Thermodynamically consistent Navier–Stokes–Cahn–Hilliard models with mass transfer and chemotaxis†

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We derive a class of Navier–Stokes–Cahn–Hilliard systems that models two-phase flows with mass transfer coupled to the process of chemotaxis. These thermodynamically consistent models can be seen as the natural Navier–Stokes analogues of earlier Cahn–Hilliard–Darcy models proposed for modelling tumour growth, and are derived based on a volume-averaged velocity, which yields simpler expressions compared to models derived based on a mass-averaged velocity. Then, we perform mathematical analysis on a simplified model variant with zero excess of total mass and equal densities. We establish the existence of global weak solutions in two and three dimensions for prescribed mass transfer terms. Under additional assumptions, we prove the global strong well-posedness in two dimensions with variable fluid viscosity and mobilities, which also includes a continuous dependence on initial data and mass transfer terms for the chemical potential and the order parameter in strong norms.

Key words: Cahn–Hilliard equation, Navier–Stokes equations, mass transfer, chemotaxis, volume-averaged velocity, mass-averaged velocity, well-posedness

1 Introduction

Biological phenomena such as tumour growth often involve complex interactions between various actors that take place over multiple spatial and temporal scales [16, 17, 22, 23, 32, 37, 50, 53]. While discrete models that track the evolution of individual cells are able to capture the biophysical rules proposed from biological intuition, the rapid increase in computational costs with the number of cells and the difficulties encountered with model calibration place limitations on discrete models. On the other hand, continuum models, which group multiple characteristic properties into one or more phenomenological parameters, tend to be too coarse at resolving the microstructures occurring at the level of individual cells, but in exchange they offer a tractable phenomenological description

of the key dynamics for which mathematical analysis and numerical simulations can be carried out.

Within the class of continuum models, we focus on the category of diffuse interface models, also commonly known as phase field models [9]. Although it is natural to view interfaces separating different components of matter as idealized hypersurfaces with zero thickness, these so-called sharp interface models break down when the interface experiences topological changes. The diffuse interface models replace this hypersurface description of the interface with a thin layer where microscopic mixing of the macroscopically distinct components of matter are allowed. This not only yields systems of equations that are better amenable to further analysis, but topological changes of the interface can also be handled naturally (see e.g., [9, 48]).

In continuum models, the ensemble of cells is often assumed to be tightly packed and move together with some averaged mixture velocity. Grounded in the observation that the morphology of certain cells and tissues behaves akin to viscous fluids [7, 10, 22, 52], many models treat the cells as viscous incompressible fluids, leading to a mathematical consideration with the Navier–Stokes equations or Darcy's law for the macroscopic cellular velocity. The latter has been a popular choice for tumour growth (see, for example, [17, 23, 32, 63] and the references therein), which treats the cells as viscous inertia-less fluids, similar in spirit to the approach taken in [8].

In this work, we include inertia effects into the modelling, and obtain a Navier–Stokes system for the mixture velocity. More precisely, we derive a thermodynamically consistent model for a two-component fluid mixture, which allows for mass transfer between the two components, in the presence of a chemical species that is subject to diffusion and other transport mechanisms such as convection and chemotaxis. The general model is a Navier–Stokes–Cahn–Hilliard system of the following form:

$$\partial_t \rho + \operatorname{div}\left(\rho \boldsymbol{v} - \frac{\overline{\rho}_2 - \overline{\rho}_1}{2} m(\varphi)\nabla\mu\right) = \Gamma_1 + \Gamma_2, \tag{1.1a}$$

$$\operatorname{div}\boldsymbol{v} = \frac{\Gamma_1}{\overline{\rho}_1} + \frac{\Gamma_2}{\overline{\rho}_2}, \tag{1.1b}$$

$$\partial_t(\rho\boldsymbol{v}) + \operatorname{div}\left(\rho\boldsymbol{v}\otimes\boldsymbol{v} - \frac{\overline{\rho}_2 - \overline{\rho}_1}{2}m(\varphi)\boldsymbol{v}\otimes\nabla\mu\right) = -\nabla p + \operatorname{div}\left(2\eta(\varphi)\mathrm{D}\boldsymbol{v}\right) \tag{1.1c}$$

$$- \operatorname{div}\left(B\nabla\varphi\otimes\nabla\varphi\right),$$

$$\partial_t\varphi + \operatorname{div}\left(\varphi\boldsymbol{v} - m(\varphi)\nabla\mu\right) = \frac{\Gamma_2}{\overline{\rho}_2} - \frac{\Gamma_1}{\overline{\rho}_1}, \tag{1.1d}$$

$$\mu = A\Psi'(\varphi) - B\Delta\varphi + N_\varphi, \tag{1.1e}$$

$$\partial_t\sigma + \operatorname{div}\left(\sigma\boldsymbol{v} - n(\varphi)\nabla N_\sigma\right) = S. \tag{1.1f}$$

Here, $\overline{\rho}_i$, $i = 1, 2$, is the constant mass density of a pure component of fluid i, $\boldsymbol{v}$ is the volume-averaged velocity of the mixture, $\mathrm{D}\boldsymbol{v} = \frac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^\top)$ is the rate of deformation tensor, p denotes the pressure and ρ and η denote the (averaged) density and viscosity of the binary mixture, respectively. The order parameter φ is the difference in volume fractions, with $\{\varphi = -1\}$ representing fluid 1 and $\{\varphi = 1\}$ representing fluid 2. In terms of φ, a typical form of the averaged density ρ is given by the affine function

$\rho(\varphi) = \frac{\bar{\rho}_2 - \bar{\rho}_1}{2}\varphi + \frac{\bar{\rho}_2 + \bar{\rho}_1}{2}$. μ stands for the chemical potential associated to φ, in which the function Ψ' is the derivative of a potential Ψ that has two equal minima at ± 1. The constants $A, B > 0$ are related to the surface tension and the thickness of the interfacial layers, while $m(\varphi)$ denotes a non-negative mobility. The functions Γ_i, $i = 1, 2$, model the mass transfer of fluid i. The density of a chemical species is given by σ, and S models a reaction term for σ. The mobility of σ is given by $n(\varphi) \geqslant 0$, and the functions N_φ and N_σ are the partial derivatives of the chemical free energy density $N(\varphi, \sigma)$ with respect to φ and σ, respectively. In the analysis below, we will take $N(\varphi, \sigma)$ as

$$N(\varphi, \sigma) = \frac{1}{2}|\sigma|^2 + \chi\sigma(1 - \varphi) \quad \Longrightarrow \quad N_\varphi = -\chi\sigma, \quad N_\sigma = \sigma + \chi(1 - \varphi), \qquad (1.2)$$

for a non-negative constant χ (see, e.g., [32]).

Equations (1.1a) and (1.1c) are the mass and momentum balance for the fluid mixture, respectively, while (1.1b) relates the divergence of the volume-averaged velocity to the mass transfer terms. Equations (1.1d) and (1.1e) constitute a convective Cahn–Hilliard system for the order parameter φ, and equation (1.1f) is a convection–reaction–diffusion equation for the chemical density σ. Under suitable boundary conditions, the above model satisfies an energy identity of the form:

$$\frac{d}{dt}\int_\Omega e \, dx + \int_\Omega \left(m(\varphi)|\nabla\mu|^2 + n(\varphi)|\nabla N_\sigma|^2 + 2\eta(\varphi)|\mathrm{D}v|^2 \right) dx$$

$$= -\int_\Omega U_v \left(\varphi\mu + \sigma N_\sigma + \frac{\bar{\rho}_2 + \bar{\rho}_1}{2}\frac{|v|^2}{2} - p - A\Psi(\varphi) - \frac{B}{2}|\nabla\varphi|^2 - N(\varphi, \sigma) \right) dx$$

$$+ \int_\Omega \left[\left(\mu - \frac{\bar{\rho}_2 - \bar{\rho}_1}{2}\frac{|v|^2}{2} \right) U_\varphi + N_\sigma S \right] dx, \qquad (1.3)$$

where the total energy density

$$e = \frac{\rho}{2}|v|^2 + A\Psi(\varphi) + \frac{B}{2}|\nabla\varphi|^2 + N(\varphi, \sigma)$$

is the sum of the kinetic energy density $\frac{\rho}{2}|v|^2$, the chemical free energy density $N(\varphi, \sigma)$ and the Ginzburg–Landau energy density $A\Psi(\varphi) + \frac{B}{2}|\nabla\varphi|^2$, and

$$U_v := \frac{\Gamma_1}{\bar{\rho}_1} + \frac{\Gamma_2}{\bar{\rho}_2}, \quad U_\varphi := \frac{\Gamma_2}{\bar{\rho}_2} - \frac{\Gamma_1}{\bar{\rho}_1}.$$

System (1.1) is a generalization of the model derived by Abels *et al.* [5] for two-phase flow with unmatched densities, where we account for mass transfer between the fluids and coupling to the dynamics of a chemical species. Furthermore, (1.1) can also be seen as the Navier–Stokes analogue of the general Cahn–Hilliard–Darcy model derived in Garcke *et al.* [32]. In Section 2.5.5, we briefly discuss how to obtain Darcy's law from the momentum equation by an averaging procedure (see [18] for a simpler situation), which allows us to formally recover Cahn–Hilliard–Darcy models from (1.1). To the authors' best knowledge, a similar Navier–Stokes–Cahn–Hilliard system was first derived by Sitka [53] for modelling tumour growth with chemotaxis, which can be obtained from the more

general model (1.1)–(1.2) with the following specific choice of mass transfer terms:

$$\Gamma_2 = -\Gamma_1, \tag{1.4}$$

so that

$$U_v = \alpha \Gamma_2, \ U_\varphi = \bar{\rho}_S \Gamma_2 \ \text{ with } \alpha := \frac{1}{\bar{\rho}_2} - \frac{1}{\bar{\rho}_1}, \ \bar{\rho}_S := \frac{1}{\bar{\rho}_2} + \frac{1}{\bar{\rho}_1}. \tag{1.5}$$

The choice $\Gamma_2 = -\Gamma_1$ stands for the case of zero excess of total mass, as any mass loss by fluid 1 is equal to the mass gained by fluid 2, and vice versa. Furthermore, by integrating (1.1a) over the physical domain Ω and under suitable boundary conditions, the total mass $\int_\Omega \rho \, dx$ is conserved. For the modelling of tumour growth, Sitka [53], and also [16, 17, 23, 32, 50, 63], considered the following biologically relevant choices:

$$\Gamma_2 = \frac{1}{2} \left(\mathcal{G}\sigma - \mathcal{A} \right) (\varphi + 1), \quad S = -\frac{1}{2} \mathcal{C}\sigma(\varphi + 1),$$

where the non-negative constants $\mathcal{G}$, $\mathcal{A}$ and $\mathcal{C}$ correspond to the constant proliferation, apoptosis and nutrient consumption rates, respectively. These mass transfer terms model the evolution of a tumour at its early stage, whose growth is proportional to the local chemical density, and experiences cell death at a constant rate. The prefactor $(\varphi + 1)$ in Γ_2 and S ensure that these source terms only affect the tumour, which is given as the region $\{\varphi = 1\}$, and not the healthy tissues $\{\varphi = -1\}$. Next, let us mention that for the choice (1.2) of the chemical free energy density N, the fluxes for φ and σ are given by

$$\boldsymbol{q}_\varphi := -m(\varphi)\nabla\mu = m(\varphi)\chi\nabla\sigma - m(\varphi)\nabla \left(A\Psi'(\varphi) - B\Delta\varphi \right),$$
$$\boldsymbol{q}_\sigma := -n(\varphi)\nabla N_\sigma = n(\varphi)\chi\nabla\varphi - n(\varphi)\nabla\sigma.$$

As pointed out in [32], the parameter χ is related to transport mechanisms such as chemotaxis (movement of fluid 2 towards high regions of σ) and active transport (establishing a persistent concentration difference between the different fluid components even against the chemical concentration gradient). The former is due to the fact that the term $m(\varphi)\chi\nabla\sigma$ in the flux $\boldsymbol{q}_\varphi$ points in the direction of increasing σ, and the latter is due to the fact that the term $n(\varphi)\chi\nabla\varphi$ in the flux $\boldsymbol{q}_\sigma$ points in the direction of increasing φ. Note that $\nabla\varphi$ is only non-zero in the vicinity of the interface between the two fluids, as the order parameter takes distinct constant values in the fluid regions, and hence we expect that the term $\text{div} \left(n(\varphi)\chi\nabla\varphi \right)$ in (1.1f), with the choice (1.2) for N, attempts to drive the chemical species towards fluid 2, i.e., the region $\{\varphi = 1\}$, leading to a higher chemical density in fluid 2 than in fluid 1 near the interface. We refer the reader to [32, Section 5, Figs. 9 and 10] for numerical simulations investigating this mechanism, and to [31, 32] for a discussion on how to decouple the chemotaxis and active transport mechanisms as at present they are both related to the parameter χ.

As a first step toward the mathematical analysis of system (1.1), we will study the special case of matched densities $\bar{\rho}_1 = \bar{\rho}_2 = 1$ and zero excess of total mass $\Gamma_1 = -\Gamma_2$,

for which the system (1.1) reduces to

$$\partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p + \operatorname{div}\left(2\eta(\varphi)\mathrm{D}\boldsymbol{v} - B\nabla\varphi \otimes \nabla\varphi\right), \tag{1.6a}$$

$$\operatorname{div} \boldsymbol{v} = 0, \tag{1.6b}$$

$$\partial_t \varphi + \boldsymbol{v} \cdot \nabla\varphi = \operatorname{div}\left(m(\varphi)\nabla\mu\right) + 2\Gamma_2, \tag{1.6c}$$

$$\mu = A\Psi'(\varphi) - B\Delta\varphi - \chi\sigma, \tag{1.6d}$$

$$\partial_t \sigma + \boldsymbol{v} \cdot \nabla\sigma = \operatorname{div}\left(n(\varphi)\nabla(\sigma - \chi\varphi)\right) + S. \tag{1.6e}$$

We remark that obtaining useful a priori estimates from the energy identity (1.3) for the general model (1.1) turns out to be rather complicated due to the source terms $U_v |\boldsymbol{v}|^2$, $U_\varphi |\boldsymbol{v}|^2$ involving the velocity and $U_v p$ involving the pressure. Unlike in the analytical treatment of the Cahn–Hilliard–Darcy variant with source terms performed in [28, 29, 40], useful estimates for the pressure p involving the left-hand side of (1.3) seem not available a priori in the case with the Navier–Stokes equations. Hence, as a first contribution to the mathematical treatment of Navier–Stokes–Cahn–Hilliard systems with mass transfer and chemotaxis, we focus on the case of zero excess of total mass (1.4) and matched densities, so that $\alpha = 0$ and $U_v = 0$ in (1.5). We leave the analysis of the general model for future research.

The above approach to model chemotaxis bears both similarities and differences to the classical approach of Keller and Segel [41]. In the context of coupling Navier–Stokes flow with chemical chemotaxis, the mathematical model by Tuval *et al.* [56] for oxygen-driven bacteria swimming in viscous incompressible fluids has been intensively studied by many authors, see for example [13, 14, 19, 20, 39, 44, 46, 60–62, 64, 65]. For the oxygen density c, bacteria density n, fluid velocity $\boldsymbol{u}$ and pressure p, the model in [56] reads as

$$\begin{aligned}
\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= -\nabla p + \eta\Delta\boldsymbol{u} + \boldsymbol{g}, \\
\operatorname{div} \boldsymbol{u} &= 0, \\
\partial_t c + \boldsymbol{u} \cdot \nabla c &= D_c \Delta c - n f(c), \\
\partial_t n + \boldsymbol{u} \cdot \nabla n &= D_n \Delta n - \operatorname{div}\left(r(c)n\nabla c\right),
\end{aligned} \tag{1.7}$$

where D_c and D_n are the diffusivities of c and n, respectively, $r(c)$ denotes the chemotactic sensitivity, $f(c)$ is a consumption rate of the chemical by the cells and $\boldsymbol{g}$ accounts for external forces such as gravity or buoyancy forces.

Writing (1.1*d*), (1.1*e*) and (1.1*f*) (with the specific choice (1.2)) as

$$\begin{aligned}
\partial_t \varphi + \operatorname{div}(\varphi\boldsymbol{v}) &= \operatorname{div}\left(m(\varphi)\nabla(A\Psi'(\varphi) - B\Delta\varphi)\right) - \operatorname{div}\left(m(\varphi)\chi\nabla\sigma\right) + U_\varphi, \\
\partial_t \sigma + \operatorname{div}(\sigma\boldsymbol{v}) &= \operatorname{div}\left(n(\varphi)\nabla\sigma\right) - \operatorname{div}\left(n(\varphi)\chi\nabla\varphi\right) + S,
\end{aligned} \tag{1.8}$$

then, we have a correspondence between σ and c as the chemoattractant, between φ and n as the variable exhibiting the chemotactic movement, between $\boldsymbol{v}$ and $\boldsymbol{u}$ as the fluid velocity and between the terms $\operatorname{div}(m(\varphi)\chi\nabla\sigma)$ and $\operatorname{div}(r(c)n\nabla c)$ as the mechanism for chemotaxis. The main differences between (1.6) (with the choice (1.2)) and (1.7) are the appearance of a cross-diffusion type term $-\operatorname{div}(n(\varphi)\chi\nabla\varphi)$ in the equation of σ accounting for the active transport mechanism, and the fact that φ is subject to a fourth-order diffusion operator. Besides, (1.6) is essentially a two-phase model.

Last, let us mention that the models derived here, and also in [5, 32], utilizes a volume-averaged velocity, which is in contrast to the models of [16,17,23,42,48,50,63] that employ a mass-averaged velocity. In the two-phase model of Lowengrub and Truskinovsky [48], the mass-averaged velocity is not solenoidal (with mismatched densities), whereas in the two-phase model of Abels, Garcke and Grün [5], the volume-averaged velocity is divergence-free, which seems to be an advantageous property in the analytical treatment as the pressure variable can be eliminated. In the present setting with mass transfer between the two components in the Navier–Stokes flow, the velocity is not divergence-free in general. However, as we will present in Section 2.8 below, the corresponding model derived with a mass-averaged velocity has a much more complicated expression than (1.1b) for the divergence equation. Another feature of the model with a mass-averaged velocity is that the pressure appears explicitly in the equation for the chemical potential μ.

The outline and contributions of the present work are as follows:

- In Section 2, we derive the general model (1.1) from balance laws. Under specific choices of the source terms and free energy we obtain the Navier–Stokes analogue of the models in Garcke *et al.* [32]. The sharp interface limit $\varepsilon \to 0$ of (1.1) with the choice $A = \frac{\beta}{\varepsilon}$, $B = \beta\varepsilon$ is also stated, which can be deduced following the formally matched asymptotic analysis presented in [5, 18, 32].

- In Section 3, we study the simplified model variant (1.6) and introduce preliminary results that will be used in our analysis, as well as a summary of our main mathematical results.

- In Section 4, we establish the global existence of weak solutions in two and three dimensions under general assumptions on the source terms Γ and S, the potential Ψ, the viscosity η and the mobilities m and n. The assertion is given in Theorem 1.

- In Section 5, under additional assumptions on the initial conditions, the source term Γ, the mobilities m and n and the viscosity η, we prove the global existence of strong solutions to (1.6) in two dimensions. The assertion is given in Theorem 2.

- In Section 6, we show the continuous dependence of strong solutions to (1.6) in two dimensions on the initial data and source terms. This is stated in Theorem 3, and the proof is based on a similar procedure used in [30, 31], which allow us to deduce continuous dependence for the chemical potential μ in $L^2(0, T; L^2(\Omega))$ and for the order parameter φ in $L^2(0, T; H^2(\Omega))$ as well. In particular, thanks to the regularities of the strong solutions, the continuous dependence result is valid for variable fluid viscosity and mobilities.

2 Model derivation

2.1 Balance laws

We begin with the following modelling assumption: in a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, there is a two-component mixture of immiscible fluids and a chemical species. Let $\bar{\rho}_i$, $i = 1, 2$, denote the constant mass density of a pure component i, while ρ_i denote the actual mass of the component i per volume in the mixture. We define the volume

fraction $u_i := \rho_i/\overline{\rho}_i$ for $i = 1, 2$, which lies in the physical interval $[0, 1]$. Assuming the excess volume of mixing is zero, i.e., the fluids behave like a simple mixture, it holds that

$$u_1 + u_2 = 1. \tag{2.1}$$

We define the mass density ρ of the mixture as $\rho = \rho_1 + \rho_2$, and introduce the volume-averaged velocity:

$$\boldsymbol{v} := u_1\boldsymbol{v}_1 + u_2\boldsymbol{v}_2, \tag{2.2}$$

where $\boldsymbol{v}_i$, $i = 1, 2$, is the individual velocity for component i.

The balance laws for mass and linear momentum are given as

$$\partial_t \rho_i + \text{div}\,(\rho_i \boldsymbol{v}_i) - \Gamma_i = 0, \tag{2.3a}$$

$$\partial_t(\rho \boldsymbol{v}) + \text{div}\,(\rho \boldsymbol{v} \otimes \boldsymbol{v}) - \text{div}\,(\boldsymbol{S} - p\text{I}) = 0, \tag{2.3b}$$

where Γ_i denotes a source term for component i, $\boldsymbol{S}$ is a tensor yet to be identified, p denotes the pressure, I is the identity tensor, and for two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$, the tensor product $\boldsymbol{a} \otimes \boldsymbol{b}$ is defined as $\boldsymbol{a} \otimes \boldsymbol{b} = (a_i b_j)_{1 \leqslant i, j \leqslant d}$. Furthermore, the divergence of a tensor A is taken row-wise. Note that upon dividing (2.3a) by $\overline{\rho}_i$ and using (2.1) and (2.2), we obtain

$$\text{div}\,\boldsymbol{v} = \frac{\Gamma_1}{\overline{\rho}_1} + \frac{\Gamma_2}{\overline{\rho}_2} =: U_v. \tag{2.4}$$

Introducing the fluxes

$$\boldsymbol{J}_i := \rho_i(\boldsymbol{v}_i - \boldsymbol{v}), \quad \boldsymbol{J} := \boldsymbol{J}_1 + \boldsymbol{J}_2, \quad \boldsymbol{J}_\varphi := \frac{\boldsymbol{J}_2}{\overline{\rho}_2} - \frac{\boldsymbol{J}_1}{\overline{\rho}_1}, \quad \boldsymbol{K} := \frac{\boldsymbol{J}_1}{\overline{\rho}_1} + \frac{\boldsymbol{J}_2}{\overline{\rho}_2}. \tag{2.5}$$

Then, by the definition of $\boldsymbol{K}$, $\boldsymbol{J}_i$ and the volume-averaged velocity $\boldsymbol{v}$, it holds that

$$\boldsymbol{K} = u_1(\boldsymbol{v}_1 - \boldsymbol{v}) + u_2(\boldsymbol{v}_2 - \boldsymbol{v}) = \boldsymbol{v} - (u_1 + u_2)\boldsymbol{v} = \boldsymbol{0}, \tag{2.6}$$

which implies

$$\boldsymbol{J} = \boldsymbol{J}_1 + \boldsymbol{J}_2 = \frac{\overline{\rho}_2 - \overline{\rho}_1}{2}\boldsymbol{J}_\varphi + \frac{\overline{\rho}_2 + \overline{\rho}_1}{2}\boldsymbol{K} = \frac{\overline{\rho}_2 - \overline{\rho}_1}{2}\boldsymbol{J}_\varphi.$$

From (2.3a), we deduce that

$$\partial_t u_i + \text{div}\,(u_i \boldsymbol{v}) + \frac{1}{\overline{\rho}_i}\text{div}\,\boldsymbol{J}_i = \frac{\Gamma_i}{\overline{\rho}_i}. \tag{2.7}$$

Defining the order parameter φ as the difference in volume fractions

$$\varphi := u_2 - u_1, \tag{2.8}$$

so that fluid 2 is represented by the region $\{\varphi = 1\}$ and fluid 1 is represented by the region $\{\varphi = -1\}$, we can deduce from subtracting (2.7) for $i = 1$ from (2.7) for $i = 2$

that

$$\partial_t\varphi + \text{div}\,(\varphi\boldsymbol{v}) + \text{div}\,\boldsymbol{J}_\varphi = \frac{\Gamma_2}{\overline{\rho}_2} - \frac{\Gamma_1}{\overline{\rho}_1} =: U_\varphi. \tag{2.9}$$

Furthermore, on the one hand, the mass density ρ can be expressed as an affine function of φ:

$$\rho(\varphi) = \overline{\rho}_1 u_1 + \overline{\rho}_2 u_2 = \frac{\overline{\rho}_2 - \overline{\rho}_1}{2}\varphi + \frac{\overline{\rho}_2 + \overline{\rho}_1}{2}, \tag{2.10}$$

and on the other hand, by adding (2.3a) for $i = 1, 2$ and using (2.5), we obtain a balance law for ρ:

$$\partial_t\rho + \text{div}\,(\rho\boldsymbol{v}) + \text{div}\,\boldsymbol{J} = \Gamma_1 + \Gamma_2 =: \Theta \quad \text{with} \quad \boldsymbol{J} = \frac{\overline{\rho}_2 - \overline{\rho}_1}{2}\boldsymbol{J}_\varphi. \tag{2.11}$$

Remark 2.1 *It is straightforward to verify that* (2.11) *is equivalent to* (2.9) *under the relation* (2.10). *Thus, in the sequel we will solve the order parameter φ and determine the mass density ρ using the relation* (2.10), *instead of considering the balance law* (2.11) *for ρ itself.*

Besides, the density of the chemical species present in Ω, denoted by σ, satisfies the following balance law:

$$\partial_t\sigma + \text{div}\,(\sigma\boldsymbol{v}) + \text{div}\,\boldsymbol{J}_\sigma = S, \tag{2.12}$$

for some source term S and flux $\boldsymbol{J}_\sigma$ that we will determine later.

