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ON THE FINE SPECTRAL EXPANSION OF JACQUET'S RELATIVE TRACE FORMULA

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Abstract The relative trace formula is a tool in the theory of automorphic forms which was invented by Jacquet in order to study period integrals and relate them to Langlands functoriality. In this paper we give an analogue of Arthur's spectral expansion of the trace formula to the relative setup in the context of GL_n. This is an important step toward application of the relative trace formula and it extends earlier work by several authors to higher rank. Our method is new and based on complex analysis and majorization of Eisenstein series. To that end we use recent lower bounds of Brumley for Rankin–Selberg L-functions at the edge of the critical strip.

Keywords: relative trace formula; Eisenstein series; L-functions

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Contents

1. Introduction

Let G be a reductive group over a number field F and let $\mathbb{A} = \mathbb{A}_F$ be the ring of adeles of F. Let H be a subgroup of G defined over F. A cuspidal automorphic representation

π of G is called *distinguished* by H if the *period* integral

$$\Pi^H(\varphi) = \int_{H(F)\backslash H(\mathbb{A})^1} \varphi(h)\,\mathrm{d}h$$

is non-zero (assuming it converges, which is often the case) on the space of π. This notion is of interest only for special classes of pairs $H \subset G$. A typical example is when H is obtained from G as the fixed points of an involution defined over F. The distinguished representations (by one or more period subgroups) are expected to be characterized as the functorial image from a third group G' (which is either linear reductive, or a cover thereof). Roughly speaking, the conjugacy classes of G' should correspond to the double cosets of H in G. The period integrals themselves are often related to special values of L-functions.

For example, let E/F be a quadratic extension and let G be the restriction of scalars of GL_n from E to F. It was conjectured by Flicker and Rallis [**12**] that the distinguished representations by $H = \mathrm{GL}_n/F$ are obtained as the functorial transfer from the quasi-split unitary group. This conjecture is motivated by analysing the poles of the Asai L-function. Dually, following the work of Jacquet and Ye, the distinguished representations by unitary groups are expected to be characterized as the base change from $G' = \mathrm{GL}_n/F$. These cases are the main focus of this paper.

Recall that in order to prove the conjectures above, Jacquet has introduced the *relative trace formula* (cf. [**38**]). It is the expression (or the sum of the expressions) of the form

$$\int_{H(F)\backslash H(\mathbb{A})^1} \int_{U^0(F)\backslash U^0(\mathbb{A})} K_f(h,u)\psi^{-1}(u)\,\mathrm{d}h\,\mathrm{d}u, \tag{1.1}$$

where $f \in C_c^\infty(G(\mathbb{A})^1)$, $K_f(\cdot\,,\cdot)$ is the automorphic kernel on G and ψ is a non-degenerate character of the maximal unipotent subgroup U^0 of G. The idea is to compare (1.1) with the so-called Kuznetsov trace formula

$$\int_{U^{0\prime}(F)\backslash U^{0\prime}(\mathbb{A})} K_{f'}(u_1,u_2)\psi'(u_1u_2^{-1})\,\mathrm{d}u_1\,\mathrm{d}u_2 \tag{1.2}$$

on G' where ψ is related to ψ' in a simple way and f and $f' \in C_c^\infty(G'(\mathbb{A})^1)$ are *matching* in the sense that they satisfy certain local compatibility conditions. On the spectral side the contribution of a cuspidal representation π' to the Kuznetsov trace formula is the so-called 'Bessel distribution'

$$\mathcal{B}'_{\pi'}(f') = \sum_{\varphi'} \mathcal{W}^{\psi'}(\pi'(f')\varphi')\overline{\mathcal{W}^{\psi'}(\varphi')},$$

where φ' ranges over an orthonormal basis of the space of π' and $\mathcal{W}^{\psi'}$ denotes the ψ'th Fourier coefficient. Similarly, the cuspidal contribution of (1.1) is the sum over π of the 'relative Bessel distribution'

$$\mathcal{B}_\pi(f) = \sum_H \sum_\varphi \Pi^H(\pi(f)\varphi)\overline{\mathcal{W}^\psi(\varphi)},$$

which is non-zero exactly when π is distinguished. The precise form of Jacquet's conjecture is the spectral identity

$$\mathcal{B}_\pi(f) = \mathcal{B}'_{\pi'}(f') \tag{1.3}$$

when π is the functorial image of π'. (We may need to sum over several π' in general.)

To obtain the spectral identity (1.3) from the equality of (1.1) and (1.2) one proceeds as in other trace formula comparisons. Roughly speaking, one expands each kernel geometrically, compares the resulting terms individually, and then infers (1.3) from the spectral expansions of the kernels. In this paper we consider the spectral expansion of (1.1). Namely, we will express the contribution of any cuspidal data in terms of integrals of relative Bessel distributions. The precise statement is Theorems 10.1 and 10.4 of § 9. It extends earlier work by several authors to higher rank (see [**12**, **19**, **29**, **36**, **39**, **65**]). The spectral expansion obtained is an analogue of Arthur's fine spectral expansion for the usual trace formula [**5**]. It relies on earlier work by Rogawski and the author on regularized periods of Eisenstein series [**48**, **49**] which had been first introduced in [**29**] and were also considered in [**42**]. A key feature of the expansion is its absolute convergence.

The results of this paper are used in recent work by Jacquet to prove that if π is a cuspidal representation of $\mathrm{GL}_n(\mathbb{A}_E)$ which is obtained as a base change from a cuspidal representation π' of $\mathrm{GL}_n(\mathbb{A}_F)$ where E/F is a quadratic extension, then π is distinguished by a unitary group with respect to E. (The converse direction follows from the argument of [**22**] and the work of Arthur and Clozel [**9**].) This was done by resolving the geometric issues arising from the relative trace formula, namely, the existence of smooth matching, and the so-called 'fundamental lemma' [**32**, **33**, **35**]. (In the function field case, the fundamental lemma had been proved earlier by Ngô in [**57**] and [**58**].)

The paper is organized as follows. Following a suggestion of Jacquet we first explain the method in the case of GL_2, for which our approach is already new (although the result is not). This can be viewed as a second, more technical, introduction to the paper. In § 3 we give a simple-minded definition of 'principal value' integrals for a certain class of functions in several variables. These integrals admit a 'residue calculus' with repeated residues. They are used in § 9, the most important section of the paper, to derive the spectral expansion.

The main technical step is the majorization of Eisenstein series (§ 6, Proposition 6.1). To that end we apply the combinatorics of (G, M)-families (§ 4) to reduce the problem to lower bounds of Rankin–Selberg L-functions (which appear in the normalization of the intertwining operators) at the edge of the critical strip (§ 5). Such bounds were recently established by Brumley [**11**]. His proof exploits the analytic properties of the Rankin–Selberg L-function and a positivity argument. A second analytic ingredient needed for uniform majorization (and hence, for the absolute convergence of the relative trace formula) is a uniform bound toward the Ramanujan hypothesis on the non-tempered parameters of cuspidal representations, which is given in [**50**]. The connection between this condition and the absolute convergence of the usual Arthur–Selberg trace formula was discovered by Müller (see [**54**], cf. [**56**]).

At this stage, we move on to set up the relative trace formula at hand (§ 7). We review the definition and the main properties of the regularized periods in the setting of quasi-split Galois pairs [**48**]. We analyse in detail the two symmetric spaces of Galois type pertaining to GL_n (§ 8). Finally, we are ready to carry out the spectral expansion in § 9. Our method is somewhat novel and differs from that of Arthur [**4**, **5**] in that instead of using Fourier analysis on Euclidean space, we use residue calculus (i.e. complex analysis). In particular, it avoids the use of a Paley–Wiener theorem and the somewhat roundabout argument of [**4**]. The combinatorics of the residue calculus is rather intricate. The analysis of § 9 culminates in Claim 9.2, which is expected to hold for any G. In the case of GL_n we explicate the result in Theorems 10.1 and 10.4 of § 10, arriving at our main result.

We mention that part of the analytic complication stems from the fact that while the right-hand side of (1.3) is of positive type, in the sense that it is non-negative for f' of the form $f'_1 \star f_1'^{\vee}$, the same is not clear, *a priori*, for the left-hand side.

From the point of view of the spectral expansion the two cases pertaining to $G = \mathrm{GL}_n/E$ are very similar. However, they differ in other respects. The case $(\mathrm{GL}_n/E, \mathrm{GL}_n/F)$ is a Gelfand pair and the period integral is related to the residue of the Asai L-function at $s = 1$ through the integral representation of the latter [**12**]. There is also an explicit 'backward' map from GL_n/E to $U(n)$. On the other hand, $(\mathrm{GL}_n/E, U(n))$ is not a Gelfand pair, but nevertheless, the period is expected to be factorizable and its value related to special values of L-functions. In fact, these expectations are intimately related to the relative trace formula and do not seem to follow from other methods. Thus, the relative trace formula is more interesting in this case.

As pointed out before, the relative trace formula had been worked out in detail for GL_2 and GL_3. An approach for the case $H = \mathrm{GL}_n/F$ is attempted in [**13**]. Other interesting cases and variants of the relative trace formulae are treated in [**14**, **25**, **26**, **28**, **37**, **41**, **43**, **44**, **51**], to mention a few. A more laid back discussion can be found in [**16**, **30**, **34**].

As already mentioned, a possible application of the relative trace formula is to the factorization of the functional $\Pi^H(\varphi)$ into local H-invariant functionals [**31**]. It is also used (in another setting) to give a formula for certain special values of L-functions [**10**, **27**]. In [**19**] the relative trace formula is applied to prove the existence of a generic representation in any tempered L-packet of $U(3)$. The relative trace formula has also applications to the recent L^∞-norm conjectures of Sarnak (cf. [**46**, **62**]). Finally, let us mention that the relative trace formula is not the only available tool to study period integrals, or even the spectral identity (1.3). This is especially true when the functoriality $\pi' \mapsto \pi$ can be constructed by other means, e.g. theta-correspondence (cf. [**18**, **34**]).

2. The GL_2-example

Let E/F be a quadratic extension of number fields and let G be the restriction of scalars of GL_2 from E to F. Thus, the F-points of G (which will also be denoted by G) are $\mathrm{GL}_2(E)$. Denote by $x \mapsto \bar{x}$ the Galois conjugation of E/F. Let $B \subset G$ be the Borel subgroup of upper triangular matrices with its Levi decomposition $B = TU$ where T is the group of diagonal matrices.

Let $\tilde{H}$ denote the group GL_2/F considered as as a subgroup of G, and set $\tilde{B} = B \cap \tilde{H}$, $\tilde{T} = T \cap \tilde{H}$.

Let $f \in C_c^\infty(G(\mathbb{A})^1)$ (with $G(\mathbb{A})^1 = \{g \in G(\mathbb{A}) : |\det(g)| = 1\}$) and let

$$K_f(x, y) = \sum_{\gamma \in G} f(x^{-1}\gamma y)$$

be the kernel of the right regular representation of $G(\mathbb{A})^1$ on $L^2(G\backslash G(\mathbb{A})^1)$. Our goal is to give the fine spectral expansion for the expression

$$\int_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1} \int_{U\backslash U(\mathbb{A})} K_f(h, u)\psi(u)\,\mathrm{d}u.$$

In this section, we will only attempt to illustrate the method and not worry about rigorous justification of the argument, which will be given below in a much more general context.

Recall that the continuous spectrum of $L^2(G\backslash G(\mathbb{A})^1)$ is spanned by Eisenstein series. These are defined for any $\varphi : U(\mathbb{A})\mathbb{R}_+T(F)\backslash G(\mathbb{A})^1 \to \mathbb{C}$ by (the meromorphic continuation of)

$$E(x, \varphi, \lambda) = \sum_{\gamma \in B\backslash G} \mathrm{e}^{(\lambda+1)H(\gamma x)}\varphi(\gamma x).$$

Here $\mathbb{R}_+$ is imbedded in $T(\mathbb{A})$ by $x \mapsto \left(\begin{smallmatrix} x & 0 \\ 0 & x^{-1} \end{smallmatrix}\right)$ (at the Archimedean places) and H is defined using the Iwasawa decomposition by

$$H\left(\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} k\right) = \tfrac{1}{2}\log\left|\frac{t_1}{t_2}\right|, \quad t_1, t_2 \in \mathbb{I}_E,\ k \in \boldsymbol{K},$$

where $\boldsymbol{K}$ is the standard maximal compact.

2.1. The regularized period (see [42])

We first recall the definition of mixed truncation—a 'relative' version of Arthur's truncation. For any automorphic form φ on $G(\mathbb{A})$ and $T \gg 0$ it is defined by

$$\Lambda_m^T\varphi(h) = \varphi(h) - \sum_{\gamma \in \tilde{B}\backslash\tilde{H}} \varphi_U(\gamma h)\chi_{\geqslant T}(H(\gamma h)), \tag{2.1}$$

where $\varphi_U(h) = \int_{U\backslash U(\mathbb{A})} \varphi(uh)\,\mathrm{d}u$ is the constant term of φ along U. The function $\Lambda_m^T\varphi$ is rapidly decreasing on $\tilde{H}\backslash\tilde{H}(\mathbb{A})^1$.

To define the regularized period we first define for any polynomial exponential

$$f(x) = \sum_{i=1}^{n} P_i(x)\mathrm{e}^{\lambda_i \cdot x}$$

with $\lambda_i \neq 0$ the regularized integral $\int_{x\geqslant T}^{\#} f(x)\,\mathrm{d}x$ to be the value at $\lambda = 0$ of the meromorphic continuation of

$$\int_{x\geqslant T} f(x)\mathrm{e}^{\lambda\cdot x}\,\mathrm{d}x$$

(convergent for $\mathrm{Re}(\lambda) \ll 0$). This is well defined since $\lambda_i \neq 0$.

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We then define $\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1}\varphi(h)\,\mathrm{d}h$ to be the expression

$$\int_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1}\Lambda^T_m\varphi(h)\,\mathrm{d}h+\int_{\tilde{\boldsymbol{K}}}\int_{\tilde{T}\backslash\tilde{T}(\mathbb{A})^1}\int^{\#}_{x\geqslant T}\mathrm{e}^{-x}\varphi_U\left(\begin{pmatrix}\mathrm{e}^{x/d} & 0\\ 0 & \mathrm{e}^{-x/d}\end{pmatrix}tk\right)\mathrm{d}x\,\mathrm{d}t\,\mathrm{d}k,$$

where $d=[E:\mathbb{Q}]$. By [**42**], this will be well defined, and independent of T, on the space of automorphic forms which do not admit 0 as an exponent along B. Moreover,

$$\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1}\varphi(h)\,\mathrm{d}h$$

is an $\tilde{H}(\mathbb{A})^1$-invariant functional on this space which agrees with the usual integral in the case where φ is integrable over $\tilde{H}\backslash\tilde{H}(\mathbb{A})^1$.

In particular, since

$$E_U(g,\varphi,\lambda)=\varphi(g)\mathrm{e}^{(\lambda+1)\cdot H(g)}+M(\lambda)\varphi(g)\mathrm{e}^{(-\lambda+1)\cdot H(g)},\tag{2.2}$$

where $M(\lambda)$ is the usual intertwining operator, we have

$$\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1}E(h,\varphi,\lambda)\,\mathrm{d}h=\int_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1}\Lambda^T_m E(h,\varphi,\lambda)\,\mathrm{d}h-\frac{\mathrm{e}^{\lambda T}}{\lambda}\int_{\tilde{\boldsymbol{K}}}\int_{\tilde{T}\backslash\tilde{T}(\mathbb{A})^1}\varphi(tk)\,\mathrm{d}t\,\mathrm{d}k$$
$$+\frac{\mathrm{e}^{-\lambda T}}{\lambda}\int_{\tilde{\boldsymbol{K}}}\int_{\tilde{T}\backslash\tilde{T}(\mathbb{A})^1}M(\lambda)\varphi(tk)\,\mathrm{d}t\,\mathrm{d}k\tag{2.3}$$

and the left-hand side is well defined whenever $E(\cdot\,,\varphi,\lambda)$ is holomorphic and $\lambda\neq 0$ (by the condition on the exponents).

Lemma 2.1. *There are three mutually disjoint possibilities.*

Case 1: $\chi_2\neq\overline{\chi_1}^{-1}$. *Then* $\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1}E(h,\varphi,\lambda)\,\mathrm{d}h$ *is* 0 *whenever defined. Moreover,*

$$\int_{\tilde{\boldsymbol{K}}}\int_{\tilde{T}\backslash\tilde{T}(\mathbb{A})^1}\varphi(tk)\,\mathrm{d}t\,\mathrm{d}k=\int_{\tilde{\boldsymbol{K}}}\int_{\tilde{T}\backslash\tilde{T}(\mathbb{A})^1}M(0)\varphi(tk)\,\mathrm{d}t\,\mathrm{d}k.$$

Case 2: $\chi_2=\overline{\chi_1}^{-1}$ but $\chi_1|_{I_F}=\chi_2^{-1}|_{I_F}\not\equiv 1$. *Then*

$$\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1}E(h,\varphi,\lambda)\,\mathrm{d}h=\int_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1}\Lambda^T_m E(h,\varphi,\lambda)\,\mathrm{d}h$$

is holomorphic for $\lambda\in\mathrm{i}\mathbb{R}$.

Case 3: $\chi_2=\chi_1$ and $\chi_1|_{I_F}\equiv 1$. *Then* $\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1}E(h,\varphi,\lambda)\,\mathrm{d}h$ *is holomorphic for* $\lambda\in\mathrm{i}\mathbb{R}$ *except for a simple pole at* 0*. On the other hand,* $E(\cdot\,,\varphi,0)\equiv 0$ *and* $M(0)=-1$.

In all cases, for any $x\in G(\mathbb{A})$,

$$\overline{E(x,\varphi,-\bar{\lambda})}\cdot\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1}E(h,\varphi,\lambda)\,\mathrm{d}h$$

is holomorphic for $\lambda\in\mathrm{i}\mathbb{R}$.

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Proof. The vanishing of $\int^*_{\tilde H\backslash\tilde H(\mathbb{A})^1}$ in the first case follows from [**42**]. The vanishing of $E(\cdot\,,\varphi,0)$ in the last case is well known. The rest of the assertions follow from the relation (2.3) and the holomorphy of the Eisenstein series and the intertwining operators on the unitary axis. □

Going back to the spectral expansion, we write

$$K_f(x,y)=K_f^{\mathrm{cusp}}(x,y)+K_f^{\mathrm{res}}(x,y)+K_f^{\mathrm{cont}}(x,y),$$

where

$$K_f^{\mathrm{cusp}}(x,y)=\sum_{\varphi}R(f)\varphi(x)\cdot\overline{\varphi(y)},$$

the sum ranging over an orthonormal basis of the cuspidal spectrum,

$$K_f^{\mathrm{res}}(x,y)=\mathrm{vol}(G\backslash G(\mathbb{A})^1)^{-1}\sum_{\chi}\int_{G(\mathbb{A})^1}f(x)\chi(\det x)\,\mathrm{d}x\cdot\overline{\chi(y)},$$

where χ ranges over Hecke characters of I_F trivial on $\mathbb{R}_+$ and

$$K_f^{\mathrm{cont}}(x,y)=\frac{1}{2}\sum_{\chi=(\chi_1,\chi_2)}\int_{\mathrm{i}\mathbb{R}}\sum_{\varphi\in\mathcal{B}(\chi)}E(x,I(f,\lambda)\varphi,\lambda)\overline{E(y,\varphi,\lambda)}\,\mathrm{d}\lambda,$$

where $\mathcal{B}(\chi)$ is an orthonormal basis of the space of functions $\varphi:U(\mathbb{A})\mathbb{R}_+T(F)\backslash G(\mathbb{A})^1\to\mathbb{C}$ satisfying

$$\varphi\left(\begin{pmatrix}t_1&*\\0&t_2\end{pmatrix}g\right)=\chi_1(t_1)\chi_2(t_2)\varphi(g)\quad\text{for all }t_1,t_2\in\mathbb{I}_F,$$

with respect to the inner product

$$(\varphi_1,\varphi_2)=\mathrm{vol}(T\backslash T(\mathbb{A})^1)\int_{\boldsymbol{K}}\varphi_1(k)\overline{\varphi_2(k)}\,\mathrm{d}k.$$

Also, the measure on $\mathrm{i}\mathbb{R}$ is the Haar-measure dual to the Lebesgue measure on $\mathbb{R}$ (i.e. under the identification $t\mapsto it$ it becomes $(2\pi)^{-1}$ times the Lebesgue measure).

Performing the integration over $U\backslash U(\mathbb{A})$ first, the residual contribution vanishes and we remain with the sum of

$$\sum_{\varphi}\int_{\tilde H\backslash\tilde H(\mathbb{A})^1}R(f)\varphi(h)\,\mathrm{d}h\cdot\overline{\varphi^{\psi}},\tag{2.4}$$

where φ^{ψ} is the ψth Fourier coefficient $\int_{U\backslash U(\mathbb{A})}\varphi(u)\overline{\psi(u)}\,\mathrm{d}u$, and

$$\frac{1}{2}\sum_{\chi=(\chi_1,\chi_2)}\int_{\tilde H\backslash\tilde H(\mathbb{A})^1}\int_{\mathrm{i}\mathbb{R}}\sum_{\varphi\in\mathcal{B}(\chi)}E(h,I(f,\lambda)\varphi,\lambda)\overline{\mathcal{W}(\varphi,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}h,\tag{2.5}$$

where $\mathcal{W}(\varphi,\lambda)=E^{\psi}(\cdot\,,\varphi,\lambda)$. The expression (2.4) is the *Bessel distribution* (on the cuspidal spectrum) with respect to the linear forms $\int_{\tilde H\backslash\tilde H(\mathbb{A})^1}$ and the ψth Fourier coefficient.

