On commensurability of right-angled Artin groups II: RAAGs defined by paths

By MONTSERRAT CASALS–RUIZ, ILYA KAZACHKOV

*Ikerbasque - Basque Foundation for Science and Matematika Saila, UPV/EHU,
Sarriena s/n, 48940, Leioa - Bizkaia, Spain.*
e-mails: montsecasals@gmail.com, ilya.kazachkov@gmail.com

AND ALEXANDER ZAKHAROV

*Chebyshev Laboratory, St Petersburg State University, 14th Line 29B,
Vasilyevsky Island, 199178, St.Petersburg, Russia, and
The Russian Foreign Trade Academy, 4a Pudovkina street, 119285, Moscow, Russia.*
e-mail: zakhar.sasha@gmail.com

(*Received* 12 *April* 2018; *revised* 28 *October* 2019)

Abstract

In this paper we continue the study of right-angled Artin groups up to commensurability initiated in [**CKZ**]. We show that RAAGs defined by different paths of length greater than 3 are not commensurable. We also characterise which RAAGs defined by paths are commensurable to RAAGs defined by trees of diameter 4. More precisely, we show that a RAAG defined by a path of length $n > 4$ is commensurable to a RAAG defined by a tree of diameter 4 if and only if $n \equiv 2 \pmod 4$. These results follow from the connection that we establish between the classification of RAAGs up to commensurability and linear integer-programming.

2010 Mathematics Subject Classification: 20E06 (Primary); 20F10, 20F65, 20E99 (Secondary)

---

1. *Introduction*

1·1. *Context of the problem*

One of the basic problems on locally compact topological groups is to classify their lattices up to commensurability. Recall that two lattices Γ_1, $\Gamma_2 < G$ are commensurable if and only if there exists $g \in G$ such that $\Gamma_1 \cap \Gamma_2^g$ has finite index in both Γ_1 and Γ_2^g. In particular, commensurable lattices have covolumes that are commensurable real numbers, that is, they have a rational ratio.

The notion of commensurability was generalised to better suit topological and large-scale geometric properties and to compare groups without requiring them to be subgroups of a common group. More precisely, we say that two groups H and K are (abstractly) commensurable if they have isomorphic finite index subgroups. In this paper, we will only be concerned with the notion of abstract commensurability and we simply refer to it as commensurability.

As we mentioned, commensurability is closely related to the large-scale geometry of the group. Indeed, any finitely generated group can be endowed with a natural word-metric which is well-defined up to quasi-isometry and since any finitely generated group is quasi-isometric to any of its finite index subgroups, it follows that commensurable groups are quasi-isometric.

Gromov suggested to study groups from this geometric point of view and understand the relation between these two concepts. More precisely, a basic problem in geometric group theory is to classify commensurability and quasi-isometry classes (perhaps within a certain class) of finitely generated groups and to understand whether or not these classes coincide.

The classification of groups up to commensurability (both in the abstract and classical case) has a long history and a number of famous solutions for very diverse classes of groups such as Lie groups, hyperbolic 3-manifold groups, pro-finite groups, Grigorchuk–Gupta–Sidki groups, etc, see for instance [**BJN09**, **DW93**, **Mar73**, **Sch95**, **Si43**, **GrW03**, **Ga16**].

In this paper, we focus on the question of classification of right-angled Artin groups, RAAGs for short, up to commensurability. Recall that a RAAG is a finitely presented group $\mathbb{G}(\Gamma)$ which can be described by a finite simplicial graph Γ, the commutation graph, in the following way: the vertices of Γ are in bijective correspondence with the generators of $\mathbb{G}(\Gamma)$ and the set of defining relations of $\mathbb{G}(\Gamma)$ consists of commutation relations, one for each pair of generators connected by an edge in Γ.

RAAGs have become central in group theory, their study interweaves geometric group theory with other areas of mathematics. This class interpolates between two of the most classical families of groups, free and free abelian groups, and its study provides uniform approaches and proofs, as well as far reaching generalisations of the results for free and free abelian groups. The study of this class from different perspectives has contributed to the development of new, rich theories such as the theory of CAT(0) cube complexes and has been an essential ingredient in Agol's solution of the Virtually Fibered Conjecture.

The commensurability classification of RAAGs has been previously solved for the following classes of RAAGs: free groups [**St68**, **Sch95**, **KPS73**], [**Gr93**, 1·C]; free Abelian groups, [**Gr81**, **B72**]; $F_m \times \mathbb{Z}^n$, [**Wh10**]; free products of free groups and free Abelian groups, [**BJN09**]; $F_m \times F_n$ with $m, n \geq 2$, [**Wi96**, **BM00**]; $\mathbb{G}(\Gamma)$, where Γ is connected, triangle- and square-free graph without any degree one vertices, [**KK14**]; Γ is star-rigid and does not have induced 4-cycles and the outer automorphism of $\mathbb{G}$ is finite, [**H16**]; and Γ is a tree of diameter ≤ 3, [**BN08**] and of diameter 4, [**CKZ**].