The prototype model consists of equations (2.3b), (2.4), (2.9) and (2.12), and we will employ the Lagrange multiplier method of Müller and Liu (see [43, Chapter 7] and [5, Section 2.2]) to determine the constitutive assumptions on S, $\boldsymbol{J}_\varphi$ and $\boldsymbol{J}_\sigma$ so that the resulting model is thermodynamically consistent.

2.2 Energy inequality

We postulate a general energy density of the form

$$e = \frac{\rho}{2}|\boldsymbol{v}|^2 + A\Psi(\varphi) + \frac{B}{2}|\nabla\varphi|^2 + N(\varphi, \sigma), \tag{2.13}$$

where A, B are positive constants and Ψ is a potential with equal minima at ± 1. The first term of e is the kinetic energy, and the second and third terms of e form the so-called Ginzburg–Landau energy, which accounts for the interfacial energy and unmixing tendencies. The last term of e is the free energy of the chemical species, in which we incorporate possible interactions with the fluids by introducing a dependence on φ. In contrast to [32], here we include the effects of inertia that leads to the appearance of the kinetic energy density in (2.13). We mention that this effect has also been considered in the thesis [53], but below we will derive a more general model.

We now derive the model based on a dissipation inequality for balance laws with source terms that has been used in [32, 34, 35, 51], see also [36, Chapter 62]. The second law of

thermodynamics in the isothermal situation can be formulated as a free energy inequality, which states that for all volumes $V(t) \subset \Omega$ transported by the fluid velocity, the following inequality has to hold:

$$\frac{d}{dt} \int_{V(t)} e \, dx + \int_{\partial V(t)} \boldsymbol{J}_e \cdot \boldsymbol{v} \, d\mathcal{H}^{d-1} - \int_{V(t)} c_\varphi U_\varphi + c_v U_v + c_S S \, dx \leqslant 0,$$

where $\boldsymbol{v}$ is the outer unit normal on $\partial V(t)$ and $\boldsymbol{J}_e$ is an energy flux yet to be determined. In the above, dx and $d\mathcal{H}^{d-1}$ denote integration with respect to the d dimensional Lebesgue measure and $(d-1)$ dimensional Hausdorff measure, respectively. The source terms U_φ, U_v and S carry with them an energy contribution of the form

$$\int_{V(t)} c_\varphi U_\varphi + c_v U_v + c_S S \, dx,$$

where the prefactors c_φ, c_v, c_S will be determined below.

By the Reynold transport theorem and divergence theorem [36], this leads to the following local inequality:

$$\partial_t e + \operatorname{div}(e\boldsymbol{v}) + \operatorname{div} \boldsymbol{J}_e - c_\varphi U_\varphi - c_v U_v - c_S S \leqslant 0. \tag{2.14}$$

Instead of asking (2.14) to hold only for variables $(\varphi, \sigma, \boldsymbol{v}, U_\varphi, U_v, S)$ that satisfy the prototype model equations (2.3b), (2.4), (2.9) and (2.12), the Lagrange multiplier method of Müller and Liu relaxes the constraint on the model equations by introducing Lagrange multipliers (see, e.g., [43, Chapter 7]), in our case λ_φ and λ_σ for (2.9) and (2.12), and instead it asks that the following inequality

$$\begin{aligned}
-\mathcal{D} := &\, \partial_t e + \operatorname{div}(e\boldsymbol{v}) + \operatorname{div} \boldsymbol{J}_e - c_\varphi U_\varphi - c_v U_v - c_S S \\
&- \lambda_\varphi \left(\partial_t \varphi + \operatorname{div}(\varphi \boldsymbol{v}) + \operatorname{div} \boldsymbol{J}_\varphi - U_\varphi \right) \\
&- \lambda_\sigma \left(\partial_t \sigma + \operatorname{div}(\sigma \boldsymbol{v}) + \operatorname{div} \boldsymbol{J}_\sigma - S \right) \leqslant 0
\end{aligned} \tag{2.15}$$

holds for arbitrary $(\varphi, \sigma, \boldsymbol{v}, U_\varphi, U_v, S, \partial_t^\bullet \varphi, \partial_t^\bullet \sigma)$, where the material derivative $\partial_t^\bullet f$ of a function f is defined as

$$\partial_t^\bullet f := \partial_t f + \boldsymbol{v} \cdot \nabla f.$$

We introduce the notations

$$N_\varphi := \frac{\partial N}{\partial \varphi}, \quad N_\sigma := \frac{\partial N}{\partial \sigma}, \quad \mu := A\Psi'(\varphi) - B\Delta\varphi + N_\varphi,$$

and recall the identities

$$\partial_t^\bullet \varphi \nabla \varphi = \partial_t \varphi \nabla \varphi + (\nabla\varphi \otimes \nabla\varphi)\boldsymbol{v}, \tag{2.16}$$

$$\nabla\varphi \cdot \partial_t^\bullet(\nabla\varphi) = \operatorname{div}(\partial_t^\bullet \varphi \nabla \varphi) - \partial_t^\bullet \varphi \Delta\varphi - \nabla\boldsymbol{v} : (\nabla\varphi \otimes \nabla\varphi), \tag{2.17}$$

where for vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$, $\nabla\boldsymbol{a} : \nabla\boldsymbol{b} := \sum_{i,j=1}^d \partial_i a_j \partial_i b_j$. Then, using (2.3b), (2.4) and (2.11), we compute that

$$\partial_t^\bullet \left(\frac{\rho}{2} |\boldsymbol{v}|^2 \right) = (\partial_t(\rho\boldsymbol{v}) + \mathrm{div}\,(\rho\boldsymbol{v} \otimes \boldsymbol{v}) - \rho(\mathrm{div}\,\boldsymbol{v})\boldsymbol{v}) \cdot \boldsymbol{v} - \frac{1}{2} |\boldsymbol{v}|^2 \partial_t^\bullet \rho$$

$$= \mathrm{div}\,(\boldsymbol{S} - p\mathrm{I}) \cdot \boldsymbol{v} + \frac{1}{2} |\boldsymbol{v}|^2 (\mathrm{div}\,\boldsymbol{J} - \Theta - \rho U_v)$$

$$= \mathrm{div}\,\left((\boldsymbol{S}^\top - p\mathrm{I})\boldsymbol{v} + \frac{1}{2} |\boldsymbol{v}|^2 \boldsymbol{J} \right) - \nabla\boldsymbol{v} : (\boldsymbol{S} + \boldsymbol{v} \otimes \boldsymbol{J}) \tag{2.18}$$

$$- \frac{1}{2} |\boldsymbol{v}|^2 (\Theta + \rho U_v) + p U_v.$$

Furthermore, using the relations

$$\Gamma_1 = \frac{\overline{\rho}_1}{2}(U_v - U_\varphi), \quad \Gamma_2 = \frac{\overline{\rho}_2}{2}(U_v + U_\varphi) \quad \text{so that} \quad \Theta = \frac{\overline{\rho}_2 + \overline{\rho}_1}{2} U_v + \frac{\overline{\rho}_2 - \overline{\rho}_1}{2} U_\varphi,$$

we can express (2.18) as

$$\partial_t^\bullet \left(\frac{\rho}{2} |\boldsymbol{v}|^2 \right) = \mathrm{div}\,\left((\boldsymbol{S} - p\mathrm{I})^\top \boldsymbol{v} + \frac{1}{2} |\boldsymbol{v}|^2 \boldsymbol{J} \right) - \nabla\boldsymbol{v} : (\boldsymbol{S} + \boldsymbol{v} \otimes \boldsymbol{J})$$

$$- \left(\left(\rho + \frac{\overline{\rho}_2 + \overline{\rho}_1}{2} \right) \frac{1}{2} |\boldsymbol{v}|^2 - p \right) U_v - \frac{1}{2} |\boldsymbol{v}|^2 \frac{\overline{\rho}_2 - \overline{\rho}_1}{2} U_\varphi.$$

Then, after a long but straightforward calculation, we obtain from (2.15) that

$$-\mathcal{D} = \mathrm{div}\,\left(\boldsymbol{J}_e - \lambda_\varphi \boldsymbol{J}_\varphi - \lambda_\sigma \boldsymbol{J}_\sigma + (\boldsymbol{S}^\top - p\mathrm{I})\boldsymbol{v} + \frac{1}{2} |\boldsymbol{v}|^2 \boldsymbol{J} + B\partial_t^\bullet \varphi \nabla\varphi \right) + \boldsymbol{J}_\varphi \cdot \nabla\lambda_\varphi$$

$$+ \boldsymbol{J}_\sigma \cdot \nabla\lambda_\sigma + (\mu - \lambda_\varphi) \partial_t^\bullet \varphi + (N_\sigma - \lambda_\sigma) \partial_t^\bullet \sigma + (\lambda_\sigma - c_S)S$$

$$+ \left(\lambda_\varphi - c_\varphi - \frac{\overline{\rho}_2 - \overline{\rho}_1}{2} \frac{1}{2} |\boldsymbol{v}|^2 \right) U_\varphi - \nabla\boldsymbol{v} : (\boldsymbol{S} + \boldsymbol{v} \otimes \boldsymbol{J} + B\nabla\varphi \otimes \nabla\varphi) \tag{2.19}$$

$$+ U_v \left(f(\varphi, \nabla\varphi, \sigma) - c_v - \lambda_\varphi \varphi - \lambda_\sigma \sigma - \frac{\overline{\rho}_2 + \overline{\rho}_1}{2} \frac{1}{2} |\boldsymbol{v}|^2 + p \right),$$

where we recall that $f(\varphi, \nabla\varphi, \sigma) = A\Psi(\varphi) + \frac{B}{2} |\nabla\varphi|^2 + N(\varphi, \sigma)$ denotes the free energy.

2.3 Constitutive assumptions and the general model

In order for $-\mathcal{D} \leqslant 0$ to hold for arbitrary $(\varphi, \sigma, \boldsymbol{v}, U_\varphi, U_v, S, \partial_t^\bullet \varphi, \partial_t^\bullet \sigma)$, where $-\mathcal{D}$ is given in (2.19), we make the following constitutive assumptions:

$$\lambda_\varphi = \mu, \quad \lambda_\sigma = N_\sigma, \tag{2.20a}$$

$$\boldsymbol{J}_e = \mu\boldsymbol{J}_\varphi + N_\sigma\boldsymbol{J}_\sigma - \boldsymbol{S}^\top\boldsymbol{v} + p\boldsymbol{v} - \frac{1}{2} |\boldsymbol{v}|^2 \frac{\overline{\rho}_2 - \overline{\rho}_1}{2} \boldsymbol{J}_\varphi - B\partial_t^\bullet \varphi \nabla\varphi, \tag{2.20b}$$

$$\boldsymbol{S} = 2\eta(\varphi)\mathrm{D}\boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{J} - B\nabla\varphi \otimes \nabla\varphi, \tag{2.20c}$$

$$\boldsymbol{J}_\varphi = -m(\varphi)\nabla\mu, \quad \boldsymbol{J}_\sigma = -n(\varphi)\nabla N_\sigma, \tag{2.20d}$$

$$c_S = N_\sigma, \quad c_\varphi = \mu - \frac{\overline{\rho}_2 - \overline{\rho}_1}{2} \frac{1}{2} |\boldsymbol{v}|^2, \tag{2.20e}$$

$$c_v = A\Psi(\varphi) + \frac{B}{2} |\nabla\varphi|^2 + N(\varphi, \sigma) - \mu\varphi - N_\sigma\sigma - \frac{\overline{\rho}_2 + \overline{\rho}_1}{2} \frac{1}{2} |\boldsymbol{v}|^2 + p, \tag{2.20f}$$

where $\mathbf{D}\boldsymbol{v} := \frac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^\top)$ is the rate of deformation tensor, $\eta(\varphi)$ is the mixture viscosity and $m(\varphi), n(\varphi)$ are positive mobilities. Equations (2.20a)–(2.20f) yield that the right-hand side of (2.19) is non-positive, namely,

$$-\mathcal{D} = -m(\varphi)\,|\nabla\mu|^2 - n(\varphi)\,|\nabla N_\sigma|^2 - 2\eta(\varphi)\,|\mathbf{D}\boldsymbol{v}|^2 \leqslant 0.$$

We remark that the constitutive assumption (2.20b) for the energy flux $\boldsymbol{J}_e$ is chosen so that the divergence term in (2.19) vanishes. The term $(p\mathbf{I} - \boldsymbol{S}^\top)\boldsymbol{v}$ accounts for energy change due to work done by macroscopic stresses (see [5]), while energy flux due to mass diffusion are described by the terms $\mu\boldsymbol{J}_\varphi$ and $N_\sigma\boldsymbol{J}_\sigma$. Changes of kinetic energy due to mass diffusion is given by $\frac{1}{2}\frac{\overline{\rho}_2-\overline{\rho}_1}{2}\,|\boldsymbol{v}|^2\,\boldsymbol{J}_\varphi = \frac{1}{2}\,|\boldsymbol{v}|^2\,\boldsymbol{J}$, and the term $\partial_t^\bullet\varphi\nabla\varphi$ arises from the moving phase boundaries.

The above constitutive assumptions lead to the following Navier–Stokes–Cahn–Hilliard model with mass transfer and chemical coupling:

$$\operatorname{div}\boldsymbol{v} = U_v = \frac{\Gamma_1}{\overline{\rho}_1} + \frac{\Gamma_2}{\overline{\rho}_2}, \tag{2.21a}$$

$$\rho(\varphi) = \frac{\overline{\rho}_2 - \overline{\rho}_1}{2}\varphi + \frac{\overline{\rho}_2 + \overline{\rho}_1}{2}, \tag{2.21b}$$

$$\partial_t(\rho\boldsymbol{v}) + \operatorname{div}\left(\rho\boldsymbol{v}\otimes\boldsymbol{v} - \frac{\overline{\rho}_2 - \overline{\rho}_1}{2}m(\varphi)\boldsymbol{v}\otimes\nabla\mu\right) = -\nabla p + \operatorname{div}(2\eta(\varphi)\mathbf{D}\boldsymbol{v}) \tag{2.21c}$$
$$- \operatorname{div}(B\nabla\varphi\otimes\nabla\varphi),$$

$$\partial_t\varphi + \operatorname{div}(\varphi\boldsymbol{v} - m(\varphi)\nabla\mu) = U_\varphi = \frac{\Gamma_2}{\overline{\rho}_2} - \frac{\Gamma_1}{\overline{\rho}_1}, \tag{2.21d}$$

$$\mu = A\Psi'(\varphi) - B\Delta\varphi + N_\varphi, \tag{2.21e}$$

$$\partial_t\sigma + \operatorname{div}(\sigma\boldsymbol{v} - n(\varphi)\nabla N_\sigma) = S. \tag{2.21f}$$

Formally, by integrating (2.19) over Ω, where we recall (2.15), the constitutive assumptions (2.20a)–(2.20f), and now φ and σ satisfy (2.9) and (2.12), we can obtain an energy identity that is given by

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \frac{\rho}{2}\,|\boldsymbol{v}|^2 + A\Psi(\varphi) + \frac{B}{2}\,|\nabla\varphi|^2 + N(\varphi,\sigma)\,dx \\
&+ \int_\Omega m(\varphi)\,|\nabla\mu|^2 + n(\varphi)\,|\nabla N_\sigma|^2 + 2\eta(\varphi)\,|\mathbf{D}\boldsymbol{v}|^2\,dx \\
&+ \int_\Omega U_v\left(\varphi\mu + \sigma N_\sigma + \frac{\overline{\rho}_2 + \overline{\rho}_1}{2}\frac{1}{2}\,|\boldsymbol{v}|^2 - p - A\Psi(\varphi) - \frac{B}{2}\,|\nabla\varphi|^2 - N(\varphi,\sigma)\right)dx \\
&- \int_\Omega \left(\mu - \frac{\overline{\rho}_2 - \overline{\rho}_1}{2}\frac{1}{2}\,|\boldsymbol{v}|^2\right)U_\varphi + N_\sigma S\,dx \\
&+ \int_{\partial\Omega}\left(p + \frac{\rho}{2}\,|\boldsymbol{v}|^2 + A\Psi(\varphi) + \frac{B}{2}\,|\nabla\varphi|^2 + N(\varphi,\sigma)\right)\boldsymbol{v}\cdot\boldsymbol{v} - n(\varphi)N_\sigma\partial_v N_\sigma\,d\mathcal{H}^{d-1} \\
&- \int_{\partial\Omega} m(\varphi)\left(\mu + \frac{1}{2}\,|\boldsymbol{v}|^2\frac{\overline{\rho}_2 - \overline{\rho}_1}{2}\right)\partial_v\mu + 2\eta(\varphi)(\mathbf{D}\boldsymbol{v})\boldsymbol{v}\cdot\boldsymbol{v} + B\partial_t\varphi\partial_v\varphi\,d\mathcal{H}^{d-1} = 0,
\end{aligned}
\tag{2.22}
$$

where $\partial_v f := \nabla f\cdot\boldsymbol{v}$ is the normal derivative of f on $\partial\Omega$.

2.4 Modified pressure and reformulation of the momentum equation

We now present three reformulations of the pressure and the corresponding momentum equation (2.21c). Associated to each reformulation is the prefactor $\lambda_v := -c_v$ that is multiplied with U_v in the energy identity (2.22). We consider, as in [32, Remark 2.1],

(1) $q := p + A\Psi(\varphi) + \dfrac{B}{2}|\nabla\varphi|^2,$

(2) $r := p + A\Psi(\varphi) + \dfrac{B}{2}|\nabla\varphi|^2 + N(\varphi,\sigma),$

(3) $s := p + A\Psi(\varphi) + \dfrac{B}{2}|\nabla\varphi|^2 + N(\varphi,\sigma) - \varphi\mu - N_\sigma\sigma,$

leading to the corresponding momentum equations

$$\partial_t(\rho\boldsymbol{v}) + \mathrm{div}\left(\rho\boldsymbol{v}\otimes\boldsymbol{v} - \frac{\overline{\rho}_2 - \overline{\rho}_1}{2}m(\varphi)\boldsymbol{v}\otimes\nabla\mu - 2\eta(\varphi)\mathrm{D}\boldsymbol{v}\right)$$
$$= \begin{cases} -\nabla q + (\mu - N_\varphi)\nabla\varphi, \\ -\nabla r + \mu\nabla\varphi + N_\sigma\nabla\sigma, \\ -\nabla s - \varphi\nabla\mu - \sigma\nabla N_\sigma, \end{cases}$$

as well as the prefactors

$$\lambda_v = \begin{cases} \varphi\mu + \sigma N_\sigma + \dfrac{\overline{\rho}_2 + \overline{\rho}_1}{2}\dfrac{1}{2}|\boldsymbol{v}|^2 - N(\varphi,\sigma) - q, \\ \varphi\mu + \sigma N_\sigma + \dfrac{\overline{\rho}_2 + \overline{\rho}_1}{2}\dfrac{1}{2}|\boldsymbol{v}|^2 - r, \\ \dfrac{\overline{\rho}_2 + \overline{\rho}_1}{2}\dfrac{1}{2}|\boldsymbol{v}|^2 - s. \end{cases} \tag{2.23}$$

2.5 Reduction to special models

2.5.1 *Absence of the chemical substance*

In the absence of chemical substances, by setting $\sigma = 0$, $N(\varphi,\sigma) = 0$ and $S = 0$, we obtain a Navier–Stokes–Cahn–Hilliard system with source terms that is composed of (2.21a)–(2.21d) and

$$\mu = A\Psi'(\varphi) - B\Delta\varphi,$$

which can be seen as the Navier–Stokes analogue of [32, (2.35)]. Furthermore, if $\Gamma_1 = \Gamma_2 = 0$, then we obtain the Navier–Stokes–Cahn–Hilliard model of Abels, Garcke and Grün [5]. For the existence of global weak solutions to the Navier–Stokes–Cahn–Hilliard models, we refer to, for instance, [2–4, 6, 11, 26, 27, 59, 66], where the reformulation of the momentum equation with the modified pressure variable r has been used.

2.5.2 *Zero excess of total mass*

We consider the case

$$\Gamma_2 = -\Gamma_1 =: \Gamma.$$

Define

$$\alpha = \frac{1}{\overline{\rho}_2} - \frac{1}{\overline{\rho}_1}, \quad \rho_S := \frac{1}{\overline{\rho}_2} + \frac{1}{\overline{\rho}_1} \implies \Theta = 0, \quad U_v = \alpha\Gamma, \quad U_\varphi = \rho_S\Gamma.$$

This leads to the Navier–Stokes analogue of [32, (2.33)] that consists of equations (2.21*b*), (2.21*c*), (2.21*e*), (2.21*f*) and

$$\operatorname{div} \boldsymbol{v} = \alpha\Gamma, \tag{2.24a}$$

$$\partial_t\varphi + \operatorname{div}(\varphi\boldsymbol{v} - m(\varphi)\nabla\mu) = \rho_S\Gamma. \tag{2.24b}$$

Furthermore, in the case of equal densities $\overline{\rho}_1 = \overline{\rho}_2 = \rho_*$ so that $\rho = \rho_*$ is constant, then

$$\alpha = 0, \quad \rho_S = \frac{2}{\rho_*},$$

and the resulting system becomes

$$\operatorname{div} \boldsymbol{v} = 0, \tag{2.25a}$$

$$\rho_*\left(\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) = -\nabla p + \operatorname{div}\left(2\eta(\varphi)\mathbf{D}\boldsymbol{v} - B\nabla\varphi \otimes \nabla\varphi\right), \tag{2.25b}$$

$$\partial_t\varphi + \boldsymbol{v}\cdot\nabla\varphi = \operatorname{div}(m(\varphi)\nabla\mu) + \frac{2}{\rho_*}\Gamma, \tag{2.25c}$$

$$\mu = A\Psi'(\varphi) - B\Delta\varphi + N_\varphi, \tag{2.25d}$$

$$\partial_t\sigma + \boldsymbol{v}\cdot\nabla\sigma = \operatorname{div}(n(\varphi)\nabla N_\sigma) + S, \tag{2.25e}$$

which is the Navier–Stokes analogue of [32, (2.34)]. If in addition, we set $\boldsymbol{v} = \boldsymbol{0}$, then neglecting the first two equations of (2.25) leads to [32, (2.36)].

2.5.3 *Scaled zero excess of total mass*

Alternatively, we consider the case

$$\frac{\Gamma_2}{\overline{\rho}_2} = -\frac{\Gamma_1}{\overline{\rho}_1},$$

then it holds that

$$U_v = 0, \quad U_\varphi = \frac{2}{\overline{\rho}_2}\Gamma_2, \quad \Theta = \left(1 - \frac{\overline{\rho}_1}{\overline{\rho}_2}\right)\Gamma_2,$$

which leads to the model consisting of (2.21*b*), (2.21*c*), (2.21*e*), (2.21*f*) and

$$\operatorname{div} \boldsymbol{v} = 0, \tag{2.26a}$$

$$\partial_t\varphi + \operatorname{div}(\varphi\boldsymbol{v} - m(\varphi)\nabla\mu) = \frac{2}{\overline{\rho}_2}\Gamma_2. \tag{2.26b}$$

An interesting feature of this model is that the densities are allowed to be different, i.e., $\overline{\rho}_1 \neq \overline{\rho}_2$ while we retain a solenoidal velocity field.

2.5.4 *No mass exchange for fluid 1*

Setting

$$\Gamma_1 = 0, \quad \Gamma := \frac{\Gamma_2}{\bar{\rho}_2}$$

leads to the model consisting of (2.21*b*), (2.21*c*), (2.21*e*), (2.21*f*) and

$$\operatorname{div} \boldsymbol{v} = \Gamma, \tag{2.27a}$$

$$\partial_t \varphi + \operatorname{div}(\varphi \boldsymbol{v} - m(\varphi)\nabla\mu) = \Gamma. \tag{2.27b}$$

Neglecting σ, the resulting model is the Navier–Stokes analogue of [32, (2.43)]. Note that the source terms for the divergence equation (2.27*a*) and the equation (2.27*b*) are equal, and this feature has also appeared in the work of [40] (see [47, 57, 58] for the case $\Gamma = 0$).

2.5.5 *Recovering Darcy's law*

Darcy's law can be obtained from the momentum equation (2.21*c*) by performing a proper averaging procedure in a Hele–Shaw geometry, that is, the domain Ω is given as $\Omega = U \times [0, \delta]$ for some positive constant $\delta \ll 1$ and a bounded domain $U \subset \mathbb{R}^{d-1}$. Rescaling the pressure p and the constant B appropriately with δ, and employing formal asymptotic expansions of the variables $(\rho, \boldsymbol{v}, \varphi, \mu, \sigma, \Gamma_i)$ in δ, by examining the equations order by order in δ, one can derive the Cahn–Hilliard–Darcy model of [32] from the Navier–Stokes–Cahn–Hilliard system (2.21). For further details, we refer to the recent work [18], in which a Hele–Shaw–Cahn–Hilliard model is derived from the Navier–Stokes–Cahn–Hilliard model of [5], see also [49, Chapter 4].