We will denote it by $\mathcal{B}^G_{\mathrm{cusp}}(f)$. We have a canonical decomposition $\mathcal{B}^G_{\mathrm{cusp}}(f) = \sum_\pi \mathcal{B}^G_\pi(f)$ according to cuspidal distinguished representations of $G(\mathbb{A})$.

Unfortunately, we cannot interchange the two integrals in (2.5) because the Eisenstein series are not integrable over $\tilde{H}\backslash\tilde{H}(\mathbb{A})^1$. Instead we express $E(\cdot\,,\varphi,\lambda)$ in terms of its mixed truncation (2.1) and use (2.2), to write each summand of (2.5) as the sum of

$$\int_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1}\int_{\mathrm{i}\mathbb{R}}\sum_{\varphi\in\mathcal{B}(\chi)} \Lambda^T_m E(h,I(f,\lambda)\varphi,\lambda)\overline{\mathcal{W}(\varphi,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}h$$

and

$$\int_{\tilde{B}\backslash\tilde{H}(\mathbb{A})^1}\int_{\mathrm{i}\mathbb{R}}\sum_{\varphi\in\mathcal{B}(\chi)}[I(f,\lambda)\varphi(h)\mathrm{e}^{(\lambda+1)\cdot H(h)} + M(\lambda)I(f,\lambda)\varphi(h)\mathrm{e}^{(-\lambda+1)\cdot H(h)}] \\ \times\chi_{\geqslant T}(H(h))\overline{\mathcal{W}(\varphi,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}h.$$

(This step deviates from the approach taken in [**19**] and [**36**].) In the first summand we can interchange the order of integration. For the second sum we use the Iwasawa decomposition for $\tilde{H}(\mathbb{A})$ (noting that the modulus function of $\tilde{B}(\mathbb{A})$ is $\mathrm{e}^{H(\bullet)}$) to write it as the sum of

$$\int_{\tilde{\boldsymbol{K}}}\int_{\tilde{T}\backslash\tilde{T}(\mathbb{A})^1}\int_{\mathbb{R}}\int_{\mathrm{i}\mathbb{R}}\sum_{\varphi\in\mathcal{B}(\chi)} I(f,\lambda)\varphi(tk)\mathrm{e}^{\lambda X}\chi_{\geqslant T}(X)\overline{\mathcal{W}(\varphi,-\bar{\lambda})}\,\mathrm{d}\lambda\,\mathrm{d}X\,\mathrm{d}t\,\mathrm{d}k \tag{2.6}$$

and

$$\int_{\tilde{\boldsymbol{K}}}\int_{\tilde{T}\backslash\tilde{T}(\mathbb{A})^1}\int_{\mathbb{R}}\int_{\mathrm{i}\mathbb{R}}\sum_{\varphi\in\mathcal{B}(\chi)} I(f,-\lambda)M(\lambda)\varphi(tk)\mathrm{e}^{-\lambda X}\chi_{\geqslant T}(X)\overline{\mathcal{W}(\varphi,-\bar{\lambda})}\,\mathrm{d}\lambda\,\mathrm{d}X\,\mathrm{d}t\,\mathrm{d}k. \tag{2.7}$$

Clearly, we cannot interchange the integrals over X and λ—the integral only converges as an iterated integral. However, up to a factor of $2\pi\mathrm{i}$ we may view the inner integral of (2.6) as a complex integration and shift the contour to $\mathrm{Re}(\lambda) = \lambda_0$ where λ_0 is negative. This is legitimate since $\mathcal{W}(\varphi,\lambda)$ is holomorphic for $\mathrm{Re}\,\lambda \geqslant 0$. Now the double integral becomes absolutely convergent and integrating first over X we obtain

$$-\int_{\tilde{\boldsymbol{K}}}\int_{\tilde{T}\backslash\tilde{T}(\mathbb{A})^1}\int_{\mathrm{Re}(\lambda)=\lambda_0}\frac{\mathrm{e}^{\lambda T}}{\lambda}\sum_{\varphi\in\mathcal{B}(\chi)} I(f,\lambda)\varphi(tk)\overline{\mathcal{W}(\varphi,-\bar{\lambda})}\,\mathrm{d}t\,\mathrm{d}\lambda\,\mathrm{d}k.$$

Using a simple variant of the residue theorem we now shift the contour of integration back to the unitary axis to obtain

$$-\,\mathrm{PV}\int_{\mathrm{i}\mathbb{R}}\frac{\mathrm{e}^{\lambda T}}{\lambda}\int_{\tilde{\boldsymbol{K}}}\int_{\tilde{T}\backslash\tilde{T}(\mathbb{A})^1}\sum_{\varphi\in\mathcal{B}(\chi)} I(f,\lambda)\varphi(tk)\overline{\mathcal{W}(\varphi,-\bar{\lambda})}\,\mathrm{d}t\,\mathrm{d}k\,\mathrm{d}\lambda \\ +\frac{1}{2}\int_{\tilde{\boldsymbol{K}}}\int_{\tilde{T}\backslash\tilde{T}(\mathbb{A})^1}\sum_{\varphi\in\mathcal{B}(\chi)} I(f,0)\varphi(tk)\overline{\mathcal{W}(\varphi,0)}\,\mathrm{d}t\,\mathrm{d}k,$$

where PV denotes the principal value of the integral (à la Cauchy) and the usual factor of $2\pi\mathrm{i}$ does not show up because of the measure taken. Similarly, we want to shift the contour of the inner integral of (2.7) to $\mathrm{Re}(\lambda) = \lambda_0$ for λ_0 positive (but small). At first glance it seems that the poles of $\mathcal{W}(\varphi, \lambda)$ for $\mathrm{Re}(\lambda) < 0$ may interfere, unless we assume a weak version of the Riemann hypothesis for L-functions of Hecke characters. However, by the functional equation of the Eisenstein series we have $\mathcal{W}(M(\lambda)\varphi, -\lambda) = \mathcal{W}(\varphi, \lambda)$, and then performing the unitary change of basis $\varphi \mapsto M(\lambda)\varphi$ for λ unitary, the integrand in (2.7) becomes

$$\sum_{\varphi \in \mathcal{B}(\chi)} I(f, -\lambda)\varphi(tk)\mathrm{e}^{-\lambda X}\chi_{\geqslant T}(X)\overline{\mathcal{W}(\varphi, \bar{\lambda})},$$

which is holomorphic for $\mathrm{Re}(\lambda)$ small and positive.

Thus, by a similar reasoning as before we obtain

$$\int_{\tilde{\boldsymbol{K}}} \int_{\mathrm{Re}(\lambda)=\lambda_0} \frac{\mathrm{e}^{-\lambda T}}{\lambda} \int_{\tilde{T}\backslash \tilde{T}(\mathbb{A})^1} \sum_{\varphi \in \mathcal{B}(\chi)} M(\lambda)I(f, \lambda)\varphi(tk)\overline{\mathcal{W}(\varphi, -\bar{\lambda})}\,\mathrm{d}t\,\mathrm{d}\lambda\,\mathrm{d}k,$$

and then by shifting back the contour as above we get

$$\begin{aligned}\mathrm{PV}\int_{\mathrm{i}\mathbb{R}} \frac{\mathrm{e}^{-\lambda T}}{\lambda} \int_{\tilde{\boldsymbol{K}}} \int_{\tilde{T}\backslash \tilde{T}(\mathbb{A})^1} \sum_{\varphi \in \mathcal{B}(\chi)} M(\lambda)I(f, \lambda)\varphi(tk)\overline{\mathcal{W}(\varphi, -\bar{\lambda})}\,\mathrm{d}t\,\mathrm{d}k\,\mathrm{d}\lambda \\ + \frac{1}{2} \int_{\tilde{\boldsymbol{K}}} \int_{\tilde{T}\backslash \tilde{T}(\mathbb{A})^1} \sum_{\varphi \in \mathcal{B}(\chi)} I(f, 0)M(0)\varphi(tk)\overline{\mathcal{W}(\varphi, 0)}\,\mathrm{d}t\,\mathrm{d}k.\end{aligned}$$

All in all, the contribution from $\chi = (\chi_1, \chi_2)$ becomes

$$\begin{aligned}\mathrm{PV}\int_{\mathrm{i}\mathbb{R}} \sum_{\varphi \in \mathcal{B}(\chi)} \overline{\mathcal{W}(\varphi, -\bar{\lambda})}\biggl(\int_{\tilde{H}\backslash \tilde{H}(\mathbb{A})^1} \Lambda_m^T E(h, I(f, \lambda)\varphi, \lambda)\,\mathrm{d}h \\ + \int_{\tilde{\boldsymbol{K}}} \int_{\tilde{T}\backslash \tilde{T}(\mathbb{A})^1} \biggl[-I(f, \lambda)\varphi(tk)\frac{\mathrm{e}^{\lambda T}}{\lambda} + M(\lambda)I(f, \lambda)\varphi(tk)\frac{\mathrm{e}^{-\lambda T}}{\lambda}\biggr]\,\mathrm{d}t\,\mathrm{d}k\biggr)\,\mathrm{d}\lambda \\ + \frac{1}{2} \int_{\tilde{\boldsymbol{K}}} \int_{\tilde{T}\backslash \tilde{T}(\mathbb{A})^1} \sum_{\varphi \in \mathcal{B}(\chi)} [I(f, 0)\varphi(tk) + I(f, 0)M(0)\varphi(tk)]\overline{\mathcal{W}(\varphi, 0)}\,\mathrm{d}t\,\mathrm{d}k.\end{aligned}$$

We finally get (using (2.3))

$$\begin{aligned}\mathrm{PV}\int_{\mathrm{i}\mathbb{R}} \sum_{\varphi \in \mathcal{B}(\chi)} \overline{\mathcal{W}(\varphi, -\bar{\lambda})} \cdot \int^{*}_{\tilde{H}\backslash \tilde{H}(\mathbb{A})^1} E(h, I(f, \lambda)\varphi, \lambda)\,\mathrm{d}h\,\mathrm{d}\lambda \\ + \frac{1}{2} \sum_{\varphi \in \mathcal{B}(\chi)} \overline{\mathcal{W}(\varphi, 0)} \int_{\tilde{\boldsymbol{K}}} \int_{\tilde{T}\backslash \tilde{T}(\mathbb{A})^1} I(f, 0)\varphi(tk)\,\mathrm{d}t\,\mathrm{d}k \\ + \frac{1}{2} \sum_{\varphi \in \mathcal{B}(\chi)} \overline{\mathcal{W}(\varphi, 0)} \int_{\tilde{\boldsymbol{K}}} \int_{\tilde{T}\backslash \tilde{T}(\mathbb{A})^1} I(f, 0)M(0)\varphi(tk)\,\mathrm{d}t\,\mathrm{d}k.\end{aligned}$$

The integrand in the first summand is the *Bessel distribution* with respect to $\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1}$ and $\mathcal{W}(\lambda)$, which will be denoted by $\mathcal{B}^G_\chi(f,\lambda)$. The second summand is the Bessel distribution with respect to $\int_{\tilde{\boldsymbol{K}}}\int_{\tilde{T}\backslash\tilde{T}(\mathbb{A})^1}$ and $\mathcal{W}(\cdot\,,0)$, which we denote by $\mathcal{B}^T_\chi(f)$. Lemma 2.1 implies that $\mathcal{B}^G_\chi(f,\lambda)$ is holomorphic on $\mathrm{i}\mathbb{R}$ and the last two terms cancel each other if $\chi_1=\chi_2$ and are equal otherwise.

Thus, we recover, insomuch as the argument can be made rigorous, the following spectral expansion (cf. [**36**, **65**]).

Scholium. *The spectral expansion for* GL_2 *is*

$$\sum_{\pi\ \mathit{cuspidal}}\mathcal{B}^G_\pi(f)+\frac{1}{2}\sum_\chi\int_{\mathrm{i}\mathbb{R}}\mathcal{B}^G_\chi(f,\lambda)\,\mathrm{d}\lambda+\frac{1}{2}\sum_{\substack{\chi_1|_{\mathbb{I}_F}=\chi_2|_{\mathbb{I}_F}=1\\ \chi_1\neq\chi_2}}\mathcal{B}^T_\chi(f).$$

In the rest of the paper we will explain how to extend the result and the argument to GL_n.

3. Improper integrals

We first define and study a rather naive notion of an improper integral for a certain class of functions in several complex variables. In particular, we obtain a 'residue calculus' for these integrals. The arguments are completely elementary.

Let V be a Euclidean space and let F be a complex-valued function on $V_{\mathbb{C}}=V\otimes\mathbb{C}$. We say that F is *tame* if there exist $c,k>0$ such that in the region

$$\|\mathrm{Re}\,u\|<c(1+\|\mathrm{Im}\,u\|)^{-k}$$

F is holomorphic and satisfies

$$|F(u)|\leqslant d(1+\|\mathrm{Im}\,u\|)^{-N}\tag{3.1}$$

for any $N>0$ (with d depending of course on N).

We note that if $F(u)$ is tame then by Cauchy's formula the same will be true for any partial derivative of F (with c replaced by $\frac{1}{2}c$ say).

Fix a finite set Λ of non-zero linear forms of V. Consider the class $\mathcal{F}$ of functions on $V_{\mathbb{C}}=V\otimes\mathbb{C}$ obtained as finite linear combinations of functions of the form

$$\frac{F(v)}{\prod_{\lambda\in\Lambda'}\lambda(v)}$$

where F is tame and Λ' is a linearly independent subset of Λ. We fix a Haar measure on V and translate it to $v_0+\mathrm{i}V$ for any $v_0\in V$. On $\mathrm{i}V^*$ we take the dual Haar measure. We say that a vector $v\in V$ is in general position if $\lambda(v)\neq0$ for all $\lambda\in\Lambda$.

Proposition 3.1. *Let* $F'\in\mathcal{F}$ *and* $v\in V$ *in general position. Then for* k *sufficiently large, the limit*

$$\int^V_{\nearrow v}F'(u)\,\mathrm{d}u=\lim_{\varepsilon\to0}\int_{\|\mathrm{Im}\,u\|\leqslant1/\varepsilon:\mathrm{Re}\,u=\varepsilon^k\cdot v}F'(u)\,\mathrm{d}u\tag{3.2}$$

exists and is independent of k.

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Definition 3.2. The left-hand side of (3.2) will be called the improper integral of F' with base point v.

Proof. It suffices to check this for

$$F'(u) = \frac{F(u)}{\prod_{\lambda\in\Lambda'}\lambda(u)}$$

and F tame. We first observe that for ε sufficiently small and k sufficiently large the inequalities (3.1) are satisfied on the domain of integration of (3.2). Set $m = |\Lambda'|$, and let $v_1,\dots,v_n$ be a basis of V such that $v_1,\dots,v_m$ forms a dual basis of Λ' and $\lambda(v_j) = 0$ for all $\lambda\in\Lambda'$ and $j > m$. Let ξ_i be the coordinates of v in this basis, and note that $\xi_i \neq 0$ for $i = 1,\dots,m$. Let κ be i^{-n} times the co-volume of the lattice $\mathcal{L}$ spanned by $v_1,\dots,v_n$. Then $\kappa = (2\pi\mathrm{i})^{-n}\operatorname{covol}(\hat{\mathcal{L}})^{-1}$, where $\hat{\mathcal{L}} \supset \Lambda'$ is the lattice dual to $\mathcal{L}$. We first claim that the limit above is equal to

$$\kappa \times \lim_{\varepsilon\to 0}\int_{\operatorname{Re} z_i=\varepsilon^k\xi_i,\ |\operatorname{Im} z_i|\leqslant 1/\varepsilon} \frac{F(\sum z_i v_i)}{z_1\cdots z_m}\,\mathrm{d}z_1\cdots\mathrm{d}z_n \tag{3.3}$$

provided that the latter exists. Indeed, for any u' in the symmetric difference

$$D = \{\operatorname{Re} u = \varepsilon^k\cdot v : \|\operatorname{Im} u\|\leqslant 1/\varepsilon\} \ominus \Big\{\sum z_i v_i : \operatorname{Re} z_i = \varepsilon^k\xi_i,\ |\operatorname{Im} z_i|\leqslant 1/\varepsilon\Big\}$$

we have $c_1\varepsilon^{-1}\leqslant\|\operatorname{Im} u'\|\leqslant c_2\varepsilon^{-1}$ for some $c_1, c_2 > 0$ and $|\lambda(u')|\geqslant|\lambda(v)|\varepsilon^k$ for any $\lambda\in\Lambda'$. Since F is tame, it is smaller than any power of ε on D, and hence, the difference between the expressions in (3.2) and (3.3) tends to zero as $\varepsilon\to 0$.

To prove that the limit (3.3) exists we will prove that for $0 < \varepsilon' < \varepsilon$ the difference between the integrals in (3.3) for ε and ε' tends to 0 as $\varepsilon\to 0$ uniformly in ε'. Applying the usual residue theorem repeatedly we write this difference as

$$\int_{T\cup T'} \frac{F(\sum z_i v_i)}{z_1\cdots z_m}\,\mathrm{d}z_1\cdots\mathrm{d}z_n, \tag{3.4}$$

where T is the union over $\emptyset\neq I\subset\{1,\dots,n\}$ of

$$\left\{(z_1,\dots,z_n) : \begin{cases}\operatorname{Im} z_i = \pm 1/\varepsilon,\ \operatorname{Re} z_i\in[\varepsilon'^k,\varepsilon^k]\xi_i & i\in I\\ |\operatorname{Im} z_i|\leqslant 1/\varepsilon,\ \operatorname{Re} z_i = \varepsilon^k\xi_i & i\notin I\end{cases}\right\}$$

and T' is the complement of the box

$$\{(z_1,\dots,z_n) : \operatorname{Re} z_i = \varepsilon'^k\xi_i,\ |\operatorname{Im} z_i|\leqslant 1/\varepsilon \text{ for all } i\}$$

in

$$\{(z_1,\dots,z_n) : \operatorname{Re} z_i = \varepsilon'^k\xi_i,\ |\operatorname{Im} z_i|\leqslant 1/\varepsilon' \text{ for all } i\}.$$

As before, the integrand in (3.4) is smaller than any power of ε as $\varepsilon\to 0$ uniformly on T. So the integral over T is uniformly small as $\varepsilon,\varepsilon'\to 0$. To bound the integral over T', we write T' as the disjoint union over $I\subset\{1,\dots,m\}$ of

$$T'_I = T'\cap\{(z_1,\dots,z_n) : |\operatorname{Im} z_i| > 1 \text{ for } i\in I,\ |\operatorname{Im} z_i|\leqslant 1 \text{ for } i\in\{1,\dots,m\}\setminus I\}.$$

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On T'_I we write the integrand as

$$\frac{1}{\prod_{i\in I} z_i} \sum_{J\subset\{1,\dots,m\}\setminus I} \frac{1}{\prod_{i\in 1,\dots,m\setminus(I\cup J)} z_i} D_J F\Bigl(\sum_{i\in I\cup J\cup\{m+1,\dots,n\}} z_i v_i\Bigr),$$

where

$$D_J F\Bigl(\sum_{i\in I\cup J\cup\{m+1,\dots,n\}} z_i v_i\Bigr) = \frac{1}{\prod_{i\in J} z_i} \sum_{J'\subset J} (-1)^{|J\setminus J'|} F\Bigl(\sum_{i\in I\cup J'\cup\{m+1,\dots,n\}} z_i v_i\Bigr).$$

We write the integral over T'_I as the sum over J of

$$\int \frac{1}{\prod_{i\in I} z_i} \frac{1}{\prod_{i\in 1,\dots,m\setminus(I\cup J)} z_i} D_J F\Bigl(\sum z_i v_i\Bigr)\, \mathrm{d}z_1\cdots \mathrm{d}z_n, \tag{3.5}$$

where the integral is taken over $(z_1,\dots,z_n)$ such that $\operatorname{Re} z_i = \varepsilon'^k \xi_i$ for all i, $|\operatorname{Im} z_i| \leqslant 1$ for $i\in\{1,\dots,m\}\setminus I$, $|\operatorname{Im} z_i| > 1$ for $i \in I$ and $(\operatorname{Im} z_i)_{i\in I\cup\{m+1,\dots,n\}}$ lies in the complement C of the box with side $2/\varepsilon$ inside the box with side $2/\varepsilon'$. The integral in (3.5) is then the product of

$$\prod_{j\in\{1,\dots,m\}\setminus(I\cup J)} \int \frac{\mathrm{d}z_j}{z_j}$$

and

$$\int \frac{1}{\prod_{i\in I} z_i} D_J F\Bigl(\sum z_i v_i\Bigr) \otimes \mathrm{d}z_i \quad (i \in I\cup J\cup\{m+1,\dots,n\}).$$

The first integral is bounded independently of k, ε, ε' because of the formula

$$\int_{-1}^{1} \frac{\mathrm{d}y}{a+\mathrm{i}y} = 2\arctan(1/a).$$

In the second integral, $|1/z_j| \leqslant 1$ for $j\in I$. Thus it suffices to prove that the integral

$$\int \Bigl| D_J F\Bigl(\sum z_i v_i\Bigr)\Bigr| \otimes \mathrm{d}z_i \quad (i\in I\cup J\cup\{m+1,\dots,n\})$$

over the domain $\operatorname{Re}(z_i) = \varepsilon'^k \xi_i$ for all i, $|\operatorname{Im}(z_i)| \leqslant 1$ for $i\in J$ and $\operatorname{Im}(z_i)_{i\in I\cup\{m+1,\dots,n\}} \in C$ tends to 0 with ε. Now we can write

$$D_J F = \int_0^1 \cdots \int_0^1 \frac{\partial^{|J|} F}{\prod_{j\in J} \partial z_j} \Bigl(\sum_{j\in J} t_j z_j v_j + \sum_{i\in I\cup\{m+1,\dots,n\}} z_i v_i\Bigr) \otimes_{j\in J} \mathrm{d}t_j.$$

Since the derivatives of F satisfy estimates similar to the ones for F, we see that the integral is majorized by

$$\int\cdots\int (1+\|((t_j y_j),(y_i))\|)^{-N} \otimes \mathrm{d}t_j \otimes \mathrm{d}y_i,$$

where

$$|t_j|\leqslant 1, \quad |y_j| \leqslant 1, \quad j\in J, \quad (y_i)_{i\in I\cup\{m+1,\dots,n\}} \in C.$$

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Since $I \cup \{m+1,\dots,n\}$ is not empty, in the integral, $|y_i| > 1/\varepsilon$ for at least one i. Thus the integral tends to 0 with ε.