In [**CKZ**], we characterise the commensurability classes of RAAGs defined by trees of diameter 4. As a consequence, we deduce the existence of infinitely many different commensurability classes, confirming a conjecture of Behrstock and Neumann, and provide first examples of RAAGs that are quasi-isometric but not commensurable.

The proof of the aforementioned results was performed in three steps. In the first step, we determine a commensurabality invariant for RAAGs defined by trees. More precisely, to a given pair of trees Δ and Γ we associate a linear system of equations $S(\Delta, \Gamma)$ and show that the existence of positive integer solutions is a commensurability invariant of $\mathbb{G}(\Delta)$ and $\mathbb{G}(\Gamma)$, i.e. we prove:

THEOREM 1·1 (see [**CKZ**]). *Let Δ and Γ be trees. If $\mathbb{G}(\Delta)$ and $\mathbb{G}(\Gamma)$ are commensurable, then the system of equations $S(\Delta, \Gamma)$ has positive integer solutions.*

The proof of this step uses methods from geometric group theory.

In the second step, we center on RAAGs defined by trees of diameter 4 and characterise the trees for which the associated linear system of equations does not have positive integer solutions. This part is the most technical, although the methods required come from linear algebra. The strategy is to locally simplify the structure of the linear system of equations, that is, to determine some subsystems of equations and show that in order for them to have positive integer solutions, they must have an "easy" form. This allows us to simplify the entire system of linear equations enough to be able to determine whether or not it has positive integer solutions. This step allows us to characterise RAAGs $\mathbb{G}(T)$ and $\mathbb{G}(T')$ defined by trees of diameter 4 for which the system of equations $S(T, T')$ does not have positive integer solutions and hence by step 1, deduce that these groups are not commensurable.

In the last step, we consider RAAGs $\mathbb{G}(T)$ and $\mathbb{G}(T')$ defined by trees of diameter 4 for which the system of linear equations $S(T, T')$ does have positive integer solutions. From a minimal solution of the system $S(T, T')$, we build explicit finite index subgroups of $\mathbb{G}(T)$ and $\mathbb{G}(T')$, show that they are isomorphic and conclude that $\mathbb{G}(T)$ and $\mathbb{G}(T')$ are commensurable. The methods used in this step come mainly from Bass-Serre theory.

1·2. *Results and strategy of the proof*

In this paper, we develop methods introduced in [**CKZ**] and study the commensurability classes of RAAGs defined by paths. More precisely, let P_n, $n \geq 1$, denote the path graph with n edges and $n + 1$ vertices and let $T_{k,k+1}$, $k \geq 1$, denote a tree of diameter 4, with the central vertex of degree 2 and such that the two vertices adjacent to the central vertex have degrees $k + 1$ and $k + 2$ correspondingly, so that $T_{k,k+1}$ has $2k + 1$ leaves, see Figure 9.

We show that different paths of length more than 4 are not commensurable.

THEOREM 1·2. *The groups* $\mathbb{G}(P_n)$ *and* $\mathbb{G}(P_m)$, $n > m \geq 0$, *are commensurable if and only if* $n = 4$ *and* $m = 3$.

We also compare the commensurability classes between paths and trees of diameter 4 and prove

THEOREM 1·3. *Let* $n > 4$. *The group* $\mathbb{G}(P_n)$ *is commensurable to* $\mathbb{G}(T)$, *where* T *is a tree of diameter* 4, *if and only if* $n = 4k + 2$, $k \geq 1$, *and* $\mathbb{G}(T)$ *is commensurable to* $\mathbb{G}(T_{k,k+1})$.

Note that in [**CKZ**] we give a complete commensurability classification of RAAGs defined by trees of diameter 4, in particular, it is described which of them are commensurable to $\mathbb{G}(T_{k,k+1})$.

The proof follows the same 3-step structure as in [**CKZ**]. Note, however, that instead of considering systems of equations as in [**CKZ**], in this paper we work with linear systems of equations and inequalities and instead of requiring that the linear system of equations have a positive integer solution (as in [**CKZ**]), we require that our system of equations and inequalities have an integer solution.

(i) In Section 2, we reduce commensurability between RAAGs defined by trees to the existence of integer solutions of a (disjunction of) linear system of equations and inequalities S, see Corollary 2·14.