2.6 Some analytical issues

2.6.1 *Restriction of the order parameter to physical values*

One of the main mathematical issues with the analysis of the general model (2.21) is to ensure that the order parameter φ stays in a physically reasonable interval, say $\varphi \in [-1, 1]$. This is due to the fact that the averaged mass density $\rho(\varphi)$, defined via the relation (2.10), can become negative if φ is outside the interval $[-1, 1]$, which in turn causes difficulties in the analysis of the momentum equation (2.21*c*). One possible approach is to consider the class of singular free energies, where the potential Ψ is non-smooth and finiteness of the integral $\int_\Omega \Psi(\varphi)\,dx$ implies that φ is indeed confined to a bounded interval. We point out that the logarithmic potential

$$\Psi_{\log}(s) = \frac{\theta}{2}((1+s)\log(1+s) + (1-s)\log(1-s)) - \frac{\theta_c}{2}s^2$$

for constants $0 < \theta_c < \theta$, and the obstacle potential

$$\Psi_{\mathrm{ob}}(s) = \begin{cases} \frac{1}{2}(1-s^2) & \text{if } s \in [-1, 1], \\ +\infty & \text{otherwise,} \end{cases}$$

would ensure φ stays in $(-1, 1)$ for the former and in $[-1, 1]$ for the latter. However, other technical issues arise with the use of a non-smooth free energy, e.g., the regularity of solutions [2, 11, 33]. Besides, a further detailed analysis is required when mass transfer terms are also included. We believe that the weak existence theory for the model (2.26) with scaled zero excess of total mass, which can be viewed as the model of [5] but including source terms for the Cahn–Hilliard system and a coupling to a convection–reaction–diffusion equation, can be treated with the techniques used in [3] by employing the logarithmic potential $\Psi_{\log}$.

2.6.2 *Boundary conditions for the velocity*

We now examine the boundary term involving the velocity $\boldsymbol{v}$ in the energy identity (2.22), which reads

$$\int_{\partial\Omega} \left(\frac{\rho}{2} |\boldsymbol{v}|^2 + p + A\Psi(\varphi) + \frac{B}{2} |\nabla\varphi|^2 + N(\varphi, \sigma) \right) \boldsymbol{v} \cdot \boldsymbol{\nu} - 2\eta(\varphi)(\mathrm{D}\boldsymbol{v})\boldsymbol{v} \cdot \boldsymbol{\nu} \, d\mathcal{H}^{d-1}. \quad (2.28)$$

Natural boundary conditions for the velocity are

$$\begin{aligned} \text{no-slip b.c.} \quad & \boldsymbol{v} = \boldsymbol{0} & \text{on } \partial\Omega, \\ \text{or free-slip b.c.} \quad & \boldsymbol{v} \cdot \boldsymbol{\nu} = 0 \quad \text{and} \quad \nabla \times (\boldsymbol{v} \times \boldsymbol{\nu}) = 0 & \text{on } \partial\Omega, \end{aligned}$$

so that (2.28) would vanish in both cases. However, integrating (2.21a) and applying the divergence theorem yields that

$$0 = \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{\nu} \, d\mathcal{H}^{d-1} = \int_{\Omega} \operatorname{div} \boldsymbol{v} \, dx = \int_{\Omega} U_{\boldsymbol{v}} \, dx, \quad (2.29)$$

which implies that $U_{\boldsymbol{v}}$ necessarily has to have zero mean over Ω. This is not an issue for the models (2.25) and (2.26), but for non-zero $U_{\boldsymbol{v}}$, analysis of the corresponding Cahn–Hilliard–Darcy model has utilized this compatibility condition, see for instance [28, 40]. In the case that the source term $U_{\boldsymbol{v}}$ is a function of the variables φ and σ, condition (2.29) may not hold in general, and thus alternative boundary conditions for the velocity have to be considered, see for example [29] for a Cahn–Hilliard–Darcy model with $U_{\boldsymbol{v}}$ depending on φ and σ, along with a Dirichlet (or Robin) boundary condition prescribed for the pressure.

2.6.3 *Remarks on the source terms*

To obtain useful a priori estimates from the energy identity (2.22), which are essential for the proof of existence of solutions, it suffices to control the source terms (neglecting the boundary term for the moment)

$$\begin{aligned} &\int_{\Omega} U_{\boldsymbol{v}} \left(\varphi\mu + \sigma N_{\sigma} + \frac{\overline{\rho}_2 + \overline{\rho}_1}{2} \frac{1}{2} |\boldsymbol{v}|^2 - p - A\Psi(\varphi) - \frac{B}{2} |\nabla\varphi|^2 - N(\varphi, \sigma) \right) dx \\ &\quad - \int_{\Omega} \left(\mu - \frac{\overline{\rho}_2 - \overline{\rho}_1}{2} \frac{1}{2} |\boldsymbol{v}|^2 \right) U_{\varphi} + N_{\sigma} S \, dx \end{aligned} \quad (2.30)$$

with the "good" part

$$\frac{d}{dt} \int_\Omega \frac{\rho}{2} |\boldsymbol{v}|^2 + A\Psi(\varphi) + \frac{B}{2} |\nabla\varphi|^2 + N(\varphi, \sigma)\, dx$$

$$+ \int_\Omega m(\varphi) |\nabla\mu|^2 + n(\varphi) |\nabla N_\sigma|^2 + 2\eta(\varphi) |\mathrm{D}\boldsymbol{v}|^2\, dx.$$

The fundamental difference between the source terms encountered in the previous works [15, 25, 28, 30, 31, 40] for Cahn–Hilliard/Cahn–Hilliard–Darcy systems and (2.30) is the appearance of $|\boldsymbol{v}|^2$ multiplied with U_v and U_φ. Suitable assumptions on the integrability of U_v and U_φ are required to control their products with the kinetic energy. Moreover, estimating (2.30) is further complicated by the presence of the term $U_v p$ involving the pressure. In contrast to Cahn–Hilliard–Darcy models, for which an estimate of the L^2-norm of the pressure can be obtained by studying Darcy's law as a second-order elliptic equation for the pressure (see [28, 29, 40]), in the present setting involving the Navier–Stokes equations, such estimates seem not available at first glance. However, we mention the work of Abels [1] for some results in this direction.

We also point out that the pressure reformulations in Section 2.4 do not entirely remove the above technical issues, as from (2.23), the source term U_v will always be multiplied with a nonlinear term containing $|\boldsymbol{v}|^2$ and the pressure. Furthermore, additional nonlinearities may appear in the boundary term (2.28) instead. For instance, consider the reformulation $s = p + A\Psi(\varphi) + \frac{B}{2} |\nabla\varphi|^2 + N(\varphi, \sigma) - \varphi\mu - N_\sigma\sigma$, so that (2.30) becomes

$$\int_\Omega U_v \left(\frac{\overline{\rho}_2 + \overline{\rho}_1}{2} \frac{1}{2} |\boldsymbol{v}|^2 - s \right) - \left(\mu - \frac{\overline{\rho}_2 - \overline{\rho}_1}{2} \frac{1}{2} |\boldsymbol{v}|^2 \right) U_\varphi - N_\sigma S\, dx,$$

but, in exchange, the boundary term (2.28) becomes

$$\int_{\partial\Omega} \left(\frac{\rho}{2} |\boldsymbol{v}|^2 + s + \varphi\mu + N_\sigma\sigma \right) \boldsymbol{v} \cdot \boldsymbol{v} - 2\eta(\varphi)(\mathrm{D}\boldsymbol{v})\boldsymbol{v} \cdot \boldsymbol{v}\, d\mathcal{H}^{d-1}.$$

In particular, if one considers non-standard boundary conditions for the velocity (so that U_v need not have mean zero), controlling terms such as $\varphi\mu$ and $N_\sigma\sigma$ become harder when they appear as part of the boundary terms. Hence, to allow for interesting effects (e.g., U_v does not have mean zero), there will be a delicate balance between choosing appropriate reformulations and boundary conditions to arrive at a tractable description of the model for which can be further analysed.

2.7 Sharp interface limit

Let us consider the general model (2.21) with the following choices:

$$A = \frac{\beta}{\varepsilon}, \quad B = \beta\varepsilon, \quad N(\varphi, \sigma) = \frac{1}{2} |\sigma|^2 + \chi\sigma(1 - \varphi), \qquad (2.31)$$

where $\beta > 0$ denotes the surface tension, $\varepsilon > 0$ is related to the thickness of the diffuse interface and $\chi \geqslant 0$ is the chemotaxis parameter. Furthermore, we allow the source terms $S = S(\varphi, \mu, \sigma)$, $\Gamma_1 = \Gamma_1(\varphi, \mu, \sigma)$ and $\Gamma_2 = \Gamma_2(\varphi, \mu, \sigma)$ to depend on φ, μ, σ but not on any

derivative. The motivation for considering a dependence on μ for the source terms comes from the works [37, 38]. Besides, we take a non-degenerate mobility $0 < n_0 \leqslant n(s)$ for all $s \in \mathbb{R}$.

From the formally matched asymptotic analysis performed in [5], different sharp interface models in the limit $\varepsilon \to 0$ can be derived from the Navier–Stokes–Cahn–Hilliard system for different choices of the mobility function $m(\varphi)$. To compare with the results in [32] we confine ourselves to the constant mobility $m(\varphi) = 1$. Furthermore, we rescale the potential Ψ such that

$$\int_{-1}^{1} \sqrt{2\Psi(s)}\, ds = 1,$$

and then the constant γ in [32, (3.17)] becomes 1.

Let $\Omega_1(t)$ denote the region for fluid 1 and $\Omega_2(t)$ denote the region for fluid 2, which are separated by a hypersurface $\Sigma(t)$. We denote the variables defined over Ω_1 with the subscript 1, and similarly for variables defined over Ω_2 with the subscript 2. For the sake of convenience, we use the following notations:

$$\rho = \begin{cases} \overline{\rho}_1 & \text{in } \Omega_1, \\ \overline{\rho}_2 & \text{in } \Omega_2, \end{cases} \quad \eta = \begin{cases} \eta(-1) & \text{in } \Omega_1, \\ \eta(+1) & \text{in } \Omega_2, \end{cases} \quad n = \begin{cases} n(-1) & \text{in } \Omega_1, \\ n(+1) & \text{in } \Omega_2, \end{cases}$$

$$\Gamma_i = \begin{cases} \Gamma_i(-1, \mu_1, \sigma_1) & \text{in } \Omega_1, \\ \Gamma_i(1, \mu_2, \sigma_2) & \text{in } \Omega_2, \end{cases} \quad S = \begin{cases} S(-1, \mu_1, \sigma_1) & \text{in } \Omega_1, \\ S(1, \mu_2, \sigma_2) & \text{in } \Omega_2, \end{cases}$$

when writing equations over the union $\Omega_1 \cup \Omega_2$. Furthermore, let $\boldsymbol{v}$ and $\mathcal{V}$ denote the unit normal (pointing into Ω_2) and the normal velocity of $\Sigma(t)$, respectively. We define the jump of a quantity f along Σ as

$$[f]_1^2(\boldsymbol{x}) := \lim_{\delta \searrow 0} \left(f_2(\boldsymbol{x} + \delta \boldsymbol{v}(\boldsymbol{x})) - f_1(\boldsymbol{x} - \delta \boldsymbol{v}(\boldsymbol{x})) \right)$$

for a point $\boldsymbol{x} \in \Sigma$ such that $\boldsymbol{x} + \delta \boldsymbol{v}(\boldsymbol{x}) \in \Omega_2$ and $\boldsymbol{x} - \delta \boldsymbol{v}(\boldsymbol{x}) \in \Omega_1$. Then, following the procedure outlined in [5, Section 4, Case I] and [32, Section 3], while considering a similar treatment for the pressure variable as in [18, Section 3], i.e., the pressure variable has a term scaling with $\frac{1}{\varepsilon}$ in its asymptotic expansion, we obtain the following sharp interface model:

$$\text{div}\,\boldsymbol{v} = \overline{\rho}_1 \Gamma_1 + \overline{\rho}_1 \Gamma_2 \qquad \text{in } \Omega_1 \cup \Omega_2, \quad (2.32a)$$

$$\rho(\partial_t \boldsymbol{v} + \text{div}\,(\boldsymbol{v} \otimes \boldsymbol{v})) - \frac{\overline{\rho}_2 - \overline{\rho}_1}{2} \text{div}\,(\boldsymbol{v} \otimes \nabla \mu) = -\nabla p + 2\eta\, \text{div}\,(\mathrm{D}\boldsymbol{v}) \quad \text{in } \Omega_1 \cup \Omega_2, \quad (2.32b)$$

$$\partial_t \sigma + \text{div}\,(\sigma \boldsymbol{v}) = n\Delta\sigma + S \qquad \text{in } \Omega_1 \cup \Omega_2, \quad (2.32c)$$

$$-\Delta \mu_2 = -2\overline{\rho}_1^{-1}\Gamma_1 \qquad \text{in } \Omega_2, \quad (2.32d)$$

$$-\Delta \mu_1 = 2\overline{\rho}_2^{-1}\Gamma_2 \qquad \text{in } \Omega_1, \quad (2.32e)$$

along with the free boundary conditions:

$$[\mu]_1^2 = 0, \quad [\sigma]_1^2 = 2\chi \quad \text{on } \Sigma, \tag{2.33a}$$

$$2\mu = \beta\kappa - \tfrac{1}{2}\left[|\sigma|^2\right]_1^2, \quad 2\chi(\boldsymbol{v}\cdot\boldsymbol{v} - \mathcal{V}) = [n\partial_v\sigma]_1^2 \quad \text{on } \Sigma, \tag{2.33b}$$

$$[\boldsymbol{v}]_1^2 = \boldsymbol{0}, \quad \boldsymbol{v}\cdot\boldsymbol{v} - \mathcal{V} = \tfrac{1}{2}\left[\partial_v\mu\right]_1^2, \quad [p\mathrm{I} - 2\eta\mathrm{D}\boldsymbol{v}]_1^2\,\boldsymbol{v} = \kappa\beta\boldsymbol{v} \quad \text{on } \Sigma, \tag{2.33c}$$

where κ is the mean curvature of the hypersurface Σ. We remark that equations (2.32a), (2.32c)–(2.33b) also appear in the sharp interface model [32, (3.49)–(3.50)], while (2.32b), (2.33c) are present in the sharp interface limit of [5, Section 4, Case I].

Remark 2.2 *We point out that it is also possible to consider mobilities scaling with ε, i.e., $m_1(\varphi) = \varepsilon$, or a two-sided degenerate mobility $m_2(\varphi) = (1 - \varphi^2)_+ := \max(1 - \varphi^2, 0)$. In these two cases, the analysis of [5, Cases II and IV] will yield a sharp interface model consisting of (2.32a), (2.32c) along with*

$$\rho(\partial_t\boldsymbol{v} + \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})) = -\nabla p + 2\eta\operatorname{div}(\mathrm{D}\boldsymbol{v}) \quad \text{in } \Omega_1\cup\Omega_2,$$

$$2\overline{\rho}_1^{-1}\Gamma_1(1,\mu,\sigma_2) = 0 \quad \text{in } \Omega_2,$$

$$2\overline{\rho}_2^{-1}\Gamma_2(-1,\mu,\sigma_1) = 0 \quad \text{in } \Omega_1,$$

$$2\mu = \beta\kappa - \frac{1}{2}\left[|\sigma|^2\right]_1^2, \quad [n\partial_v\sigma]_1^2 = 0, \quad [\sigma]_1^2 = 2\chi \quad \text{on } \Sigma,$$

$$[\boldsymbol{v}]_1^2 = \boldsymbol{0}, \quad [p\mathrm{I} - 2\eta\mathrm{D}\boldsymbol{v}]_1^2\,\boldsymbol{v} = \kappa\beta\boldsymbol{v}, \quad \mathcal{V} = \boldsymbol{v}\cdot\boldsymbol{v} \quad \text{on } \Sigma.$$

In particular, the choices of m_1 and m_2 lead to the relation $\mathcal{V} = \boldsymbol{v}\cdot\boldsymbol{v}$, which means that the interface is passively transported by the fluid velocity. Note that there are no differential operators acting on the variable μ, and the source terms Γ_1, Γ_2 must be chosen such that $\Gamma_1 = 0$ in Ω_2 and $\Gamma_2 = 0$ in Ω_1 in order for the above sharp interface model to be well defined.

2.8 A model with a mass-averaged velocity

In the modelling, a mass averaged velocity could also be used (see for instance, [17, 42, 48] and the references cited therein). In this section, we will sketch the derivation of a model using a mass-averaged velocity that includes mass transfer between the two fluid components, which serves to compare with our model (2.21) with a volume-averaged velocity.

Our starting point is the balance law (2.3a) for ρ_i, $i = 1, 2$. Let $u_i = \frac{\rho_i}{\overline{\rho}_i}$ be the volume fraction of component i, and $\rho = \rho_1 + \rho_2$ as the mass density of the fluid mixture. We now consider the mass concentration $c_i := \frac{\rho_i}{\rho} = \frac{\rho_i}{\rho_1 + \rho_2} \in [0, 1]$, $i = 1, 2$, which satisfies the relation:

$$c_i \frac{\rho}{\overline{\rho}_i} = u_i.$$

The assumption on zero excess volume of mixing (2.1) then implies that the mass density ρ as a function of the order parameter $c := c_2 - c_1 \in [-1, 1]$ is

$$\frac{1}{\rho(c)} = \frac{c_1}{\overline{\rho}_1} + \frac{c_2}{\overline{\rho}_2} = \frac{1-c}{2\overline{\rho}_1} + \frac{1+c}{2\overline{\rho}_2}, \tag{2.34}$$

due to the relation $c_1 + c_2 = 1$. Let $\boldsymbol{v}_i$, $i = 1, 2$, denote the individual velocity of fluid i, then the mass-averaged velocity $\boldsymbol{w}$ is defined as

$$\rho(c)\boldsymbol{w} = \rho_1 \boldsymbol{v}_1 + \rho_2 \boldsymbol{v}_2. \tag{2.35}$$

Adding (2.3a) for $i = 1, 2$ and employing the relation (2.35) yields the mass balance

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{w}) = \Gamma_1 + \Gamma_2 = \Theta \quad \Longrightarrow \quad \operatorname{div} \boldsymbol{w} = \frac{1}{\rho}(\Theta - \partial_t^{\bullet} \rho), \tag{2.36}$$

where $\partial_t^{\bullet} f = \partial_t f + \boldsymbol{w} \cdot \nabla f$ now denotes the material derivative with respect to $\boldsymbol{w}$. To obtain an equation for the order parameter c, interpreted as the difference in mass concentrations, we introduce the fluxes:

$$\boldsymbol{L}_i := \rho_i(\boldsymbol{v}_i - \boldsymbol{w}) \quad \Longrightarrow \quad \partial_t(\rho c_i) + \operatorname{div}(\rho c_i \boldsymbol{w}) + \operatorname{div} \boldsymbol{L}_i = \Gamma_i. \tag{2.37}$$

Furthermore, using (2.35) and the relation $\rho = \rho_1 + \rho_2$, it is easy to see that $\boldsymbol{L}_1 = -\boldsymbol{L}_2$. Setting $\boldsymbol{L} := \boldsymbol{L}_2 - \boldsymbol{L}_1$, we then obtain from (2.36) and (2.37) that

$$\partial_t(\rho c) + \operatorname{div}(\rho c \boldsymbol{w}) + \operatorname{div} \boldsymbol{L} = \Gamma_2 - \Gamma_1,$$

which implies

$$\rho \partial_t^{\bullet} c + \operatorname{div} \boldsymbol{L} = \Gamma_2(1 - c) - \Gamma_1(1 + c) =: \Upsilon. \tag{2.38}$$

For the linear momentum, we consider a balance law of the following form:

$$\partial_t(\rho \boldsymbol{w}) + \operatorname{div}(\rho \boldsymbol{w} \otimes \boldsymbol{w}) = \rho(\partial_t \boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{w}) + \Theta \boldsymbol{w} = \operatorname{div}(\boldsymbol{T} - \hat{p}\boldsymbol{I}), \tag{2.39}$$

for a symmetric tensor $\boldsymbol{T}$ and a pressure function $\hat{p}$, while for the nutrient density σ we have analogously

$$\partial_t \sigma + \operatorname{div}(\sigma \boldsymbol{w}) + \operatorname{div} \boldsymbol{J}_\sigma = S. \tag{2.40}$$

To derive the constitutive assumptions for $\boldsymbol{T}$, $\boldsymbol{J}_\sigma$ and $\boldsymbol{L}$ so that the resulting model is thermodynamically consistent, we postulate a general energy density of the form

$$e_{\boldsymbol{w}} = \frac{\rho(c)}{2} |\boldsymbol{w}|^2 + A\rho(c)\Psi(c) + \rho(c)\frac{B}{2} |\nabla c|^2 + N(\sigma, c), \tag{2.41}$$

where, as before, A and B are positive constants and Ψ is a potential with equal minima at ± 1. We note that the difference between (2.41) and (2.13) is that the

Ginzburg–Landau energy density is now multiplied with $\rho(c)$ (cf. [48, (3.36)]). Similar to Section 2.2, we use the Lagrange multiplier method of Müller and Liu to investigate the local inequality

$$-\mathcal{D} := \partial_t e_w + \operatorname{div}(e_w \boldsymbol{w}) + \operatorname{div} \boldsymbol{J}_e - c_\Upsilon \Upsilon - c_\Theta \Theta - c_S S$$
$$- \lambda_c \left(\rho \partial_t^\bullet c + \operatorname{div} \boldsymbol{L} - \Upsilon \right) - \lambda_\sigma \left(\partial_t^\bullet \sigma + \sigma \operatorname{div} \boldsymbol{w} + \operatorname{div} \boldsymbol{J}_\sigma - S \right) \leqslant 0$$

holding for arbitrary $(c, \sigma, \boldsymbol{w}, \partial_t^\bullet c, \partial_t^\bullet \sigma, \Upsilon, \Theta, S)$ with an energy flux $\boldsymbol{J}_e$ and prefactors c_Υ, c_Θ and c_S yet to be determined. Denoting $E := \frac{1}{2} |\boldsymbol{w}|^2 + A\Psi(c) + \frac{B}{2} |\nabla c|^2$, and the partial derivatives of $N(\sigma, c)$ with respect to σ and c as N_σ and N_c, respectively, then a long computation shows that

$$-\mathcal{D} = \operatorname{div}\left(\boldsymbol{J}_e - \lambda_c \boldsymbol{L} - \lambda_\sigma \boldsymbol{J}_\sigma + (\boldsymbol{T} - \hat{p}\boldsymbol{I})\boldsymbol{w} + B\rho \partial_t^\bullet c \nabla c \right) + (\lambda_c - c_\Upsilon)\Upsilon$$
$$+ S(\lambda_\sigma - c_S) + \boldsymbol{L} \cdot \nabla \lambda_c + \boldsymbol{J}_\sigma \cdot \nabla \lambda_\sigma - \nabla \boldsymbol{w} : (\boldsymbol{T} + B\rho \nabla c \otimes \nabla c)$$
$$+ \partial_t^\bullet c \left(\rho'(c)E + A\rho\Psi'(c) - \operatorname{div}(B\rho \nabla c) - \rho\lambda_c + N_c \right) + \partial_t^\bullet \sigma (N_\sigma - \lambda_\sigma)$$
$$+ (\hat{p} + \rho E - \lambda_\sigma \sigma) \operatorname{div} \boldsymbol{w} - (c_\Theta + |\boldsymbol{w}|^2)\Theta.$$

Using the expression in (2.36) for $\operatorname{div} \boldsymbol{w}$ leads to a simplification of the above equation:

$$-\mathcal{D} = \operatorname{div}\left(\boldsymbol{J}_e - \lambda_c \boldsymbol{L} - \lambda_\sigma \boldsymbol{J}_\sigma + (\boldsymbol{T} - \hat{p}\boldsymbol{I})\boldsymbol{w} + B\rho \partial_t^\bullet c \nabla c \right) + (\lambda_c - c_\Upsilon)\Upsilon$$
$$+ S(\lambda_\sigma - c_S) + \boldsymbol{L} \cdot \nabla \lambda_c + \boldsymbol{J}_\sigma \cdot \nabla \lambda_\sigma - \nabla \boldsymbol{w} : (\boldsymbol{T} + B\rho \nabla c \otimes \nabla c)$$
$$+ \partial_t^\bullet c \left(A\rho\Psi'(c) - \operatorname{div}(B\rho \nabla c) + N_c - \rho\lambda_c - \frac{\rho'}{\rho}(\hat{p} - \lambda_\sigma \sigma) \right) + \partial_t^\bullet \sigma (N_\sigma - \lambda_\sigma)$$
$$- \left(c_\Theta + |\boldsymbol{w}|^2 - \frac{1}{\rho}\hat{p} - E + \frac{1}{\rho}\lambda_\sigma \sigma \right) \Theta.$$

In choosing the constitutive assumptions

$$\lambda_\sigma = N_\sigma, \quad c_S = \lambda_\sigma,$$
$$\lambda_c = \mu := A\Psi'(c) - \frac{B}{\rho(c)} \operatorname{div}(\rho(c)\nabla c) + \frac{1}{\rho(c)} N_c(\sigma, c) - \frac{\rho'(c)}{\rho^2(c)} (\hat{p} - N_\sigma(\sigma, c)\sigma),$$
$$c_\Upsilon = \lambda_c, \quad \boldsymbol{L} = -m(c)\nabla \mu, \quad \boldsymbol{J}_\sigma = -n(c)\nabla N_\sigma,$$
$$\boldsymbol{T} = 2\eta(c)\mathrm{D}\boldsymbol{w} + \theta(c)\operatorname{div} \boldsymbol{w} \boldsymbol{I} - B\rho(c)\nabla c \otimes \nabla c,$$
$$\boldsymbol{J}_e = \lambda_c \boldsymbol{L} + \lambda_\sigma \boldsymbol{J}_\sigma - (\boldsymbol{T} - \hat{p}\boldsymbol{I})\boldsymbol{w} - B\rho(c)\partial_t^\bullet c \nabla c,$$
$$c_\Theta = \frac{1}{\rho(c)}(\hat{p} - N_\sigma \sigma) - \frac{1}{2} |\boldsymbol{w}|^2 + A\Psi(c) + \frac{B}{2} |\nabla c|^2,$$

for non-negative functions $\eta(c)$, $\theta(c)$, $m(c)$ and $n(c)$, we obtain the analogue of system (2.21) with a mass-averaged velocity $\boldsymbol{w}$:

$$\operatorname{div} \boldsymbol{w} = \frac{1}{\rho(c)}(\Gamma_1 + \Gamma_2) - \frac{\rho'(c)}{\rho^2(c)}(\Gamma_2(1-c) - \Gamma_1(1+c) + \operatorname{div}(m(c)\nabla\mu)), \tag{2.42a}$$

$$\frac{1}{\rho(c)} = \frac{1-c}{2\bar{\rho}_1} + \frac{1+c}{2\bar{\rho}_2}, \tag{2.42b}$$

$$\partial_t(\rho\boldsymbol{w}) + \operatorname{div}(\rho\boldsymbol{w}\otimes\boldsymbol{w}) + \nabla\hat{p} = \operatorname{div}(2\eta(c)\mathrm{D}\boldsymbol{w} + \theta(c)\operatorname{div}\boldsymbol{w}\boldsymbol{I} - B\rho(c)\nabla c\otimes\nabla c), \tag{2.42c}$$

$$\rho(c)\partial_t^\bullet c = \operatorname{div}(m(c)\nabla\mu) + \Gamma_2(1-c) - \Gamma_1(1+c), \tag{2.42d}$$

$$\mu = A\Psi'(c) - \frac{1}{\rho(c)}B\operatorname{div}(\rho(c)\nabla c) + \frac{1}{\rho(c)}N_c(\sigma,c) - \frac{\rho'(c)}{\rho^2(c)}(\hat{p} - N_\sigma(\sigma,c)\sigma), \tag{2.42e}$$

$$\partial_t^\bullet \sigma = \operatorname{div}(n(c)\nabla N_\sigma(\sigma,c)) - \sigma\operatorname{div}\boldsymbol{w} + S. \tag{2.42f}$$

Let us point out that (2.42) can be seen as a generalization of the quasi-incompressible model of Lowengrub and Truskinovsky [48, (4.14)] to include mass transfer and chemotaxis effects. In particular, by setting $\Gamma_1 = \Gamma_2 = \sigma = N(\sigma,c) = 0$, we recover [48, (4.14)]. Furthermore, in comparison with the volume-averaged variant (2.21) obtained above, we observe that the equation for the divergence of the volume-averaged velocity (2.21a) is considerably simpler than the corresponding equation (2.42a) for the mass-averaged velocity, and the pressure does not explicitly appear in the equation for the chemical potential (2.21e), unlike in (2.42e).