The independence on k is proved in a similar vein. □

We remark that if F is actually holomorphic and rapidly decreasing for $\operatorname{Re} u \in [0,v]$ then a similar argument shows that

$$\int_{\nearrow v}^{V} \frac{F(u)}{\prod_{\lambda\in\Lambda'} \lambda(u)}\,\mathrm{d}u = \int_{\operatorname{Re} u = v} \frac{F(u)}{\prod_{\lambda\in\Lambda'} \lambda(u)}\,\mathrm{d}u. \tag{3.6}$$

Let now F be tame and Λ' as before. We want to analyse the effect of the base point on the improper integral. For any $S \subset \Lambda'$ let $V^S = \{v \in V : \lambda(v) = 0 \text{ for all } \lambda \in S\}$. Choose a dual basis $\{\lambda^\vee\}$ of Λ' in V. Denote by pr^S the projection of V onto V^S defined by

$$\mathrm{pr}^S(v) = v - \sum_{\lambda\in S} \lambda(v)\lambda^\vee.$$

Note that this projection depends on the choice of the $\lambda^\vee$, and in particular it is not necessarily an orthogonal projection. However, we have $\lambda(\mathrm{pr}^S v) = \lambda(v)$ for all $\lambda \in \Lambda' \setminus S$. We claim the following lemma.

Lemma 3.3. *Suppose that $v, v' \in V$ are in general position. Then*

$$\int_{\nearrow v}^{V} \frac{F(u)}{\prod_{\lambda\in\Lambda'} \lambda(u)}\,\mathrm{d}u = \sum_S \operatorname{sgn}\Bigl(\prod_{\lambda\in S} \lambda(v)\Bigr) \operatorname{vol}_S^{-1} \int_{\nearrow \mathrm{pr}^S v'}^{V^S} \frac{F(u)}{\prod_{\lambda\in\Lambda'\setminus S} l(u)}\,\mathrm{d}u, \tag{3.7}$$

where S ranges over the subsets of $S_0 = \{\lambda \in \Lambda' : \lambda(v)\lambda(v') < 0\}$ and vol_S is the co-volume of the lattice spanned by S. In particular,

$$\int_{\nearrow v}^{V} \frac{F(u)}{\prod_{\lambda\in\Lambda'} l(u)}\,\mathrm{d}u$$

depends only on the signs of $\lambda(v)$, $\lambda \in \Lambda'$.

Note that by the remark above, the terms on the right-hand side of (3.7) are independent of the choice of basis.

Proof. We choose a basis $v_1,\dots,v_n$ containing $\lambda^\vee$, $\lambda \in \Lambda'$ as in the proof of Proposition 3.1. Using the formula (3.3) we will prove (3.7) by induction on $\#\{i : \xi_i \neq \xi_i'\}$ where ξ_i' are the coordinates of v'. The case $v = v'$ is trivial. Let l be the last index so that $\xi_l \neq \xi_l'$. We shift the integral in (3.3) from $\operatorname{Re} z_l = \varepsilon^k \xi_l$ to $\operatorname{Re} z_l = \varepsilon^k \xi_l'$ without changing the other variables. By the residue theorem the difference between the two integrals is

$$\int_{\substack{\operatorname{Re} z_i = \varepsilon^k \xi_i,\ |\operatorname{Im} z_i| \leqslant 1/\varepsilon,\ i\neq l \\ \operatorname{Im} z_l = \pm 1/\varepsilon,\ \operatorname{Re} z_l \in [\xi_l,\xi_l']\varepsilon^k}} \frac{F(\sum z_i v_i)}{z_1\cdots z_m}\,\mathrm{d}z_1\cdots\mathrm{d}z_n \tag{3.8}$$

plus an additional term

$$2\pi\mathrm{i}\,\operatorname{sgn}(\xi_l)\cdot \int_{\operatorname{Re} z_i = \varepsilon^k \xi_i,\ |\operatorname{Im} z_i| \leqslant 1/\varepsilon,\ i\neq l} \frac{F(\sum_{i\neq l} z_i v_i)}{\prod_{i=1,\, i\neq l}^{m} z_i} \prod_{i\neq l} \mathrm{d}z_i \tag{3.9}$$

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if $l \leqslant m$ and $\xi_l \xi_l' < 0$. The integrand, and hence the integral, of (3.8) is smaller than any power of ε as $\varepsilon \to 0$. We now apply the induction hypothesis to the shifted integral and to (3.9). □

In particular, we note that if F is tame, then for any v

$$\int_{\nearrow v}^{V} F(u)\,\mathrm{d}u = \int_{\operatorname{Re} u=0} F(u)\,\mathrm{d}u. \tag{3.10}$$

4. (G, M)-families

Let G be a reductive group over F. We recall the setup of (G, M)-families [**3**]. We fix a maximal split torus T^0 of G. Denote by $W = W_G$ the Weyl group $N_G(T^0)/M^0$ of G where M^0 is the centralizer of T^0 in G (which is a minimal Levi subgroup). A parabolic subgroup P containing T^0 admits a unique Levi part M containing T^0 (and even M^0). Such P and M are called semi-standard. Henceforth, all parabolic and Levi subgroups will always be assumed to be semi-standard. We denote by T_M the split part of the centre of M and by W_M the Weyl group of M.

We will use the notation of [**5**]. In particular, $\mathfrak{a}_M^*$ denotes the vector space

$$X^*(M) \otimes_{\mathbb{Z}} \mathbb{R} \simeq X^*(T_M) \otimes_{\mathbb{Z}} \mathbb{R}$$

where $X^*(\cdot)$ denotes the lattice of rational characters over F. Set $\mathfrak{a}_{M,\mathbb{C}}^* = \mathfrak{a}_M^* \otimes_{\mathbb{R}} \mathbb{C}$. If P has Levi subgroup M, we denote by $\Sigma_P \subset \mathfrak{a}_M^*$ the set of reduced roots of T_M in the radical $U = U_P$ of P and by $\Delta_P \subset \Sigma_P$ the subset of simple roots. The dual space $\mathfrak{a}_M$ of $\mathfrak{a}_M^*$ contains the co-roots $\alpha^\vee$, $\alpha \in \Sigma_P$. The set of Levi subgroups containing M will be denoted by $\mathcal{L}(M)$ and the set of parabolic subgroups whose Levi part equals (respectively, contains) M will be denoted by $\mathcal{P}(M)$ (respectively, $\mathcal{F}(M)$). The parabolic subgroup opposite to P containing M is denoted by $\bar{P}$. Occasionally, we set $\mathfrak{a}_P = \mathfrak{a}_M$ for any $P \in \mathcal{L}(M)$. We set $\mathfrak{a}_0 = \mathfrak{a}_{M^0}$ and fix a positive definite W-invariant scalar product on $\mathfrak{a}_0$. This defines a measure on any subspace of $\mathfrak{a}_0$.

Recall that a (G, M)-family [**3**] is a family of smooth functions $c_P(\Lambda)$ on $\mathrm{i}\mathfrak{a}_M^*$, one for each $P \in \mathcal{P}(M)$, satisfying the compatibility relations

$$c_P(\Lambda) = c_{P'}(\Lambda)$$

on $\mathrm{i}\mathfrak{a}_L^*$ whenever P, P' are adjacent along the root α, i.e. when $\Sigma_{\bar{P}} \cap \Sigma_{P'} = \{\alpha\}$, and where L is the Levi subgroup of $P \cdot P'$. For such a (G, M)-family one defines

$$c_M(\Lambda) = \sum_{P \in \mathcal{P}(M)} \frac{c_P(\Lambda)}{\theta_P(\Lambda)},$$

where

$$\theta_P(\Lambda) = v_M^{-1} \prod_{\alpha \in \Delta_P} \Lambda(\alpha^\vee)$$

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and v_M is the co-volume of the lattice spanned by the co-roots in $\mathfrak{a}_M$. The basic result is that the function c_M is smooth on $i\mathfrak{a}_M^*$.

Suppose that $L \in \mathcal{L}(M)$ and $Q \in \mathcal{P}(L)$. For any $R \in \mathcal{F}^L(M)$ we denote by $Q(R) = R \cdot \mathrm{rad}(Q)$ the unique parabolic subgroup of G contained in Q such that $Q(R) \cap L = R$. We have $M_{Q(R)} = M_R$.

If $c_Q(\Lambda)$ is a (G, M)-family and $S \in \mathcal{F}(M)$ we may consider the (M_S, M)-family

$$c_R^S(\Lambda) = c_{S(R)}(\Lambda).$$

Similarly, if $L \in \mathcal{L}(M)$ we may consider the (G, L)-family

$$c_Q(\Lambda) = c_{Q_1}(\Lambda),$$

where $Q_1 \subset Q$ and $Q_1 \in \mathcal{P}(M)$. (Since $\Lambda \in i\mathfrak{a}_L^*$, $c_Q(\Lambda)$ is independent of the choice of Q_1.)

Suppose that $c_P(\Lambda)$ and $d_Q(\Lambda)$ are two (G, M)-families. Then we have a product formula [**7**, Corollary 7.4]*

$$(cd)_M = \sum_{Q_1, Q_2 \in \mathcal{F}(M)} \alpha_{Q_1,Q_2} c_M^{Q_1} d_M^{Q_2}, \tag{4.1}$$

where α_{Q_1,Q_2} are certain constants, which are non-zero only if

$$\mathfrak{a}_M^G = \mathfrak{a}_M^{L_1} \oplus \mathfrak{a}_M^{L_2} = \mathfrak{a}_{L_1}^G \oplus \mathfrak{a}_{L_2}^G.$$

The constants α_{Q_1,Q_2} depend on a certain choice. Their exact value will be immaterial to us.

We consider now (G, M)-families of a special form. Let c be a meromorphic function on $\mathbb{C}$, holomorphic on the imaginary axis such that $c(0) = 1$. Fix a reduced root β of (T_M, G) and consider the (G, M)-family

$$c_Q(\Lambda) = \begin{cases} c(\langle \Lambda, \beta^\vee \rangle) & \text{if } \beta \in \Sigma_Q, \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 4.1. *We have*

$$c_M(\Lambda) = \begin{cases} 1 & \textit{if } M = G, \\ \dfrac{c(\langle \Lambda, \beta^\vee \rangle) - 1}{\langle \Lambda, \beta^\vee \rangle} & \textit{if } M \textit{ is of co-rank one in } G, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The first two cases are clear. For the last case, observe that

$$c_M(\Lambda) = c(\langle \Lambda, \beta^\vee \rangle) \sum_{P \in \mathcal{P}(M): \beta \in \Sigma_P} \theta_P(\Lambda)^{-1} + \sum_{P \in \mathcal{P}(M): \beta \notin \Sigma_P} \theta_P(\Lambda)^{-1}.$$

* Strictly speaking this is stated only for $\Lambda = 0$ but the proof carries over verbatim for any Λ (cf. [**7**, Proposition 7.1]).

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Since

$$\sum_{P\in\mathcal{P}(M)} \theta_P(\Lambda)^{-1} = 0,$$

it remains to prove that

$$\sum_{P\in\mathcal{P}(M):\beta\in\Sigma_P} \theta_P(\Lambda)^{-1} = 0.$$

Observe that the function

$$A(\Lambda) = \langle\Lambda,\beta^\vee\rangle \cdot \sum_{P\in\mathcal{P}(M):\beta\in\Sigma_P} \theta_P(\Lambda)^{-1}$$

has no singularities. Indeed, let $\alpha \in \Sigma(T_M, G)$ and consider the hyperplane $L_\alpha = \{\Lambda : \langle\Lambda,\alpha^\vee\rangle = 0\}$. If $\alpha = \pm\beta$ then the singularity along L_α is cancelled by the factor $\langle\Lambda,\beta^\vee\rangle$. If $\alpha \neq \pm\beta$ then whenever $P, P' \in \mathcal{P}(M)$ are adjacent with $\Sigma_{\bar{P}} \cap \Sigma_{P'} = \{\alpha\}$ we have $\theta_P(\Lambda) = -\theta_{P'}(\Lambda)$ on L_α. Hence, there is no singularity along L_α since $\beta \in \Sigma_P$ if and only if $\beta \in \Sigma_{P'}$.

It follows that $A(\lambda)$ is a polynomial. However, since $\dim \mathfrak{a}_M^G > 1$, it is easy to see that $\lim_{t\to\infty} A(t\vec{v}) \to 0$ for any $\vec{v}$ in general position. Hence $A = 0$. $\square$

Let $S \in \mathcal{P}(L)$ with $L \supset M$. The lemma immediately extends to the (L, M)-family c_R^S. We get

$$c_M^S(\Lambda) = \begin{cases} \dfrac{c(\langle\Lambda,\beta^\vee\rangle) - 1}{\langle\Lambda,\beta^\vee\rangle} & \text{if } M \text{ is of co-rank one in } L \text{ and } \beta \in \Sigma(T_M, L), \\ c(\langle\Lambda,\beta^\vee\rangle) & \text{if } M = L \text{ and } \beta \in \Sigma(T_M, \mathrm{rad}(S)), \\ 1 & \text{if } M = L \text{ and } \beta \in \Sigma(T_M, \mathrm{rad}(\bar{S})), \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Consider now a more general family (G, M)-family of the form

$$c_Q(\Lambda) = \prod_{\beta\in\Sigma_Q} c_\beta(\langle\Lambda,\beta^\vee\rangle) \tag{4.3}$$

as in [**5**, § 7]. Here, for any reduced root β of (T_M, G) c_β is a meromorphic functions on $\mathbb{C}$ holomorphic in a neighbourhood of $\mathrm{i}\mathbb{R}$ such that $c_\beta(0) = 1$. We can write c_Q as a product of (G, M)-families of the previous type. Applying the product formula (4.1) and using induction we get

$$c_M(\Lambda) = \sum_{\mathcal{B}_1,\mathcal{B}_2} \alpha_{\mathcal{B}_1,\mathcal{B}_2} \prod_{\beta\in\mathcal{B}_1} \frac{c_\beta(\langle\Lambda,\beta^\vee\rangle) - 1}{\langle\Lambda,\beta^\vee\rangle} \prod_{\beta\in\mathcal{B}_2} c_\beta(\langle\Lambda,\beta^\vee\rangle),$$

where $\mathcal{B}_1$, $\mathcal{B}_2$ range over pairs of subsets of $\Sigma(T_M, G)$ such that $\mathcal{B}_1$ forms a basis of $(\mathfrak{a}_M^G)^*$ and $\mathcal{B}_2$ is contained in the complement of $\mathcal{B}_1$, and $\alpha_{\mathcal{B}_1,\mathcal{B}_2}$ are certain explicit constants whose exact value is unimportant for us.

Let now $L \in \mathcal{L}(M)$, $S \in \mathcal{P}(L)$ and $c_Q(\Lambda)$ as before. The (L,M)-family $c_R^S(\Lambda)$ is the product of

$$\prod_{\beta \in \Sigma(T_M, \operatorname{rad}(S))} c_\beta(\langle \Lambda, \beta^\vee \rangle)$$

with an (L,M)-family of the same type (using the c_β for $\beta \in \Sigma(T_M, L)$). It follows that

$$c_M^Q(\Lambda) = \sum_{\mathcal{B}_1, \mathcal{B}_2} \alpha_{\mathcal{B}_1, \mathcal{B}_2} \prod_{\beta \in \mathcal{B}_1} \frac{c_\beta(\langle \Lambda, \beta^\vee \rangle) - 1}{\langle \Lambda, \beta^\vee \rangle} \prod_{\beta \in \mathcal{B}_2} c_\beta(\langle \Lambda, \beta^\vee \rangle), \tag{4.4}$$

where now $\mathcal{B}_1$, $\mathcal{B}_2$ ranges over the pairs of subset of $\Sigma(T_M, G)$ such that $\mathcal{B}_1$ forms a basis for $(\mathfrak{a}_M^L)^*$ and $\mathcal{B}_2 \supset \Sigma(\operatorname{rad} Q)$ satisfies $\mathcal{B}_2 \cap (\Sigma(\operatorname{rad} \bar{Q}) \cup \mathcal{B}_1) = \emptyset$.

5. Eisenstein series and intertwining operators

By our convention, whenever X is a variety over F, we denote its F-points by X as well.

For any (semi-standard) M choose an isomorphism $T_M \simeq \mathbb{G}_m^l$ and let A_M be the image of $\mathbb{R}_+^l$ in $T_M(F_\infty)$ where $\mathbb{R} \hookrightarrow F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$ by $x \mapsto 1 \otimes x$. We set $M(\mathbb{A})^1 = \cap \ker |\chi|$ where χ ranges over all characters of M over F. We have $M(\mathbb{A}) = M(\mathbb{A})^1 \times A_M$.

We choose a maximal compact $\boldsymbol{K}$ of $G(\mathbb{A})$ which is admissible relative to M^0 (see [**3**, § 1]) and define a height function $H_P : G(\mathbb{A}) \to \mathfrak{a}_M$ which is right-$\boldsymbol{K}$-invariant, left-$U(\mathbb{A})M(\mathbb{A})^1$-invariant and gives isomorphisms of A_M with $\mathfrak{a}_M$. It would be harmless to assume that representatives of the Weyl group W can be chosen to lie in $\boldsymbol{K}$ (as in the case of GL_n). Otherwise, the distinguished point $T_0 \in \mathfrak{a}_0$ of [**3**, Lemma 1.1] has to be taken into account. Let δ_P be the modulus function of $P(\mathbb{A})$ and let $\rho_P \in \mathfrak{a}_M^*$ be the weight corresponding to $\delta_P^{1/2}$.

Let π be an automorphic representation of $M(\mathbb{A})$ (which will always be assumed to be trivial on A_M). We define $\mathcal{A}_P$ (respectively, $\mathcal{A}_P^2$, $\mathcal{A}_P^{\mathrm{c}}$, $\mathcal{A}_P^\pi$, $\mathcal{A}_P^{\mathrm{r}}$) to be the space of automorphic forms on $U(\mathbb{A})M\backslash G(\mathbb{A})$ such that $\varphi(ag) = \mathrm{e}^{\langle \rho_P, H_P(a) \rangle} \varphi(g)$ for any $a \in A_M$, $g \in G(\mathbb{A})$ (and for any $k \in \boldsymbol{K}$ the function $m \to \varphi(mk)$ on $M\backslash M(\mathbb{A})^1$ is square-integrable, cuspidal, belongs to the space of π, or is residual, respectively). The unitary structure on $\mathcal{A}_P^2$ is given by integration over $A_M U(\mathbb{A}) M \backslash G(\mathbb{A})$. For any $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ define

$$I_P(g, \lambda)\varphi(x) = \mathrm{e}^{-\langle \lambda, H_P(x) \rangle} \mathrm{e}^{\langle \lambda, H_P(xg) \rangle} \varphi(xg)$$

for any $g, x \in G(\mathbb{A})$ and $\varphi \in \mathcal{A}_P$. If $Q \supset P$, and $\varphi \in \mathcal{A}_P$ the Eisenstein series is defined by

$$E^Q(g, \varphi, \lambda) = \sum_{\gamma \in P \backslash Q} \mathrm{e}^{\langle \lambda, H_P(\gamma g) \rangle} \varphi(\gamma g)$$

for $\lambda \in \mathfrak{a}_{M,\mathbb{C}}^*$ with $\operatorname{Re} \lambda$ sufficiently positive. If $Q = G$ we set $E(g, \varphi, \lambda) = E^G(g, \varphi, \lambda)$

We may identify $\mathcal{A}_P^\pi$ with the $\boldsymbol{K}$-finite part of the induced representation $I_P(\pi, \lambda)$. Whenever regular, the Eisenstein series defines a morphism from the $\boldsymbol{K}$-finite part of $I_P(\pi, \lambda)$ to the space of automorphic forms of G. By the 'automatic continuity theorem' of Casselman–Wallach [**64**, Theorem 11.6.7] this extends to a continuous map from the smooth part of $I_P(\pi, \lambda)$ to the space of smooth functions of moderate growth on $G\backslash G(\mathbb{A})$

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(which are eigenfunctions for the centre of the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ with the appropriate character). This means that there exists $N \geqslant 0$ such that for any $X \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ we have

$$|\rho(X)E(g,\varphi,\lambda)| \leqslant \|g\|^N \nu(\varphi) \tag{5.1}$$

for some continuous semi-norm ν on the space of smooth vectors. We will still use $E(g,\varphi,\lambda)$ to denote the resulting extension.

Recall the intertwining operators $M_{Q|P}(\lambda)$ defined in [**5**, § 1]. They extend to continuous maps between the appropriate spaces of smooth vectors. We normalize the intertwining operators as in [**5**, § 6] (cf. [**8**, Theorem 2.1]). Thus, we write

$$M_{Q|P}(\lambda) = n_{Q|P}(\pi,\lambda)N_{Q|P}(\pi,\lambda)$$

on $\mathcal{A}_P^{\pi}$ where $N_{Q|P}$ are the normalized intertwining operators satisfying the properties [**5**, (6.3)–(6.6)] and $n_{Q|P}$ are the normalizing factors which are given by

$$\prod_{\beta \in \Sigma_Q \cap \Sigma_{\bar{P}}} n_\beta(\pi, \langle \lambda, \beta^\vee \rangle) = \prod_{\beta \in \Sigma_{\bar{Q}} \cap \Sigma_P} n_\beta(\pi, \langle \lambda, \beta^\vee \rangle)^{-1}.$$

We will only consider the case where π is cuspidal. The functions $n_\beta(\pi, z)$ are meromorphic on $\mathbb{C}$, holomorphic on the imaginary axis. They are also holomorphic on the *left* half-plane $\mathrm{Re}(z) < 0$ except for finitely many simple poles, all of them on the real line. We have the functional equations

$$n_{-\beta}(\pi,z)n_\beta(\pi,-z) = 1,$$
$$\overline{n_\beta(\pi,z)} = n_{-\beta}(\pi,\bar{z}).$$

Thus, $n_\beta(\pi,z)^{-1} = \overline{n_\beta(\pi,-\bar{z})}$ so that $|n_\beta(\pi,\mathrm{i}t)| = 1$ for $t \in \mathbb{R}$.

