| \le 1$ for $n \ge 2$. In [10], corresponding inequalities for functions in $\mathcal{S}^*(\alpha)$ were obtained in the case $n = 2$.

The problem of extending these inequalities to the class $\mathcal{S}^*_p(\alpha)$ has been considered by Arora *et al.* [1], and earlier by Li [8]. In [1, 8], using Leung's method, it was shown that Leung's result remains valid in $\mathcal{S}^*_p(\alpha)$ for all $n \ge 2$ when $\alpha = 0$. Also in [8], Li gave a partial solution to finding sharp upper and lower bounds for $|a_3| - |a_2|$ when $0 \le \alpha < 1$ and $\gamma \in (-\pi/2, \pi/2)$. In this paper we give the complete solution to finding sharp upper and lower bounds for $|a_3| - |a_2|$ in $\mathcal{S}^*_p(\alpha)$, together with corresponding sharp upper and lower bounds for $|A_3| - |A_2|$.

3. Preliminary lemma

Denote by $\mathcal{P}$ the class of analytic functions p with positive real part on $\mathbb{D}$ given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{3.1}$$

In the subsequent sections we note that the classes $\mathcal{P}$, $\mathcal{S}^*_p(\alpha)$ and the functionals $|a_3| - |a_2|$ and $|A_3| - |A_2|$ are rotationally invariant.

We will use the following lemma given in [11], concerning the coefficients of functions in $\mathcal{P}$, given by (3.1), noting that for our application it is critical that $B_2 \in \mathbb{C}$. In [11] the general nature of Lemma 3.1 enabled sharp upper and lower bounds to be found for $|A_3| - |A_2|$ for a variety of classes of convex and starlike functions.

LEMMA 3.1. *Let B_1, B_2 and B_3 be numbers such that $B_1 \ge 0$, $B_2 \in \mathbb{C}$ and $B_3 \in \mathbb{R}$. Let $p \in \mathcal{P}$ and be given by (3.1). Define $\Psi_+(p_1, p_2)$ and $\Psi_-(p_1, p_2)$ by*

$$\Psi_+(p_1, p_2) = |B_2 p_1^2 + B_3 p_2| - |B_1 p_1| \quad \text{and} \quad \Psi_-(p_1, p_2) = -\Psi_+(p_1, p_2).$$

Then

$$\Psi_+(p_1, p_2) \le \begin{cases} |4B_2 + 2B_3| - 2B_1, & \text{when } |2B_2 + B_3| \ge |B_3| + B_1, \\ 2|B_3|, & \text{otherwise,} \end{cases} \tag{3.2}$$

and

$$\Psi_-(p_1, p_2) \le \begin{cases} 2B_1 - B_4, & \text{when } B_1 \ge B_4 + 2|B_3|, \\ 2B_1 \sqrt{\dfrac{2|B_3|}{B_4 + 2|B_3|}}, & \text{when } B_1^2 \le 2|B_3|(B_4 + 2|B_3|), \\ 2|B_3| + \dfrac{B_1^2}{B_4 + 2|B_3|}, & \text{otherwise,} \end{cases} \tag{3.3}$$

where $B_4 = |4B_2 + 2B_3|$. All inequalities in (3.2) and (3.3) are sharp.

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4. Bounds for $|a_3| - |a_2|$

We prove the following inequalities for functions in $\mathcal{S}_p^*(\alpha)$.

THEOREM 4.1. *Let $f \in \mathcal{S}_p^*(\alpha)$ be given by (1.1). Then for $0 \le \alpha < 1$,*

$$-\frac{2(1-\alpha)\cos\gamma}{\sqrt{1+\sqrt{1+4(2-\alpha)(1-\alpha)\cos^2\gamma}}} \le |a_3| - |a_2| \le (1-\alpha)\cos\gamma. \tag{4.1}$$

Both inequalities are sharp.

PROOF. First note that from (2.1) we can write

$$e^{-i\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right) = \cos\gamma\,(\alpha + (1-\alpha)p(z) - 1)\,, \tag{4.2}$$

for some $p \in \mathcal{P}$. Thus, equating coefficients, we obtain

$$\begin{aligned} a_2 &= (1-\alpha)e^{i\gamma}p_1\cos\gamma, \\ a_3 &= \tfrac{1}{2}(1-\alpha)(e^{i\gamma}p_2\cos\gamma + (1-\alpha)e^{2i\gamma}p_1^2\cos^2\gamma). \end{aligned} \tag{4.3}$$