3 Analysis of the simplified model with zero excess of total mass and equal densities

3.1 Problem setting

For the rest of the paper, we analyse the model (2.25) with zero excess of total mass and equal densities, and consider the following typical form of the chemical free energy N:

$$N(\varphi,\sigma) = \frac{1}{2}|\sigma|^2 + \chi\sigma(1-\varphi) \quad \text{for } \chi \geq 0, \tag{3.1}$$

which has also been considered in [28–31]. Then, we have

$$N_\varphi = -\chi\sigma, \quad N_\sigma = \sigma + \chi(1-\varphi). \tag{3.2}$$

For any given $T \in (0,+\infty)$, we set $\rho_* = 1$ and absorb the prefactor 2 into the source term Γ, leading to the system

$$\operatorname{div}\boldsymbol{v} = 0 \qquad\qquad \text{in } \Omega \times (0,T), \tag{3.3a}$$

$$\partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla q + \operatorname{div}(2\eta(\varphi)\mathrm{D}\boldsymbol{v}) + (\mu + \chi\sigma)\nabla\varphi \quad \text{in } \Omega \times (0,T), \tag{3.3b}$$

$$\partial_t\varphi + \boldsymbol{v}\cdot\nabla\varphi = \operatorname{div}(m(\varphi)\nabla\mu) + \Gamma \qquad\qquad \text{in } \Omega \times (0,T), \tag{3.3c}$$

$$\mu = A\Psi'(\varphi) - B\Delta\varphi - \chi\sigma \qquad\qquad \text{in } \Omega \times (0,T), \tag{3.3d}$$

$$\partial_t\sigma + \boldsymbol{v}\cdot\nabla\sigma = \operatorname{div}(n(\varphi)\nabla(\sigma + \chi(1-\varphi))) + S \qquad \text{in } \Omega \times (0,T), \tag{3.3e}$$

where we have also reformulated the momentum equation with the modified pressure variable q. We now prescribe the following initial-boundary conditions:

$$v(0) = v_0, \quad \varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0 \quad \text{in } \Omega, \tag{3.4a}$$

$$v = 0, \quad m(\varphi)\partial_\nu\mu = \partial_\nu\varphi = n(\varphi)\partial_\nu N_\sigma = 0 \quad \text{on } \partial\Omega \times (0, T), \tag{3.4b}$$

so that the energy identity (2.22) reduces to

$$
\frac{d}{dt} \int_\Omega \left[\frac{1}{2}|v|^2 + A\Psi(\varphi) + \frac{B}{2}|\nabla\varphi|^2 + \frac{1}{2}|\sigma|^2 + \chi\sigma(1-\varphi) \right] dx
$$
$$
+ \int_\Omega \left[m(\varphi)|\nabla\mu|^2 + n(\varphi)|\nabla(\sigma + \chi(1-\varphi))|^2 + 2\eta(\varphi)|\mathrm{D}v|^2 \right] dx \tag{3.5}
$$
$$
- \int_\Omega \left[(\sigma + \chi(1-\varphi))S + \mu\Gamma \right] dx = 0.
$$

3.2 Preliminaries

Assume that $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain with smooth boundary $\partial\Omega$. We denote by $Q = \Omega \times (0, T)$ the space–time cylinder with $\Sigma := \partial\Omega \times (0, T)$. For the standard Lebesgue and Sobolev spaces, we use the notations $L^p := L^p(\Omega)$ and $W^{k,p} := W^{k,p}(\Omega)$ for any $p \in [1, +\infty]$, $k > 0$ equipped with the norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{k,p}}$. In the case $p = 2$, we use $H^k := W^{k,2}$ and the norm $\|\cdot\|_{H^k}$. The L^2-scalar product between two functions f and g is denoted by (f, g). The dual space of a Banach space X is denoted by X', and the duality pairing between X and its dual will be denoted by $\langle\cdot, \cdot\rangle_X$.

We express $\mathbb{R}^d$-valued functions and function spaces consisting of vector-valued/tensor-valued functions in boldface. The matrix of second derivatives of a scalar function f is denoted by $\nabla^2 f$, and similarly the tensor of second derivatives of a vector $\boldsymbol{f}$ is denoted by $\nabla^2\boldsymbol{f}$. For Bochner spaces, we will often use the isometric isomorphism between $L^p(0, T; L^p(\Omega))$ and $L^p(Q)$ for $1 \leqslant p < \infty$. Similarly, we reuse the notation $L^p(Q)$ for $L^p(0, T; \boldsymbol{L}^p(\Omega))$ for $1 \leqslant p < \infty$. Since the pressure is determined up to a time-dependent constant, we ask that the pressure q belongs to the space $L_0^2 := \{f \in L^2(\Omega) : \overline{f} = 0\}$, where $\overline{f} := \frac{1}{|\Omega|} \int_\Omega f\, dx$ denotes the mean of f. Furthermore, we set

$$H_N^2 := \{f \in H^2 : \partial_\nu f = 0 \text{ on } \partial\Omega\}.$$

We now introduce the classical function spaces for the Navier–Stokes equations [54]. For a vector-valued/tensor-valued Banach space $\boldsymbol{X}$, we define $\boldsymbol{X}_{0,\sigma}$ as the completion of $\boldsymbol{C}_{0,\sigma}^\infty := \{\boldsymbol{f} \in (C_0^\infty(\Omega))^d : \mathrm{div}\,\boldsymbol{f} = 0\}$ with respect to the $\boldsymbol{X}$-norm. In the case $\boldsymbol{X} = \boldsymbol{L}^2$, we use the notation $\boldsymbol{L}_\sigma^2 := \boldsymbol{L}_{0,\sigma}^2$. The space $\boldsymbol{H}_{0,\sigma}^1$ is endowed with the scalar product

$$(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{H}_{0,\sigma}^1} := (\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{H}_{0,\sigma}^1,$$

and we denote its topological dual as $\boldsymbol{H}^{-1}$. It is well known that $\boldsymbol{L}^2$ can be decomposed into the sum $\boldsymbol{L}_\sigma^2 \oplus \boldsymbol{G}(\Omega)$, where $\boldsymbol{G}(\Omega) := \{\boldsymbol{f} \in \boldsymbol{L}^2 : \exists z \in L^2 \text{ with } \boldsymbol{f} = \nabla z\}$, i.e., the space $\boldsymbol{G}(\Omega)$ is the orthogonal complement of $\boldsymbol{L}_\sigma^2$. Then, for every $\boldsymbol{f} \in \boldsymbol{L}^2$, we have the unique decomposition (up to an additive constant for z)

$$\boldsymbol{f} = \boldsymbol{f}_0 + \nabla z, \quad \text{where } \boldsymbol{f}_0 \in \boldsymbol{L}_\sigma^2, \quad \nabla z \in \boldsymbol{G}(\Omega),$$

and as a consequence, we obtain a bounded linear operator $P_H : L^2 \to L^2_\sigma$ defined by $P_H(f) = f_0$, which is more commonly known as the Leray projection onto the space of divergence-free functions. We recall the Stokes operator $\mathbb{S} : D(\mathbb{S}) \to L^2_\sigma$ defined as

$$(\mathbb{S}u, \zeta) = (\nabla u, \nabla \zeta) \quad \forall \zeta \in H^1_{0,\sigma}$$

with domain $D(\mathbb{S}) = H^2 \cap H^1_{0,\sigma}$. Then, the following estimates hold (see [45, Lemma 3.4] and [54]):

Lemma 3.1 *For any* $u \in D(\mathbb{S})$, *consider the Helmholtz decomposition* $\mathbb{S}u = -\Delta u + \nabla \pi$ *where the pressure-like function* π *is taken such that* $\int_\Omega \pi \, dx = 0$. *Then, for any* $v > 0$, *there exists a positive constant* C_v *independent of* u *such that*

$$\|\pi\|_{L^2 \setminus \mathbb{R}} \leqslant v\|\mathbb{S}u\|_{L^2} + C_v\|\nabla u\|_{L^2}. \tag{3.6}$$

Moreover, there exists a positive constant $c = c(d, \Omega)$ *such that*

$$\|u\|_{H^2} + \|\pi\|_{H^1 \setminus \mathbb{R}} \leqslant c\|\mathbb{S}u\|_{L^2}. \tag{3.7}$$

Furthermore, we state some inequalities that will be useful in the subsequent analysis:

- *Poincaré's inequality*: There exists a positive constant C depending only on Ω such that, for all $f \in W^{1,p}$, $p \in [1, \infty]$,

$$\left\| f - \overline{f} \right\|_{L^p} \leqslant C\|\nabla f\|_{L^p}. \tag{3.8}$$

- *The Gagliardo–Nirenberg interpolation inequality* in dimension d: Let Ω be a bounded domain with Lipschitz boundary, and $f \in W^{m,r} \cap L^q$, $1 \leqslant q, r \leqslant \infty$. For any integer j, $0 \leqslant j < m$, suppose there is $\alpha \in \mathbb{R}$ such that

$$\frac{1}{p} = \frac{j}{d} + \left(\frac{1}{r} - \frac{m}{d}\right)\alpha + \frac{1-\alpha}{q}, \quad \frac{j}{m} \leqslant \alpha \leqslant 1. \tag{3.9}$$

If $r \in (1, \infty)$ and $m - j - \frac{d}{r}$ is a non-negative integer, we additionally assume $\alpha \neq 1$. Under these assumptions, there exists a positive constant C depending only on Ω, m, j, q, r and α such that

$$\|D^j f\|_{L^p} \leqslant C\|f\|^\alpha_{W^{m,r}}\|f\|^{1-\alpha}_{L^q}, \tag{3.10}$$

where $D^j f$ denotes the jth weak partial derivatives of f. An alternate variant of the Gagliardo–Nirenberg inequality we will use is

$$\|D^j f\|_{L^p} \leqslant C\|D^m f\|^\alpha_{L^r}\|f\|^{1-\alpha}_{L^q} + C\|f\|_{L^s},$$

where p, j, m, r, q and α satisfy (3.9), and $s > 0$ is arbitrary.

- *Korn's inequality*: For any $\boldsymbol{u} \in \boldsymbol{H}_0^1$, there exists a positive constant C, depending only on Ω such that

$$\|\nabla \boldsymbol{u}\|_{L^2} \leqslant C\|\mathrm{D}\boldsymbol{u}\|_{L^2}, \text{ where } \mathrm{D}\boldsymbol{u} := \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top). \qquad (3.11)$$

- *The Brézis–Gallouet interpolation inequality* for $d = 2$ (cf. [12,21]): There exists a positive constant C depending only on Ω such that

$$\|g\|_{L^\infty} \leqslant C\|g\|_{H^1}\sqrt{\ln(1 + \|g\|_{H^2})} + C\|g\|_{H^1} \quad \forall g \in H^2(\Omega). \qquad (3.12)$$

- Elliptic estimates: If $f \in H^2(\Omega)$ satisfies $\partial_\nu f = 0$ on $\partial\Omega$, then there exists a positive constant C depending only on Ω such that

$$\|f\|_{H^2} \leqslant C\left(\|\Delta f\|_{L^2} + \|f\|_{L^2}\right). \qquad (3.13)$$

If $f \in H^4(\Omega)$ satisfies $\partial_\nu f = \partial_\nu(\Delta f) = 0$ on $\partial\Omega$, then there exists a positive constant C depending only on Ω such that

$$\|f\|_{H^4} \leqslant C\left(\|\Delta^2 f\|_{L^2} + \|f\|_{L^2}\right). \qquad (3.14)$$

3.3 Main results

Our first result concerns the existence of global weak solutions to the system (3.3)–(3.4) in both two and three dimensions.

Theorem 1 (Global weak solutions in 2D and 3D) *We assume that*
(1) $m, n, \eta \in C^0(\mathbb{R})$ *and satisfy*

$$m_0 \leqslant m(s) \leqslant m_1, \quad n_0 \leqslant n(s) \leqslant n_1, \quad \eta_0 \leqslant \eta(s) \leqslant \eta_1 \quad \forall s \in \mathbb{R},$$

where $m_0, m_1, n_0, n_1, \eta_0$ and η_1 are given positive constants.
(2) *The external source terms $S \in L^2(Q)$ and $\Gamma \in L^2(0, T; L^\infty(\Omega))$ are prescribed functions.*
(3) *The potential $\Psi \in C^2(\mathbb{R})$ is non-negative and satisfies*

$$|\Psi''(s)| \leqslant C_0(1 + |s|^r), \quad |\Psi'(s)| \leqslant C_1\Psi(s) + C_2, \quad \Psi(s) \geqslant C_3|s|^2 - C_4, \qquad (3.15)$$

for all $s \in \mathbb{R}$ with positive constants C_0, C_1, C_2, C_3 and C_4 that are independent of s. If $d = 3$, $r \in [0, 4)$; if $d = 2$, $r \in [0, +\infty)$.
(4) *The coefficients $A > 0$, $B > 0$ and $\chi \geqslant 0$ are given constants and the following condition holds:*

$$A > \frac{2\chi^2}{C_3}. \qquad (3.16)$$

Let $T \in (0, +\infty)$ be an arbitrary but fixed terminal time. Then, for any initial data $\varphi_0 \in H^1(\Omega)$, $\sigma_0 \in L^2(\Omega)$ and $\boldsymbol{v}_0 \in \boldsymbol{L}^2(\Omega)$, there exists at least one global weak solution $(\varphi, \mu, \sigma, \boldsymbol{v})$ to problem (3.3)–(3.4) on $[0, T]$ such that

$$\varphi \in L^\infty(0, T; H^1) \cap L^2(0, T; H^3) \cap H^1(0, T; (H^1)'), \quad \mu \in L^2(0, T; H^1),$$
$$\sigma \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1) \cap W^{1,y}(0, T; (H^1)'),$$
$$\boldsymbol{v} \in L^\infty(0, T; \boldsymbol{L}^2_\sigma) \cap L^2(0, T; \boldsymbol{H}^1_{0,\sigma}) \cap W^{1,y}(0, T; \boldsymbol{H}^{-1}),$$

where $y = 2$ if $d = 2$, $y = \frac{4}{3}$ if $d = 3$. Moreover, the quadruple $(\varphi, \mu, \sigma, \boldsymbol{v})$ satisfies for a.e. $t \in (0, T)$

$$0 = \langle \partial_t \varphi, \zeta \rangle_{H^1} + (\boldsymbol{v} \cdot \nabla \varphi, \zeta) + (m(\varphi)\nabla\mu, \nabla\zeta) - (\Gamma, \zeta), \tag{3.17a}$$
$$0 = (\mu - A\Psi'(\varphi) + \chi\sigma, \zeta) - (B\nabla\varphi, \nabla\zeta), \tag{3.17b}$$
$$0 = \langle \partial_t \sigma + \boldsymbol{v} \cdot \nabla\sigma, \zeta \rangle_{H^1} + (n(\varphi)\nabla(\sigma + \chi(1 - \varphi)), \nabla\zeta) - (S, \zeta), \tag{3.17c}$$
$$0 = \langle \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}, \xi \rangle_{H^1} + (2\eta(\varphi)\mathrm{D}\boldsymbol{v}, \mathrm{D}\xi) - ((\mu + \chi\sigma)\nabla\varphi, \xi), \tag{3.17d}$$

and

$$\varphi(0) = \varphi_0, \quad \langle \sigma(0), \zeta \rangle_{H^1} = \langle \sigma_0, \zeta \rangle_{H^1}, \quad \langle \boldsymbol{v}(0), \xi \rangle_{H^1} = \langle \boldsymbol{v}_0, \xi \rangle_{H^1},$$

for all $\zeta \in H^1(\Omega)$ and $\xi \in \boldsymbol{H}^1_{0,\sigma}$.

Remark 3.1 (1). *The initial value for φ is attained precisely due to the (compact) embedding $L^\infty(0, T; H^1) \cap H^1(0, T; (H^1)') \subset\subset C^0([0, T]; L^2)$. Meanwhile, for σ and $\boldsymbol{v}$, in two dimensions we use the continuous embedding $L^2(0, T; H^1) \cap H^1(0, T; (H^1)') \subset C^0([0, T]; L^2)$ to deduce that $\sigma(0) = \sigma_0$ and $\boldsymbol{v}(0) = \boldsymbol{v}_0$. In three dimensions, we use the continuous embedding $W^{1,\frac{4}{3}}(0, T; (H^1)') \subset C^0([0, T]; (H^1)')$ to deduce that the initial conditions for σ and $\boldsymbol{v}$ are attained.*

(2). *To recover the pressure, we argue as follows. The distribution $\boldsymbol{F}_v$ defined as*

$$\boldsymbol{F}_v := \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \mathrm{div}\,(2\eta(\varphi)\mathrm{D}\boldsymbol{v}) - (\mu + \chi\sigma)\nabla\varphi$$

belongs to $L^y(0, T; \boldsymbol{H}^{-1})$ and vanishes on the subspace $C_0^\infty([0, T]; C_{0,\sigma}^\infty)$ by (3.17d). Applying [54, Lemma IV.1.4.1] allows us to deduce that there exists a function $q \in L^y(0, T; L_0^2)$ such that $\boldsymbol{F}_v = -\nabla q$ holds as an equality in the sense of distributions for a.e. $t \in (0, T)$.

Our next result yields the existence of global strong solutions to problem (3.3)–(3.4) when the spatial dimension is two.

Theorem 2 (Global strong solutions in 2D) *In addition to the assumptions of Theorem 1, we assume that*

(1) *$m \in C_b^2(\mathbb{R})$, $n \in C_b^1(\mathbb{R})$, $\eta \in C_b^1(\mathbb{R})$ with bounded derivatives.*
(2) *The potential $\Psi \in C^3(\mathbb{R})$ satisfies*

$$|\Psi'''(s)| \leqslant C_0(1 + |s|^{r-1}) \quad \text{for } r \in [1, +\infty), \tag{3.18}$$

instead of the first assumption in (3.15).

(3) *The source term Γ belongs to $L^2(0, T; H^1(\Omega) \cap L^\infty(\Omega))$.*

Then, for any $T > 0$ and arbitrary initial data $\boldsymbol{v}_0 \in \boldsymbol{H}^1(\Omega)$, $\sigma_0 \in H^1(\Omega)$ and $\varphi_0 \in H^3(\Omega) \cap H^2_N(\Omega)$, there exists at least one global strong solution $(\varphi, \mu, \sigma, \boldsymbol{v})$ to problem (3.3)–(3.4) such that

$$\varphi \in L^\infty(0, T; H^3 \cap H^2_N) \cap L^2(0, T; H^4) \cap H^1(0, T; H^1) \cap C^0([0, T]; C^{1,\delta}(\overline{\Omega})),$$

$$\mu \in L^\infty(0, T; H^1) \cap L^2(0, T; H^3 \cap H^2_N),$$

$$\sigma \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2_N) \cap H^1(0, T; L^2),$$

$$\boldsymbol{v} \in L^\infty(0, T; \boldsymbol{H}^1_0) \cap L^2(0, T; \boldsymbol{H}^2) \cap H^1(0, T; \boldsymbol{L}^2_\sigma),$$

$$q \in L^2(0, T; H^1 \cap L^2_0),$$

for some $0 < \delta < 1$. For a.e. $(x, t) \in Q$, the quadruple $(\varphi, \mu, \sigma, \boldsymbol{v})$ satisfies

$$0 = \partial_t \varphi + \boldsymbol{v} \cdot \nabla \varphi - \operatorname{div}(m(\varphi)\nabla\mu) - \Gamma, \tag{3.19a}$$

$$0 = \mu - A\Psi'(\varphi) + B\Delta\varphi + \chi\sigma, \tag{3.19b}$$

$$0 = \partial_t \sigma + \boldsymbol{v} \cdot \nabla \sigma - \operatorname{div}(n(\varphi)\nabla(\sigma + \chi(1 - \varphi))) - S, \tag{3.19c}$$

$$0 = \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \operatorname{div}(2(\eta(\varphi)D\boldsymbol{v}) + \nabla q - (\mu + \chi\sigma)\nabla\varphi. \tag{3.19d}$$

Furthermore, it holds

$$\varphi(0) = \varphi_0, \quad \sigma(0) = \sigma_0, \quad \boldsymbol{v}(0) = \boldsymbol{v}_0, \quad a.e. \text{ in } \Omega.$$

Remark 3.2

(1) *We used the compact embedding $L^\infty(0, T; H^3) \cap H^1(0, T; H^1) \subset\subset C^0([0, T]; C^{1,\delta}(\overline{\Omega}))$ to deduce the Hölder spatial regularity for φ.*

(2) *The assumption $\varphi_0 \in H^2_N(\Omega)$ implies that $\mu_0 := A\Psi'(\varphi_0) - B\Delta\varphi_0 - \chi\sigma_0 \in L^2(\Omega)$, and if $\varphi_0 \in H^3(\Omega)$, then $\mu_0 \in H^1(\Omega)$.*

(3) *We point out that one can weaken the assumptions to $m \in C^1_b(\mathbb{R})$ with bounded derivatives and $\varphi_0 \in H^2_N(\Omega)$ to derive a strong solution with less regularities:*

$$\varphi \in L^\infty(0, T; H^3 \cap H^2_N) \cap L^2(0, T; H^4) \cap H^1(0, T; L^2),$$

$$\mu \in L^\infty(0, T; L^2) \cap L^2(0, T; H^2_N).$$

In essence, we neglect the fourth a priori estimate in Section 5.

The last theorem gives a continuous dependence on the initial data and source terms for global strong solutions in two dimensions.

Theorem 3 (Continuous dependence in 2D) *Let $\{\varphi_i, \mu_i, \sigma_i, \boldsymbol{v}_i, q_i\}_{i=1,2}$ denote two global strong solutions to problem (3.3)–(3.4) corresponding to initial data $\{\varphi_{0,i}, \sigma_{0,i}, \boldsymbol{v}_{0,i}\}_{i=1,2}$ and source terms $\{\Gamma_i, S_i\}_{i=1,2}$ obtained in Theorem 2. Besides, we assume that $m(\cdot)$, $\eta(\cdot)$ and $n(\cdot)$*

are Lipschitz continuous, and Ψ satisfies

$$|\Psi'(s_1) - \Psi'(s_2)| \leqslant C_5 \left(1 + |s_1|^r + |s_2|^r\right) |s_1 - s_2| \quad \forall s_1, s_2 \in \mathbb{R}, \tag{3.20}$$

for some $r > 0$ and constant $C_5 > 0$. Then, there exists a positive constant C, depending on T, Ω, χ, η_0, n_0, m_0, $\|\eta'\|_{L^\infty(\mathbb{R})}$, $\|n'\|_{L^\infty(\mathbb{R})}$, $\|m'\|_{L^\infty(\mathbb{R})}$, A, B, C_i ($i = 1, \ldots, 5$), $\|\nabla\sigma_i\|_{L^2(0,T;L^4)}$, $\|\varphi_i\|_{L^{2r}(0,T;L^\infty)}$, $\|\mu_i + \chi\sigma_i\|_{L^2(0,T;L^\infty)}$, $\|\nabla\varphi_i\|_{L^\infty(0,T;L^\infty)}$, $\|\mu_i\|_{L^2(0,T;H^3)}$, $\|\nabla(\sigma_i - \chi\varphi_i)\|_{L^4(Q)}$ and $\|\mathbf{D}\mathbf{v}_i\|_{L^4(Q)}$ such that

$$
\begin{aligned}
\sup_{t \in (0,T]} & \left(\|(\varphi_1 - \varphi_2)(t)\|_{L^2}^2 + \|(\mathbf{v}_1 - \mathbf{v}_2)(t)\|_{\mathbf{L}^2}^2 + \|(\sigma_1 - \sigma_2)(t)\|_{L^2}^2 \right) + \|\mu_1 - \mu_2\|_{L^2(Q)}^2 \\
& + \|\nabla(\sigma_1 - \sigma_2)\|_{L^2(Q)}^2 + \|\nabla(\mathbf{v}_1 - \mathbf{v}_2)\|_{L^2(Q)}^2 + \|\varphi_1 - \varphi_2\|_{L^2(0,T;H^2)}^2 \\
& \leqslant C \left(\|\sigma_{0,1} - \sigma_{0,2}\|_{L^2}^2 + \|\mathbf{v}_{0,1} - \mathbf{v}_{0,2}\|_{\mathbf{L}^2}^2 + \|\varphi_{0,1} - \varphi_{0,2}\|_{L^2}^2 \right) \\
& + C \left(\|\Gamma_1 - \Gamma_2\|_{L^2(Q)}^2 + \|S_1 - S_2\|_{L^2(Q)}^2 \right).
\end{aligned}
\tag{3.21}
$$

In particular, the global strong solution to problem (3.3)–(3.4) is unique under the above assumptions.