Suppose that $G = \mathrm{GL}_n$ and $\pi = \pi_1 \otimes \pi_2$ is a cuspidal representation of the Levi part of a maximal parabolic P of G. Then

$$n_\beta(\pi,-z) = \frac{L(z,\pi_1 \times \tilde{\pi}_2)}{\varepsilon(z,\pi_1 \times \tilde{\pi}_2)L(z+1,\pi_1 \times \tilde{\pi}_2)} = \frac{L(z,\pi_1 \times \tilde{\pi}_2)}{L(-z,\tilde{\pi}_1 \times \pi_2)},$$

where $\Sigma_{\bar{P}} = \{\beta\}$ and $L(z,\pi_1 \times \tilde{\pi}_2)$, $\varepsilon(z,\pi_1 \times \tilde{\pi}_2)$ are the Rankin–Selberg L-functions and epsilon factors defined in [**40**] and we are using the functional equation

$$L(z,\pi_1 \times \tilde{\pi}_2) = \varepsilon(z,\pi_1 \times \tilde{\pi}_2)\overline{L(1-\bar{z},\pi_1 \times \tilde{\pi}_2)}.$$

Also, we recall that

$$\varepsilon(z,\pi_1 \times \tilde{\pi}_2) = \varepsilon_0 q^{(1/2)-z}, \tag{5.2}$$

where q is the conductor (a positive integer) and $|\varepsilon_0| = 1$. The function

$$[z(z-1)]^{\delta_{\pi_1,\pi_2}} L(z,\pi_1 \times \tilde{\pi}_2)$$

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is an entire function of order 1 where $\delta_{\pi_1,\pi_2} = 1$ if $\pi_2 = \pi_1$ and 0 otherwise (see, for example, [**61**]). We set $\xi(z) = n_\beta(\pi, -z)$. The function

$$\xi(z)\left[\frac{z-1}{z+1}\right]^{\delta\pi_1,\pi_2}$$

is holomorphic for $\mathrm{Re}(z) \geqslant 0$.

For a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ we denote by $\mathfrak{c}(\pi)$ the analytic conductor of π which combines the conductor appearing in the functional equation for $L(s,\pi)$ and the 'Archimedean size' of π (cf. [**24**]). Actually, if we fix a compact open subgroup K_{fin} of $G(\mathbb{A}_{\mathrm{fin}})$ and assume that π has a K_{fin}-fixed vector, then the analytic conductor is bounded by the Archimedean size of π up to a constant which depends only on K_{fin} (see [**56**, § 2]). More generally, if $\pi = \pi_1 \otimes \cdots \otimes \pi_r$ we set $\mathfrak{c}(\pi) = \prod_i \mathfrak{c}(\pi_i)$.

Notation

If Y is a quantity depending on X and π we will use the notation $Y \ll \mathcal{O}_\pi(X)$ to mean that there exist constants c, $\alpha > 0$, depending only on n and F, such that

$$|Y| \leqslant c(\mathfrak{c}(\pi)(1 + |X|))^\alpha.$$

For example it is well known (e.g. [**23**]) that the conductor in (5.2) satisfies

$$q \ll \mathcal{O}_\pi(1),$$

where $\pi = \pi_1 \otimes \pi_2$.

The known analytic properties of the Rankin–Selberg L-function and the Phragmen–Lindelof principle yield the standard bound

$$(z-1)^{\delta\pi_1,\pi_2} L^\infty(z, \pi_1 \times \tilde{\pi}_2) \ll \mathcal{O}_\pi(\mathrm{Im}\, z) \tag{5.3}$$

uniformly in any (fixed) vertical strip $a \leqslant \mathrm{Re}(z) \leqslant b$, where L^∞ denotes the partial L-function outside the Archimedean places (i.e. the corresponding Dirichlet series). A similar estimate holds for the derivatives of $L^\infty(z, \pi_1 \times \tilde{\pi}_2)$.

We are now going to use the following result due to Brumley (cf. [**11**], where the implied constants are made explicit):

$$[t^{\delta\pi_1,\pi_2} L^\infty(\sigma + \mathrm{i}t, \pi_1 \times \tilde{\pi}_2)]^{-1} \ll \mathcal{O}_\pi(t) \tag{5.4}$$

for all $t \in \mathbb{R}$ and $\sigma \geqslant 1$. (It is enough to consider $\sigma = 1$ by Phragmen–Lindelof.) This is closely related to a coarse zero-free region for $L(s, \pi_1 \times \tilde{\pi}_2)$. The simplest case of (5.4) is the lower bound $|L(1,\chi)| \geqslant c/\sqrt{q}$ for a Dirichlet character χ of conductor q. (For odd quadratic characters this amounts to the trivial inequality $h(d) \geqslant 1$ for the class number of an imaginary quadratic field.)

We denote by $\mathfrak{I}(\pi)$ a region in the complex plane of the form

$$\{z \in \mathbb{C} : -d(\mathfrak{c}(\pi)(1 + |\mathrm{Im}\, z|))^{-\beta} < \mathrm{Re}(z) < \tfrac{1}{2}\},$$

where $d, \beta > 0$ depend only on n and F.

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Lemma 5.1. *For an appropriate choice of $\mathfrak{I}(\pi)$ (i.e. of d, β) we have*

$$\xi(z) \ll \mathcal{O}_\pi(\operatorname{Im} z) \tag{5.5}$$

for all $z \in \mathfrak{I}(\pi)$. In particular, $\xi(z)$ is holomorphic on $\mathfrak{I}(\pi)$.

Proof. The function $\xi(z)$ differs from $L^\infty(z, \pi_1 \times \tilde{\pi}_2)/\varepsilon(z, \pi_1 \times \tilde{\pi}_2) L^\infty(z+1, \pi_1 \times \tilde{\pi}_2)$ by the factor $L_\infty(z, \pi_{1,\infty} \times \tilde{\pi}_{2,\infty})/L_\infty(z+1, \pi_{1,\infty} \times \tilde{\pi}_{2,\infty})$, where

$$L_\infty(z, \pi_{1,\infty} \times \tilde{\pi}_{2,\infty}) = \prod_{v|\infty} L_v(z, \pi_{1v} \times \tilde{\pi}_{2v})$$

can be written as $\prod_{j=1}^m \Gamma_{\mathbb{R}}(z - \alpha_j)$ for certain parameters $\{\alpha_j\}$ where $\Gamma_{\mathbb{R}}(z) = \pi^{-z/2}\Gamma(z/2)$ and $m = \dim U/\mathbb{Q}$. Thus, we have

$$L_\infty(z, \pi_{1,\infty} \times \tilde{\pi}_{2,\infty}) = P(z)^{-1} L_\infty(z+2, \pi_{1,\infty} \times \tilde{\pi}_{2,\infty}),$$

where

$$P(z) = \prod_j \frac{z - \alpha_j}{2\pi}.$$

Now, $(z(z-1))^{\delta_{\pi_1,\pi_2}} P(z)^{-1} L^\infty(z, \pi_1 \times \tilde{\pi}_2)$ is entire and $\ll \mathcal{O}_\pi(\operatorname{Im} z)$ in any vertical strip. On the other hand, by Jacquet–Shalika, $L_\infty(z, \pi_{1,\infty} \times \tilde{\pi}_{2,\infty})$ is holomorphic for $\operatorname{Re}(z) \geqslant 1$, and hence, $\operatorname{Re}(\alpha_j) < 1$. By elementary properties of the Γ-function we have

$$|\Gamma(z + \tfrac{1}{2})/\Gamma(z)| \leqslant 100(1 + |z|)^5$$

for $\operatorname{Re}(z) \geqslant -\frac{1}{4}$ (see, for example, [**6**, p. 33]). It follows that

$$\frac{L_\infty(z+2, \pi_{1,\infty} \times \tilde{\pi}_{2,\infty})}{L_\infty(z+1, \pi_{1,\infty} \times \tilde{\pi}_{2,\infty})} \ll \mathcal{O}_\pi(|z|)$$

for $-\frac{1}{2} < \operatorname{Re}(z) < \frac{1}{2}$. Finally, it easily follows from (5.4) and standard upper bounds on the derivative that

$$[z^{\delta_{\pi_1,\pi_2}} L^\infty(z+1, \pi_1 \times \tilde{\pi}_2)]^{-1} \ll \mathcal{O}_\pi(|z|)$$

on $\mathfrak{I}(\pi)$ for a certain choice of d, $\beta > 0$. All in all,

$$\begin{aligned}\xi(z) = (z-1)^{-\delta_{\pi_1,\pi_2}} \cdot \varepsilon_0 q^{z-(1/2)} \cdot (z(z-1))^{\delta_{\pi_1,\pi_2}} P(z)^{-1} L^\infty(z, \pi_1 \times \tilde{\pi}_2) \\ \cdot \frac{L_\infty(z+2, \pi_{1,\infty} \times \tilde{\pi}_{2,\infty})}{L_\infty(z+1, \pi_{1,\infty} \times \tilde{\pi}_{2,\infty})} \cdot [z^{\delta_{\pi_1,\pi_2}} L^\infty(z+1, \pi_1 \times \tilde{\pi}_2)]^{-1}\end{aligned}$$

and the lemma follows. □

Corollary 5.2.

(1) $\xi'(z) \ll \mathcal{O}_\pi(\operatorname{Im} z)$ *on* $\mathfrak{I}(\pi)$.

(2) $\dfrac{\xi(z') - \xi(z)}{z' - z} \ll \mathcal{O}_\pi(|\operatorname{Im} z| + |\operatorname{Im} z'|)$ *for* $z, z' \in \mathfrak{I}(\pi)$.

(3) $\dfrac{\xi(-z)^{-1}\xi(z') - 1}{z' + z} \ll \mathcal{O}_\pi(|\operatorname{Im} z| + |\operatorname{Im} z'|)$ *for* $z, z' \in \mathfrak{I}(\pi)$.

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Proof. The first part follows from Cauchy's formula, while the second part follows from the first by the mean value theorem. To see the last part we write, for $\mathrm{Re}(z) < 0$ (respectively, $\mathrm{Re}(z') < 0$),

$$\frac{\xi(-z)^{-1}\xi(z')-1}{z'+z} = \overline{\xi(\bar{z})}\frac{\xi(z')-\xi(-z)}{z'+z} \quad \left(\text{respectively, } \xi(z')\overline{\frac{\xi(\bar{z})-\xi(-\bar{z}')}{z+z'}}\right)$$

and use the previous bound. On the other hand, if $\mathrm{Re}(z), \mathrm{Re}(z') > 0$ then we write (for $\bar{z} = \sigma + \mathrm{i}t$)

$$\xi(-z)^{-1}\xi(z') - 1 = \overline{\xi(\bar{z})}(\xi(z') - \xi(\bar{z})) + \overline{\xi(\bar{z})}(\xi(\bar{z}) - \xi(\mathrm{i}t)) + \xi(\mathrm{i}t)\cdot\overline{\xi(\bar{z}) - \xi(\mathrm{i}t)}$$

and observe that $|z + z'| \geqslant |z' - \bar{z}|, \sigma$. □

6. Majorization of Eisenstein series

Let M be a Levi subgroup of G and π a cuspidal representation of $M(\mathbb{A})^1$. We fix a large compact subset C of $G(\mathbb{A})$ and a small compact open subgroup K_{fin} of $G(\mathbb{A}_{\mathrm{fin}})$ and consider the space of sufficiently differentiable bi-K_{fin}-invariant functions f on $G(\mathbb{A})$ which are supported in C. On this space we consider the semi-norms obtained as finite sums of $\|\mathcal{X}_i * f * \mathcal{Y}_i\|_\infty$, where $\mathcal{X}_i, \mathcal{Y}_i \in \mathcal{U}(\mathfrak{g}_\mathbb{C})$. Extending the notation of the last section, we write $Y \ll \mathfrak{o}_{\pi,f}(X, Z)$ if there exists a semi-norm μ and $\alpha > 0$, depending only on n and F such that for any N there exists a constant c' so that for all f as above

$$|Y| \leqslant c'\mu(f)(1 + |X|)^\alpha((1 + \Lambda_\pi)(1 + Z))^{-N}.$$

The invariant Λ_π is defined in [**54**, p. 695]. We denote by $\mathfrak{I}_M(\pi)$ a region in $\mathfrak{a}^*_{M,\mathbb{C}}$ of the form

$$\{\lambda \in \mathfrak{a}^*_{M,\mathbb{C}},\ \|\mathrm{Re}\,\lambda\| < \delta : \text{either } \mathrm{Re}\,\lambda \in \overline{(\mathfrak{a}^*_M)_+} \text{ or } \|\mathrm{Re}\,\lambda\| < d(\mathfrak{c}(\pi)(1 + \|\mathrm{Im}\,\lambda\|))^{-\beta}\},$$

where again d, β, $\delta > 0$ depend only on n and F.

Assume now that $G = \mathrm{GL}_n$. It is shown in [**55**] that $\mathfrak{c}(\pi) \leqslant \Lambda_\pi$. (In fact, the proof carries over for general G.)

Proposition 6.1. *With the above notation we have* (*for an appropriate choice of* $\mathfrak{I}_M(\pi)$)

$$\left(\sum_{\varphi\in\mathcal{B}_P(\pi)} |E(g, I(f,\lambda)\varphi, \lambda)|^2\right)^{1/2} \ll \mathfrak{o}_{\pi,f}(\|g\|, \|\lambda\|) \tag{6.1}$$

for all $g \in G(\mathbb{A})^1$ *and* $\lambda \in \mathfrak{I}_M(\pi)$.

The proof of Proposition 6.1 occupies the rest of this section. We will fix a minimal parabolic P^0, with Levi part M^0, and assume, as we may, that $P \supset P^0$. We set $H_0 = H_{P^0}$ and similarly for other notation. Recall the definition of a Siegel set $\mathfrak{S}$ [**53**, I.2] and Arthur's truncation operator Λ^T [**2**]. Let $\hat{\Delta}_0$ be the dual basis of the co-roots in $\mathfrak{a}^*_0 = \mathfrak{a}^*_{P^0}$. More generally, if $P \supset P_0$, denote by $\hat{\Delta}_P$ the dual basis of $\Delta^\vee_P = \{\alpha^\vee : \alpha \in \Delta_P\}$ in $\mathfrak{a}^*_M$.

We first prove the following lemma (which is valid for any G).

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Lemma 6.2. *There exists a constant $c < 0$ depending only on G such that*

$$\Lambda^T \varphi(g) = \varphi(g)$$

for any automorphic form φ and any $g \in \mathfrak{S}$ with $\langle \varpi, H_0(g) - T \rangle < c$ for all $\varpi \in \hat{\Delta}_0$.

Proof. We have to show all the terms in Λ^T coming from proper parabolic subgroups vanish. Thus, we will show that $\hat{\tau}_P(H_P(\gamma g) - T) = 0$ for all $P \supset P^0$ and $\gamma \in G$ where, as in [**1**], $\hat{\tau}_P$ is the characteristic function of the obtuse cone in $\mathfrak{a}_P$ spanned by the co-roots.

Suppose on the contrary that $\hat{\tau}_P(H_P(\gamma g) - T) = 1$ and write $\gamma = pwu$ with $p \in P$, $w \in W$ and $u \in U^0$ (Bruhat decomposition). Also write $g = p_0 a k$ with $p_0 \in P^0(\mathbb{A})^1$, $a \in A_0$ and $k \in \boldsymbol{K}$ using the Iwasawa decomposition. Then

$$H_P(\gamma g) = H_P(wug) = (wH_0(g) + H_0(wu'))_P$$

with $u' \in U^0(\mathbb{A})$. Thus,

$$\begin{aligned}\langle \varpi, T \rangle &\leqslant \langle \varpi, H_0(\gamma g) \rangle \\ &= \langle \varpi, wH_0(g) \rangle + \langle \varpi, H_0(wu') \rangle \\ &= \langle \varpi, H_0(g) \rangle + \langle w^{-1}\varpi - \varpi, H_0(g) \rangle + \langle \varpi, H_0(wu') \rangle\end{aligned}$$

for any $\varpi \in \hat{\Delta}_P$. It follows that

$$\langle \varpi, H_0(g) - T \rangle > \langle \varpi - w^{-1}\varpi, H_0(g) \rangle - \langle \varpi, H_0(wu') \rangle.$$

However, $\varpi - w^{-1}\varpi$ is a linear combination of roots with non-negative coefficients, while $\langle \varpi, H_0(wu') \rangle$ is bounded above uniformly in $u' \in U^0(\mathbb{A})$. The lemma follows. □

We remark that at the cost of changing T, the lemma and its proof still hold if we replace g by gx where x is confined to a fixed compact subset of $G(\mathbb{A})$. Let now $g \in \mathfrak{S}$. It follows that for T which is a fixed translate T of $H_0(g)$ (depending on the support of f) we have

$$\begin{aligned}E(g, I(f,\lambda)\varphi, \lambda) &= \int_{G(\mathbb{A})^1} f(x) E(gx, \varphi, \lambda)\,\mathrm{d}x \\ &= \int_{G(\mathbb{A})^1} f(x) \Lambda^T E(gx, \varphi, \lambda)\,\mathrm{d}x \\ &= \int_{G(\mathbb{A})^1} f(g^{-1}x) \Lambda^T E(x, \varphi, \lambda)\,\mathrm{d}x \\ &= \int_{G\backslash G(\mathbb{A})^1} K_f(g,x) \Lambda^T E(x, \varphi, \lambda)\,\mathrm{d}x.\end{aligned}$$

By [**53**, I.2.4], $|K_f(g,x)| \leqslant c|f|_\infty \|g\|^N$ uniformly in x for some $c, N > 0$ (depending only on the support of f). Hence, the Cauchy–Schwarz inequality gives

$$|E(g, I(f,\lambda)\varphi, \lambda)|^2 \leqslant c' \|g\|^{2N} \|f\|_\infty^2 \int_{G\backslash G(\mathbb{A})^1} |\Lambda^T E(x, \varphi, \lambda)|^2\,\mathrm{d}x. \tag{6.2}$$

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As in [**4**] let $\Omega_P^T(\lambda)$ be the operator on $\mathcal{A}_P$ which is given by

$$\langle \Omega_P^T(\lambda)\varphi, \varphi' \rangle_{U(\mathbb{A})MA_M\backslash G(\mathbb{A})^1} = \langle \Lambda^T E(\varphi, \lambda), E(\varphi, \lambda) \rangle_{G\backslash G(\mathbb{A})^1}.$$

By [**1**, § 4] we may write $f = g_1 \star f_1 + g_2 \star f_2$, where $f_1 = f$, $f_2 = Z \star f$, $Z \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$, g_1 is smooth and of compact support and g_2 is compactly supported of class C^m (m is as large as we please). The g_i and Z are independent of f. For the proof of Proposition 6.1 we can assume that $g \in \mathfrak{S}$. Applying (6.2) we obtain

$$\begin{aligned}\sum_{\varphi} |E(g, I(f, \lambda)\varphi, \lambda)|^2 &\leqslant \sum_{j=1}^{2} c' \|g\|^{2N} \|g_j\|_\infty^2 \|I_P(f_j, \lambda)^* \Omega^T(\lambda) I_P(f_j, \lambda)\|_{\pi,1} \\ &\leqslant \sum_{j=1}^{2} c'' \|g\|^{2N} \|f_j\|_\infty \|\Omega^T(\lambda) I_P(f_j, \lambda)\|_{\pi,1}, \end{aligned} \tag{6.3}$$

where $\|\cdot\|_{\pi,1}$ denotes the trace norm on $\overline{\mathcal{A}_P^\pi}$. Here we use the fact that $\|I_P(f_j, \lambda)\|_{\overline{\mathcal{A}_P^2}}$ is bounded by a constant multiple of $\|f_j\|_\infty$ in every vertical strip.

Next, we bound the right-hand side of (6.3). For simplicity of notation we replace f_j by f.