We first use Lemma 3.1 to find the upper bound for $|a_3| - |a_2|$. From (4.3), we obtain

$$|a_3| - |a_2| = \tfrac{1}{2}(1-\alpha)\cos\gamma(|p_2 + (1-\alpha)e^{i\gamma}p_1^2\cos\gamma| - |2p_1|),$$

so that in Lemma 3.1 we take $B_1 = 2$, $B_2 = (1-\alpha)e^{i\gamma}\cos\gamma$ and $B_3 = 1$. A simple calculation shows that the condition $|2B_2 + B_3| \ge |B_3| + B_1$ is equivalent to

$$\sqrt{1 + 4(1-\alpha)\cos^2\gamma + 4(1-\alpha)^2\cos^2\gamma} \ge 3,$$

which is easily seen to be false unless $\cos\gamma = 1$ and $\alpha = 0$, which reduces to the case of starlike functions. Thus from Lemma 3.1 we deduce that $|a_3| - |a_2| \le (1-\alpha)\cos\gamma$ as required.

It is easy to see that equality holds in the right-hand inequality in (4.1) when $f \in \mathcal{S}_p^*(\alpha)$ satisfying (4.2) with $p(z) = (1+z^2)/(1-z^2)$.

For the lower bound we use Lemma 3.1 and write

$$|a_2| - |a_3| = (1-\alpha)\cos\gamma(|p_1| - \tfrac{1}{2}|p_2 + (1-\alpha)e^{i\gamma}p_1^2\cos\gamma|),$$

so that in Lemma 3.1 we take $B_1 = 1$, $B_2 = ((1-\alpha)e^{i\gamma}\cos\gamma)/2$ and $B_3 = 1/2$. Checking the conditions for $\Psi_-(p_1, p_2)$, we find that with these values of B_1, B_2 and B_3, the second condition is satisfied, which after some simplification gives the bound

$$|a_2| - |a_3| \le \frac{2(1-\alpha)\cos\gamma}{\sqrt{1+\sqrt{1+4(2-\alpha)(1-\alpha)\cos^2\gamma}}},$$

as required.

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To see that this lower bound is sharp, consider $f \in \mathcal{S}_p^*(\alpha)$ defined by (4.2) with

$$p(z) = \frac{1 + \zeta_1(\zeta_2 + 1)z + \zeta_2 z^2}{1 + \zeta_1(\zeta_2 - 1)z - \zeta_2 z^2},$$

where

$$\zeta_1 = \frac{1}{\sqrt{1 + |\mu|}}, \quad \zeta_2 = -e^{i \arg \mu} \quad \text{and} \quad \mu = 1 + 2(1-\alpha)e^{i\gamma}\cos\gamma.$$

Then

$$a_2 = \frac{2(1-\alpha)e^{i\gamma}\cos\gamma}{\sqrt{1+|\mu|}} \quad \text{and} \quad a_3 = 0,$$

which gives equality in the left-hand inequality in (4.1), and so completes the proof of Theorem 4.1. □

REMARK 4.2. When $\alpha = 0$, we obtain the result obtained by Li in [8].

5. Bounds for the difference $|A_3| - |A_2|$ of inverse coefficients

We first note that since $f(f^{-1}(\omega)) = \omega$; equating coefficients gives

$$A_2 = -a_2 \quad \text{and} \quad A_3 = 2a_2^2 - a_3. \tag{5.1}$$

We first find the upper bounds.

THEOREM 5.1. *Let $f \in \mathcal{S}_p^*(\alpha)$, with inverse coefficients given by (1.2). Then*

$$|A_3| - |A_2| \le (1-\alpha)\cos\gamma\Big(-2 + \sqrt{1 - 12(1-\alpha)(3\alpha-2)\cos^2\gamma}\Big) \tag{5.2}$$

when

$$\frac{1}{\sqrt{3}} < \cos\gamma \le 1 \quad \textit{and} \quad 0 \le \alpha \le \frac{5}{6} - \frac{1}{6}\sqrt{1 + 8\sec^2\gamma},$$

and

$$|A_3| - |A_2| \le (1-\alpha)\cos\gamma$$

when

$$\frac{1}{\sqrt{3}} < \cos\gamma \le 1 \quad \textit{and} \quad \frac{5}{6} - \frac{1}{6}\sqrt{1 + 8\sec^2\gamma} < \alpha < 1,$$

and also when

$$0 < \cos\gamma \le \frac{1}{\sqrt{3}} \quad \textit{and} \quad 0 \le \alpha < 1.$$

All the inequalities are sharp.