Remark 3.3 It is worth mentioning that the above results concerning global weak existence and strong well-posedness also hold when the mass transfer terms Γ and S are functions of φ and σ. For example, one may consider the typical choice

$$\Gamma = h(\varphi)P(\sigma), \quad S = -h(\varphi)\mathcal{C}\sigma,$$

where $h(s)$ is a non-negative, bounded, Lipschitz function such that $h(-1) = 0$ and $h(1) = 1$, $\mathcal{C} \geqslant 0$ can be seen as a constant consumption rate, and P is a non-negative, bounded, Lipschitz function modelling growth by consuming the chemical species. Similar types of mass transfer terms have been studied in [30, 31] for the Cahn–Hilliard system modelling tumour growth.

4 Proof of Theorem 1: global weak solutions

Theorem 1 can be proved by using a suitable Galerkin approximation scheme based on the energy identity (3.5).

4.1 Galerkin approximation

We recall that the set of eigenfunctions $\{w_j\}_{j\in\mathbb{N}}$ of the Neumann–Laplacian forms an orthonormal basis of L^2 and is also an orthogonal basis of H_N^2. Furthermore, we can choose $w_1 = 1$. Meanwhile, by the Lax–Milgram theorem, for every $\mathbf{f} \in \mathbf{L}_\sigma^2$ there exists a unique $\mathbf{v} \in \mathbf{H}_{0,\sigma}^1$ satisfying

$$(\nabla\mathbf{v}, \nabla\boldsymbol{\zeta}) = (\mathbf{f}, \boldsymbol{\zeta}) \quad \forall \boldsymbol{\zeta} \in \mathbf{H}_{0,\sigma}^1. \tag{4.1}$$

Defining the operator $\mathbb{A} : \mathbf{L}_\sigma^2 \to \mathbf{H}_{0,\sigma}^1$ by $\mathbb{A}(\mathbf{f}) = \mathbf{v}$, where $\mathbf{v}$ solves (4.1) with data $\mathbf{f}$. The operator $\mathbb{A}$ can also be identified as the inverse to the Stokes operator $\mathbb{S}$. It is clear that $\mathbb{A}$

is a linear, continuous and self-adjoint operator. Furthermore, by the compact embedding $H_{0,\sigma}^1 \subset\subset L_\sigma^2$ it holds that $\mathbb{A}$ is a compact operator, and thus by spectral theory there exists an increasing sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ of positive real numbers and a family $\{y_j\}_{j\in\mathbb{N}} \subset H^2 \cap H_{0,\sigma}^1$ of eigenfunctions that forms an orthonormal basis of L_σ^2 and an orthogonal basis of $H_{0,\sigma}^1$.

We define the finite-dimensional subspaces

$$W_n := \text{span}\{w_1, \dots, w_n\} \quad \text{and} \quad Y_n := \text{span}\{y_1, \dots, y_n\}$$

with corresponding orthogonal projections $\mathbb{P}_{W_n}$ and $\mathbb{P}_{Y_n}$, and look for functions of the form

$$\varphi_n(x,t) := \sum_{i=1}^n \alpha_{n,i}(t) w_i(x), \qquad \mu_n(x,t) := \sum_{i=1}^n \beta_{n,i}(t) w_i(x),$$

$$\sigma_n(x,t) := \sum_{i=1}^n \gamma_{n,i}(t) w_i(x), \qquad v_n(x,t) := \sum_{i=1}^n \delta_{n,i}(t) y_i(x)$$

that solve the following approximating problem:

$$0 = \partial_t \varphi_n + \mathbb{P}_{W_n}(v_n \cdot \nabla \varphi_n) - \mathbb{P}_{W_n}(\text{div}(m(\varphi_n)\nabla\mu_n)) - \mathbb{P}_{W_n}(\Gamma), \tag{4.2a}$$

$$0 = \partial_t \sigma_n + \mathbb{P}_{W_n}(v_n \cdot \nabla \sigma_n) - \mathbb{P}_{W_n}(\text{div}(n(\varphi_n)\nabla(\sigma_n + \chi(1-\varphi_n)))) - \mathbb{P}_{W_n}(S), \tag{4.2b}$$

$$0 = \partial_t v_n + \mathbb{P}_{Y_n}((v_n \cdot \nabla)v_n) - \mathbb{P}_{Y_n}(\text{div}(2\eta(\varphi_n)Dv_n)) - \mathbb{P}_{Y_n}((\mu_n + \chi\sigma_n)\nabla\varphi_n), \tag{4.2c}$$

$$0 = \mu_n - A\mathbb{P}_{W_n}(\Psi'(\varphi_n)) + B\Delta\varphi_n + \chi\sigma_n, \tag{4.2d}$$

subject to the initial data

$$\varphi_n(0) = \mathbb{P}_{W_n}(\varphi_0), \quad \sigma_n(0) = \mathbb{P}_{W_n}(\sigma_0), \quad v_n(0) = \mathbb{P}_{Y_n}(v_0).$$

It is easy to see that (4.2) is equivalent to

$$0 = (\partial_t \varphi_n, w_i) + (v_n \cdot \nabla\varphi_n, w_i) + (m(\varphi_n)\nabla\mu_n, \nabla w_i) - (\Gamma, w_i), \tag{4.3a}$$

$$0 = (\partial_t \sigma_n, w_i) + (v_n \cdot \nabla\sigma_n, w_i) + (n(\varphi_n)\nabla(\sigma_n + \chi(1-\varphi_n)), \nabla w_i) - (S, w_i), \tag{4.3b}$$

$$0 = (\partial_t v_n, y_i) + ((v_n \cdot \nabla)v_n, y_i) + (2\eta(\varphi_n)Dv_n, Dy_i) - ((\mu_n + \chi\sigma_n)\nabla\varphi_n, y_i), \tag{4.3c}$$

$$0 = (\mu_n, w_i) - (A\Psi'(\varphi_n), w_i) - (B\nabla\varphi_n, \nabla w_i) + (\chi\sigma_n, w_i), \tag{4.3d}$$

for all $i \in \mathbb{N}$.

The approximating problem (4.2) is equivalent to solving a system of nonlinear ordinary differential equations in the $4n$ unknowns $\{\alpha_{n,i}, \beta_{n,i}, \gamma_{n,i}, \delta_{n,i}\}_{i=1}^n$. The continuity of $\Psi'(\cdot)$, $m(\cdot)$, $n(\cdot)$, $\eta(\cdot)$ and the Cauchy–Peano theorem allow us to deduce the existence of a local solution $(\alpha, \beta, \gamma, \delta) \in C^1([0, t_n], \mathbb{R}^{4n})$ for some $t_n \in (0, T]$.

4.2 A priori estimates and passage to the limit as $n \to +\infty$

We now derive a priori estimates to show that $t_n = T$ for all $n \in \mathbb{N}$ and the approximate solutions $(\varphi_n, \mu_n, \sigma_n, v_n)$ are uniformly bounded with respect to n in suitable function spaces. Below the symbol C denotes a generic positive constant that is independent of the parameter n and may vary from line to line.

First estimate: We obtain from testing (4.2a) with μ_n, testing (4.2b) with $\sigma_n + \chi(1 - \varphi_n)$, testing (4.2c) with $\boldsymbol{v}_n$ and testing (4.2d) with $\partial_t \varphi_n$ an analogous energy identity to (3.5):

$$
\frac{d}{dt} \int_\Omega \frac{1}{2} |\boldsymbol{v}_n|^2 + A\Psi(\varphi_n) + \frac{B}{2} |\nabla \varphi_n|^2 + \frac{1}{2} |\sigma_n|^2 + \chi \sigma_n(1 - \varphi_n) \, dx
$$

$$
+ \int_\Omega m(\varphi_n) |\nabla \mu_n|^2 + n(\varphi_n) |\nabla(\sigma_n + \chi(1 - \varphi_n))|^2 + 2\eta(\varphi_n) |\mathbf{D}\boldsymbol{v}_n|^2 \, dx \qquad (4.4)
$$

$$
= \int_\Omega (\sigma_n + \chi(1 - \varphi_n))S + \mu_n \Gamma \, dx.
$$

Here, we use the fact $\partial_t \varphi_n \in W_n$ so that

$$
(\mathbb{P}_{W_n}(\Psi'(\varphi_n)), \partial_t \varphi_n) = (\Psi'(\varphi_n), \mathbb{P}_{W_n}(\partial_t \varphi_n)) = (\Psi'(\varphi_n), \partial_t \varphi_n).
$$

Similarly, during the testing procedure, we shift the projection operators from the nonlinearities onto $\mu_n, \sigma_n + \chi(1 - \varphi_n)$ and $\boldsymbol{v}_n$, and this leads to (4.4).

Besides, by the Sobolev embedding theorem $H^1 \subset L^s$ for $s \in [1, +\infty)$ if $d = 2$ and for $s \in [1, 6]$ if $d = 3$, and assumption (3.15), we see that the initial approximating energy satisfies

$$
\int_\Omega \frac{1}{2} |\boldsymbol{v}_n(0)|^2 + A\Psi(\varphi_n(0)) + \frac{B}{2} |\nabla \varphi_n(0)|^2 + \frac{1}{2} |\sigma_n(0)|^2 + \chi \sigma_n(0)(1 - \varphi_n(0)) \, dx
$$

$$
\leqslant C \left(1 + \|\boldsymbol{v}_0\|_{L^2}^2 + \|\varphi_0\|_{H^1}^{r+2} + \|\sigma_0\|_{L^2}^2 \right)
$$

with a constant C that is independent of n due to the fact that

$$
\|\mathbb{P}_{W_n}(f)\|_X \leqslant C\|f\|_X \text{ for any } X \supseteq H_N^2, \quad \|\mathbb{P}_{Y_n}(\boldsymbol{f})\|_{\boldsymbol{Z}} \leqslant C\|\boldsymbol{f}\|_{\boldsymbol{Z}} \text{ for } \boldsymbol{Z} = \boldsymbol{L}_\sigma^2 \text{ or } \boldsymbol{H}_{0,\sigma}^1.
$$

By the assumptions on Γ and S, we see that the right-hand side of (4.4) can be estimated as

$$
|\mathrm{RHS}| \leqslant \|S\|_{L^2} \left(\|\sigma_n\|_{L^2} + \chi \|\varphi_n\|_{L^2} + C \right) + \|\mu_n\|_{L^1} \|\Gamma\|_{L^\infty}
$$
$$
\leqslant C \left(1 + \|\varphi_n\|_{L^2} + \|\sigma_n\|_{L^2} \right) \|S\|_{L^2} + \|\Gamma\|_{L^\infty} \|\mu_n - \overline{\mu_n}\|_{L^1} + |\overline{\mu_n}| \|\Gamma\|_{L^\infty}. \qquad (4.5)
$$

Note that by assumption (3.15) it holds that

$$
|\overline{\mu_n}| \leqslant C \left(\|\Psi'(\varphi_n)\|_{L^1} + \|\sigma_n\|_{L^1} \right) \leqslant C \left(1 + \|\Psi(\varphi_n)\|_{L^1} + \|\sigma_n\|_{L^1} \right). \qquad (4.6)
$$

Then, applying Poincaré's inequality in L^1, Hölder's inequality and Young's inequality to (4.5), we have

$$
|\mathrm{RHS}| \leqslant C \left(1 + \|\varphi_n\|_{L^2}^2 + \|\sigma_n\|_{L^2}^2 \right) + \|S\|_{L^2}^2 + C\|\Gamma\|_{L^\infty}^2 + \frac{m_0}{2} \|\nabla \mu_n\|_{L^2}^2
$$
$$
+ C\|\Gamma\|_{L^\infty}(1 + \|\Psi(\varphi_n)\|_{L^1} + \|\sigma_n\|_{L^1}). \qquad (4.7)
$$

Employing the lower bounds of η, m and n, the inequality

$$
\frac{1}{2} \|\nabla \sigma_n\|_{L^2}^2 \leqslant \|\nabla(\sigma_n - \varphi_n)\|_{L^2}^2 + \|\nabla \varphi_n\|_{L^2}^2
$$

and the quadratic lower bound of Ψ in (3.15), we obtain from (4.4) the inequality

$$
\begin{aligned}
& \frac{d}{dt} \int_\Omega \frac{1}{2} |\boldsymbol{v}_n|^2 + A\Psi(\varphi_n) + \frac{B}{2} |\nabla\varphi_n|^2 + \frac{1}{2} |\sigma_n|^2 + \chi\sigma_n(1-\varphi_n) \, dx \\
& \quad + \int_\Omega \frac{m_0}{2} |\nabla\mu_n|^2 + \frac{n_0}{2} |\nabla\sigma_n|^2 + 2\eta_0 |\mathrm{D}\boldsymbol{v}_n|^2 \, dx \\
& \quad \leqslant \|S\|_{L^2}^2 + C \left(1 + \|\Gamma\|_{L^\infty}^2\right) \left(1 + \|\Psi(\varphi_n)\|_{L^1} + \|\nabla\varphi_n\|_{L^2}^2 + \|\sigma_n\|_{L^2}^2\right).
\end{aligned}
\tag{4.8}
$$

Applying Young's inequality and (3.15), we see that (cf. [28, (4.37)] or [31])

$$
\begin{aligned}
& \int_\Omega A\Psi(\varphi_n) + \frac{1}{2} |\sigma_n|^2 + \chi\sigma_n(1-\varphi_n) \, dx \\
& \quad \geqslant \left(A - \frac{2\chi^2}{C_3}\right) \|\Psi(\varphi_n)\|_{L^1} + \frac{1}{4} \|\sigma_n\|_{L^2}^2 - C.
\end{aligned}
\tag{4.9}
$$

Then, integrating (4.8) in time from 0 to arbitrary $s \in (0, T]$, by assumption (3.16) and estimate (4.9), we find that

$$
\begin{aligned}
& \left(\|\boldsymbol{v}_n(s)\|_{L^2}^2 + \|\Psi(\varphi_n(s))\|_{L^1} + \|\nabla\varphi_n(s)\|_{L^2}^2 + \|\sigma_n(s)\|_{L^2}^2\right) \\
& \quad + \|\nabla\mu_n\|_{L^2(0,s;L^2)}^2 + \|\nabla\sigma_n\|_{L^2(0,s;L^2)}^2 + \|\mathrm{D}\boldsymbol{v}_n\|_{L^2(0,s;L^2)}^2 \\
& \quad \leqslant \|S\|_{L^2(0,T;L^2)}^2 + C \int_0^s \left(1 + \|\Gamma\|_{L^\infty}^2\right) \left(1 + \|\Psi(\varphi_n)\|_{L^1} + \|\nabla\varphi_n\|_{L^2}^2 + \|\sigma_n\|_{L^2}^2\right) \, dt + C.
\end{aligned}
$$

Applying Gronwall's lemma in integral form (see e.g., [31, Lemma 3.1]) leads to

$$
\begin{aligned}
& \sup_{t\in[0,T]} \left(\|\boldsymbol{v}_n(t)\|_{L^2}^2 + \|\Psi(\varphi_n(t))\|_{L^1} + \|\nabla\varphi_n(t)\|_{L^2}^2 + \|\sigma_n(t)\|_{L^2}^2\right) \\
& \quad + \|\nabla\mu_n\|_{L^2(Q)}^2 + \|\nabla\sigma_n\|_{L^2(Q)}^2 + \|\mathrm{D}\boldsymbol{v}_n\|_{L^2(Q)}^2 \leqslant C.
\end{aligned}
$$

Using Korn's inequality, Poincaré's inequality, (3.15) and (4.6), we see that

$$
\begin{aligned}
& \|\boldsymbol{v}_n\|_{L^\infty(0,T;L^2)\cap L^2(0,T;H^1)} + \|\varphi_n\|_{L^\infty(0,T;H^1)} + \|\Psi(\varphi_n)\|_{L^\infty(0,T;L^1)} \\
& \quad + \|\sigma_n\|_{L^\infty(0,T;L^2)\cap L^2(0,T;H^1)} + \|\mu_n\|_{L^2(0,T;H^1)} \leqslant C.
\end{aligned}
\tag{4.10}
$$

In particular, the uniform estimate (4.10) ensures that we can extend $(\varphi_n, \mu_n, \sigma_n, \boldsymbol{v}_n)$ to the full time interval $[0, T]$, and so $t_n = T$ for all $n \in \mathbb{N}$.

Second estimate: In order to obtain higher order estimates for φ_n, we view (4.2d) as an elliptic equation for φ_n subject to a homogeneous Neumann boundary condition. For $d = 2$, we see that

$$
\begin{aligned}
\|\mathbb{P}_{W_n}(\Psi'(\varphi_n))\|_{H^1} & \leqslant C \left(\|\nabla(\Psi'(\varphi_n))\|_{L^2} + \|\Psi'(\varphi_n)\|_{L^2}\right) \\
& \leqslant C\|\Psi''(\varphi_n)\|_{L^4}\|\nabla\varphi_n\|_{L^4} + C\|\Psi'(\varphi_n)\|_{L^2} \\
& \leqslant C \left(1 + \|\varphi_n\|_{L^{4r}}^r\right) \|\varphi_n\|_{H^2}^{\frac{1}{2}} \|\nabla\varphi_n\|_{L^2}^{\frac{1}{2}} + C \left(1 + \|\varphi_n\|_{L^{2(r+1)}}^{r+1}\right) \\
& \leqslant C\|\varphi_n\|_{H^2}^{\frac{1}{2}} + C,
\end{aligned}
$$

where we have used (3.15), the Gagliardo–Nirenberg inequality $\|f\|_{L^4} \leqslant C\|f\|_{H^1}^{\frac{1}{2}}\|f\|_{L^2}^{\frac{1}{2}}$ for two dimensions, and the fact that (4.10) implies $\varphi_n \in L^\infty(0, T; L^s)$ for all $s \in [1, +\infty)$. Then, by the elliptic estimate for (4.2*d*) and Young's inequality, we get

$$\|\varphi_n\|_{H^3} \leqslant C \left(\|\mu_n\|_{H^1} + A\|\mathbb{P}_{W_n}(\Psi'(\varphi_n))\|_{H^1} + \chi\|\sigma_n\|_{H^1} + \|\varphi_n\|_{L^2} \right)$$
$$\leqslant \frac{1}{2}\|\varphi_n\|_{H^2} + C \left(\|\mu_n\|_{H^1} + \|\sigma_n\|_{H^1} + 1 \right), \tag{4.11}$$

which together with (4.10) yields the higher order estimate

$$\|\varphi_n\|_{L^2(0,T;H^3)} + \|\Psi'(\varphi_n)\|_{L^2(0,T;H^1)} \leqslant C. \tag{4.12}$$

In the case $d = 3$, since the potential Ψ has at most polynomial growth of order 6, we can appeal to the bootstrapping argument used in [28, Section 4.2] and [30, Section 3.3] to obtain the higher order estimate (4.12).

Third estimate: Using the facts that $\nabla\varphi_n \in L^2(0, T; \mathbf{H}^2) \subset L^2(0, T; \mathbf{L}^\infty)$ and $\mathbf{v}_n \in L^\infty(0, T; \mathbf{L}^2)$, we have

$$\|\mathbf{v}_n \cdot \nabla\varphi_n\|_{L^2(Q)} \leqslant C. \tag{4.13}$$

Then, by testing (4.2*a*) with an arbitrary test function $\zeta \in L^2(0, T; H^1)$, we easily see from (4.10) and (4.13) that

$$\|\partial_t\varphi_n\|_{L^2(0,T;(H^1)')} \leqslant C. \tag{4.14}$$

Fourth estimate: We now estimate the convection term for σ. In two dimensions, we recall the following Gagliardo–Nirenberg inequalities

$$\|f\|_{L^{2s}} \leqslant C\|f\|_{H^1}^{\frac{s-1}{s}}\|f\|_{L^2}^{\frac{1}{s}}, \quad \|f\|_{L^{\frac{2s}{s-1}}} \leqslant C\|f\|_{H^1}^{\frac{1}{s}}\|f\|_{L^2}^{\frac{s-1}{s}},$$

for any $s > 1$. Using boundedness of $\mathbf{v}_n$ and σ_n in $L^\infty(0, T; L^2) \cap L^2(0, T; H^1)$, for arbitrary $\zeta \in L^2(0, T; H^1)$, we see that

$$\left| \int_0^T \int_\Omega \sigma_n \mathbf{v}_n \cdot \nabla\zeta \, dx \, dt \right|$$
$$\leqslant \int_0^T \|\sigma_n\|_{L^{2s}}\|\mathbf{v}_n\|_{\mathbf{L}^{\frac{2s}{s-1}}}\|\nabla\zeta\|_{L^2} \, dt$$
$$\leqslant C\|\sigma_n\|_{L^\infty(0,T;L^2)}^{\frac{1}{s}}\|\mathbf{v}_n\|_{\mathbf{L}^\infty(0,T;\mathbf{L}^2)}^{\frac{s-1}{s}}\|\sigma_n\|_{L^2(0,T;H^1)}^{\frac{s-1}{s}}\|\mathbf{v}_n\|_{\mathbf{L}^2(0,T;\mathbf{H}^1)}^{\frac{1}{s}}\|\nabla\zeta\|_{L^2(Q)}$$

for any $s > 1$. This implies that $\mathbf{v}_n \cdot \nabla\sigma_n$ is uniformly bounded in $L^2(0, T; (H^1)')$. Then, from equation (4.2*b*), we find that

$$\|\partial_t\sigma_n\|_{L^2(0,T;(H^1)')} + \|\mathbf{v}_n \cdot \nabla\sigma_n\|_{L^2(0,T;(H^1)')} \leqslant C. \tag{4.15}$$

In three dimensions, the Gagliardo–Nirenberg inequality $\|f\|_{L^3} \leqslant C\|f\|_{H^1}^{\frac{1}{2}}\|f\|_{L^2}^{\frac{1}{2}}$ yields that for arbitrary $\zeta \in L^4(0, T; H^1)$,

$$\left| \int_0^T \int_\Omega (\boldsymbol{v}_n \cdot \nabla \sigma_n)\zeta \, dx \, dt \right|$$

$$\leqslant \int_0^T \|\boldsymbol{v}_n\|_{\boldsymbol{L}^3} \|\nabla \sigma_n\|_{L^2} \|\zeta\|_{L^6} \, dt$$

$$\leqslant C \|\boldsymbol{v}_n\|_{L^\infty(0,T;\boldsymbol{L}^2)}^{\frac{1}{2}} \|\sigma_n\|_{L^2(0,T;H^1)} \|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{H}^1)}^{\frac{1}{2}} \|\zeta\|_{L^4(0,T;H^1)},$$

which implies

$$\|\partial_t \sigma_n\|_{L^{\frac{4}{3}}(0,T;(H^1)')} + \|\boldsymbol{v}_n \cdot \nabla \sigma_n\|_{L^{\frac{4}{3}}(0,T;(H^1)')} \leqslant C. \tag{4.16}$$

Fifth estimate: We proceed to estimate the convection term in the momentum equation. Note that, by the boundedness of $\mu_n + \chi\sigma_n$ in $L^2(0, T; H^1)$ and $\nabla\varphi_n$ in $L^\infty(0, T; \boldsymbol{L}^2)$, it holds

$$\|(\mu_n + \chi\sigma_n)\nabla\varphi_n\|_{L^2(0,T;\boldsymbol{L}^{\frac{3}{2}})} \leqslant \|\nabla\varphi_n\|_{L^\infty(0,T;\boldsymbol{L}^2)} \|\mu_n + \chi\sigma_n\|_{L^2(0,T;L^6)},$$

and so we find that $(\mu_n + \chi\sigma_n)\nabla\varphi_n$ is bounded in $L^2(0, T; \boldsymbol{L}^{\frac{3}{2}}) \subset L^2(0, T; \boldsymbol{H}^{-1})$. In two dimensions, thanks to the boundedness of $\boldsymbol{v}_n$ in $L^\infty(0, T; \boldsymbol{L}^2) \cap L^2(0, T; \boldsymbol{H}^1)$ and the Gagliardo–Nirenberg inequality $\|f\|_{L^4} \leqslant C\|f\|_{L^2}^{\frac{1}{2}}\|f\|_{H^1}^{\frac{1}{2}}$, we see that for arbitrary $\zeta \in L^2(0, T; \boldsymbol{H}^1)$,

$$\left| \int_0^T \int_\Omega (\boldsymbol{v}_n \otimes \boldsymbol{v}_n) : \nabla\zeta \, dx \, dt \right| \leqslant \int_0^T \|\boldsymbol{v}_n\|_{\boldsymbol{L}^4}^2 \|\nabla\zeta\|_{L^2} \, dt$$

$$\leqslant C \|\boldsymbol{v}_n\|_{L^\infty(0,T;\boldsymbol{L}^2)} \|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{H}^1)} \|\zeta\|_{L^2(0,T;\boldsymbol{H}^1)}.$$

This implies

$$\|\partial_t \boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{H}^{-1})} + \|(\boldsymbol{v}_n \cdot \nabla)\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{H}^{-1})} \leqslant C. \tag{4.17}$$

In three dimensions, the Gagliardo–Nirenberg inequality $\|f\|_{L^4} \leqslant C\|f\|_{L^2}^{\frac{1}{4}}\|f\|_{H^1}^{\frac{3}{4}}$ yields for arbitrary $\zeta \in L^4(0, T; \boldsymbol{H}^1)$,

$$\left| \int_0^T \int_\Omega (\boldsymbol{v}_n \otimes \boldsymbol{v}_n) : \nabla\zeta \, dx \, dt \right| \leqslant \int_0^T \|\boldsymbol{v}_n\|_{\boldsymbol{L}^4}^2 \|\nabla\zeta\|_{\boldsymbol{L}^2} \, dx$$

$$\leqslant C \|\boldsymbol{v}_n\|_{L^\infty(0,T;\boldsymbol{L}^2)}^{\frac{1}{2}} \|\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{H}^1)}^{\frac{3}{2}} \|\zeta\|_{L^4(0,T;\boldsymbol{H}^1)}.$$

Hence, we obtain

$$\|\partial_t \boldsymbol{v}_n\|_{L^{\frac{4}{3}}(0,T;\boldsymbol{H}^{-1})} + \|(\boldsymbol{v}_n \cdot \nabla)\boldsymbol{v}_n\|_{L^{\frac{4}{3}}(0,T;\boldsymbol{H}^{-1})} \leqslant C. \tag{4.18}$$

Passage to the limit as $n \to +\infty$: The above a priori estimates (4.10)–(4.18) are sufficient to obtain compactness results that allow us to obtain a convergent subsequence of the approximate solutions $(\varphi_n, \mu_n, \sigma_n, \boldsymbol{v}_n)$, whose limit denoted by $(\varphi, \mu, \sigma, \boldsymbol{v})$ satisfies the weak formulation (3.17) as well as the initial conditions. Since this procedure is standard (see, e.g., [11] for the Navier–Stokes–Cahn–Hilliard system), we omit the details.