By [**5**, pp. 1295, 1296] the operator $\Omega^T(\lambda)$ is given by the sum over the representatives $s \in W/W_M$ such that $sMs^{-1} = M$ of the value at $\lambda' = \lambda$ of

$$\sum_{Q \in \mathcal{P}(M_P)} M_{Q|P}(-\bar{\lambda})^{-1} M_{Q|P}(s\lambda') M_{P|P}(s, \lambda') \mathrm{e}^{\langle s\lambda' + \bar{\lambda}, Y_Q(T) \rangle} \theta_Q(s\lambda' + \bar{\lambda})^{-1},$$

where $Y_Q(T)$ is the projection of $t^{-1}T$ to $\mathfrak{a}_M$ and t is such that tQ is standard. ($Y_Q(T)$ depends only on Q since t is uniquely determined up to right multiplication by W_M.) Unlike in [**5**], we do not assume that $\lambda \in \mathrm{i}\mathfrak{a}_M^*$. We also recall that in the notation of [**5**] $T_0 = 0$ because we are working with GL_n. Recall [**5**, p. 1310] the (G, M)-families (in Λ)

$$\begin{aligned} \mathcal{M}_Q(P, \lambda, \Lambda) &= M_{Q|P}(\lambda)^{-1} M_{Q|P}(\lambda + \Lambda), \\ c_Q(T, \Lambda) &= \mathrm{e}^{\langle \Lambda, Y_Q(T) \rangle}, \\ \mathcal{M}_Q^T(P, \lambda, \Lambda) &= c_Q(T, \Lambda) \mathcal{M}_Q(P, \lambda, \Lambda). \end{aligned}$$

Then $\Omega^T(\lambda)$ is the sum over s of the value at $\lambda' = \lambda$ of

$$\sum_{Q \in \mathcal{P}(M_P)} \mathcal{M}_Q(P, -\bar{\lambda}, s\lambda' + \bar{\lambda}) c_Q(T, s\lambda' + \bar{\lambda}) M_{P|P}(s, \lambda') \theta_Q(s\lambda' + \bar{\lambda})^{-1}$$

which is $\mathcal{M}_M^T(P, -\bar{\lambda}, s\lambda + \bar{\lambda}) M_{P|P}(s, \lambda)$. We now use the product formula (4.1). For any Q the function $c_M^Q(T, \Lambda)$ is the Fourier transform of a characteristic function of a compact domain determined by the points $Y_Q(T)$ [**5**, (3.1)]. Hence, it is bounded by a fixed power of $\mathrm{e}^{\|T\|}$, and hence, by a fixed power of $\|g\|$. It remains to bound

$$\|\mathcal{M}_M^Q(P, -\bar{\lambda}, s\lambda + \bar{\lambda})_\pi M_{P|P}(s, \lambda) I_P(f, \lambda)\|_{\pi,1}$$

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for any Q. Using the normalization of the intertwining operators we may write

$$\mathcal{M}_Q(P,\lambda,\Lambda)_\pi = \nu_Q(P,\pi,\lambda,\Lambda)\mathfrak{N}_Q(P,\pi,\lambda,\Lambda),$$

where $\nu_Q(P,\pi,\lambda,\Lambda)$ and $\mathfrak{N}_Q(P,\pi,\lambda,\Lambda)$ are the (G,M)-families given by

$$\begin{aligned}\nu_Q(P,\pi,\lambda,\Lambda) &= n_{Q|P}(\pi,\lambda)^{-1}n_{Q|P}(\pi,\lambda+\Lambda),\\ \mathfrak{N}_Q(P,\pi,\lambda,\Lambda) &= N_{Q|P}(\pi,\lambda)^{-1}N_{Q|P}(\pi,\lambda+\Lambda).\end{aligned}$$

Also, using [**5**, (1.4)] we may write

$$M_{P|P}(s,\lambda) = M_{P|sP}(s\lambda)s = n_{P|sP}(s\lambda)N_{P|sP}(s\lambda)s,$$

where s denotes the unitary intertwining operator $\mathcal{A}_P \to \mathcal{A}_{sP}$ obtained by conjugation by s. We apply the product formula (4.1) once again to

$$\mathcal{M}_R^Q(P,\lambda,\Lambda) = \nu_R^Q(P,\pi,\lambda,\Lambda)\mathfrak{N}_R^Q(P,\pi,\lambda,\Lambda).$$

We first estimate

$$\nu_M^R(P,\pi,-\bar{\lambda},s\lambda+\bar{\lambda})n_{P|sP}(s\lambda) = \nu_M^R(P,\pi,-\bar{\lambda},s\lambda+\bar{\lambda})\cdot\prod_{\beta\in\Sigma_{\bar{P}}\cap s\Sigma_P} n_\beta(\pi,\langle s\lambda,\beta^\vee\rangle)^{-1} \quad (6.4)$$

for any R. The family $\nu_Q(P,\pi,\lambda,\Lambda)$ is of the type (4.3) where

$$c_\beta(z) = \begin{cases} n_\beta(\pi,\langle\lambda,\beta^\vee\rangle)^{-1}n_\beta(\pi,\langle\lambda,\beta^\vee\rangle+z) & \text{if } \beta\in\Sigma_{\bar{P}},\\ 1 & \text{otherwise.}\end{cases}$$

Using (4.4), (6.4) can be expressed in terms of

$$\begin{gathered} n_\beta(\pi,\langle s\lambda,\beta^\vee\rangle)^{-1},\\ n_\beta(\pi,-\langle\bar{\lambda},\beta^\vee\rangle)^{-1},\\ \frac{n_\beta(\pi,-\langle\bar{\lambda},\beta^\vee\rangle)^{-1} - n_\beta(\pi,\langle s\lambda,\beta^\vee\rangle)^{-1}}{\langle s\lambda+\bar{\lambda},\beta^\vee\rangle},\\ n_\beta(\pi,\langle s\lambda,\beta^\vee\rangle)n_\beta(\pi,-\langle\bar{\lambda},\beta^\vee\rangle)^{-1},\\ \frac{n_\beta(\pi,\langle s\lambda,\beta^\vee\rangle)n_\beta(\pi,-\langle\bar{\lambda},\beta^\vee\rangle)^{-1}-1}{\langle s\lambda+\bar{\lambda},\beta^\vee\rangle}, \end{gathered}$$

where $\beta\in\Sigma_{\bar{P}}$ and in addition, $s^{-1}\beta\in\Sigma_P$ in the first three cases, while $s^{-1}\beta\in\Sigma_{\bar{P}}$ in the last two cases. We may use Corollary 5.2 to bound each of these functions by $\mathcal{O}_\pi(\|\lambda\|)$ on $\mathfrak{I}_M(\pi)$.

Finally, to complete the proof of Proposition 6.1 we will prove that

$$\|\mathfrak{N}_M^S(P,\pi,\lambda,\Lambda)N_{P|sP}(s\lambda)I_P(f,\lambda)\|_{\pi,1} \ll \mathfrak{o}_{\pi,f}(1,\|\lambda\|) \quad (6.5)$$

for $\|\mathrm{Re}\,\lambda\|, \|\mathrm{Re}\,\Lambda\| < \delta$ (δ depending only on n). Indeed, the proof of [**54**, Lemma 6.2] (the rapid decay of $\|I_P(f,\lambda)\|$ on each K_∞-type) applies word by word, except that in

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the inequality (6.16) of [**54**] we have to multiply the right-hand side by a constant, since λ is not assumed to be unitary. At any rate, the argument reduces (6.5) to the following estimate: there exists a constant c such that for any cuspidal π and $\sigma \in \hat{K}_\infty$

$$\|\mathfrak{N}_M^S(P,\pi,\lambda,\Lambda)N_{P|sP}(s\lambda)\|_{\pi,\sigma,K_f} \leqslant c(1+\|\sigma\|)^k.$$

This statement is proved in § 4 of [**56**] using bounds toward the Ramanujan hypothesis proved by Luo, Rudnick and Sarnak [**50**]. This finishes the proof of Proposition 6.1.

Remark 6.3. Majorization for Eisenstein series on GL_n induced from the Borel subgroup was obtained in [**66**]. We expect that Proposition 6.1 remains true for any G. There are two stumbling blocks. The first one is lower bounds for the L-functions appearing in Langlands's formula for the constant term of Eisenstein series. In the case where π is generic this is established in [**17**, **20**]. However, the bounds obtained are not known to be uniform in the Archimedean size of π. The other subtle point is the analysis of the local intertwining operators (cf. [**56**]) which in the GL_n case requires uniform bounds toward the Ramanujan hypothesis.

Change of setup

Fix a minimal parabolic P^0 as before and consider standard parabolic subgroups, i.e. those containing P_0. The Levi part containing M_0 of a standard parabolic is called a standard Levi. A standard parabolic is determined by its Levi part. Henceforth, all parabolic and Levi subgroups are implicitly assumed to be standard. We set $\Delta_M = \Delta_P$, and $\Delta_0 = \Sigma_{P^0}$.

We also denote by Δ_0^M the set of simple roots of T^0 in M with respect to $P^0 \cap M$. Denote by $(\mathfrak{a}_M^*)_+$ the positive Weyl chamber defined by $\{X \in \mathfrak{a}_M^* : \langle X, \alpha^\vee\rangle > 0 \text{ for all } \alpha \in \Delta_M\}$. It is the open cone spanned by the weights $\hat{\Delta}_P$. Set $W(M) = \{w \in W : w\Delta_0^M \subset \Delta_0\}$; it is in natural one-to-one correspondence with $\mathcal{P}(M)$. This correspondence, which we write as $w \mapsto w(P)$, is defined by the property that if P' is the standard parabolic with Levi wMw^{-1} then $w(P) = w^{-1}P'w$. For example, $w(P_0) = w^{-1}P_0w$, but in general $w^{-1}Pw$ may not belong to $\mathcal{P}(M)$. To each $w \in W(M)$ and P' as above we denote by $M(w,\lambda) : \mathcal{A}_P \to \mathcal{A}_{P'}$ the intertwining operator $wM_{w(P)|P}(\lambda)$.

More generally, the sets $W_L(M) = W(M) \cap W_L$ are defined for any $L \supset M$. We denote by w_M^L the longest element in $W_L(M)$. If M, L are two standard Levi subgroups we define

$$W(M;L) = \{w \in W : w^{-1}\alpha > 0 \text{ for all } \alpha \in \Delta_0^L,\ wMw^{-1} \subset L\}.$$

It is a subset of $W(M)$. We will need the following lemma.

Lemma 6.4. *Let M, L and L' be Levi subgroups of G with $L' \subset L$. Then any element $w' \in W(M;L')$ can be expressed uniquely as w_1w where $w_1 \in W_L(M_1;L')$, $w \in W(M;L)$ and $M_1 = wMw^{-1}$.*

Proof. The uniqueness is clear since the conditions imply that w is the element of minimal length in W_Lw'. With this choice of w and with w_1 so that $w' = w_1w$, Lemma I.1.9 of [**53**] (applied to w'^{-1}) shows that $w \in W(M)$ and $w_1 \in W_L(M_1)$ with $M_1 = wMw^{-1}$.

By the choice of w we infer that $w \in W(M;L)$. Moreover, since $w' = w_1 w$ is a reduced decomposition and w' is left-$W_{L'}$-reduced, w_1 is also left-$W_{L'}$ reduced. Hence, $w_1 \in W_L(M_1; L')$. □

If $\varphi \in \mathcal{A}_P^{\mathrm{c}}$ then the constant term of $E(\cdot\,,\varphi,\lambda)$ along the (standard) parabolic $Q = LV$ is given by

$$E_Q(\cdot\,,\varphi,\lambda) = \sum_{w\in W(M;L)} E^Q(\cdot\,, M(w,\lambda)\varphi, w\lambda). \tag{6.6}$$

7. Galois pairs and regularized periods

In this section we will summarize the main results of [**48**]. For a thorough discussion and proofs the reader should consult [**49**].

The setup consists of a quasi-split group H_0 over F and G which is obtained from H_0 by restrictions of scalars from a quadratic extension E/F. Thus, G is quasi-split and (G,θ) is a relatively quasi-split Galois pair in the sense of [**48**] where θ denotes the Galois involution. We choose P^0, as well as $\boldsymbol{K}$, to be θ-invariant. The symmetric space $\mathcal{C}' = \{\varepsilon \in G : \varepsilon\theta(\varepsilon) = 1\}$ has a natural action of G denoted by $\star$. Given a G-orbit $\mathcal{C}$ we choose a representative $\varepsilon_0 \in \mathcal{C}$ to lie in the defect M^{00} of $\mathcal{C}$ (cf. [**49**, § 4.5]). We take $\tilde{H}$ to be the stabilizer of ε_0. Then $\tilde{H}$ is an inner form of H_0 and it is the fixed point subgroup of the Galois involution $\tilde{\theta} = \mathrm{Ad}(\varepsilon_0)\circ\theta$. The map $P \to \tilde{P} = P \cap \tilde{H}$ defines a bijection between $\tilde{\theta}$-stable parabolic subgroups of G and parabolic subgroups of $\tilde{H}$. The $\tilde{\theta}$-stable parabolic subgroups of G are the θ-stable parabolic subgroups containing P^{00}. The notation pertaining to $\tilde{H}$ will always be appended by a tilde. We have

$$\delta_{\tilde{P}} = \delta_P^{1/2}|_{\tilde{P}(\mathbb{A})} \tag{7.1}$$

for any parabolic $\tilde{P}$ of $\tilde{H}$.

Definition 7.1.

(1) A Levi subgroup M of G is called θ-elliptic, if $w_M M w_M^{-1} = \theta(M)$ and $w_M\alpha = -\theta\alpha$ for all $\alpha \in \Delta_M$.

(2) Let π be a cuspidal representation of $M(\mathbb{A})$. We say that (π, M) is $\tilde{\theta}$-elliptic with respect to G if M is θ-elliptic in G and π is distinguished with respect to M_x for some $x \in \mathcal{C} \cap M w_M^{-1}$, where M_x is the stabilizer of the Galois involution $\theta_x = \mathrm{Ad}(x)\circ\theta$ of M.

In [**49**, § 8] we defined the regularized $\tilde{H}$-period of (certain) automorphic forms on G and we denoted it by

$$\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1} \varphi(h)\,\mathrm{d}h.$$

It is an $\tilde{H}(\mathbb{A})^1$-invariant functional defined by a certain regularization procedure and agrees with the usual integral if φ is integrable over $\tilde{H}\backslash\tilde{H}(\mathbb{A})^1$.

Let π be a cuspidal representation of $M(\mathbb{A})$. By the main result (Theorem 9.1.1) of [**49**] $\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1} E(h,\varphi,\lambda) = 0$ unless (π, M) is $\tilde{\theta}$-elliptic, in which case

$$\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1} E(h,\varphi,\lambda)\,\mathrm{d}h = J(w_{\theta M},\varphi,\lambda),$$

where the right-hand side is the *intertwining period* defined by the sum over the M-orbits $\mathcal{O}$ in $\mathcal{C}\cap M w_M^{-1}$ of

$$\int_{\tilde{H}_\eta\backslash\tilde{H}(\mathbb{A})^1} \mathrm{e}^{\langle\lambda, H_P(\eta h)\rangle}\varphi(\eta h)\,\mathrm{d}h = \int_{\tilde{H}_\eta(\mathbb{A})\backslash\tilde{H}(\mathbb{A})^1}\int_{M_x\backslash M_x(\mathbb{A})^1}\varphi(m\eta h)\,\mathrm{d}m\,\mathrm{d}h,$$

where $x = \eta\star\varepsilon_0$ is a representative of $\mathcal{O}$ and $\tilde{H}_\eta = \tilde{H}\cap\eta^{-1}P\eta$.

Also defined in [**49**, § 8] was a relative variant of Arthur's truncation operator which we called *mixed truncation* and denoted it Λ^T_m. Here T is a sufficiently positive element of $\tilde{\mathfrak{a}}_0$ which will be fixed throughout. The mixed truncation defines a map from the space of smooth functions on $G\backslash G(\mathbb{A})^1$ which together with their derivatives have uniform moderate growth, into the space of rapidly decreasing functions of $\tilde{H}\backslash\tilde{H}(\mathbb{A})^1$. This map is in fact continuous in the usual topologies. (This is an adaptation of [**2**, Lemma 1.4], which is proved in a similar vein (cf. [**49**, Lemma 8.2.1]).)

If L is $\tilde{\theta}$-stable, then $(L, \tilde{\theta}|_L)$ is also a quasi-split Galois pair and, in particular, the regularized $\tilde{L}(\mathbb{A})^1$-periods are defined for automorphic forms of $L(\mathbb{A})$. The orbit of ε_0 under L is $\mathcal{C}^L = \mathcal{C}\cap L$. If φ is an automorphic form on $V(\mathbb{A})L\backslash G(\mathbb{A})$ which satisfies $\varphi(ag) = \delta_Q^{1/2}(a)\varphi(g)$ for all $a\in A_Q$ then we can define its regularized integral over $A_{\tilde{Q}}V(\mathbb{A})\tilde{L}\backslash\tilde{H}(\mathbb{A})^1$ by

$$\int^*_{A_{\tilde{Q}}V(\mathbb{A})\tilde{L}\backslash\tilde{H}(\mathbb{A})^1}\varphi(h)\,\mathrm{d}h = \int_{\boldsymbol{K}_H}\int^*_{\tilde{L}\backslash\tilde{L}(\mathbb{A})^1}\varphi(lk)\,\mathrm{d}l\,\mathrm{d}k.$$

Then

$$\int^*_{A_{\tilde{Q}}V(\mathbb{A})\tilde{L}\backslash\tilde{H}(\mathbb{A})^1} E^Q(h,\varphi,\lambda)\,\mathrm{d}h = 0$$

unless (M,π) is $\tilde{\theta}$-elliptic in L, in which case

$$\int^*_{A_{\tilde{Q}}V(\mathbb{A})\tilde{L}\backslash\tilde{H}(\mathbb{A})^1} E^Q(h,\varphi,\lambda)\,\mathrm{d}h = J(w^L_{\theta M},\varphi,\lambda). \tag{7.2}$$

Similarly, there is a notion of a mixed truncation relative to Q, denoted by $\Lambda^{T,Q}_m$, for automorphic forms on $V(\mathbb{A})L\backslash G(\mathbb{A})$.

The regularized period is defined in general using mixed truncation (with respect to $\tilde{\theta}$-stable parabolic subgroups). In particular, for cuspidal Eisenstein series, the definition and the formula (6.6) give that $\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1} E(h,\varphi,\lambda)\,\mathrm{d}h$ is equal to

$$\sum_{\tilde{L}<\tilde{H}}\sum_{w\in W(M;L)}\hat{v}_{\tilde{L}}\frac{\mathrm{e}^{\langle w\lambda,T\rangle}}{\prod_{\varpi\in\hat{\tilde{\Delta}}_{\tilde{L}}}\langle -w\lambda,\varpi^\vee\rangle}\cdot\int_{A_{\tilde{L}}\tilde{L}\tilde{V}(\mathbb{A})\backslash\tilde{H}(\mathbb{A})}\Lambda^{T,Q}_m E^Q(h, M(w,\lambda)\varphi,(w\lambda)^L)\,\mathrm{d}h,$$

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where $\hat{v}_{\tilde{L}}$ is the co-volume of the lattice spanned by the weights $\hat{\tilde{\Delta}}_{\tilde{L}}$ in $\tilde{\mathfrak{a}}_{\tilde{L}}$. (We could have used the usual truncation Λ^T instead [**49**, § 12].) It follows that for a generic value of λ the form

$$\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1} E(h,\varphi,\lambda)\,\mathrm{d}h$$

lies in the topological dual of the smooth part of $I(\pi,\lambda)$. Indeed, it follows from (5.1) and the properties of the mixed truncation that $\Lambda^T_m E(h,\varphi,\lambda)$ is bounded by a semi-norm of φ, uniformly in $h\in\tilde{H}(\mathbb{A})^1$. Similarly for $\Lambda^{T,Q}_m E^Q(h,\varphi',\lambda')$ for $h\in\tilde{L}(\mathbb{A})^1\cdot\tilde{\boldsymbol{K}}$. Since $M(w,\lambda)$ is continuous for smooth vectors, we obtain the required property.

More generally, it follows from the definition that

$$\int^*_{A_{\tilde{Q}}V(\mathbb{A})\tilde{L}\backslash\tilde{H}(\mathbb{A})^1} E^Q(h,\varphi,\lambda)\,\mathrm{d}h = \sum_{\tilde{L}'<\tilde{L}}\sum_{w'\in W_L(M;L')}\hat{v}^{\tilde{L}}_{\tilde{L}'}\frac{\mathrm{e}^{\langle w'\lambda,T\rangle}}{\prod_{\varpi\in\hat{\tilde{\Delta}}^{\tilde{L}}_{\tilde{L}'}}\langle -w'\lambda,\varpi^\vee\rangle} \times\int_{A_{\tilde{L}'}\tilde{L}'\tilde{V}'(\mathbb{A})\backslash\tilde{H}(\mathbb{A})}\Lambda^{T,Q'}_m E^{Q'}(h,M(w,\lambda)\varphi,(w\lambda)^{L'})\,\mathrm{d}h. \quad (7.3)$$

We can also invert the process and express, under some mild restrictions on the exponents, the periods of truncated automorphic forms in terms of regularized periods. In particular, in the case of cuspidal Eisenstein we obtain an equality of meromorphic functions

$$\int_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1}\Lambda^T_m E(h,\varphi,\lambda)\,\mathrm{d}h = \sum_{(\tilde{L},w)} v_{\tilde{L}}\cdot\frac{\mathrm{e}^{\langle (w\lambda)_L,T\rangle}}{\prod_{\tilde{\alpha}\in\tilde{\Delta}_{\tilde{L}}}\langle w\lambda,\tilde{\alpha}^\vee\rangle}J(w^L_{\theta M'},M(w,\lambda)\varphi,(w\lambda)^L), \quad (7.4)$$

where $v_{\tilde{L}}$ is the co-volume of the lattice spanned by the co-roots $(\Delta_{\tilde{L}})^\vee$ in $\tilde{\mathfrak{a}}_{\tilde{L}}$ and $(\tilde{L},w)$ ranges over pairs of Levi subgroup $\tilde{L}$ of $\tilde{H}$ and $w\in W(M;L)$ such that $(w\pi, M'=wMw^{-1})$ is $\tilde{\theta}$-elliptic in L.

We will often use the notation $\mathfrak{a}^{\tilde{L}}_M$ to denote the space $\mathfrak{a}_M\oplus(\mathfrak{a}_L)^-_\theta$. Similarly for $(\mathfrak{a}^{\tilde{L}}_M)^*$.

Suppose now that G is the restriction of scalars of GL_n from E to F. This case will be considered more thoroughly in the next section. Whenever (π,M) is $\tilde{\theta}$-elliptic in L we set

$$\mathcal{B}^{\tilde{L}}_{(M,\pi)}(f,\lambda)=\sum_{\varphi\in\mathcal{B}_P(\pi)}J(w^L_{\theta M},I(f,\lambda)\varphi,\lambda)\overline{\mathcal{W}(\varphi,-\bar{\lambda})},$$

where $\mathcal{B}_P(\pi)$ is an orthonormal basis of $\mathcal{A}^\pi_P$. For $\lambda\in\mathrm{i}\mathfrak{a}^*_M$ this is the 'Bessel distribution' with respect to the forms $J(w^L_{\theta M},\cdot,\lambda)$ and $\mathcal{W}(\cdot,\lambda)$ (cf. [**43**]). Since both linear forms are continuous (for J, see above; for $\mathcal{W}$ this follows from [**64**, Theorem 15.4.1]), the Bessel distribution is defined and continuous in f (at least for generic λ). We will show in the next section that $\mathcal{B}^{\tilde{L}}_{(M,\pi)}(f,\lambda)$ is holomorphic for $\lambda\in\mathrm{i}\mathfrak{a}^*_M$.