PROOF. Using (4.3) and (5.1), a simple calculation gives

$$|A_3| - |A_2| = \tfrac{1}{2}(1-\alpha)\cos\gamma(|-3(1-\alpha)e^{i\gamma}\cos\gamma\, p_1^2 + p_2| - 2|p_1|).$$

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We apply Lemma 3.1, with $B_1 = 2$, $B_2 = -3(1-\alpha)e^{i\gamma}\cos\gamma$ and $B_3 = 1$, and check the conditions for the bound $\Psi_+(p_1, p_2)$.

The condition $|2B_2 + B_3| \geq |B_3| + B_1$ is satisfied when $|1 - 6\cos\gamma(1-\alpha)e^{i\gamma}| \geq 3$, which is equivalent to $\sqrt{1 - 12(1-\alpha)\cos^2\gamma + 36(1-\alpha)^2\cos^2\gamma} \geq 3$. A calculation shows that this condition is valid when

$$\frac{1}{\sqrt{3}} < \cos\gamma \leq 1 \quad \text{and} \quad 0 \leq \alpha \leq \frac{5}{6} - \frac{1}{6}\sqrt{1 + 8\sec^2\gamma}.$$

Thus substituting the chosen values of B_1, B_2 and B_3 into Lemma 3.1 gives

$$|A_3| - |A_2| \leq (1-\alpha)\cos\gamma\Big(-2 + \sqrt{1 - 12(1-\alpha)(3\alpha - 2)\cos^2\gamma}\Big)$$

when

$$\frac{1}{\sqrt{3}} < \cos\gamma \leq 1 \quad \text{and} \quad 0 \leq \alpha \leq \frac{5}{6} - \frac{1}{6}\sqrt{1 + 8\sec^2\gamma},$$

and

$$|A_3| - |A_2| \leq (1-\alpha)\cos\gamma$$

when

$$\frac{1}{\sqrt{3}} < \cos\gamma \leq 1 \quad \text{and} \quad \frac{5}{6} - \frac{1}{6}\sqrt{1 + 8\sec^2\gamma} < \alpha < 1,$$

and also when

$$0 < \cos\gamma \leq \frac{1}{\sqrt{3}} \quad \text{and} \quad 0 \leq \alpha < 1.$$

The first bound in (5.2) is sharp when $p(z) = (1+z)/(1-z)$, and the second bound is sharp when $p(z) = (1+z^2)/(1-z^2)$. □

We next find the lower bounds.

THEOREM 5.2. *Let $f \in \mathcal{S}_p^*(\alpha)$, with inverse coefficients given by (1.2). Then*

$$|A_2| - |A_3| \leq \frac{2(1-\alpha)\cos\gamma}{\sqrt{1 + \sqrt{1 + 12(1-\alpha)(3\alpha-2)\cos^2\gamma}}}. \tag{5.3}$$

PROOF. We again use Lemma 3.1, this time with $B_1 = 2$, $B_2 = -3(1-\alpha)e^{i\gamma}\cos\gamma$ and $B_3 = 1$, so that $B_1^2 \leq 2|B_3|(|4B_2 + 2B_3| + 2|B_1|)$ is true for all $\alpha \in [0, 1)$ and $\gamma \in (-\pi/2, \pi/2)$, which on substitution gives the required inequality (5.3).

To see that this lower bound is sharp, consider $f \in \mathcal{S}_p^*(\alpha)$ defined by (4.2) with

$$p(z) = \frac{1 + \zeta_1(\zeta_2 + 1)z + \zeta_2 z^2}{1 + \zeta_1(\zeta_2 - 1)z - \zeta_2 z^2},$$

where

$$\zeta_1 = \frac{1}{\sqrt{1+|\nu|}}, \quad \zeta_2 = -e^{i\arg\nu} \quad \text{and} \quad \nu = 1 - 6(1-\alpha)e^{i\gamma}\cos\gamma.$$

Then

$$A_2 = \frac{-2(1-\alpha)e^{i\gamma}\cos\gamma}{\sqrt{1+|\nu|}} \quad \text{and} \quad A_3 = 0,$$

which gives equality in (5.3). □

REMARK 5.3. It is easy to see that when $\gamma = 0$ in all of the above results, we obtain the corresponding inequalities obtained in [10, 11].

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Y. J. SIM, Department of Mathematics,
Kyungsung University, Busan 48434, Korea
e-mail: yjsim@ks.ac.kr

D. K. THOMAS, Department of Mathematics,
Swansea University, Bay Campus, Swansea SA1 8EN, UK
e-mail: d.k.thomas@swansea.ac.uk

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