5 Proof of Theorem 2: global strong solutions in two dimensions

Keeping the a priori estimates (4.10)–(4.14) in mind, we proceed to derive further higher order estimates for the approximate solutions $(\varphi_n, \mu_n, \sigma_n, \boldsymbol{v}_n)$ obtained via the Galerkin scheme.

First estimate: Testing (4.2a) by $\Delta^2 \varphi_n \in W_n$ and using (4.2d), we get

$$
\begin{aligned}
\frac{d}{dt} \int_\Omega \frac{1}{2} |\Delta \varphi_n|^2 \, dx &+ B \int_\Omega m(\varphi_n) \left| \Delta^2 \varphi_n \right|^2 \, dx \\
&= - \int_\Omega (\boldsymbol{v}_n \cdot \nabla \varphi_n) \Delta^2 \varphi_n \, dx + A \int_\Omega m(\varphi_n) \Delta (\mathbb{P}_{W_n}(\Psi'(\varphi_n))) \Delta^2 \varphi_n \, dx \\
&\quad + \int_\Omega m'(\varphi_n)(\nabla \varphi_n \cdot \nabla \mu_n) \Delta^2 \varphi_n \, dx - \chi \int_\Omega m(\varphi_n) \Delta \sigma_n \Delta^2 \varphi_n \, dx \\
&\quad + \int_\Omega \Gamma \Delta^2 \varphi_n \, dx \\
&=: R_1 + R_2 + R_3 + R_4 + R_5.
\end{aligned}
\tag{5.1}
$$

The terms on the right-hand side of (5.1) can be estimated as follows. Using Hölder's inequality, Young's inequality and the Gagliardo–Nirenberg inequality (for $d = 2$)

$$
\|\nabla \varphi_n\|_{\boldsymbol{L}^\infty} \leqslant C \|\nabla \varphi_n\|_{\boldsymbol{H}^3}^{\frac{1}{3}} \|\nabla \varphi_n\|_{\boldsymbol{L}^2}^{\frac{2}{3}},
\tag{5.2}
$$

the boundedness of $\boldsymbol{v}_n$ in $L^\infty(0, T; \boldsymbol{L}^2)$ and φ_n in $L^\infty(0, T; H^1)$, and the elliptic estimate (3.14), we have

$$
\begin{aligned}
R_1 &\leqslant \|\boldsymbol{v}_n\|_{\boldsymbol{L}^2} \|\nabla \varphi_n\|_{\boldsymbol{L}^\infty} \|\Delta^2 \varphi_n\|_{L^2} \\
&\leqslant C \|\nabla \varphi_n\|_{\boldsymbol{H}^3}^{\frac{1}{3}} \|\nabla \varphi_n\|_{\boldsymbol{L}^2}^{\frac{2}{3}} \|\Delta^2 \varphi_n\|_{L^2} \\
&\leqslant C \left(\|\Delta^2 \varphi_n\|_{L^2} + \|\varphi_n\|_{L^2} \right)^{\frac{1}{3}} \|\varphi_n\|_{H^1}^{\frac{2}{3}} \|\Delta^2 \varphi_n\|_{L^2} \\
&\leqslant \varepsilon \|\Delta^2 \varphi_n\|_{L^2}^2 + C,
\end{aligned}
$$

for some positive constant ε yet to be determined. Next, noticing that from integration by parts and using $\partial_\nu \varphi_n = \partial_\nu (\Delta \varphi_n) = 0$ on $\partial \Omega$, it holds that

$$
\|\Delta \varphi_n\|_{L^2}^2 = \int_\Omega \varphi_n \Delta^2 \varphi_n \, dx \leqslant \|\Delta^2 \varphi_n\|_{L^2} \|\varphi_n\|_{L^2}.
$$

Then, together with estimates (4.10) and (3.14), Young's inequality and the Gagliardo–Nirenberg inequality $\|\nabla\varphi_n\|_{L^4} \leqslant C\|\varphi_n\|_{H^4}^{\frac{1}{6}}\|\nabla\varphi_n\|_{L^2}^{\frac{5}{6}}$, we have

$$
\begin{aligned}
&\|\Delta(\mathbb{P}_{W_n}(\Psi'(\varphi_n)))\|_{L^2} \\
&\quad \leqslant C\left(\|\Delta(\Psi'(\varphi_n))\|_{L^2} + \|\Psi'(\varphi_n)\|_{L^2}\right) \\
&\quad \leqslant C\|\Psi'''(\varphi_n)\|_{L^\infty}\|\nabla\varphi_n\|_{L^4}^2 + C\|\Psi''(\varphi_n)\|_{L^\infty}\|\Delta\varphi_n\|_{L^2} + C\|\Psi'(\varphi_n)\|_{L^2} \\
&\quad \leqslant C\left(1 + \|\varphi_n\|_{L^\infty}^{r-1}\right)\|\varphi_n\|_{H^4}^{\frac{1}{3}}\|\nabla\varphi_n\|_{L^2}^{\frac{5}{3}} + C\left(1 + \|\varphi_n\|_{L^\infty}^r\right)\|\Delta^2\varphi_n\|_{L^2}^{\frac{1}{2}}\|\varphi\|_{L^2}^{\frac{1}{2}} \\
&\quad\quad + C\left(1 + \|\varphi_n\|_{L^\infty}^{r+1}\right) \\
&\quad \leqslant C\left(1 + \|\varphi_n\|_{L^\infty}^{r-1}\right)\|\Delta^2\varphi_n\|_{L^2}^{\frac{1}{3}} + C\left(1 + \|\varphi_n\|_{L^\infty}^r\right)\|\Delta^2\varphi_n\|_{L^2}^{\frac{1}{2}} + C\left(1 + \|\varphi_n\|_{L^\infty}^{r+1}\right) \\
&\quad \leqslant C\left(1 + \|\varphi_n\|_{L^\infty}^{3r}\right)\|\Delta^2\varphi_n\|_{L^2}^{\frac{1}{2}} + C\left(1 + \|\varphi_n\|_{L^\infty}^{r+1}\right).
\end{aligned}
$$

Applying the Brézis–Gallouet inequality (3.12) for φ_n, using estimates (3.13) and (4.10) and Young's inequality, we deduce that

$$
\begin{aligned}
&\|\Delta(\mathbb{P}_{W_n}(\Psi'(\varphi_n)))\|_{L^2} \\
&\quad \leqslant C\left[1 + (\ln(1 + \|\Delta\varphi_n\|_{L^2}))^{\frac{3r}{2}}\right]\|\Delta^2\varphi_n\|_{L^2}^{\frac{1}{2}} + C\left[1 + (\ln(1 + \|\Delta\varphi_n\|_{L^2}))^{\frac{r+1}{2}}\right] \\
&\quad \leqslant C\left(1 + \|\Delta\varphi_n\|_{L^2}^{\frac{1}{2}}\right)\|\Delta^2\varphi_n\|_{L^2}^{\frac{1}{2}} + C\|\Delta\varphi_n\|_{L^2} + C.
\end{aligned}
\tag{5.3}
$$

For the second inequality in (5.3), we used the fact that for any $k \geqslant 1$,

$$
z(x) := \frac{(\ln(1+x))^k}{x^{\frac{1}{2}}} \to 0 \quad \text{as } x \to +\infty,
$$

as its derivative $z'(x) = \frac{(\ln(1+x))^{k-1}}{x^{\frac{1}{2}}}\left(\frac{k}{1+x} - \frac{\ln(1+x)}{2x}\right)$ becomes negative for sufficiently large x. As a consequence,

$$
\begin{aligned}
R_2 &\leqslant Am_1\|\Delta(\mathbb{P}_{W_n}(\Psi'(\varphi_n)))\|_{L^2}\|\Delta^2\varphi_n\|_{L^2} \\
&\leqslant \varepsilon\|\Delta^2\varphi_n\|_{L^2}^2 + C\|\Delta\varphi_n\|_{L^2}^2 + C.
\end{aligned}
\tag{5.4}
$$

Next, using the following estimate

$$
\begin{aligned}
\|\Delta\mu_n\|_{L^2} &\leqslant A\|\Delta(\mathbb{P}_{W_n}(\Psi'(\varphi_n)))\|_{L^2} + B\|\Delta^2\varphi_n\|_{L^2} + \chi\|\Delta\sigma_n\|_{L^2} \\
&\leqslant B\|\Delta^2\varphi_n\|_{L^2} + \chi\|\Delta\sigma_n\|_{L^2} + C\|\Delta\varphi_n\|_{L^2}^{\frac{1}{2}}\|\Delta^2\varphi_n\|_{L^2}^{\frac{1}{2}} + C\|\Delta\varphi_n\|_{L^2} + C \\
&\leqslant C\left(1 + \|\Delta^2\varphi_n\|_{L^2} + \|\Delta\varphi_n\|_{L^2} + \|\Delta\sigma_n\|_{L^2}\right),
\end{aligned}
$$

we obtain from (3.13) and Young's inequality that R_3 can be controlled as follows:

$$
\begin{aligned}
R_3 &\leqslant \|m'(\varphi_n)\|_{L^\infty} \|\nabla\varphi_n\|_{L^4} \|\nabla\mu_n\|_{L^4} \|\Delta^2\varphi_n\|_{L^2} \\
&\leqslant C\|\varphi_n\|_{H^1}^{\frac{1}{2}} \|\varphi_n\|_{H^2}^{\frac{1}{2}} \|\mu_n\|_{H^1}^{\frac{1}{2}} \|\mu_n\|_{H^2}^{\frac{1}{2}} \|\Delta^2\varphi_n\|_{L^2} \\
&\leqslant C\left(1+\|\Delta\varphi_n\|_{L^2}^{\frac{1}{2}}\right) \|\mu_n\|_{H^1}^{\frac{1}{2}} \left(\|\mu_n\|_{H^1}^{\frac{1}{2}} + \|\Delta\mu_n\|_{L^2}^{\frac{1}{2}}\right) \|\Delta^2\varphi_n\|_{L^2} \\
&\leqslant C\left(1+\|\Delta\varphi_n\|_{L^2}^{\frac{1}{2}}\right) \|\mu_n\|_{H^1} \|\Delta^2\varphi_n\|_{L^2} \\
&\quad + C\|\mu_n\|_{H^1}^{\frac{1}{2}} \left(1+\|\Delta^2\varphi_n\|_{L^2}+\|\Delta\varphi_n\|_{L^2}+\|\Delta\sigma_n\|_{L^2}\right)^{\frac{1}{2}} \|\Delta^2\varphi_n\|_{L^2} \\
&\quad + C\|\mu_n\|_{H^1}^{\frac{1}{2}} \|\Delta\varphi_n\|_{L^2}^{\frac{1}{2}} \left(1+\|\Delta^2\varphi_n\|_{L^2}+\|\Delta\varphi_n\|_{L^2}+\|\Delta\sigma_n\|_{L^2}\right)^{\frac{1}{2}} \|\Delta^2\varphi_n\|_{L^2} \\
&\leqslant \varepsilon\|\Delta^2\varphi_n\|_{L^2}^2 + \varepsilon\|\Delta\sigma_n\|_{L^2}^2 + C\left(1+\|\mu_n\|_{H^1}^2\right)\|\Delta\varphi_n\|_{L^2}^2 + C\|\mu_n\|_{H^1}^2 + C.
\end{aligned}
\tag{5.5}
$$

By Young's inequality, we easily have

$$
R_4 \leqslant \chi m_1 \|\Delta\sigma_n\|_{L^2} \|\Delta^2\varphi_n\|_{L^2} \leqslant \varepsilon\|\Delta^2\varphi_n\|_{L^2}^2 + \frac{\chi^2 m_1^2}{4\varepsilon}\|\Delta\sigma_n\|_{L^2}^2
\tag{5.6}
$$

and

$$
R_5 \leqslant \|\Gamma\|_{L^2} \|\Delta^2\varphi_n\|_{L^2} \leqslant \varepsilon\|\Delta^2\varphi_n\|_{L^2}^2 + C\|\Gamma\|_{L^2}^2.
\tag{5.7}
$$

Now testing (4.2*b*) by $-\Delta\sigma_n \in W_n$, we get

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega \frac{1}{2}|\nabla\sigma_n|^2 \, dx &+ \int_\Omega n(\varphi_n)|\Delta\sigma_n|^2 \, dx \\
&= \int_\Omega (\boldsymbol{v}_n\cdot\nabla\sigma_n)\Delta\sigma_n \, dx + \chi\int_\Omega n(\varphi_n)\Delta\varphi_n\Delta\sigma_n \, dx \\
&\quad - \int_\Omega n'(\varphi_n)(\nabla\varphi_n\cdot\nabla(\sigma_n-\chi\varphi_n))\Delta\sigma_n \, dx - \int_\Omega S\Delta\sigma_n \, dx \\
&:= R_6 + R_7 + R_8 + R_9.
\end{aligned}
\tag{5.8}
$$

The terms on the right-hand side of (5.8) can be estimated as follows. By the Gagliardo–Nirenberg inequality $\|f\|_{L^4} \leqslant C\|f\|_{H^1}^{\frac{1}{2}}\|f\|_{L^2}^{\frac{1}{2}}$, (3.13) and the estimate (4.10), it holds

$$
\begin{aligned}
R_6 &\leqslant C\|\boldsymbol{v}_n\|_{L^4}\|\nabla\sigma_n\|_{L^4}\|\Delta\sigma_n\|_{L^2} \\
&\leqslant C\|\boldsymbol{v}_n\|_{L^2}^{\frac{1}{2}}\|\nabla\boldsymbol{v}_n\|_{L^2}^{\frac{1}{2}}\|\nabla\sigma_n\|_{L^2}^{\frac{1}{2}}\|\sigma_n\|_{H^2}^{\frac{1}{2}}\|\Delta\sigma_n\|_{L^2} \\
&\leqslant C\|\nabla\boldsymbol{v}_n\|_{L^2}^{\frac{1}{2}}\|\nabla\sigma_n\|_{L^2}^{\frac{1}{2}}\left(\|\sigma_n\|_{L^2}^{\frac{1}{2}}+\|\Delta\sigma_n\|_{L^2}^{\frac{1}{2}}\right)\|\Delta\sigma_n\|_{L^2} \\
&\leqslant \varepsilon\|\Delta\sigma_n\|_{L^2}^2 + C\left(1+\|\nabla\boldsymbol{v}_n\|_{L^2}^2\right)\|\nabla\sigma_n\|_{L^2}^2.
\end{aligned}
$$

By Young's inequality, we easily have

$$
R_7 \leqslant \varepsilon\|\Delta\sigma_n\|_{L^2}^2 + C\|\Delta\varphi_n\|_{L^2}^2, \quad R_9 \leqslant \varepsilon\|\Delta\sigma_n\|_{L^2}^2 + C\|S\|_{L^2}^2.
\tag{5.9}
$$

Meanwhile, for R_8, we have by applying estimates (4.10) and (3.13),

$$
\begin{aligned}
R_8 &\leqslant \|n'(\varphi_n)\|_{L^\infty} \|\nabla\varphi_n\|_{\boldsymbol{L}^4} \|\nabla\sigma_n\|_{\boldsymbol{L}^4} \|\Delta\sigma_n\|_{L^2} + \chi\|n'(\varphi_n)\|_{L^\infty} \|\nabla\varphi_n\|_{\boldsymbol{L}^4}^2 \|\Delta\sigma_n\|_{L^2} \\
&\leqslant C\|\varphi_n\|_{H^2}^{\frac{1}{2}} \|\nabla\varphi_n\|_{\boldsymbol{L}^2}^{\frac{1}{2}} \|\sigma_n\|_{H^2}^{\frac{1}{2}} \|\nabla\sigma_n\|_{\boldsymbol{L}^2}^{\frac{1}{2}} \|\Delta\sigma_n\|_{L^2} + C\|\varphi_n\|_{H^2} \|\nabla\varphi_n\|_{\boldsymbol{L}^2} \|\Delta\sigma_n\|_{L^2} \\
&\leqslant C\left(\|\varphi_n\|_{L^2}^{\frac{1}{2}} + \|\Delta\varphi_n\|_{L^2}^{\frac{1}{2}}\right) \|\nabla\varphi_n\|_{\boldsymbol{L}^2}^{\frac{1}{2}} \left(\|\sigma_n\|_{L^2}^{\frac{1}{2}} + \|\Delta\sigma_n\|_{L^2}^{\frac{1}{2}}\right) \|\nabla\sigma_n\|_{\boldsymbol{L}^2}^{\frac{1}{2}} \|\Delta\sigma_n\|_{L^2} \\
&\quad + C\left(\|\varphi_n\|_{L^2} + \|\Delta\varphi_n\|_{L^2}\right) \|\nabla\varphi_n\|_{\boldsymbol{L}^2} \|\Delta\sigma_n\|_{L^2} \\
&\leqslant C\left(1 + \|\Delta\varphi_n\|_{L^2}^{\frac{1}{2}}\right)\left(1 + \|\Delta\sigma_n\|_{L^2}^{\frac{1}{2}}\right) \|\nabla\sigma_n\|_{\boldsymbol{L}^2}^{\frac{1}{2}} \|\Delta\sigma_n\|_{L^2} \\
&\quad + C\left(1 + \|\Delta\varphi_n\|_{L^2}\right) \|\Delta\sigma_n\|_{L^2} \\
&\leqslant \varepsilon\|\Delta\sigma_n\|_{L^2}^2 + C\left(1 + \|\nabla\sigma_n\|_{\boldsymbol{L}^2}^2\right) \|\Delta\varphi_n\|_{L^2}^2 + C\|\nabla\sigma_n\|_{\boldsymbol{L}^2}^2 + C.
\end{aligned}
\tag{5.10}
$$

Then, multiplying (5.8) by ε^{-2}, adding the resultant with (5.1), we deduce from the above estimates (5.3)–(5.7) and (5.9)–(5.10) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Delta\varphi_n\|_{L^2}^2 + \varepsilon^{-2}\|\nabla\sigma_n\|_{L^2}^2\right) + (Bm_0 - 5\varepsilon)\|\Delta^2\varphi_n\|_{L^2}^2 \\
&\quad + \left(\frac{n_0}{\varepsilon^2} - \varepsilon - \frac{4}{\varepsilon} - \frac{\chi^2 m_1^2}{4\varepsilon}\right) \|\Delta\sigma_n\|_{L^2}^2 \\
&\leqslant C\left(1 + \|\mu_n\|_{H^1}^2 + \|\nabla\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 + \|\nabla\sigma_n\|_{\boldsymbol{L}^2}^2\right)\left(\|\Delta\varphi_n\|_{L^2}^2 + \|\nabla\sigma_n\|_{\boldsymbol{L}^2}^2\right) \\
&\quad + C\left(1 + \|\mu_n\|_{H^1}^2 + \|\nabla\sigma_n\|_{\boldsymbol{L}^2}^2 + \|\Gamma\|_{L^2}^2 + \|S\|_{L^2}^2\right).
\end{aligned}
\tag{5.11}
$$

We can choose

$$
\varepsilon = \min\left\{1,\ \frac{Bm_0}{10},\ \frac{2n_0}{20 + \chi^2 m_1^2}\right\},
$$

then, by Gronwall's lemma, (5.11) and the lower order estimate (4.10), we obtain

$$
\sup_{t\in[0,T]}\left(\|\Delta\varphi_n(t)\|_{L^2}^2 + \|\nabla\sigma_n(t)\|_{\boldsymbol{L}^2}^2\right) + \|\Delta^2\varphi_n\|_{L^2(Q)}^2 + \|\Delta\sigma_n\|_{L^2(Q)}^2 \leqslant C,
$$

which further implies

$$
\|\varphi_n\|_{L^\infty(0,T;H^2)\cap L^2(0,T;H^4)} + \|\sigma_n\|_{L^\infty(0,T;H^1)\cap L^2(0,T;H^2)} \leqslant C.
\tag{5.12}
$$

Besides, we infer from (4.2d), (5.3) and (5.12) that

$$
\|\mu_n\|_{L^2(0,T;H^2)} \leqslant C.
\tag{5.13}
$$

Second estimate: By definition of the Stokes operator $\mathbb{S}$, we can write $\mathbb{S}\boldsymbol{v}_n = -\Delta\boldsymbol{v}_n + \nabla\pi_n$, which implies that (cf. [11, (3.30)])

$$
-\int_\Omega \eta(\varphi_n)\Delta\boldsymbol{v}_n \cdot \mathbb{S}\boldsymbol{v}_n\, dx = \int_\Omega \eta(\varphi_n)\,|\mathbb{S}\boldsymbol{v}_n|^2\, dx - \int_\Omega \eta(\varphi_n)\nabla\pi_n \cdot \mathbb{S}\boldsymbol{v}_n\, dx.
$$

Then, testing (4.2c) by $\mathbb{S}v_n \in Y_n$, using (4.2d) and the above fact, we get

$$
\begin{aligned}
\frac{d}{dt} &\int_\Omega \frac{1}{2} |\nabla v_n|^2 \, dx + \int_\Omega \eta(\varphi_n) |\mathbb{S}v_n|^2 \, dx \\
&= -\int_\Omega [(v_n \cdot \nabla)v_n] \cdot \mathbb{S}v_n \, dx + \int_\Omega 2\eta'(\varphi_n)\nabla\varphi_n \cdot (\mathbb{D}v_n \mathbb{S}v_n) \, dx \\
&\quad + \int_\Omega \eta(\varphi_n)\nabla\pi_n \cdot \mathbb{S}v_n \, dx - B\int_\Omega \Delta\varphi_n \nabla\varphi_n \cdot \mathbb{S}v_n \, dx \\
&\quad + A\int_\Omega \mathbb{P}_{W_n}(\Psi'(\varphi_n))\nabla\varphi_n \cdot \mathbb{S}v_n \, dx \\
&=: R_{10} + R_{11} + R_{12} + R_{13} + R_{14}.
\end{aligned}
\tag{5.14}
$$

Keeping in mind the estimate (5.12), the reminder terms on the right-hand side of (5.14) can be estimated as follows. Using (3.7), we have

$$
\begin{aligned}
R_{10} &\leqslant \|v_n\|_{L^4}\|\nabla v_n\|_{L^4}\|\mathbb{S}v_n\|_{L^2} \\
&\leqslant C\|v_n\|_{L^2}^{\frac{1}{2}}\|\nabla v_n\|_{L^2}\|\nabla v_n\|_{H^1}^{\frac{1}{2}}\|\mathbb{S}v_n\|_{L^2} \\
&\leqslant C\|v_n\|_{L^2}^{\frac{1}{2}}\|\nabla v_n\|_{L^2}\|\mathbb{S}v_n\|_{L^2}^{\frac{3}{2}} \\
&\leqslant \varepsilon\|\mathbb{S}v_n\|_{L^2}^2 + C\|\nabla v_n\|_{L^2}^4.
\end{aligned}
$$

Using (5.2) and Young's inequality, it holds

$$
\begin{aligned}
R_{11} &\leqslant 2\|\eta'(\varphi_n)\|_{L^\infty}\|\nabla\varphi_n\|_{L^\infty}\|\mathbb{D}v_n\|_{L^2}\|\mathbb{S}v_n\|_{L^2} \\
&\leqslant C\|\nabla\varphi_n\|_{H^3}^{\frac{1}{3}}\|\nabla\varphi_n\|_{L^2}^{\frac{2}{3}}\|\nabla v_n\|_{L^2}\|\mathbb{S}v_n\|_{L^2} \\
&\leqslant C\left(\|\Delta^2\varphi_n\|_{L^2} + \|\varphi_n\|_{L^2}\right)^{\frac{1}{3}}\|\nabla v_n\|_{L^2}\|\mathbb{S}v_n\|_{L^2} \\
&\leqslant \varepsilon\|\mathbb{S}v_n\|_{L^2}^2 + C\|\nabla v_n\|_{L^2}^3 + C\|\Delta^2\varphi_n\|_{L^2}^2 + C.
\end{aligned}
$$