It will be useful to introduce the following auxiliary functions:

$$\hat{\Xi}_{M,\pi}(f,\lambda)(h)=\sum_{\varphi\in\mathcal{B}_P(\pi)}E(h,I(f,\lambda)\varphi,\lambda)\cdot\overline{\mathcal{W}(\varphi,-\bar{\lambda})},$$

and

$$\Xi^T_{M,\pi}(f,\lambda) = \int_{\tilde H\backslash \tilde H(\mathbb{A})} \Lambda^T_m(\hat{\Xi}_{M,\pi}(f,\lambda))(h)\,\mathrm{d}h$$
$$= \sum_{\varphi\in\mathcal{B}_P(\pi)} \overline{\mathcal{W}(\varphi,-\bar\lambda)} \times \int_{\tilde H\backslash \tilde H(\mathbb{A})} \Lambda^T_m E(h, I(f,\lambda)\varphi,\lambda)\,\mathrm{d}h.$$

The majorization of Eisenstein series (§ 6) give the following estimates.

Lemma 7.2.

(1) $\hat{\Xi}_{M,\pi}(f,\lambda)(h) \ll \mathfrak{o}_{\pi,f}(\|h\|,\|\lambda\|)$ *on* $\mathfrak{I}(\pi)$

(2) $\Xi^T_{M,\pi}(f,\lambda) \ll \mathfrak{o}_{\pi,f}(1,\|\lambda\|)$ *on* $\mathfrak{I}(\pi)$. *In particular,* $\Xi^T_{M,\pi}(f,\lambda)$ *is tame.*

Proof. Writing $f = f_1 \star g_1 + f_2 \star g_2$ as in § 6, and using the Cauchy–Schwarz inequality, $\hat{\Xi}_{M,\pi}(f,\lambda)(h)$ is bounded by the sum of

$$\sum_{j=1}^{2}\Bigg(\sum_{\varphi\in\mathcal{B}_P(\pi)} |E(h, I(f_j,\lambda)\varphi,\lambda)|^2\Bigg)^{1/2}$$

and

$$\sum_{j=1}^{2} \sup_{u\in U^0\backslash U^0(\mathbb{A})} \Bigg(\sum_{\varphi\in\mathcal{B}_P(\pi)} |E(u, I(g_j,\lambda)\varphi,-\bar\lambda)|^2\Bigg)^{1/2}.$$

We can apply Proposition 6.1 to conclude the first part of the lemma. Upon replacing f by $\rho(X)f$ with $X \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ we get a similar estimate for the derivatives (in the group variable) of $\hat{\Xi}_{M,\pi}(f,\lambda)$. By the properties of the mixed truncation the second part of the lemma follows. Note that by approximating f by bi-K_∞-finite functions we can assure that $\Xi^T_{M,\pi}(f,\lambda)$ is analytic on $\mathfrak{I}(\pi)$. □

More generally, for any $\tilde\theta$-stable L we set

$$\hat{\Xi}^{\tilde L}_{M,\pi}(f,\lambda)(h) = \sum_{\varphi\in\mathcal{B}_P(\pi)} E^Q(h, I(f,\lambda)\varphi,\lambda^L)\cdot\overline{\mathcal{W}(\varphi,-\bar\lambda)},$$

and

$$\Xi^{\tilde L,T}_{M,\pi}(f,\lambda) = \int_{A_{\tilde L}\tilde V(\mathbb{A})\tilde L\backslash \tilde H(\mathbb{A})} \Lambda^{T,Q}_m(\hat{\Xi}^{\tilde L}_{M,\pi}(f,\lambda))(h)\,\mathrm{d}h$$
$$= \sum_{\varphi\in\mathcal{B}_P(\pi)} \overline{\mathcal{W}(\varphi,-\bar\lambda)} \times \int_{A_{\tilde L}\tilde V(\mathbb{A})\tilde L\backslash \tilde H(\mathbb{A})} \Lambda^{T,Q}_m E^Q(h, I(f,\lambda)\varphi,\lambda^L)\,\mathrm{d}h.$$

The analogue of Lemma 7.2 for $\hat{\Xi}^{\tilde L}_{M,\pi}$ and $\Xi^{\tilde L,T}_{M,\pi}$ is still valid.

Remark 7.3. The proof shows in fact that

$$\int_{A_{\tilde L}\tilde V(\mathbb{A})\tilde L\backslash \tilde H(\mathbb{A})} |\Lambda^{T,Q}_m(\hat{\Xi}^{\tilde L}_{M,\pi}(f,\lambda))(h)|\,\mathrm{d}h \ll \mathfrak{o}_{\pi,f}(1,\|\lambda\|)$$

on $\mathfrak{I}(\pi)$.

Lemma 7.4. *$\mathcal{B}_M^{\tilde{L}}(\pi, f, \lambda) \ll \mathfrak{o}_{\pi,f}(1, \|\lambda\|)$ on $\mathfrak{I}(\pi)$. In particular, $\mathcal{B}_M^{\tilde{L}}(\pi, f, \lambda)$ is tame.*

Proof. We write $\mathcal{B}_M^{\tilde{L}}(\pi, f, \lambda)$ as

$$\sum_{\tilde{L}'<\tilde{L}} \sum_{w \in W_L(M;L')} \hat{v}_{\tilde{L}'}^{\tilde{L}} \frac{\mathrm{e}^{\langle w\lambda, T\rangle}}{\prod_{\varpi \in \tilde{\hat{\Delta}}_{\tilde{Q}'}^{\tilde{Q}}} \langle -w\lambda, \varpi^\vee\rangle} \Xi_{M,\pi}^{\tilde{L}',T}(f, \lambda, w)\, \mathrm{d}\lambda.$$

Using a common denominator we have

$$\mathcal{B}_M^{\tilde{L}}(\pi, f, \lambda) = \frac{F_{M,\pi,L}(\lambda)}{\prod_{\varpi \in \tilde{\hat{\Delta}}^{\tilde{Q}},\ w \in W_L} \langle -\lambda, w\varpi^\vee\rangle}, \tag{7.5}$$

where $F_{M,\pi,L}(\lambda)$ is some linear combination of products of $\Xi_{M,\pi}^{\tilde{L}',T}(f, \lambda, w)$ with linear forms. Thus, $F_{M,\pi,L} \ll \mathfrak{o}_{\pi,f}(1, \|\lambda\|)$ on $\mathfrak{I}(\pi)$, and the same is true for its derivatives. Since $\mathcal{B}_M^{\tilde{L}}(\pi, f, \lambda)$, and hence (7.5) is holomorphic on $\mathrm{i}\mathfrak{a}_M^*$ (see below) the lemma follows from Cauchy's formula. □

Remark 7.5. Even without assuming that $\mathcal{B}_M^{\tilde{L}}(\pi, f, \lambda)$ is holomorphic on the unitary axis, Lemma 7.2 and (7.5) already show that $P(\lambda) \cdot \mathcal{B}_M^{\tilde{L}}(\pi, f, \lambda)$ is holomorphic on $\mathfrak{I}(\pi)$ for some fixed polynomial $P(\lambda)$. Moreover, if $f_n \to f$, then

$$P(\lambda)\mathcal{B}_M^{\tilde{L}}(\pi, f, \lambda) = \lim_n P(\lambda)\mathcal{B}_M^{\tilde{L}}(\pi, f_n, \lambda)$$

uniformly on $\mathfrak{I}(\pi)$.

8. The GL_n cases

We are now going to examine the case $G = \mathrm{GL}_n/E$ more carefully. There are two possibilities for the Galois involution, corresponding to the two quasi-split forms of GL_n.

8.1. The split case

Consider first the case where θ is the Galois action induced from $\tilde{H} = \mathrm{GL}_n/F$. This was the case considered in [**42**]. Note that there is only one G-orbit in the symmetric space $\mathcal{C}'$ by Hilbert 90, and hence $\tilde{\theta} = \theta$. Also, θ acts trivially on the roots and there is a one-to-one correspondence between parabolic subgroups of G and $\tilde{H}$.

If π is a cuspidal representation of $G(\mathbb{A})$, we denote by $\bar{\pi}$ the Galois conjugate of π and by π^* the contragredient of $\bar{\pi}$. We recall that if π is distinguished by $\tilde{H}$, then the Asai L-function has a pole at $s = 1$, and in particular $\pi^* \simeq \pi$ [**15**].

If (π, M) is θ-elliptic, then n is even, M is maximal of type $(\frac{1}{2}n, \frac{1}{2}n)$ and π is of the form $\sigma \otimes \sigma^*$.

Finally, we recall that if π is a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ then the Eisenstein series on $\mathrm{GL}_{2n}(\mathbb{A})$ induced from $\pi \otimes \pi$ vanishes at 0, because the intertwining operator is -1 there [**45**, Proposition 6.3].

We also note that if L is θ-stable then the Galois pair $(L, \theta|_L)$ consists of a product of smaller Galois pairs of split GL_m-type.

The following lemma is a generalization of Lemma 2.1.

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Lemma 8.1. *Consider the case where M is a maximal parabolic of G and $\pi = \pi_1 \otimes \pi_2$. There are three mutually disjoint possibilities.*

Case 1: $\pi_2 \not\simeq \pi_1^*$. *Then $\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1} E(h,\varphi,\lambda)\,\mathrm{d}h$ is 0 whenever defined. Moreover,*

$$J(1,\varphi,0) = J(1, M(w_M,0)\varphi, 0).$$

Case 2: $\pi_2 \simeq \pi_1^*$ but π_1 (or, equivalently, π_2) is not distinguished. *Then*

$$\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1} E(h,\varphi,\lambda)\,\mathrm{d}h = \int_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1} \Lambda_m^T E(h,\varphi,\lambda)\,\mathrm{d}h$$

is holomorphic for $\lambda \in \mathrm{i}\mathfrak{a}_M^*$.

Case 3: $\pi_2 = \pi_1$ and π_1 is distinguished. *Then $\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1} E(h,\varphi,\lambda)\,\mathrm{d}h$ is holomorphic for $\lambda \in \mathfrak{a}_M^*$ except for a simple pole at 0. On the other hand, $E(\cdot\,,\varphi,0) \equiv 0$ and $M(w_M,0) = -1$.*

Proof. Applying (7.4) to this case we get

$$\int_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1} \Lambda_m^T E(h,\varphi,\lambda)\,\mathrm{d}h = \int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1} E(h,\varphi,\lambda)\,\mathrm{d}h + \|\alpha^\vee\| \frac{\mathrm{e}^{\langle\lambda,T\rangle}}{\langle\lambda,\alpha^\vee\rangle} J(1,\varphi,0) - \|\alpha^\vee\| \frac{\mathrm{e}^{\langle w\lambda,T\rangle}}{\langle\lambda,\alpha^\vee\rangle} J(1, M(w,\lambda)\varphi, 0), \quad (8.1)$$

where $\Delta_M = \{\alpha\}$ and $w = w_M$. If the π_i are not distinguished, then the last two terms in (8.1) disappear. On the other hand, $\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1}$ vanishes unless $\pi_2 \simeq \pi_1^*$ [**42**]. The remaining assertions follow from the holomorphy of the left-hand side of (8.1) on the unitary axis. □

Proposition 8.2. *The distribution $\mathcal{B}^{\tilde{L}'}_{(M',\pi')}(f,\lambda')$ is holomorphic for $\lambda' \in \mathrm{i}(\mathfrak{a}^{L'}_{M'})^*$. Moreover, if $\mathcal{B}^{\tilde{L}'}_{(M',\pi')}(f,\lambda') \not\equiv 0$, then there exist Levi subgroups $M \subset L$, a cuspidal representation π of $M(\mathbb{A})$ and an element $w \in W$ such that*

(1) *M is of type $(n_1, n_1, \dots, n_k, n_k, m_1, \dots, m_l)$;*

(2) *L is the Levi of type $(2n_1, \dots, 2n_k, m_1, \dots, m_l)$;*

(3) *π is of the form*

$$\pi = \sigma_1 \otimes \sigma_1^* \otimes \cdots \otimes \sigma_k \otimes \sigma_k^* \otimes \tau_1 \otimes \cdots \otimes \tau_l,$$

where the τ_j are distinguished and mutually inequivalent;

(4) *w conjugates (M',L',π') to (M,L,π);*

(5) *$\mathcal{B}^{\tilde{L}'}_{(M',\pi')}(f,\lambda) = \mathcal{B}^{\tilde{L}}_{(M,\pi)}(f,w\lambda)$ for all $\lambda \in (\mathfrak{a}^{L'}_{M'})^*_{\mathbb{C}}$.*

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Finally, if w_1 conjugates (M, L, π) to (M_1, L_1, π_1) in such a way that it permutes the σ_i and the τ_j among themselves (or permutes σ_i with σ_i^*), then

$$\mathcal{B}^{\tilde{L}_1}_{(M_1,\pi_1)}(f, w_1\lambda) = \mathcal{B}^{\tilde{L}}_{(M,\pi)}(f, \lambda).$$

Proof. By Remark 7.5 it is enough to consider the case where f is bi-K_∞-finite. Suppose that the term $\mathcal{B}^{\tilde{L}'}_{(M',\pi')}(f, \lambda) \not\equiv 0$ for some elliptic Levi M' of L'. Then, by [**42**, Theorem 23], L' is of type $(N_1, \dots, N_r)$ and $\pi' = \Pi_1 \otimes \cdots \otimes \Pi_r$ where each Π_i is either a distinguished cuspidal representation of GL_{N_i} or $\Pi_i = \sigma_i \otimes \sigma_i^*$ for a cuspidal representation σ_i of $\mathrm{GL}_{N_i/2}$. Let us say that Π_i is a singleton in the first case, or a pair in the second case. By Lemma 2.1, the singularities of $J(w^L_M, \varphi, \lambda)$ on $\mathrm{i}\mathfrak{a}^*_M$ are (at most) simple poles along the hyperplanes $\langle \lambda, \alpha^\vee \rangle = 0$ for the roots $\alpha \in \Delta^{L'}_{M'}$ pertaining to those σ_i which are distinguished. On these hyperplanes, the Eisenstein series, hence also $\mathcal{W}(\varphi, \lambda)$, vanishes. Thus, $\mathcal{B}^{\tilde{L}}_{(M,\pi)}(f, \lambda)$ is holomorphic on $\mathrm{i}\mathfrak{a}^*_M$. Moreover, if $\Pi_i \simeq \Pi_j$ for some $i \neq j$ and Π_i is a singleton, then once again, $\mathcal{W}(\varphi, \lambda)$ will vanish on the hyperplane $\langle \lambda, \beta^\vee \rangle = 0$ for an appropriate root β which is orthogonal to $\mathfrak{a}^L$. Hence $\mathcal{B}^{\tilde{L}}_{(M,\pi)}(f, \lambda) \equiv 0$ on $\mathrm{i}(\mathfrak{a}^L_M)^*$.

In order to complete the proof of the first part we need to show that we can interchange Π_i and Π_j of different types (i.e. a singleton and a pair). We may assume that $\operatorname{Re}\lambda = 0$. Let w be the Weyl group element corresponding to such a permutation. Suppose that w conjugates (π, M', L') to (π_1, M_1, L_1). By the functional equations of [**42**, Theorem 33] we have

$$J(w^{L_1}_{M_1}, M(w, \lambda)\varphi, w\lambda) = J(w^{L'}_{M'}, \varphi, \lambda).$$

By using a unitary change of basis $\varphi \mapsto M(w, \lambda)\varphi$ and the functional equation for the Eisenstein series we obtain

$$\mathcal{B}^{\tilde{L}'}_{(M',\pi')}(f, \lambda) = \mathcal{B}^{\tilde{L}_1}_{(M_1,\pi_1)}(f, w\lambda).$$

To prove the last statement we need to show, by a similar reasoning, that

$$J(w^{L_1}_{M_1}, M(w_1, \lambda)\varphi, w_1\lambda) = J(w^L_M, \varphi, \lambda),$$

where w corresponds to a permutation of two σ_i, σ_i with σ_i^*, or two τ_j. For the first two cases the functional equations of [**42**, Theorem 33] still apply. The last case boils down to the functional equation described in Case 1 of Lemma 2.1. □

8.2. The unitary case

Consider now the case where $\mathcal{C}'$ is the symmetric space of Hermitian forms. Thus, $\tilde{\theta}$ is of the form $\Phi^{-1}\,{}^{\mathrm{t}}\bar{g}^{-1}\Phi$ for some Hermitian form Φ of the form

$$\Phi = \begin{pmatrix} & & D \\ & \Phi_1 & \\ {}^{\mathrm{t}}\bar{D} & & \end{pmatrix},$$

where D is anti-diagonal of size t (the Whit index of Φ) and Φ_1 is anisotropic of size $d = n - 2t$.

In this case θ acts as the principal involution on the root system and thus any M is θ-elliptic. Moreover, P^{00} is the parabolic subgroup of type

$$\overbrace{1,\dots,1}^{t}, d, \overbrace{1,\dots,1}^{t}.$$

Recall that by the argument of [**22**], if π is distinguished by $\tilde{H}$, then $\bar{\pi} \simeq \pi$.

Let $\pi = \pi_1 \otimes \cdots \otimes \pi_r$ be a cuspidal representation of $M(\mathbb{A})$ and identify $\mathfrak{a}_M^*$ with $\mathbb{R}^r$ in the usual way, writing λ_i, $i = 1,\dots,r$, for the coordinates of λ.

If (π, M) is $\tilde{\theta}$-elliptic, then each π_i is distinguished with respect to some unitary groups of GL_{n_i}.

We note that if L is $\tilde{\theta}$-stable, that is L is of the form $(n_1,\dots,n_k,m,n_k,\dots,n_1)$ with $m \geqslant d$, then the pair $(L, \tilde{\theta}|_L)$ consists of the product of a non-split Galois pair of type GL_m with products of Galois pairs of the type $(\mathrm{GL}_{n_i} \times \mathrm{GL}_{n_i})$ with θ acting by $(x,y) \mapsto (\theta'(y), \theta'(x))$ for an appropriate Galois action θ'. So in principle, we need also to consider the case of (G,θ) where $G = \mathrm{GL}_m \times \mathrm{GL}_m$ and $\theta(x,y) = (\theta'(y), \theta'(x))$. However, this case is rather trivial since the only θ-elliptic Levi is G itself.

Next, we prove the following lemma.

Lemma 8.3. *The singularities of $\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})} E(h,\varphi,\lambda)\,\mathrm{d}h$ on $\mathrm{i}\mathfrak{a}_M^*$ for $\varphi \in \mathcal{A}_P^\pi$ are (at most) simple poles along the hyperplanes $\lambda_i = \lambda_j$ whenever $\pi_i \simeq \pi_j$.*

Proof. We may assume of course that $\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1} E(h,\varphi,\lambda)\,\mathrm{d}h$ does not vanish. In this case, all the π_i are distinguished. To prove the lemma we will use the formula (7.4) and induction on n. Note that the left-hand side of (7.4) is holomorphic on $\mathrm{i}\mathfrak{a}_M^*$. One of the terms in the right-hand side is $\int^*_{\tilde{H}\backslash\tilde{H}(\mathbb{A})} E(h,\varphi,\lambda)\,\mathrm{d}h$. The other terms are of the form

$$\Psi_{w,\tilde{L}} = v_{\tilde{L}} \frac{\mathrm{e}^{\langle (w\lambda)_L, T\rangle}}{\prod_{\tilde{\alpha}\in\tilde{\Delta}_{\tilde{L}}} \langle w\lambda, \tilde{\alpha}^\vee\rangle} J(w^L_{\theta M'}, M(w,\lambda)\varphi, (w\lambda)^L), \tag{8.2}$$

where $\tilde{L} \subsetneq \tilde{H}$ and $w \in W(M;L)$ is such that $M' = wMw^{-1}$ is θ-elliptic in L. Concretely, if $\tilde{L}$ is of co-rank k in $\tilde{H}$ and M is of type $(n_1,\dots,n_r)$, then identifying the set $W(M)$ with the set of permutations of $\{1,\dots,r\}$ in the usual way, we have $n_{w(i)} = n_{w(r+1-i)}$ for $i = 1,\dots,k$ and L is of type $(n_{w(1)},\dots,n_{w(k)},m,n_{w(k)},\dots,n_{w(1)})$ with $m = n_{w(k+1)} + \cdots + n_{w(r-k)} \geqslant d$. If (8.2) is non-zero, then $\pi_{w(i)} = \bar{\pi}_{w(r+1-i)} = \pi_{w(r+1-i)}$ for $i = 1,\dots,k$. By the induction hypothesis, the singularities of $J(w^L_{\theta M'}, M(w,\lambda)\varphi, (w\lambda)^L)$ satisfy the conclusion of the lemma. It thus remains to show that the singularities of the $\Psi_{w,\tilde{L}}$ along the hyperplanes $\mathbb{H}_{w,\tilde{\alpha}}$, $\tilde{\alpha} \in \tilde{\Delta}_{\tilde{L}}$ defined by $\langle w\lambda, \tilde{\alpha}^\vee\rangle = 0$ cancel. We write $\tilde{\Delta}_{\tilde{L}} = \{\tilde{\alpha}_1,\dots,\tilde{\alpha}_k\}$ in the usual order. If $\tilde{\alpha} = \tilde{\alpha}_k$, then $\mathbb{H}_{w,\tilde{\alpha}}$ is given by $\lambda_{w(k)} = \lambda_{w(r+1-k)}$ and, by the above, $\pi_{w(k)} = \pi_{w(r+1-k)}$. Suppose that $\tilde{\alpha} \neq \tilde{\alpha}_k$. Then the element $\tilde{s}_{\tilde{\alpha}} \in W_{\tilde{H}} \subset W_G^\theta$ belongs to $W(L)$. Moreover, if $\tilde{L}_1 = \tilde{s}_{\tilde{\alpha}}\tilde{L}\tilde{s}_{\tilde{\alpha}}^{-1}$ (or, equivalently, $L_1 = \tilde{s}_{\tilde{\alpha}}L\tilde{s}_{\tilde{\alpha}}^{-1}$), then $w_1 = \tilde{s}_{\tilde{\alpha}}w \in W(M;L_1)$, $M'' = \tilde{s}_{\tilde{\alpha}}M'\tilde{s}_{\tilde{\alpha}}^{-1}$ is θ-elliptic in L_1 and $w^{L_1}_{\theta M''} = \tilde{s}_{\tilde{\alpha}} \star w^L_{\theta M'}$. We may write $\tilde{s}_{\tilde{\alpha}} = s_\beta s_{\theta\beta}$ for $\beta \in \Delta_L$ with $\theta(\beta) \neq \beta$. Thus, $w^L_{\theta M'}\theta\beta = \theta\beta \neq \beta$. Let $(w\lambda)^{\tilde{L}}$ denote the orthogonal projection of $w\lambda$ on $(\mathfrak{a}^{\tilde{L}}_{M'})^*$. Then on $\mathbb{H}_{w,\tilde{\alpha}}$ we have

$\langle (w\lambda), \beta^\vee \rangle = \langle (w\lambda)^{\tilde{L}}, \beta^\vee \rangle$ and similarly for $\theta\beta$. By the properties of the intertwining period, and, in particular, the functional equation [**48**, Theorem 5], we have

$$\begin{aligned} J(w^L_{\theta M'}, M(w,\lambda)\varphi, (w\lambda)^L) &= J(w^L_{\theta M'}, M(w,\lambda)\varphi, (w\lambda)^{\tilde{L}}) \\ &= J(\tilde{s}_{\tilde{\alpha}} \star w^L_{\theta M'}, M(\tilde{s}_{\tilde{\alpha}}, (w\lambda)^{\tilde{L}})M(w,\lambda)\varphi, \tilde{s}_{\tilde{\alpha}}(w\lambda)^{\tilde{L}}) \\ &= J(w^{L_1}_{\theta M''}, M(\tilde{s}_{\tilde{\alpha}}, w\lambda)M(w,\lambda)\varphi, (w_1\lambda)^{\tilde{L}_1}) \\ &= J(w^{L_1}_{\theta M''}, M(w_1,\lambda)\varphi, (w_1\lambda)^{L_1}) \end{aligned}$$

on $\mathbb{H}_{w,\tilde{\alpha}^\vee}$. Thus, the singularity of the term $\Psi_{w,\tilde{L}}$ along $\mathbb{H}_{w,\tilde{\alpha}^\vee}$ cancels with that of $\Psi_{w_1,\tilde{L}_1}$ (along the identical hyperplane $\mathbb{H}_{w_1,-\tilde{s}_{\tilde{\alpha}}\tilde{\alpha}^\vee}$). □

Proposition 8.4. *The distribution $\mathcal{B}^{\tilde{L}'}_{(M',\pi')}(f,\lambda')$ is holomorphic for $\lambda' \in \mathrm{i}(\mathfrak{a}^{\tilde{L}'}_{M'})^*$. Moreover, if $\mathcal{B}^{\tilde{L}'}_{(M',\pi')}(f,\lambda') \not\equiv 0$ then there exist Levi subgroups $M \subset L$, a cuspidal representation π of $M(\mathbb{A})$ and an element $w \in W$ such that*

(1) *M is of type $(n_1,\dots,n_k,m_1,\dots,m_l,n_k,\dots,n_1)$;*

(2) *L is the Levi of type $(n_1,\dots,n_k,m_1+\dots+m_l,n_k,\dots,n_1)$ with $m_1+\dots+m_l \geqslant d$;*

(3) *$\pi = \sigma_1 \otimes \cdots \sigma_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \overline{\sigma_k} \otimes \cdots \otimes \overline{\sigma_1}$, where $\overline{\sigma_i} \not\simeq \sigma_i$, $i = 1,\dots,k$, and τ_j is distinguished by some unitary group for $j = 1,\dots,l$;*

(4) *w conjugates (M',L',π') to (M,L,π);*

(5) *$\mathcal{B}^{\tilde{L}'}_{(M',\pi')}(f,\lambda) = \mathcal{B}^{\tilde{L}}_{(M,\pi)}(f,w\lambda)$ for all $\lambda \in (\mathfrak{a}^{\tilde{L}'}_{M'})^*_{\mathbb{C}}$.*

Finally, if w_1 conjugates (M,L,π) to (M_1,L_1,π_1) by permuting the σ_i and the τ_j among themselves (or permuting σ_i with $\overline{\sigma_i}$), then

$$\mathcal{B}^{\tilde{L}_1}_{(M_1,\pi_1)}(f,w_1\lambda) = \mathcal{B}^{\tilde{L}}_{(M,\pi)}(f,\lambda).$$

Proof. As before, we may assume that f is bi-K_∞-finite. By Lemma 8.3, $\mathcal{B}^{\tilde{H}}_{(M,\pi)}(f,\lambda)$ is holomorphic, since $E(\cdot,\varphi,\lambda)$, and hence, $\mathcal{W}(\varphi,\lambda)$ is zero along the hyperplane singularities of $\Pi^{\tilde{H}}E(\cdot,\varphi,\lambda)$. More generally, if M is θ-elliptic in L then $\mathcal{B}^{\tilde{L}}_{(M,\pi)}(f,\lambda)$ is holomorphic on $(\mathfrak{a}^{\tilde{L}}_M)^*$. Suppose that M is of type $(n_1,\dots,n_k,m_1,\dots,m_l,n_k,\dots,n_1)$ and L is of type $(n_1,\dots,n_k,m_1+\dots+m_l,n_k,\dots,n_1)$. If $\mathcal{B}^{\tilde{L}}_{(M,\pi)}(f,\lambda) \not\equiv 0$ on $(\mathfrak{a}^{\tilde{L}}_M)^*_{\mathbb{C}}$, then π is necessarily of the form $\sigma_1 \otimes \cdots \sigma_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \overline{\sigma_k} \otimes \cdots \otimes \overline{\sigma_1}$, where the τ_j are distinguished by some unitary group. By our identification

$$(\mathfrak{a}^{\tilde{L}}_M)^* = \{(\lambda_1,\dots,\lambda_k,\mu_1,\dots,\mu_l,\lambda_k,\dots,\lambda_1)\}.$$

Thus, if $\overline{\sigma_i} \simeq \sigma_i$ for any i, then $E(\varphi,\lambda) = 0$ on $(\mathfrak{a}^{\tilde{L}}_M)^*_{\mathbb{C}}$ as before, and again by Lemma 8.3 we have $\mathcal{B}^{\tilde{L}}_{(M,\pi)}(f,\lambda) \equiv 0$.

The last part follows from the functional equations of [**48**, Theorem 5] as in the previous case. □

9. Spectral expansion

Let $f \in C_c^\infty(G(\mathbb{A})^1)$ and let $K_f(x,y) = \sum_{\gamma \in G} f(x^{-1}\gamma y)$ be the kernel of the right regular representation of $G(\mathbb{A})^1$ on $L^2(G\backslash G(\mathbb{A})^1)$. Our goal is to give the fine spectral expansion for the expression

$$\int_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1} \int_{U^0\backslash U^0(\mathbb{A})} K_f(h,u)\psi(u)\,\mathrm{d}u\,\mathrm{d}h. \tag{9.1}$$

As in § 6 we will assume that f is bi-K_{fin}-invariant for a fixed compact subgroup K_{fin} of $G(\mathbb{A}_{\mathrm{fin}})$ and the support of f is contained in a fixed left and right K_∞-invariant compact set C. We will assume, to begin with, that f is bi-K_∞-finite. More importantly, *we assume that* $G = \mathrm{GL}_n$ as before.

9.1. First step: vanishing of residual contribution (cf. [29])

Following [**1**, § 4] we write

$$K(x,y) = \sum_\chi K_\chi(x,y),$$

where χ runs over the set of cuspidal data, and

$$K_\chi(x,y) = \sum_M \frac{|W_M|}{|W|} \sum_\pi \int_{\mathrm{i}\mathfrak{a}_M^*} \sum_{\varphi \in \mathcal{B}_P(\pi)} E(x, I(f,\lambda)\varphi,\lambda)\overline{E(y,\varphi,\lambda)}\,\mathrm{d}\lambda,$$

where π runs over the discrete spectrum of $L^2(M\backslash M(\mathbb{A})^1)_\chi$ (a finite sum) and $\mathcal{B}_P(\pi)$ is an orthonormal basis of $\mathcal{A}_P^\pi$. Here $\mathrm{d}\lambda$ is a Haar measure on $\mathrm{i}\mathfrak{a}_M^*$ dual to that of $\mathfrak{a}_M$.

By Proposition 2.1 of [**29**], for any n there exists c such that

$$\sum_\chi |K_\chi(x,y)| \leqslant c\|x\|^{-n}$$

provided that y is in a fixed compact set. Thus, we can write (9.1) as the sum over χ of

$$\int_{\tilde{H}\backslash\tilde{H}(\mathbb{A})^1} \int_{U^0\backslash U^0(\mathbb{A})} K_\chi(h,u)\psi(u)\,\mathrm{d}u\,\mathrm{d}h.$$

Using Lemma 4.4 of [**1**] we can write the inner integral as the sum over π and φ of

$$\int_{\mathrm{i}\mathfrak{a}_M^*} E(h, I(f,\lambda)\varphi,\lambda)\overline{\mathcal{W}(\varphi,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}h, \tag{9.2}$$

where

$$\mathcal{W}(\varphi,\lambda) = \int_{U^0\backslash U^0(\mathbb{A})} E(u,\varphi,\lambda)\overline{\psi(u)}\,\mathrm{d}u.$$

Lemma 9.1 (cf. p. 13 of [29]). *If* $\varphi \in \mathcal{A}_P^{\mathrm{r}}$, *then* $\mathcal{W}(\varphi,\lambda) = 0$.

Proof. It follows from the description of the discrete spectrum of GL_n [**52**] that no residual representation of GL_n is generic. (This is expected to hold for any reductive algebraic group.) Thus the lemma is true for $P = G$. To see the general case we may write $\mathcal{W}(\varphi, \lambda)$ as

$$\sum_{w \in W_M \backslash W} \int_{w^{-1}U^0 w \cap U^0 \backslash U^0(\mathbb{A})} \varphi(wu) \mathrm{e}^{\langle \lambda, H_P(wu) \rangle} \psi(u) \, \mathrm{d}u$$

in the range of absolute convergence of the Eisenstein series (cf. [**63**]). If $w \neq w_M^{-1}$ then the integral is zero since it factors through

$$\int_{w^{-1}Uw \cap U^0 \backslash w^{-1}U(\mathbb{A}) w \cap U^0(\mathbb{A})} \psi(u) \, \mathrm{d}u.$$

If $w = w_M^{-1}$ the integral factors through a non-degenerate Fourier coefficient of φ which is zero since $\varphi \in \mathcal{A}_P^{\mathrm{r}}$. □

It follows that only cuspidal π contribute to (9.2). All in all (9.1) is equal to the sum over pairs (M_0, π_0), up to conjugation, of a Levi subgroup M_0 and a cuspidal representation π_0 of $M_0(\mathbb{A})^1$, of a simple combinatorial constant times

$$\int_{\tilde{H} \backslash \tilde{H}(\mathbb{A})^1} \int_{\mathrm{i}\mathfrak{a}_{M_0}^*} \sum_{\varphi \in \mathcal{B}_{P_0}(\pi_0)} E(h, I(f, \lambda)\varphi, \lambda) \overline{\mathcal{W}(\varphi, \lambda)} \, \mathrm{d}\lambda \, \mathrm{d}h. \tag{9.3}$$

9.2. Second step: dissecting the Eisenstein series and shifting contours

The heart of the matter is to appropriately interchange the two integrals in (9.3). This is subtle because the Eisenstein series are not integrable over $\tilde{H} \backslash \tilde{H}(\mathbb{A})^1$. In principle, one can, following Arthur, truncate the kernel and let the truncation parameter go to infinity. This is the approach of [**19**]. We opt for a slightly different approach.

Fixing T we have the inversion formula

$$\phi(h) = \sum_{\tilde{P} \supset \tilde{P}^{00}} \sum_{\gamma \in \tilde{P} \backslash \tilde{H}} \Lambda_m^{T,P} \phi_P(\gamma h) \tau_P(H_P(\gamma h) - T) \tag{9.4}$$

valid for any automorphic form ϕ on G [**49**, Lemma 8.2.1].

Applying it to the Eisenstein series we can write (9.3) as the sum over $\tilde{Q}'$ of

$$\int_{\tilde{Q}' \backslash \tilde{H}(\mathbb{A})^1} \biggl(\int_{\mathrm{i}\mathfrak{a}_{M_0}^*} \sum_{\varphi} \Lambda_m^{T,Q'} E(h, I(f, \lambda)\varphi, \lambda \biggr) \tau_{Q'}(H_{Q'}(h) - T) \overline{\mathcal{W}(\varphi, \lambda)} \, \mathrm{d}\lambda) \, \mathrm{d}h \tag{9.5}$$

provided that the latter converges. By the formula (6.6) for the constant term,

$$\Lambda_m^{T,Q'} E(h, \varphi, \lambda) = \sum_{w' \in W(M_0; L')} \Lambda_m^{T,Q'} E^{Q'}(h, M(w', \lambda)\varphi, w'\lambda).$$

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Using the Iwasawa decomposition together with (7.1) we get, assuming convergence, the sum over $w' \in W(M_0; L')$ of

$$\int_{\tilde{\mathfrak{a}}_{\tilde{L}'}} \int_{A_{\tilde{L}'}\tilde{V}'(\mathbb{A})\tilde{L}'\backslash\tilde{H}(\mathbb{A})} \int_{\mathrm{i}\mathfrak{a}^*_{M_0}} \sum_{\varphi} \Lambda_m^{T,Q'} E^{Q'}(h, I(f, w'\lambda)M(w',\lambda)\varphi, (w'\lambda)^{L'}) \\ \times \mathrm{e}^{\langle w'\lambda, X\rangle} \tau_{Q'}(X-T)\overline{\mathcal{W}(\varphi,\lambda)}\,\mathrm{d}\lambda\,\mathrm{d}h\,\mathrm{d}X.$$

Interchanging the order of integration we can write this as

$$\int_{\tilde{\mathfrak{a}}_{\tilde{L}'}} \int_{\mathrm{i}\mathfrak{a}^*_{M_0}} \Xi^{\tilde{L}',T}_{M_0,\pi_0}(f,\lambda,w')\mathrm{e}^{\langle w'\lambda, X\rangle}\tau_{Q'}(X-T)\,\mathrm{d}\lambda\,\mathrm{d}X, \tag{9.6}$$

where for any M, a $\tilde{\theta}$-stable L and $w \in W(M; L)$ we define $\Xi^{\tilde{L},T}_{M,\pi}(f,\lambda,w)$ to be the sum over $\varphi \in \mathcal{B}_P(\pi)$ of

$$\overline{\mathcal{W}(\varphi,-\bar{\lambda})} \times \int_{A_{\tilde{L}}\tilde{V}(\mathbb{A})\tilde{L}\backslash\tilde{H}(\mathbb{A})} \Lambda_m^{T,Q} E^{Q}(h, I(f, w\lambda)M(w,\lambda)\varphi, (w\lambda)^{L})\,\mathrm{d}h.$$

By using the functional equation $\mathcal{W}(\varphi,\lambda) = \mathcal{W}(M(w,\lambda)\varphi, w\lambda)$ and a unitary change of basis $\varphi \mapsto M(w,\lambda)\varphi$ we get

$$\Xi^{\tilde{L},T}_{M,\pi}(f,\lambda,w) = \Xi^{\tilde{L},T}_{wMw^{-1},w\pi}(f,w\lambda)$$

for $\lambda \in \mathrm{i}\mathfrak{a}^*_M$ (and hence, for all λ), where the right-hand side is defined in § 7. Thus, using the substitution $\lambda' = w'\lambda$ we can rewrite (9.6) as

$$\int_{\tilde{\mathfrak{a}}_{\tilde{L}'}} \tau_{Q'}(X-T)\int_{\mathrm{i}\mathfrak{a}^*_{M'}} \mathrm{e}^{\langle \lambda', X\rangle}\Xi^{\tilde{L}',T}_{M',\pi'}(f,\lambda')\,\mathrm{d}\lambda'\,\mathrm{d}X, \tag{9.7}$$

where $M' = w'M_0w'^{-1}$, $\pi' = w'\pi$.

The integral (9.7) does not converge as a double integral. However, by Lemma 7.2 we may shift the inner integral to $\operatorname{Re}\lambda' = -\lambda'_0$ where $\lambda'_0 \in (\mathfrak{a}^*_{L'})_+$ is sufficiently close to 0. For such λ',

$$\int_{\tilde{\mathfrak{a}}_{\tilde{L}'}} \mathrm{e}^{\langle \lambda', X\rangle}\tau_{Q'}(X-T)\,\mathrm{d}X = \hat{v}_{\tilde{L}'}\frac{\mathrm{e}^{\langle \lambda', T\rangle}}{\prod_{\varpi\in\hat{\tilde{\Delta}}_{\tilde{Q}'}}\langle -\lambda', \varpi^\vee\rangle} \tag{9.8}$$

and hence the double integral converges and gives

$$\hat{v}_{\tilde{L}'}\int_{\operatorname{Re}\lambda'=-\lambda'_0} \frac{\mathrm{e}^{\langle \lambda', T\rangle}}{\prod_{\varpi\in\hat{\tilde{\Delta}}_{\tilde{Q}'}}\langle -\lambda', \varpi^\vee\rangle}\Xi^{\tilde{L}',T}_{M',\pi'}(f,\lambda')\,\mathrm{d}\lambda'. \tag{9.9}$$

To justify the above steps we note that once again by Lemma 7.2 it is possible to shift the inner integral in

$$\int_{\tilde{\mathfrak{a}}_{\tilde{L}}} \tau_Q(X-T)\int_{A_{\tilde{L}}\tilde{V}(\mathbb{A})\tilde{L}\backslash\tilde{H}(\mathbb{A})} \left|\int_{\mathrm{i}\mathfrak{a}^*_{M'}} \Lambda_m^{T,Q}(\hat{\Xi}^{\tilde{L}}_{M',\pi'}(f,\lambda'))(h)\mathrm{e}^{\langle \lambda', X\rangle}\,\mathrm{d}\lambda'\right|\mathrm{d}h\,\mathrm{d}X \tag{9.10}$$

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to $\operatorname{Re}\lambda' = -\lambda'_0$ with $\lambda'_0 \in (\mathfrak{a}^*_{L'})_+$ sufficiently close to 0. After interchanging the order of integration and using (9.8) we may bound the integral by a constant multiple of

$$\int_{\operatorname{Re}\lambda'=-\lambda'_0} \int_{A_{\tilde L}\tilde V(\mathbb{A})\tilde L\backslash \tilde H(\mathbb{A})} |\Lambda^{T,Q}_m(\hat{\Xi}^{\tilde L}_{M',\pi'}(f,\lambda'))(h)| \, d\lambda' \, dh,$$

which converges by Remark 7.3. The convergence of (9.10) justifies the derivation of (9.5), (9.7) and (9.9).

We remark that up to now we did not really use the full power of (5.4). We could have used an argument similar to that of [**54**].