Applying integration by parts and the estimate for the Stokes problem (see Lemma 3.1), and the facts that $\mathrm{div}(\mathbb{S}v_n) = 0$ as well as $\mathbb{S}v_n \cdot v = 0$ on $\partial\Omega$, we infer from the higher order estimate (5.12) that

$$
\begin{aligned}
R_{12} &= -\int_\Omega \eta'(\varphi_n)\pi_n\nabla\varphi_n \cdot \mathbb{S}v_n \, dx \\
&\leqslant \|\eta'(\varphi_n)\|_{L^\infty}\|\pi_n\|_{L^4}\|\nabla\varphi_n\|_{L^4}\|\mathbb{S}v_n\|_{L^2} \\
&\leqslant C\|\pi_n\|_{L^2}^{\frac{1}{2}}\|\pi_n\|_{H^1}^{\frac{1}{2}}\|\nabla\varphi_n\|_{L^2}^{\frac{1}{2}}\|\varphi_n\|_{H^2}^{\frac{1}{2}}\|\mathbb{S}v_n\|_{L^2} \\
&\leqslant C_*\left(\nu\|\mathbb{S}v_n\|_{L^2}^{\frac{1}{2}} + C_\nu\|\nabla v_n\|_{L^2}^{\frac{1}{2}}\right)\|\mathbb{S}v_n\|_{L^2}^{\frac{3}{2}} \\
&\leqslant C_*\nu\|\mathbb{S}v_n\|_{L^2}^2 + \varepsilon\|\mathbb{S}v_n\|_{L^2}^2 + \frac{C_*^2 C_\nu^2}{4\varepsilon}\|\nabla v_n\|_{L^2}^2,
\end{aligned}
$$

where the constant C_* is independent of ε and ν. Next, for R_{13}, we deduce from (5.2) and (5.12) that

$$R_{13} \leqslant B\|\varDelta\varphi_n\|_{L^2}\|\nabla\varphi_n\|_{\boldsymbol{L}^\infty}\|\mathbb{S}\boldsymbol{v}_n\|_{L^2}$$

$$\leqslant \|\varDelta\varphi_n\|_{L^2}\|\nabla\varphi_n\|_{\boldsymbol{H}^3}^{\frac{1}{3}}\|\nabla\varphi_n\|_{\boldsymbol{L}^2}^{\frac{2}{3}}\|\mathbb{S}\boldsymbol{v}_n\|_{L^2}$$

$$\leqslant \varepsilon\|\mathbb{S}\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 + C\|\varDelta^2\varphi_n\|_{L^2}^2 + C.$$

Finally, thanks to the fact that $\varphi_n \in L^\infty(0,T;H^2)$, we observe

$$R_{14} \leqslant A\|\mathbb{P}_{W_n}(\Psi'(\varphi_n))\|_{L^2}\|\nabla\varphi_n\|_{\boldsymbol{L}^\infty}\|\mathbb{S}\boldsymbol{v}_n\|_{L^2}$$

$$\leqslant C\left(1 + \|\varphi_n\|_{L^{2r+2}}^{r+1}\right)\|\nabla\varphi_n\|_{\boldsymbol{H}^3}^{\frac{1}{3}}\|\nabla\varphi_n\|_{\boldsymbol{L}^2}^{\frac{2}{3}}\|\mathbb{S}\boldsymbol{v}_n\|_{L^2}$$

$$\leqslant \varepsilon\|\mathbb{S}\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 + C\|\varDelta^2\varphi_n\|_{L^2}^2 + C.$$

Collecting the above estimates, we infer from (5.14) that

$$\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 + (\eta_0 - 5\varepsilon - C_*\nu)\|\mathbb{S}\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2$$

$$\leqslant \left(C\|\nabla\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 + \frac{C_*^2C_\nu^2}{4\varepsilon}\right)\|\nabla\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 + C\|\varDelta^2\varphi_n\|_{L^2}^2 + C.$$

In the above inequality, choosing

$$\varepsilon = \frac{\eta_0}{20} \quad\text{and}\quad \nu = \frac{\eta_0}{4C_*},$$

then, by Gronwall's lemma and (5.12), we get

$$\sup_{t\in[0,T]}\|\nabla\boldsymbol{v}_n(t)\|_{\boldsymbol{L}^2}^2 + \|\mathbb{S}\boldsymbol{v}_n\|_{L^2(0,T;L^2)}^2 \leqslant C,$$

which together with Lemma 3.1 yields that

$$\|\boldsymbol{v}_n\|_{L^\infty(0,T;\boldsymbol{H}^1)\cap L^2(0,T;\boldsymbol{H}^2)} \leqslant C. \tag{5.15}$$

Third estimate: We now obtain further estimates for the time derivatives $\partial_t\varphi_n$, $\partial_t\sigma_n$ and $\partial_t\boldsymbol{v}_n$. Thanks to the estimates

$$\|\mathbb{P}_{W_n}(\boldsymbol{v}_n\cdot\nabla\varphi_n)\|_{L^2} \leqslant \|\boldsymbol{v}_n\cdot\nabla\varphi_n\|_{L^2} \leqslant \|\boldsymbol{v}_n\|_{\boldsymbol{L}^4}\|\nabla\varphi_n\|_{\boldsymbol{L}^4}$$

and

$$\|\mathbb{P}_{W_n}(\text{div}\,(m(\varphi_n)\nabla\mu_n))\|_{L^2} \leqslant \|\text{div}\,(m(\varphi_n)\nabla\mu_n)\|_{L^2}$$

$$\leqslant \|m'(\varphi_n)\|_{L^\infty}\|\nabla\varphi_n\|_{\boldsymbol{L}^4}\|\nabla\mu_n\|_{\boldsymbol{L}^4} + \|m(\varphi_n)\|_{L^\infty}\|\varDelta\mu_n\|_{L^2},$$

we infer from equation (4.2*a*) and estimates (5.12), (5.13) and (5.15) that

$$\|\partial_t\varphi_n\|_{L^2(0,T;L^2)} \leqslant C. \tag{5.16}$$

In a similar manner, from the estimates

$$\|\mathbb{P}_{W_n}(\boldsymbol{v}_n\cdot\nabla\sigma_n)\|_{L^2} \leqslant \|\boldsymbol{v}_n\cdot\nabla\sigma_n\|_{L^2} \leqslant \|\boldsymbol{v}_n\|_{\boldsymbol{L}^4}\|\nabla\sigma_n\|_{\boldsymbol{L}^4}$$

and

$$\|\mathbb{P}_{W_n}(\mathrm{div}\,(n(\varphi_n)\nabla(\sigma_n - \chi\varphi_n)))\|_{L^2}$$
$$\leqslant \|\mathrm{div}\,(n(\varphi_n)\nabla(\sigma_n - \chi\varphi_n))\|_{L^2}$$
$$\leqslant \|n'(\varphi_n)\|_{L^\infty}\|\nabla\varphi_n\|_{L^4}\left(\|\nabla\sigma_n\|_{L^4} + \chi\|\nabla\varphi_n\|_{L^4}\right)$$
$$+ \|n(\varphi_n)\|_{L^\infty}\left(\|\Delta\sigma_n\|_{L^2} + \chi\|\Delta\varphi_n\|_{L^2}\right)$$

as well as (4.2*b*), we conclude

$$\|\partial_t\sigma_n\|_{L^2(0,T;L^2)} \leqslant C. \tag{5.17}$$

Finally, testing (4.2*c*) with $\partial_t\boldsymbol{v}_n$ (again $\partial_t\boldsymbol{v}_n \in \boldsymbol{Y}_n$ and thus the projection operator $\mathbb{P}_{\boldsymbol{Y}_n}$ can be shifted to $\partial_t\boldsymbol{v}_n$) and using (5.12) and (5.15), we get

$$\|\partial_t\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2$$
$$= \int_\Omega \mathrm{div}\,(2\eta(\varphi_n)\mathrm{D}\boldsymbol{v}_n)\cdot\partial_t\boldsymbol{v}_n - (\boldsymbol{v}_n\cdot\nabla)\boldsymbol{v}_n\cdot\partial_t\boldsymbol{v}_n + (\mu_n + \chi\sigma_n)\nabla\varphi_n\cdot\partial_t\boldsymbol{v}_n\,dx$$
$$\leqslant 2\|\eta(\varphi_n)\|_{L^\infty}\|\Delta\boldsymbol{v}_n\|_{\boldsymbol{L}^2}\|\partial_t\boldsymbol{v}_n\|_{\boldsymbol{L}^2} + 2\|\eta'(\varphi_n)\|_{L^\infty}\|\nabla\varphi_n\|_{\boldsymbol{L}^\infty}\|\nabla\boldsymbol{v}_n\|_{\boldsymbol{L}^2}\|\partial_t\boldsymbol{v}_n\|_{\boldsymbol{L}^2}$$
$$+ \|\boldsymbol{v}_n\|_{\boldsymbol{L}^4}\|\nabla\boldsymbol{v}_n\|_{\boldsymbol{L}^4}\|\partial_t\boldsymbol{v}_n\|_{\boldsymbol{L}^2} + \|(\mu_n + \chi\sigma_n)\nabla\varphi_n\|_{\boldsymbol{L}^2}\|\partial_t\boldsymbol{v}_n\|_{\boldsymbol{L}^2}$$
$$\leqslant \frac{1}{2}\|\partial_t\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 + C\|\Delta\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 + C\|\nabla\varphi_n\|_{\boldsymbol{L}^\infty}^2\|\nabla\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2 + C\|\boldsymbol{v}_n\|_{\boldsymbol{L}^4}^2\|\nabla\boldsymbol{v}_n\|_{\boldsymbol{L}^4}^2$$
$$+ C\|(\mu_n + \chi\sigma_n)\|_{\boldsymbol{L}^2}^2\|\nabla\varphi_n\|_{\boldsymbol{L}^\infty}^2. \tag{5.18}$$

Using the facts that $\mu_n + \chi\sigma_n \in L^\infty(0,T;L^2)$ and $\nabla\varphi \in L^2(0,T;\boldsymbol{H}^2) \subset L^2(0,T;\boldsymbol{L}^\infty)$, the last term on the right-hand side of (5.18) belongs to $L^1(0,T)$. Furthermore, by the Gagliardo–Nirenberg inequality

$$\|\boldsymbol{v}_n\|_{\boldsymbol{L}^4}^2\|\nabla\boldsymbol{v}_n\|_{\boldsymbol{L}^4}^2 \leqslant C\|\boldsymbol{v}_n\|_{\boldsymbol{L}^2}\|\nabla\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^2\|\nabla\boldsymbol{v}_n\|_{\boldsymbol{H}^1}$$

and estimate (5.15), we can conclude that

$$\|\partial_t\boldsymbol{v}_n\|_{L^2(0,T;\boldsymbol{L}^2)} \leqslant C. \tag{5.19}$$

Fourth estimate: Taking the time derivative of (4.2*d*) leads to

$$\partial_t\mu_n = A\mathbb{P}_{W_n}(\Psi''(\varphi_n)\partial_t\varphi_n) - B\Delta\partial_t\varphi_n - \chi\partial_t\sigma_n. \tag{5.20}$$

Then, testing (5.20) with $-\Delta\mu_n$ and using integration by parts and (4.2a) yields that

$$
\begin{aligned}
\frac{d}{dt} \int_\Omega \frac{1}{2} |\nabla\mu_n|^2 \, dx &+ B \int_\Omega m(\varphi_n) |\nabla\Delta\mu_n|^2 \, dx \\
&= -A \int_\Omega \Psi''(\varphi_n)\partial_t\varphi_n\Delta\mu_n \, dx + \chi \int_\Omega \partial_t\sigma_n\Delta\mu_n \, dx - B \int_\Omega \nabla\Gamma \cdot \nabla\Delta\mu_n \, dx \\
&\quad - B \int_\Omega m'(\varphi_n)\Delta\mu_n\nabla\varphi_n \cdot \nabla\Delta\mu_n \, dx - B \int_\Omega \nabla\left(m'(\varphi_n)\nabla\mu_n \cdot \nabla\varphi_n\right) \cdot \nabla\Delta\mu_n \, dx \\
&\quad + B \int_\Omega \nabla\left(\boldsymbol{v}_n \cdot \nabla\varphi_n\right) \cdot \nabla\Delta\mu_n \, dx \\
&=: R_{15} + R_{16} + R_{17} + R_{18} + R_{19} + R_{20},
\end{aligned}
\tag{5.21}
$$

where we have used the fact that $\Delta\mu_n \in W_n$ and thus

$$
(\mathbb{P}_{W_n}(\Psi''(\varphi_n)\partial_t\varphi_n), \Delta\mu_n) = (\Psi''(\varphi_n)\partial_t\varphi_n, \mathbb{P}_{W_n}(\Delta\mu_n)) = (\Psi''(\varphi_n)\partial_t\varphi_n, \Delta\mu_n).
$$

Here, we point out that the assumption $m \in C_b^2(\mathbb{R})$ is used for this estimate. It follows from (5.12) and Young's inequality that

$$
\begin{aligned}
R_{15} &\leqslant A\|\Psi''(\varphi_n)\|_{L^\infty}\|\partial_t\varphi_n\|_{L^2}\|\Delta\mu_n\|_{L^2} \\
&\leqslant C\left(1 + \|\varphi_n\|_{L^\infty}^r\right)\|\partial_t\varphi_n\|_{L^2}\|\nabla\mu_n\|_{\boldsymbol{L}^2}^{\frac{1}{2}}\|\nabla\Delta\mu_n\|_{\boldsymbol{L}^2}^{\frac{1}{2}} \\
&\leqslant \varepsilon\|\nabla\Delta\mu_n\|_{\boldsymbol{L}^2}^2 + C\|\partial_t\varphi_n\|_{L^2}^2 + C\|\nabla\mu_n\|_{\boldsymbol{L}^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
R_{16} &\leqslant \chi\|\partial_t\sigma_n\|_{L^2}\|\Delta\mu_n\|_{L^2} \\
&\leqslant \varepsilon\|\nabla\Delta\mu_n\|_{\boldsymbol{L}^2}^2 + C\|\partial_t\sigma_n\|_{L^2}^2 + C\|\nabla\mu_n\|_{\boldsymbol{L}^2}^2,
\end{aligned}
$$

$$
R_{17} \leqslant B\|\nabla\Gamma\|_{\boldsymbol{L}^2}\|\nabla\Delta\mu_n\|_{\boldsymbol{L}^2} \leqslant \varepsilon\|\nabla\Delta\mu_n\|_{\boldsymbol{L}^2}^2 + C\|\nabla\Gamma\|_{\boldsymbol{L}^2}^2,
$$

where we use the fact that $\partial_\nu(\Delta\mu_n) = 0$ on $\partial\Omega$ and integration by parts to deduce that

$$
\|\Delta\mu_n\|_{L^2}^2 = \int_\Omega \nabla\Delta\mu_n \cdot \nabla\mu_n \, dx \leqslant \|\nabla\Delta\mu_n\|_{\boldsymbol{L}^2}\|\nabla\mu_n\|_{\boldsymbol{L}^2}.
$$

By Agmon's inequality $\|f\|_{L^\infty} \leqslant C\|f\|_{H^2}^{\frac{1}{2}}\|f\|_{L^2}^{\frac{1}{2}}$, we see that

$$
\begin{aligned}
R_{18} &\leqslant B\|m'(\varphi_n)\|_{L^\infty}\|\Delta\mu_n\|_{L^2}\|\nabla\varphi_n\|_{\boldsymbol{L}^\infty}\|\nabla\Delta\mu_n\|_{\boldsymbol{L}^2} \\
&\leqslant C\|\nabla\varphi_n\|_{\boldsymbol{L}^2}^{\frac{1}{2}}\|\varphi_n\|_{H^3}^{\frac{1}{2}}\|\nabla\mu_n\|_{\boldsymbol{L}^2}^{\frac{1}{2}}\|\nabla\Delta\mu_n\|_{\boldsymbol{L}^2}^{\frac{3}{2}} \\
&\leqslant \varepsilon\|\nabla\Delta\mu_n\|_{\boldsymbol{L}^2}^2 + C\|\varphi_n\|_{H^3}^2\|\nabla\mu_n\|_{\boldsymbol{L}^2}^2.
\end{aligned}
$$

In a similar manner, expanding the gradient term in R_{19} leads to

$$
\begin{aligned}
R_{19} &\leqslant B\|m''(\varphi_n)\|_{L^\infty}\|\nabla\varphi_n\|_{\boldsymbol{L}^\infty}^2\|\nabla\mu_n\|_{\boldsymbol{L}^2}\|\nabla\varDelta\mu_n\|_{\boldsymbol{L}^2} \\
&\quad + B\|m'(\varphi_n)\|_{L^\infty}\|\nabla^2\varphi_n\|_{\boldsymbol{L}^\infty}\|\nabla\mu_n\|_{\boldsymbol{L}^2}\|\nabla\varDelta\mu_n\|_{\boldsymbol{L}^2} \\
&\quad + B\|m'(\varphi_n)\|_{L^\infty}\|\nabla\varphi_n\|_{\boldsymbol{L}^\infty}\|\nabla^2\mu_n\|_{\boldsymbol{L}^2}\|\nabla\varDelta\mu_n\|_{\boldsymbol{L}^2} \\
&\leqslant C\|\nabla\varphi_n\|_{\boldsymbol{L}^2}\|\varphi_n\|_{H^3}\|\nabla\mu_n\|_{\boldsymbol{L}^2}\|\nabla\varDelta\mu_n\|_{\boldsymbol{L}^2} \\
&\quad + C\|\varphi_n\|_{H^2}^{\frac{1}{2}}\|\varphi_n\|_{H^4}^{\frac{1}{2}}\|\nabla\mu_n\|_{\boldsymbol{L}^2}\|\nabla\varDelta\mu_n\|_{\boldsymbol{L}^2} \\
&\quad + C\|\nabla\varphi_n\|_{\boldsymbol{L}^2}^{\frac{1}{2}}\|\varphi_n\|_{H^3}^{\frac{1}{2}}\|\nabla\mu_n\|_{\boldsymbol{L}^2}^{\frac{1}{2}}\|\nabla\mu_n\|_{\boldsymbol{H}^2}^{\frac{1}{2}}\|\nabla\varDelta\mu_n\|_{\boldsymbol{L}^2} \\
&\leqslant \varepsilon\|\nabla\varDelta\mu_n\|_{\boldsymbol{L}^2}^2 + C\left(1+\|\varphi_n\|_{H^4}^2\right)\|\nabla\mu_n\|_{\boldsymbol{L}^2}^2.
\end{aligned}
$$

Last, for R_{20}, we have

$$
\begin{aligned}
R_{20} &\leqslant C\|\nabla\boldsymbol{v}_n\|_{\boldsymbol{L}^2}\|\nabla\varphi_n\|_{\boldsymbol{L}^\infty}\|\nabla\varDelta\mu_n\|_{\boldsymbol{L}^2} + C\|\boldsymbol{v}_n\|_{\boldsymbol{L}^4}\|\nabla^2\varphi_n\|_{\boldsymbol{L}^4}\|\nabla\varDelta\mu_n\|_{\boldsymbol{L}^2} \\
&\leqslant C\|\nabla\boldsymbol{v}_n\|_{\boldsymbol{L}^2}\|\varphi_n\|_{H^3}\|\nabla\varDelta\mu_n\|_{\boldsymbol{L}^2} \\
&\quad + C\|\boldsymbol{v}_n\|_{\boldsymbol{L}^2}^{\frac{1}{2}}\|\boldsymbol{v}_n\|_{\boldsymbol{H}^1}^{\frac{1}{2}}\|\varphi_n\|_{H^2}^{\frac{1}{2}}\|\varphi_n\|_{H^3}^{\frac{1}{2}}\|\nabla\varDelta\mu_n\|_{\boldsymbol{L}^2} \\
&\leqslant \varepsilon\|\nabla\varDelta\mu_n\|_{\boldsymbol{L}^2}^2 + C\|\boldsymbol{v}_n\|_{\boldsymbol{H}^1}^2 + C\|\varphi_n\|_{H^3}^2.
\end{aligned}
$$

Collecting the above estimates together, we infer from (5.21) that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\|\nabla\mu_n\|_{\boldsymbol{L}^2}^2 + (Bm_0 - 6\varepsilon)\|\nabla\varDelta\mu_n\|_{\boldsymbol{L}^2}^2 \\
&\leqslant C\left(1+\|\varphi_n\|_{H^4}^2\right)\|\nabla\mu_n\|_{\boldsymbol{L}^2}^2 + C\|\partial_t\varphi_n\|_{\boldsymbol{L}^2}^2 + C\|\partial_t\sigma_n\|_{\boldsymbol{L}^2}^2 \\
&\quad + C\|\boldsymbol{v}_n\|_{\boldsymbol{H}^1}^2 + C\|\varphi_n\|_{H^3}^2 + C\|\nabla\Gamma\|_{\boldsymbol{L}^2}^2.
\end{aligned}
$$

Taking $\varepsilon = \frac{Bm_0}{12}$, then it follows from (5.12), (5.15) and Gronwall's lemma that

$$
\sup_{t\in[0,T]}\|\nabla\mu_n(t)\|_{\boldsymbol{L}^2}^2 + \|\nabla\varDelta\mu_n\|_{L^2(0,T;\boldsymbol{L}^2)}^2 \leqslant C.
$$

The above estimate combined with (4.6) and Poincaré's inequality yields

$$
\|\mu_n\|_{L^\infty(0,T;H^1)\cap L^2(0,T;H^3)} \leqslant C. \tag{5.22}
$$

Going back to (4.2a), we further deduce that

$$
\|\partial_t\varphi_n\|_{L^2(0,T;H^1)} \leqslant C \tag{5.23}
$$

and by (5.12), it follows

$$
\|\varphi_n\|_{L^\infty(0,T;H^3)} \leqslant C. \tag{5.24}
$$

Passing to the limit as $n\to+\infty$: By the above higher order a priori estimates, there exists a quadruple of functions $(\varphi,\mu,\sigma,\boldsymbol{v})$ that satisfies the regularities stated in Theorem

2 and

$$0 = \partial_t\varphi + \boldsymbol{v}\cdot\nabla\varphi - \operatorname{div}(m(\varphi)\nabla\mu) - \Gamma, \tag{5.25a}$$

$$0 = \mu - A\Psi'(\varphi) + B\Delta\varphi + \chi\sigma, \tag{5.25b}$$

$$0 = \partial_t\sigma + \boldsymbol{v}\cdot\nabla\sigma - \operatorname{div}(n(\varphi)\nabla(\sigma + \chi(1-\varphi))) - S, \tag{5.25c}$$

$$\boldsymbol{0} = \partial_t\boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \operatorname{div}(2\eta(\varphi)\mathrm{D}\boldsymbol{v}) - (\mu + \chi\sigma)\nabla\varphi, \tag{5.25d}$$

hold a.e in Q. Thus, it remains to derive estimates for the pressure. From (5.25d), there exists a function $q \in L^2(0, T; L_0^2)$ such that

$$\partial_t\boldsymbol{v}(t) + (\boldsymbol{v}(t)\cdot\nabla)\boldsymbol{v}(t) - \operatorname{div}(2\eta(\varphi(t))\mathrm{D}\boldsymbol{v}(t)) - (\mu(t) + \chi\sigma(t))\nabla\varphi(t) = -\nabla q(t)$$

holds as an equality in the sense of distribution for a.e. $t \in (0, T)$. Since the left-hand side belongs to $\boldsymbol{L}^2$ for a.e. $t \in (0, T)$, we obtain that $\nabla q \in L^2(0, T; \boldsymbol{L}^2)$.

The proof is complete. $\qquad\qquad\square$

Remark 5.1 (Further Hölder regularity for σ) *In the case $\chi = 0$ and $S = 0$, equation (5.25c) reduces to*

$$\partial_t\sigma + \boldsymbol{v}\cdot\nabla\sigma - \operatorname{div}(n(\varphi)\nabla\sigma) = 0. \tag{5.26}$$

If in addition, $\sigma_0 \in L^\infty(\Omega)$, then by a weak comparison principle, i.e., testing (5.26) with $(\sigma - l)_- = \max(l - \sigma, 0)$, $l = \inf_{x\in\Omega}\sigma_0(x)$, and testing (5.26) with $(\sigma - m)_+ = \max(\sigma - m, 0)$, $m = \sup_{x\in\Omega}\sigma_0(x)$, it follows that $\sigma \in L^\infty(0, T; L^\infty)$ and $\|\sigma\|_{L^\infty(0,T;L^\infty)} \leqslant \|\sigma_0\|_{L^\infty}$. For more details, see for instance [29, 45]. We just remark that the convection term can be handled as follows:

$$(\boldsymbol{v}\cdot\nabla(\sigma - l), (\sigma - l)_-) = -\frac{1}{2}(\boldsymbol{v}, \nabla|(\sigma - l)_-|^2) = \frac{1}{2}(\operatorname{div}\boldsymbol{v}, |(\sigma - l)_-|^2) = 0.$$

Then, the divergence-free property of $\boldsymbol{v}$ and the fact that $\boldsymbol{v} \in L^4(Q)$ allow us to deduce the existence of constants $C > 0$ and $\alpha \in (0, 1)$, depending on $\|\sigma\|_{L^\infty(0,T;L^\infty)}$ and $\|\boldsymbol{v}\|_{L^4(Q)}$ such that

$$|\sigma(x, t) - \sigma(y, s)| \leqslant C\left(|x - y|^\alpha + |t - s|^{\frac{\alpha}{2}}\right) \quad \forall(x, t), (y, s) \in \overline{Q}, \; x \neq y, \; s \neq t.$$

The above estimate follows from similar arguments to the proof of [55, Lemma 3.2] (see also [24, Lemma 2]).