To summarize, we expressed (9.3) as the sum over $\tilde Q' \supset \tilde P^{00}$ and over $w' \in \overline{W(M_0; L')}$ of (9.9). Since $\Xi^{\tilde L,T}_{M,\pi}(f,\lambda)$ is holomorphic and rapidly decreasing for $\operatorname{Re}\lambda \in -\overline{(\mathfrak{a}^*_L)_+}$ near 0 we may write (9.9) as an improper integral (cf. 3.6):

$$\hat v_{\tilde L'} \int_{\nearrow -\lambda'_0} \frac{e^{\langle\lambda',T\rangle}}{\prod_{\varpi\in\tilde{\hat\Delta}_{\tilde Q'}} \langle -\lambda', \varpi^\vee\rangle} \Xi^{\tilde L',T}_{M',\pi'}(f,\lambda') \, d\lambda'.$$

9.3. Third step: back to the unitary axis

The integrals (9.9) are obtained by a change of variable from integrals on the line $\operatorname{Re}\lambda = -w'^{-1}\lambda'_0$. We want to shift the contour of integration to the unitary axis. However, the integrands may have singularities there. Instead, we fix a base point $\lambda_0 \in (\mathfrak{a}^*_{M_0})_+$ sufficiently close to 0 and in general position, and try to 'align' the integrals to $\operatorname{Re}\lambda = \lambda_0$. Thus, we want to apply Lemma 3.3 with

$$V = \mathfrak{a}^*_{M'}, \qquad \Lambda' = \Big\{\varpi^\vee : \varpi \in \tilde{\hat\Delta}_{\tilde Q'}\Big\}, \qquad v = -\lambda'_0, \qquad v' = -w'\lambda_0$$

and the dual basis $\tilde\Delta_{\tilde Q'}$. The subsets of $\tilde{\hat\Delta}_{\tilde Q'}$ are the sets $\tilde{\hat\Delta}_{\tilde L_1}$, where $\tilde L_1 \supset \tilde L$. We will denote by $\mathrm{pr}^{\tilde L_1}$ the corresponding projection from V to $(\mathfrak{a}^{\tilde L_1}_{M'})^*$ where the latter was defined in § 7 (cf. § 3). The upshot of Lemma 3.3 is the equality

$$\int_{\nearrow -\lambda'_0} \frac{e^{\langle\lambda',T\rangle}}{\prod_{\varpi\in\tilde{\hat\Delta}_{\tilde Q'}} \langle -\lambda', \varpi^\vee\rangle} \Xi^{\tilde L',T}_{M',\pi'}(f,\lambda') \, d\lambda' = \sum_{\tilde L_1 \supset \tilde L_{w'}} \hat v^{-1}_{\tilde L_1} \int^{(\mathfrak{a}^{\tilde L_1}_{M'})^*}_{\nearrow -\mathrm{pr}^{\tilde L_1}(w'\lambda_0)} \frac{e^{\langle\lambda',T\rangle}}{\prod_{\varpi\in\tilde{\hat\Delta}^{\tilde L_1}_{\tilde L'}} \langle -\lambda', \varpi^\vee\rangle} \Xi^{\tilde L',T}_{M',\pi'}(f,\lambda') \, d\lambda',$$

where $\tilde L_{w'} \supset \tilde L'$ is the Levi subgroup of $\tilde H$ defined by

$$\tilde{\hat\Delta}_{\tilde L_{w'}} = \Big\{\varpi \in \tilde{\hat\Delta}_{\tilde L'} : \langle w'\lambda_0, \varpi^\vee\rangle < 0\Big\}.$$

Note that the condition $\tilde L_1 \supset \tilde L_{w'}$, i.e. that $\langle w'\lambda_0, \varpi^\vee\rangle < 0$ for all $\varpi \in \tilde{\hat\Delta}_{\tilde L_1}$, depends only on the coset $W_{L_1}w'$, since $w\varpi^\vee = \varpi^\vee$ for all $w \in W_{L_1}$, $\varpi \in \tilde{\hat\Delta}_{\tilde L_1}$. Summing over $\tilde L'$,

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$w' \in W(M_0; L')$, multiplying by the factor $\hat{v}_{\tilde{L}'}$ and using Lemma 6.4 (with $M = M_0$ and $L = L_1$) we get that (9.3) is equal to

$$\sum_{(\tilde{L}_1, w_1, \tilde{L}', w')} \hat{v}_{\tilde{L}'}^{\tilde{L}_1} \int_{\nearrow - \mathrm{pr}^{\tilde{L}_1}(w' w_1 \lambda_0)}^{(\mathfrak{a}_{M'}^{\tilde{L}_1})^*} \frac{\mathrm{e}^{\langle \lambda', T \rangle}}{\prod_{\varpi \in \hat{\tilde{\Delta}}_{\tilde{L}'}^{\tilde{L}_1}} \langle -\lambda', \varpi^\vee \rangle} \Xi_{M', \pi'}^{\tilde{L}', T}(f, \lambda') \, \mathrm{d}\lambda', \tag{9.11}$$

where $(\tilde{L}_1, w_1, \tilde{L}', w')$ ranges over Levi subgroups $\tilde{L}_1 \supset \tilde{L}'$ of $\tilde{H}$ and elements $w_1 \in W(M_0; L_1)$, $w' \in W_{L_1}(M_1; L')$ with $M_1 = w_1 M_0 w_1^{-1}$ such that $\langle w_1 \lambda_0, \varpi^\vee \rangle < 0$ for all $\varpi^\vee \in \hat{\tilde{\Delta}}_{\tilde{L}_1}$. Here $M' = w' M_1 w'^{-1}$ and $\pi' = w' w_1 \pi$.

More generally, we claim that, for any $k = 1, 2, \dots$, (9.3) equals

$$\sum_{\mathfrak{M}'} \hat{v}_{\tilde{L}'}^{\tilde{L}_k} \varepsilon_{\mathfrak{M}'} \int_{\nearrow - \lambda'_k}^{(\mathfrak{a}_{M'}^{\tilde{L}_k})^*} \frac{\mathrm{e}^{\langle \lambda', T \rangle}}{\prod_{\varpi \in \hat{\tilde{\Delta}}_{\tilde{L}'}^{\tilde{L}_k}} \langle -\lambda', \varpi^\vee \rangle} \Xi_{M', \pi'}^{\tilde{L}', T}(f, \lambda') \, \mathrm{d}\lambda', \tag{9.12}$$

where $\mathfrak{M}'$ ranges over all tuples $(L_1, w_1, \dots, L_k, w_k, L', w')$ with the following properties.

(1) $\tilde{L}_1 \supset \tilde{L}_2 \supset \cdots \supset \tilde{L}_k \supset \tilde{L}'$.

(2) $w_j \in W_{L_{j-1}}(M_{j-1}; L_j)$ and $(M_j, \pi_j) = w_j (M_{j-1}, \pi_{j-1}) w_j^{-1}$, for all $j = 1, \dots, k$ (setting $L_0 = G$). Also, $w' \in W_{L_k}(M_k; L')$ and $(M', \pi') = w'(M_k, \pi_k) w'^{-1}$.

(3) The λ_j are defined inductively by

$$\lambda_j = \mathrm{pr}^{\tilde{L}_j}(w_j \lambda_{j-1}).$$

We set $\lambda'_k = \mathrm{pr}^{\tilde{L}_k}(w' w_k \lambda_{k-1})$.

(4) $\langle w_1 \lambda_0, \varpi^\vee \rangle < 0$ for all $\varpi^\vee \in \hat{\tilde{\Delta}}_{\tilde{L}_1}$ and for each $j = 2, \dots, k$ we have

$$\langle w_j \lambda_{j-1}, \varpi^\vee \rangle \langle w_j w_{j-1} \lambda_{j-2}, \varpi^\vee \rangle < 0$$

for all $\varpi^\vee \in \hat{\tilde{\Delta}}_{\tilde{L}_j}^{\tilde{L}_{j-1}}$.

Finally,

$$\varepsilon_{\mathfrak{M}'} = \prod_{i=0}^{k-1} \prod_{\varpi^\vee \in \hat{\tilde{\Delta}}_{\tilde{L}_{i+1}}^{\tilde{L}_i}} \mathrm{sgn}(\langle \lambda_i, \varpi^\vee \rangle).$$

We prove (9.12) by induction on k, the case $k = 1$ being the expression (9.11) above. To carry out the induction step, we use Lemma 3.3 to shift the base point of the improper integral in (9.12) from $-\lambda'_k$ to $-w' \lambda_k$. As before we obtain the sum over the Levi subgroups $\tilde{L}_{k+1}$ of $\tilde{L}_k$ containing $\tilde{L}'$ such that

$$\langle w' \lambda_k, \varpi^\vee \rangle \langle w' w_k \lambda_{k-1}, \varpi^\vee \rangle < 0 \quad \text{for all } \varpi^\vee \in \hat{\tilde{\Delta}}_{\tilde{L}_{k+1}}^{\tilde{L}_k}$$

of

$$(\hat{v}_{\tilde{L}_{k+1}}^{\tilde{L}_k})^{-1} \prod_{\varpi \in \hat{\tilde{\Delta}}_{\tilde{L}_{k+1}}^{\tilde{L}_k}} \operatorname{sgn}(\langle \lambda_k, \varpi^\vee \rangle) \int_{\nearrow - \operatorname{pr}^{\tilde{L}_{k+1}}(w'\lambda_k)}^{(\mathfrak{a}_{M'}^{\tilde{L}_{k+1}})^*} \frac{\mathrm{e}^{\langle \lambda', T \rangle}}{\prod_{\varpi \in \hat{\tilde{\Delta}}_{\tilde{L}'}^{\tilde{L}_{k+1}}} \langle -\lambda', \varpi^\vee \rangle} \varXi_{M',\pi'}^{\tilde{L}',T}(f, \lambda') \,\mathrm{d}\lambda'.$$

Using Lemma 6.4 (with $G = L_k$, $M = M_k$, $L = L_{k+1}$) and observing that the condition on $\tilde{L}_{k+1}$ depends only on $W_{L_{k+1}} w'$ we obtain the induction step as before.

We will use (9.12) with $k = \operatorname{rank}(G) + 1$. Given $\mathfrak{M}'$ let m be the first index so that $L_m = L_{m+1}$. (Obviously, m exists.) Then it is easy to see that $w_{m+1} = \cdots = w_k = 1$ and $L_m = L_{m+1} = \cdots = L_k$. Therefore, we can write the sum over $\mathfrak{M}'$ as the sum over $\mathfrak{M}$ of

$$\varepsilon_{\mathfrak{M}} \sum_{\tilde{L}' \subset \tilde{L}_{\mathfrak{M}},\ w' \in W_{L_{\mathfrak{M}}}(M_{\mathfrak{M}}; L')} \hat{v}_{\tilde{L}'}^{\tilde{L}_{\mathfrak{M}}} \int_{\nearrow - w'\lambda_{\mathfrak{M}}}^{(\mathfrak{a}_{M'}^{\tilde{L}_{\mathfrak{M}}})^*} \frac{\mathrm{e}^{\langle \lambda', T \rangle}}{\prod_{\varpi \in \hat{\tilde{\Delta}}_{\tilde{L}'}^{\tilde{L}_{\mathfrak{M}}}} \langle -\lambda', \varpi^\vee \rangle} \varXi_{M',\pi'}^{\tilde{L}',T}(f, \lambda') \,\mathrm{d}\lambda',$$

where $\mathfrak{M}$ ranges over $(L_1, w_1, \dots, L_m = L_{\mathfrak{M}}, w_m)$ such that

- $\tilde{L}_1 \supsetneq \tilde{L}_2 \supsetneq \cdots \supsetneq \tilde{L}_m$;
- $w_j \in W_{L_{j-1}}(M_{j-1}; L_j)$ and $(M_j, \pi_j) = w_j(M_{j-1}, \pi_{j-1}) w_j^{-1}$, for all $j = 1, \dots, m$ (setting $L_0 = G$);
- $\langle w_1 \lambda_0, \varpi^\vee \rangle < 0$ for all $\varpi^\vee \in \hat{\tilde{\Delta}}_{\tilde{L}_1}$, and for each $j = 2, \dots, m$ we have

  $$\langle w_j \lambda_{j-1}, \varpi^\vee \rangle \langle w_j w_{j-1} \lambda_{j-2}, \varpi^\vee \rangle < 0 \quad \text{for all } \varpi^\vee \in \hat{\tilde{\Delta}}_{\tilde{L}_j}^{\tilde{L}_{j-1}},$$

  where the λ_j are defined by $\lambda_j = \operatorname{pr}^{\tilde{L}_j}(w_j \lambda_{j-1})$ as before;

and where we set $(M', \pi') = w'(M_m, \pi_m) w'^{-1}$, $M_{\mathfrak{M}} = M_m$, $\lambda_{\mathfrak{M}} = \lambda_m$ and

$$\varepsilon_{\mathfrak{M}} = \prod_{j=0}^{m-1} \prod_{\varpi^\vee \in \hat{\tilde{\Delta}}_{\tilde{L}_{j+1}}^{\tilde{L}_j}} \operatorname{sgn}(\langle \lambda_j, \varpi^\vee \rangle).$$

Using the substitution $\lambda' = w'\lambda$, the contribution from $\mathfrak{M}$ becomes $\varepsilon_{\mathfrak{M}}$ times

$$\sum_{\substack{\tilde{L}' \subset \tilde{L}_{\mathfrak{M}} \\ w' \in W_{L_{\mathfrak{M}}}(M_{\mathfrak{M}}; L')}} \hat{v}_{\tilde{L}'}^{\tilde{L}_{\mathfrak{M}}} \int_{\nearrow - \lambda_{\mathfrak{M}}}^{(\mathfrak{a}_{M_{\mathfrak{M}}}^{\tilde{L}_{\mathfrak{M}}})^*} \frac{\mathrm{e}^{\langle w'\lambda, T \rangle}}{\prod_{\varpi \in \hat{\tilde{\Delta}}_{\tilde{L}'}^{\tilde{L}_{\mathfrak{M}}}} \langle -w'\lambda, \varpi^\vee \rangle} \varXi_{M_{\mathfrak{M}},\pi_{\mathfrak{M}}}^{\tilde{L}',T}(f, \lambda, w') \,\mathrm{d}\lambda.$$

By (7.3) this is the integral over λ of the sum over $\varphi \in \mathcal{B}(\pi_{\mathfrak{M}})$ of

$$\overline{\mathcal{W}(\varphi, -\bar{\lambda})} \times \int_{A_{\tilde{Q}_{\mathfrak{M}}} \tilde{V}_{\mathfrak{M}}(\mathbb{A}) \tilde{L}_{\mathfrak{M}} \backslash \tilde{H}(\mathbb{A})}^{*} E^{Q_{\mathfrak{M}}}(h, I(f, \lambda)\varphi, \lambda^{L_{\mathfrak{M}}}) \,\mathrm{d}h,$$

or, simply, of $\mathcal{B}_{(M_{\mathfrak{M}}, \pi_{\mathfrak{M}})}^{\tilde{L}_{\mathfrak{M}}}(f, \lambda)$ by (7.2).

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Using the holomorphy of $\mathcal{B}^{\tilde{L}_{\mathfrak{M}}}_{(M_{\mathfrak{M}},\pi_{\mathfrak{M}})}(f,\lambda)$ on the unitary axis and Lemma 7.4 we finally obtain (cf. (3.10))

$$\sum_{\mathfrak{M}} \varepsilon_{\mathfrak{M}} \int_{\mathrm{i}(\mathfrak{a}^{\tilde{L}_{\mathfrak{M}}}_{M_{\mathfrak{M}}})^*} \mathcal{B}^{\tilde{L}_{\mathfrak{M}}}_{(M_{\mathfrak{M}},\pi_{\mathfrak{M}})}(f,\lambda)\,\mathrm{d}\lambda,$$

where the sum is over $\mathfrak{M}$ such that $M_{\mathfrak{M}}$ is θ-elliptic in $L_{\mathfrak{M}}$.

Summarizing, we have the following claim.

Claim 9.2. *The expression (9.1) is equal to*

$$\sum_{M,\pi,L} \mathfrak{c}(M,\pi,L) \int_{\mathrm{i}(\mathfrak{a}^{\tilde{L}}_{M})^*} \mathcal{B}^{\tilde{L}}_{(M,\pi)}(f,\lambda)\,\mathrm{d}\lambda \tag{9.13}$$

for some constants $\mathfrak{c}(M,\pi,L)$, where the sum is over all triplets (M,π,L) consisting of a θ-elliptic Levi subgroup M of L and a cuspidal representation π of $M(\mathbb{A})$.

The combinatorial constants $\mathfrak{c}(M,\pi,L)$ (which could possibly be zero) can be computed explicitly, at least in principle. However, since they are not so important for our immediate applications we refrain from doing it. It suffices to note that $\mathfrak{c}(M,\pi,L)$ takes only finitely many values (given G) and that $\mathfrak{c}(G,\pi,G)=1$.

We expect Claim 9.2 to hold for general quasi-split Galois pairs.

10. Conclusion

We specialize Claim 9.2 to the case where $G=\mathrm{GL}_n(E)$ using the analysis of § 8.

Theorem 10.1. *Let $G=\mathrm{GL}_n/E$ and $\tilde{H}=\mathrm{GL}_n/F$. Then (9.1) can be expressed as*

$$\sum_{(M,\pi)} \mathfrak{c}(M,\pi) \int_{\mathrm{i}(\mathfrak{a}^{L}_{M})^*} \mathcal{B}^{\tilde{L}}_{(M,\pi)}(f,\lambda)\,\mathrm{d}\lambda \tag{10.1}$$

for some constants $\mathfrak{c}(M,\pi)$ (which are bounded independently of π). Here (M,π) range over pairs consisting of a Levi subgroup M of type $(n_1,n_1,\dots,n_k,n_k,m_1,\dots,m_l)$ and a cuspidal representation π of $M(\mathbb{A})$ of the form

$$\pi=\sigma_1\otimes\sigma_1^*\otimes\cdots\otimes\sigma_k\otimes\sigma_k^*\otimes\tau_1\otimes\cdots\otimes\tau_l,$$

where the τ_j are distinguished (by $\mathrm{GL}_{m_j}(F)$) and mutually inequivalent; L is the Levi subgroup of type $(2n_1,\dots,2n_k,m_1,\dots,m_l)$.

Corollary 10.2. *The discrete part of the relative trace formula is*

$$\sum_{(M,\pi)} \mathfrak{c}(M,\pi)\mathcal{B}^{\tilde{H}}_{(M,\pi)}(f,0),$$

where (M,π) range over pairs consisting of a Levi subgroup M and a cuspidal representation $\pi=\tau_1\otimes\cdots\otimes\tau_l$ of $M(\mathbb{A})$ where the τ_j are distinguished and mutually inequivalent.

Remark 10.3. The relative trace formula for the pair $(\mathrm{GL}_n/E, \theta)$ is compared with the Kuznetsov trace formula for the quasi-split $U(n)$. On the spectral side the term considered above should match the contribution from the cuspidal representation

$$\sigma_1 \otimes \cdots \otimes \sigma_k \otimes \tau'$$

on the parabolic subgroup of $U(n)$ whose Levi part is isomorphic to $\mathrm{GL}_{n_1}(E) \times \cdots \times \mathrm{GL}_{n_k}(E) \times U(m')$. Here $m' = \sum m_j$ and τ' is the generic cuspidal representation on $U(m')$ whose functorial transfer to $\mathrm{GL}_{m'}(\mathbb{A}_E)$ is $\tau_1 \times \cdots \times \tau_l$. (The existence and uniqueness of such τ' is analogous to the case of $\mathrm{SO}(2n+1)$ which was considered in [**21**]. If $l > 1$ then τ' is endoscopic; cf. [**60**] for the case $n = 3$.) In particular, the discrete contribution comes from the case $k = 0$.

Consider now the case where $\mathcal{C}'$ is the symmetric space of Hermitian forms. Let $d = n - 2t$ where t is the Witt index of the Hermitian form defining $\tilde{H}$.

Theorem 10.4. *The relative trace formula (9.1) can be expressed in this case as*

$$\sum_{(M,\pi)} \mathfrak{c}'(M,\pi) \int_{\mathrm{i}(\mathfrak{a}_M^{\tilde{L}})^*} \mathcal{B}_{(M,\pi)}^{\tilde{L}}(f,\lambda)\,\mathrm{d}\lambda,$$

where (M,π) range over the cuspidal representations of $M(\mathbb{A})$ of the form

$$\pi = \sigma_1 \otimes \cdots \sigma_k \otimes \tau_1 \otimes \cdots \otimes \tau_l \otimes \overline{\sigma_k} \otimes \cdots \otimes \overline{\sigma_1},$$

where $\overline{\sigma_i} \not\simeq \sigma_i$, $i = 1, \dots, k$, and τ_j is distinguished by some unitary group for $j = 1, \dots, l$; if M is of type $(n_1, \dots, n_k, m_1, \dots, m_l, n_k, \dots, n_1)$ then L is the Levi subgroup of type $(n_1, \dots, n_k, m_1 + \cdots + m_l, n_k, \dots, n_1)$ and $m_1 + \cdots + m_l \geqslant d$.

Corollary 10.5. *For odd n, the discrete part of the relative trace formula is*

$$\sum_{\pi} \mathcal{B}_\pi(f),$$

where the sum is over distinguished cuspidal representations of $G(\mathbb{A})$. For even n we have in addition to that the sum over $\mathrm{Gal}(E/F)$-orbits of size two $\{\sigma, \bar{\sigma}\}$ of cuspidal representations of $\mathrm{GL}_{n/2}(\mathbb{A}_E)$ of

$$\mathcal{B}_{(M,\sigma\otimes\bar{\sigma})}^{\tilde{M}}(f,0),$$

where M is of type $(n/2, n/2)$.

Remark 10.6. The matching terms in the Kuznetsov trace formula for GL_n come from the cuspidal data

$$\mathrm{AI}_E^F(\sigma_1) \otimes \cdots \otimes \mathrm{AI}_E^F(\sigma_k) \otimes \pi_1 \otimes \cdots \otimes \pi_l,$$

where $\mathrm{BC}_F^E(\pi_j) = \tau_j$. Here AI and BC denote automorphic induction and base change, respectively. In particular, the discrete part corresponds to the cases where either $k = 0$

and $l = 1$, or (if n is even) $k = 1$ and $l = 0$. The fact that $\mathfrak{c}'(M, \pi) = 1$ in this case (if we count $\sigma \otimes \bar{\sigma}$ and $\bar{\sigma} \otimes \sigma$ together) follows directly from the analysis of § 9 (cf. § 2).

Recall also that the base change is not one to one, and hence, it may be necessary to sum over more than one cuspidal data in GL_n/F to get the required contribution. This corresponds to the fact that the intertwining periods are not Eulerian in this case and it is necessary to stabilize them first. For the case $M = T$ this was done in [**47**] and [**59**].

Absolute convergence

An important feature of the expansions described in Theorems 10.1 and 10.4 is their absolute convergence, in the sense that

$$\sum_{(M,\pi)} \int_{\mathrm{i}(\mathfrak{a}_M^{\tilde{L}})^*} |\mathcal{B}_{(M,\pi)}^{\tilde{L}}(f, \lambda)| \, \mathrm{d}\lambda < \infty.$$

Indeed, taking into account Lemma 7.4, the absolute convergence follows from the fact [**54**, § 6] that

$$\sum_{M,\pi} (1 + \Lambda_\pi)^{-N} < \infty \tag{10.2}$$

for sufficiently large N, where π ranges over the cuspidal representations of M with non-zero K_{fin}-fixed vectors.

The statement of Theorems 10.1 and 10.4 is an equality between two continuous distributions in f. Indeed, (9.1) is bounded by a constant multiple of $\|f\|_\infty$, while for the sum of Bessel distributions this follow from Lemma 7.4, (10.2) and Lebesgue's dominated convergence theorem. Thus, Theorems 10.1 and 10.4 hold without the K_∞-finiteness assumption on f.

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