6 Proof of Theorem 3: continuous dependence in two dimensions

Let $\{\varphi_i, \mu_i, \sigma_i, \boldsymbol{v}_i, q_i\}_{i=1,2}$ denote two global strong solutions to problem (3.3)–(3.4) corresponding to initial data $\{\varphi_{0,i}, \sigma_{0,i}, \boldsymbol{v}_{0,i}\}_{i=1,2}$ and source terms $\{\Gamma_i, S_i\}_{i=1,2}$. Denoting the differences as $\hat{f}$ for $f \in \{\varphi, \mu, \sigma, \boldsymbol{v}, q, \varphi_0, \sigma_0, \boldsymbol{v}_0, \Gamma, S\}$, we obtain the system satisfied by the

difference of solutions

$$0 = \partial_t \hat{\varphi} + \hat{\boldsymbol{v}} \cdot \nabla \varphi_1 + \boldsymbol{v}_2 \cdot \nabla \hat{\varphi} - \operatorname{div}\left((m(\varphi_1) - m(\varphi_2))\nabla \mu_1 + m(\varphi_2)\nabla \hat{\mu}\right) - \hat{\Gamma}, \tag{6.1a}$$

$$0 = \hat{\mu} - A(\Psi'(\varphi_1) - \Psi'(\varphi_2)) + B\Delta\hat{\varphi} + \chi\hat{\sigma}, \tag{6.1b}$$

$$0 = \partial_t \hat{\sigma} + \hat{\boldsymbol{v}} \cdot \nabla \sigma_1 + \boldsymbol{v}_2 \cdot \nabla \hat{\sigma} - \hat{S} \tag{6.1c}$$
$$\qquad - \operatorname{div}\left(n(\varphi_1)\nabla(\hat{\sigma} - \chi\hat{\varphi}) + (n(\varphi_1) - n(\varphi_2))\nabla(\sigma_2 + \chi(1 - \varphi_2))\right),$$

$$\boldsymbol{0} = \partial_t \hat{\boldsymbol{v}} + (\hat{\boldsymbol{v}} \cdot \nabla)\boldsymbol{v}_1 + (\boldsymbol{v}_2 \cdot \nabla)\hat{\boldsymbol{v}} + \nabla \hat{q} - (\hat{\mu} + \chi\hat{\sigma})\nabla \varphi_1 - (\mu_2 + \chi\sigma_2)\nabla \hat{\varphi} \tag{6.1d}$$
$$\qquad - 2\operatorname{div}\left(\eta(\varphi_1)\mathrm{D}\hat{\boldsymbol{v}} + (\eta(\varphi_1) - \eta(\varphi_2))\mathrm{D}\boldsymbol{v}_2\right),$$

with

$$\hat{\varphi}(0) = \hat{\varphi}_0, \quad \hat{\sigma}(0) = \hat{\sigma}_0, \quad \hat{\boldsymbol{v}}(0) = \hat{\boldsymbol{v}}_0.$$

Testing (6.1a) with $B\hat{\varphi}$, (6.1b) with $m(\varphi_2)\hat{\mu}$, (6.1c) with $\hat{\sigma}$, (6.1d) with $\hat{\boldsymbol{v}}$ and (6.1b) with $K\hat{\varphi}$ for some positive constant K yet to be determined, after integration by parts and summing the resultants, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(B\|\hat{\varphi}\|_{L^2}^2 + \|\hat{\boldsymbol{v}}\|_{L^2}^2 + \|\hat{\sigma}\|_{L^2}^2\right)$$
$$\qquad + \|m^{\frac{1}{2}}(\varphi_2)\hat{\mu}\|_{L^2}^2 + 2\|\eta^{\frac{1}{2}}(\varphi_1)\mathrm{D}\hat{\boldsymbol{v}}\|_{L^2}^2 + \|n^{\frac{1}{2}}(\varphi_1)\nabla\hat{\sigma}\|_{L^2}^2 + BK\|\nabla\hat{\varphi}\|_{L^2}^2$$
$$= \int_\Omega -B\hat{\varphi}\hat{\boldsymbol{v}} \cdot \nabla\varphi_1 + B\hat{\Gamma}\hat{\varphi} + m(\varphi_2)\left(A(\Psi'(\varphi_1) - \Psi'(\varphi_2))\hat{\mu} - \chi\hat{\sigma}\hat{\mu}\right) dx$$
$$\qquad + \int_\Omega Bm'(\varphi_2)\hat{\mu}\nabla\varphi_2 \cdot \nabla\hat{\varphi}\, dx + \int_\Omega (\hat{\mu} + \chi\hat{\sigma})\nabla\varphi_1 \cdot \hat{\boldsymbol{v}} + (\mu_2 + \chi\sigma_2)\nabla\hat{\varphi} \cdot \hat{\boldsymbol{v}}\, dx$$
$$\qquad + \int_\Omega 2(\eta(\varphi_2) - \eta(\varphi_1))\mathrm{D}\boldsymbol{v}_2 \cdot \mathrm{D}\hat{\boldsymbol{v}} - (\hat{\boldsymbol{v}} \cdot \nabla)\boldsymbol{v}_1 \cdot \hat{\boldsymbol{v}}\, dx$$
$$\qquad + \int_\Omega \hat{S}\hat{\sigma} + (n(\varphi_2) - n(\varphi_1))\nabla(\sigma_2 - \chi\varphi_2) \cdot \nabla\hat{\sigma}\, dx$$
$$\qquad - \int_\Omega \chi n(\varphi_1)\nabla\hat{\varphi} \cdot \nabla\hat{\sigma} + \hat{\sigma}\hat{\boldsymbol{v}} \cdot \nabla\sigma_1\, dx - \int_\Omega B(m(\varphi_1) - m(\varphi_2))\nabla\mu_1 \cdot \nabla\hat{\varphi}\, dx$$
$$\qquad + K\int_\Omega (\hat{\mu} + \chi\hat{\sigma})\hat{\varphi} - A\left(\Psi'(\varphi_1) - \Psi'(\varphi_2)\right)\hat{\varphi}\, dx$$
$$=: \sum_{j=1}^{8} I_j, \tag{6.2}$$

where we have used the following facts:

$$(\hat{\varphi}, \boldsymbol{v}_2 \cdot \nabla\hat{\varphi}) = \frac{1}{2}(\boldsymbol{v}_2, \nabla|\hat{\varphi}|^2) = -\frac{1}{2}(\operatorname{div}\boldsymbol{v}_2, |\hat{\varphi}|^2) = 0,$$

$$((\boldsymbol{v}_2 \cdot \nabla)\hat{\boldsymbol{v}}, \hat{\boldsymbol{v}}) = -\frac{1}{2}(\operatorname{div}\boldsymbol{v}_2, |\hat{\boldsymbol{v}}|^2) = 0,$$

$$(\nabla\hat{q}, \hat{\boldsymbol{v}}) = -(\operatorname{div}\hat{\boldsymbol{v}}, q) = 0, \quad (\boldsymbol{v}_2 \cdot \nabla\hat{\sigma}, \hat{\sigma}) = 0.$$

We now proceed to estimate the right-hand side of (6.2) term by term. First, assumption (3.20) yields that

$$
\begin{aligned}
I_1 + I_2 &\leqslant B\|\hat{\varphi}\|_{L^2}\|\hat{\boldsymbol{v}}\|_{\boldsymbol{L}^2}\|\nabla\varphi_1\|_{\boldsymbol{L}^\infty} + B\|\hat{\Gamma}\|_{L^2}\|\hat{\varphi}\|_{L^2} + m_1\chi\|\hat{\sigma}\|_{L^2}\|\hat{\mu}\|_{L^2} \\
&\quad + m_1 A\left(1 + \|\varphi_1\|_{L^\infty}^r + \|\varphi_2\|_{L^\infty}^r\right)\|\hat{\varphi}\|_{L^2}\|\hat{\mu}\|_{L^2} \\
&\quad + B\|m'\|_{L^\infty}\|\nabla\varphi_2\|_{\boldsymbol{L}^\infty}\|\hat{\mu}\|_{L^2}\|\nabla\hat{\varphi}\|_{L^2} \\
&\leqslant \delta\|\hat{\mu}\|_{L^2}^2 + C_\delta\|\hat{\sigma}\|_{L^2}^2 + C\|\hat{\Gamma}\|_{L^2}^2 + \|\hat{\boldsymbol{v}}\|_{\boldsymbol{L}^2}^2 + C\|\nabla\varphi_2\|_{\boldsymbol{L}^\infty}^2\|\nabla\hat{\varphi}\|_{L^2}^2 \\
&\quad + C\left(1 + \|\varphi_1\|_{L^\infty}^{2r} + \|\varphi_2\|_{L^\infty}^{2r} + \|\nabla\varphi_1\|_{\boldsymbol{L}^\infty}^2\right)\|\hat{\varphi}\|_{L^2}^2,
\end{aligned}
$$

for some $\delta > 0$ yet to be determined. Similarly, we have

$$
\begin{aligned}
I_8 &\leqslant K\left(\|\hat{\mu}\|_{L^2} + \chi\|\hat{\sigma}\|_{L^2}\right)\|\hat{\varphi}\|_{L^2} + KA\left(1 + \|\varphi_1\|_{L^\infty}^r + \|\varphi_2\|_{L^\infty}^r\right)\|\hat{\varphi}\|_{L^2}^2 \\
&\leqslant \delta\|\hat{\mu}\|_{L^2}^2 + \|\hat{\sigma}\|_{L^2}^2 + \left(CK^2 + CK\left(1 + \|\varphi_1\|_{L^\infty}^r + \|\varphi_2\|_{L^\infty}^r\right)\right)\|\hat{\varphi}\|_{L^2}^2.
\end{aligned}
$$

Next, thanks to the Lipschitz continuity of m, we have

$$
I_7 \leqslant C\|\nabla\mu_1\|_{\boldsymbol{L}^\infty}\|\hat{\varphi}\|_{L^2}\|\nabla\hat{\varphi}\|_{L^2} \leqslant \|\nabla\hat{\varphi}\|_{L^2}^2 + C\|\mu_1\|_{H^3}^2\|\hat{\varphi}\|_{L^2}^2.
$$

Similarly, the Lipschitz continuity of n, Young's inequality and Korn's inequality allow us to deduce that

$$
\begin{aligned}
I_5 + I_6 &\leqslant \|\hat{S}\|_{L^2}\|\hat{\sigma}\|_{L^2} + C\|\nabla(\sigma_2 - \chi\varphi_2)\|_{\boldsymbol{L}^4}\|\hat{\varphi}\|_{L^4}\|\nabla\hat{\sigma}\|_{\boldsymbol{L}^2} \\
&\quad + C\|\nabla\hat{\varphi}\|_{L^2}\|\nabla\hat{\sigma}\|_{\boldsymbol{L}^2} + \|\hat{\boldsymbol{v}}\|_{\boldsymbol{L}^4}\|\hat{\sigma}\|_{L^2}\|\nabla\sigma_1\|_{\boldsymbol{L}^4} \\
&\leqslant \|\hat{\sigma}\|_{L^2}^2 + C\|\hat{S}\|_{L^2}^2 + \varepsilon\|\nabla\hat{\sigma}\|_{\boldsymbol{L}^2}^2 + C_\varepsilon\left(\|\nabla(\sigma_2 - \chi\varphi_2)\|_{\boldsymbol{L}^4}^2\|\hat{\varphi}\|_{L^4}^2 + \|\nabla\hat{\varphi}\|_{\boldsymbol{L}^2}^2\right) \\
&\quad + \theta\|D\hat{\boldsymbol{v}}\|_{\boldsymbol{L}^2}^2 + C_\theta\|\hat{\boldsymbol{v}}\|_{\boldsymbol{L}^2}^2 + C_\theta\|\nabla\sigma_1\|_{\boldsymbol{L}^4}^2\|\hat{\sigma}\|_{L^2}^2,
\end{aligned}
$$

for some positive constants ε, θ yet to be determined. By the Gagliardo–Nirenberg inequality $\|f\|_{L^4} \leqslant C\|f\|_{L^2}^{\frac{1}{2}}\|\nabla f\|_{\boldsymbol{L}^2}^{\frac{1}{2}} + C\|f\|_{L^2}$ in two dimensions, we have

$$
\begin{aligned}
\|\nabla(\sigma_2 - \chi\varphi_2)\|_{\boldsymbol{L}^4}^2\|\hat{\varphi}\|_{L^4}^2 &\leqslant C\|\nabla(\sigma_2 - \chi\varphi_2)\|_{\boldsymbol{L}^4}^2\left(\|\hat{\varphi}\|_{L^2}\|\nabla\hat{\varphi}\|_{L^2} + \|\hat{\varphi}\|_{L^2}^2\right) \\
&\leqslant \|\nabla\hat{\varphi}\|_{L^2}^2 + C\left(1 + \|\nabla(\sigma_2 - \chi\varphi_2)\|_{\boldsymbol{L}^4}^4\right)\|\hat{\varphi}\|_{L^2}^2.
\end{aligned}
$$

Then, it follows that

$$
\begin{aligned}
I_5 + I_6 &\leqslant \varepsilon\|\nabla\hat{\sigma}\|_{L^2}^2 + C_\varepsilon\|\nabla\hat{\varphi}\|_{L^2}^2 + \theta\|D\hat{\boldsymbol{v}}\|_{\boldsymbol{L}^2}^2 + \left(1 + C_\theta\|\nabla\sigma_1\|_{\boldsymbol{L}^4}^2\right)\|\hat{\sigma}\|_{L^2}^2 \\
&\quad + C_\varepsilon\left(1 + \|\nabla(\sigma_2 - \chi\varphi_2)\|_{\boldsymbol{L}^4}^4\right)\|\hat{\varphi}\|_{L^2}^2 + C_\theta\|\hat{\boldsymbol{v}}\|_{\boldsymbol{L}^2}^2 + C\|\hat{S}\|_{L^2}^2.
\end{aligned}
$$

Last, thanks to the Lipschitz continuity of η, it holds that

$$
\begin{aligned}
I_3 + I_4 &\leqslant \|\nabla\varphi_1\|_{\boldsymbol{L}^\infty}(\|\hat{\mu}\|_{L^2} + \chi\|\hat{\sigma}\|_{L^2})\|\hat{\boldsymbol{v}}\|_{\boldsymbol{L}^2} + \|\mu_2 + \chi\sigma_2\|_{L^\infty}\|\hat{\boldsymbol{v}}\|_{\boldsymbol{L}^2}\|\nabla\hat{\varphi}\|_{L^2} \\
&\quad + C\|D\boldsymbol{v}_2\|_{\boldsymbol{L}^4}\|D\hat{\boldsymbol{v}}\|_{\boldsymbol{L}^2}\|\hat{\varphi}\|_{L^4} + \|\nabla\boldsymbol{v}_1\|_{\boldsymbol{L}^2}\|\hat{\boldsymbol{v}}\|_{\boldsymbol{L}^4}^2 \\
&\leqslant \delta\|\hat{\mu}\|_{L^2}^2 + \|\hat{\sigma}\|_{L^2}^2 + \|\nabla\hat{\varphi}\|_{L^2}^2 + \left(C_\delta\|\nabla\varphi_1\|_{\boldsymbol{L}^\infty}^2 + C\|\mu_2 + \chi\sigma_2\|_{L^\infty}^2\right)\|\hat{\boldsymbol{v}}\|_{\boldsymbol{L}^2}^2 \\
&\quad + \theta\|D\hat{\boldsymbol{v}}\|_{\boldsymbol{L}^2}^2 + C_\theta\|D\boldsymbol{v}_2\|_{\boldsymbol{L}^4}^2\|\hat{\varphi}\|_{L^4}^2 + \|\nabla\boldsymbol{v}_1\|_{\boldsymbol{L}^2}\|\hat{\boldsymbol{v}}\|_{\boldsymbol{L}^4}^2.
\end{aligned}
$$

By the Gagliardo–Nirenberg inequality $\|f\|_{L^4} \leqslant C\|f\|_{L^2}^{\frac{1}{2}}\|\nabla f\|_{L^2}^{\frac{1}{2}} + C\|f\|_{L^2}$ in two dimensions and Korn's inequality, we have

$$
\begin{aligned}
\|\nabla \boldsymbol{v}_1\|_{L^2}\|\hat{\boldsymbol{v}}\|_{L^4}^2 &\leqslant C\|\nabla \boldsymbol{v}_1\|_{L^2}\left(\|\hat{\boldsymbol{v}}\|_{L^2}\|\mathrm{D}\hat{\boldsymbol{v}}\|_{L^2} + \|\hat{\boldsymbol{v}}\|_{L^2}^2\right)\\
&\leqslant \theta\|\mathrm{D}\hat{\boldsymbol{v}}\|_{L^2}^2 + (C_\theta\|\nabla \boldsymbol{v}_1\|_{L^2}^2 + C\|\nabla \boldsymbol{v}_1\|_{L^2})\|\hat{\boldsymbol{v}}\|_{L^2}^2,
\end{aligned}
$$

$$
\begin{aligned}
C_\theta\|\mathrm{D}\boldsymbol{v}_2\|_{L^4}^2\|\hat{\varphi}\|_{L^4}^2 &\leqslant C_\theta\|\mathrm{D}\boldsymbol{v}_2\|_{L^4}^2(\|\hat{\varphi}\|_{L^2}\|\nabla\hat{\varphi}\|_{L^2} + \|\hat{\varphi}\|_{L^2}^2)\\
&\leqslant \|\nabla\hat{\varphi}\|_{L^2}^2 + C_\theta(1 + \|\mathrm{D}\boldsymbol{v}_2\|_{L^4}^4)\|\hat{\varphi}\|_{L^2}^2.
\end{aligned}
$$

As a consequence,

$$
\begin{aligned}
I_3 + I_4 &\leqslant 2\theta\|\mathrm{D}\hat{\boldsymbol{v}}\|_{L^2}^2 + \delta\|\hat{\mu}\|_{L^2}^2 + 2\|\nabla\hat{\varphi}\|_{L^2}^2 + \|\hat{\sigma}\|_{L^2}^2 + C_\theta\left(1 + \|\mathrm{D}\boldsymbol{v}_2\|_{L^4}^4\right)\|\hat{\varphi}\|_{L^2}^2\\
&\quad + \left(C_\delta\|\nabla\varphi_1\|_{L^\infty}^2 + C\|\mu_2 + \chi\sigma_2\|_{L^\infty}^2 + C_\theta\|\nabla \boldsymbol{v}_1\|_{L^2}^2 + C\right)\|\hat{\boldsymbol{v}}\|_{L^2}^2.
\end{aligned}
$$

In summary, we infer from (6.2) and the above estimates that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(B\|\hat{\varphi}\|_{L^2}^2 + \|\hat{\boldsymbol{v}}\|_{L^2} + \|\hat{\sigma}\|_{L^2}^2\right)\\
&\quad + (m_0 - 3\delta)\|\hat{\mu}\|_{L^2}^2 + (n_0 - \varepsilon)\|\nabla\hat{\sigma}\|_{L^2}^2 + (2\eta_0 - 3\theta)\|\mathrm{D}\hat{\boldsymbol{v}}\|_{L^2}^2\\
&\quad + \left(BK - 3 - C_\varepsilon - C_\delta\|\nabla\varphi_2\|_{L^\infty(0,T;L^\infty)}^2\right)\|\nabla\hat{\varphi}\|_{L^2}^2\\
&\leqslant C\left(\|\hat{\Gamma}\|_{L^2}^2 + \|\hat{S}\|_{L^2}^2\right) + h_0(t)\|\hat{\sigma}\|_{L^2}^2 + h_1(t)\|\hat{\boldsymbol{v}}\|_{L^2}^2 + h_2(t)\|\hat{\varphi}\|_{L^2}^2,
\end{aligned}
\tag{6.3}
$$

where

$$
\begin{aligned}
h_0(t) &:= C\left(1 + \|\nabla\sigma_1\|_{L^4}^2\right),\\
h_1(t) &:= C\left(1 + \|\nabla\varphi_1\|_{L^\infty}^2 + \|\mu_2 + \chi\sigma_2\|_{L^\infty}^2 + \|\nabla \boldsymbol{v}_1\|_{L^2}^2\right),\\
h_2(t) &:= C\left(1 + \|\varphi_1\|_{L^\infty}^{2r} + \|\varphi_2\|_{L^\infty}^{2r} + \|\nabla\varphi_1\|_{L^\infty}^2\right)\\
&\quad + C\left(\|\nabla(\sigma_2 - \chi\varphi_2)\|_{L^4}^4 + \|\mathrm{D}\boldsymbol{v}_2\|_{L^4}^4 + \|\mu_1\|_{H^3}^2\right).
\end{aligned}
$$

In the expression of h_0, h_1 and h_2, the constant C may depend on the parameters δ, θ, ε and K.

Since $\|\varphi_2\|_{L^\infty(0,T;H^3)}$ is bounded (see (5.24)), so is $\|\nabla\varphi_2\|_{L^\infty(0,T;L^\infty)}$ by the Sobolev embedding theorem. Then, in (6.3), we can choose

$$
\delta = \frac{m_0}{6}, \quad \varepsilon = \frac{n_0}{2}, \quad \theta = \frac{\eta_0}{3}, \quad K = \frac{1}{B}\left(4 + C_\varepsilon + C_\delta\|\nabla\varphi_2\|_{L^\infty(0,T;L^\infty)}^2\right).
$$

On the other hand, by regularies of the global strong solutions $\{\varphi_i, \mu_i, \sigma_i, \boldsymbol{v}_i\}_{i=1,2}$ stated in Theorem 2 and the continuous embedding $L^\infty(0,T;L^2) \cap L^2(0,T;H^1) \subset L^4(Q)$ in two dimensions, it holds that $h_0, h_1, h_2 \in L^1(0,T)$. Then, applying Gronwall's lemma to the

differential inequality (6.3), and then using Korn's inequality, we have

$$
\sup_{t \in (0,T]} \left(\|\hat{\varphi}(t)\|_{L^2}^2 + \|\hat{\boldsymbol{v}}(t)\|_{\boldsymbol{L}^2}^2 + \|\hat{\sigma}(t)\|_{L^2}^2 \right)
$$

$$
+ \int_0^T \|\hat{\mu}\|_{L^2}^2 + \|\nabla \hat{\sigma}\|_{L^2}^2 + \|\nabla \hat{\boldsymbol{v}}\|_{\boldsymbol{L}^2}^2 + \|\nabla \hat{\varphi}\|_{L^2}^2 \, dt
$$

$$
\leqslant C \left(\|\hat{\Gamma}\|_{L^2(Q)}^2 + \|\hat{S}\|_{L^2(Q)}^2 + \|\hat{\sigma}_0\|_{L^2}^2 + \|\hat{\boldsymbol{v}}_0\|_{\boldsymbol{L}^2}^2 + \|\hat{\varphi}_0\|_{L^2}^2 \right) =: C \mathcal{Y}.
$$

To obtain continuous dependence of $\hat{\varphi}$ in the $L^2(0, T; H^2)$-norm, first we see that by (3.20)

$$
\|\Psi'(\varphi_1) - \Psi'(\varphi_2)\|_{L^2(Q)}^2 \leqslant C \left(1 + \|\varphi_1\|_{L^\infty(Q)}^{2r} + \|\varphi_2\|_{L^\infty(Q)}^{2r} \right) \|\hat{\varphi}\|_{L^2(Q)}^2 \leqslant C \mathcal{Y},
$$

then viewing (6.1*b*) as an elliptic problem for $\hat{\varphi}$ and applying the elliptic regularity theory we obtain

$$
\|\hat{\varphi}\|_{L^2(0,T;H^2)} \leqslant C \left(\|\hat{\mu}\|_{L^2(Q)} + \|\Psi'(\varphi_1) - \Psi'(\varphi_2)\|_{L^2(Q)} + \|\hat{\sigma}\|_{L^2(Q)} \right) \leqslant C \mathcal{Y}.
$$

The proof of Theorem 3 is complete. $\qquad\qquad\square$

7 Conclusions

In this work, we derive a class of thermodynamically consistent Navier–Stokes–Cahn–Hilliard system for two-phase fluid flows with density contrast, based on a volume-averaged velocity in the spirit of [5], which also allows for mass transfer between the fluids and chemotactic response to a chemical species present in the physical domain. The presence of mass transfer leads to a non-solenoidal velocity field, and while models derived with a mass-averaged velocity also involve non-solenoidal velocity fields, it turns out that the model equations in the volume-averaged variant are considerably simpler than the mass-averaged variant. Besides, we state the sharp interface limit and present various simplified versions of the general volume-averaged velocity model. As far as the mathematical analysis is concerned, for the general model some difficulties will be encountered in deriving a priori estimates due to the mass transfer terms, especially when the volume-averaged velocity is not divergence-free. Hence, we consider the simplest model variant as a starting point, which has a divergence-free velocity and matched densities, but allows the fluid viscosity and mobility to be dependent on the order parameter φ. By a suitable Galerkin approximation, we establish global weak existence in both two and three dimensions for prescribed mass transfer terms, and then under additional assumptions we show global strong well-posedness as well as continuous dependence in two dimensions. We believe that the a priori estimates derived here will also be useful to study long-time behaviour of the system. Furthermore, we expect that global weak existence to the more general model variant (2.21*c*), (2.21*e*), (2.21*f*) and (2.26) that has a solenoidal velocity but allows for density contrast can be shown by employing the approach of [3].

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