(ii) In Sections 3, 4 and 6, we analyse the system S, characterise when it has no integer solutions and deduce when two RAAGs from our class are not commensurable. In Section 3, we study the system S defined by paths of length 3 and $m \geq 5$ and show

that it never has integer solutions. This is the simplest case and it introduces the techniques and ideas for the other cases. In Section 4, we study the system S defined by a path of length $n > 4$ and a tree of diameter 4 and show that if $n \not\equiv 2 \pmod 4$ then the system does not have integer solutions. Finally in Section 6, we address the system defined by paths of different length greater than 4 and again show that it never has integer solutions.

(iii)   In Section 5, we show that when the path is of length $4k + 2$ and the tree of diameter 4 is $T_{k,k+1}$ we can exhibit isomorphic finite index subgroups and conclude that the corresponding RAAGs are commensurable.

1·3. *Related problems and further research*

As we already mentioned in [**CKZ**], it is our belief that the general strategy of the proof can be used to study commensurability classes of RAAGs defined by trees and more generally all RAAGs.

Corollary 2·14 reduces commensurability between two RAAGs defined by trees to the existence of integer solutions of a linear system of equations and inequalities. This brings up a natural question of whether or not this necessary condition is also sufficient.

Question 1·4. Let $\mathbb{G}(\Gamma_1)$ and $\mathbb{G}(\Gamma_2)$ be RAAGs defined by trees. Is it true that $\mathbb{G}(\Gamma_1)$ and $\mathbb{G}(\Gamma_2)$ are commensurable if and only if the system $S_i(\Gamma_1, \Gamma_2)$ defined by the product graph (see Section 2·5) has integer solutions?

In all the cases we studied so far, solutions of the (linear) system (of equations and inequalities) have guided the construction of the subgroups which witness commensurability. In essence, Question 1·4 asks whether one can build a finite cover of the Salvetti complex of a RAAG from local covers of the complexes associated to the centralisers of generators.

As pointed out to us by Henry Wilton, this question may be approached using techniques introduced by Ian Agol in his solution of the virtual Haken conjecture. In [**A13**], the author constructs a finite-sheeted cover which is modelled on some hierarchy. In order to do it, he constructs a measure on the space of colorings of a wall graph and then refines the colors to reflect how each wall is cut up by previous stages of the hierarchy. He then uses the measure to find a solution to certain gluing equations on the colored cubical polyhedra defined by the refined colorings, and uses solutions to these equations to get the base case of the hierarchy and glue up successively each stage of the hierarchy using a gluing theorem to glue at each stage after passing to a finite-sheeted cover.

Solutions of the linear system are, in some sense, values necessary for the consistent gluing of the local covers. The goal would be to generalise Agol's Gluing Theorem to build the finite index cover from the local ones and so describe the finite index subgroup that witnesses commensurability.

In the same way we speculate that trees and, more generally, 2-dimensional RAAGs could play the role of the base of induction for a hierarchy. (Recall that a RAAG is 2-dimensional if and only if its commutation graph is triangle-free.) Centralisers in 2-dimensional RAAGs are of the form $\mathbb{Z} \times F_n$. If the answer to Question 1·4 is positive, then given a solution of the system, one can build the finite index subgroup from local covers of free groups. In the general case, commensurability of RAAGs would imply compatible commensurable centralisers of certain elements and centralisers are again RAAGs of lower complexity. By induction, one

then could build finite index subgroups for the centralisers and, using a gluing theorem, extend them to a finite cover of the group. This brings us to the following question:

Question 1·5. Let $\mathbb{G}(\Gamma_1)$ and $\mathbb{G}(\Gamma_2)$ be RAAGs. Is it true that $\mathbb{G}(\Gamma_1)$ and $\mathbb{G}(\Gamma_2)$ are commensurable if and only if the system $S_i(\Gamma_1, \Gamma_2)$ defined by the product graph (see Section 2·5) has integer solutions?

This is just a rough strategy to approach the general problem. A good starting point is to understand whether or not the reduction from commensurability to integer solutions of a linear system generalises from trees to 2-dimensional RAAGs. More precisely, we expect that Corollary 2·14 can be generalised as follows

CONJECTURE 1·6. *Let* $\mathbb{G}(\Gamma_1)$ *and* $\mathbb{G}(\Gamma_2)$ *be 2-dimensional RAAGs. If* $\mathbb{G}(\Gamma_1)$ *and* $\mathbb{G}(\Gamma_2)$ *are commensurable, then the system* $S_i(\Gamma_1, \Gamma_2)$ *defined by the product graph* (*see* Section 2·5) *has integer solutions.*

The existence of integer solutions of a linear system of equations and inequalities can be interpreted as a syntactic fragment of the Presburger arithmetic (with order) and so in particular, it is a decidable problem. The Presburger arithmetic has quantifier elimination if we add predicates for division. Hence, the existence of integer solutions is equivalent to a boolean combination of atomic formulas in the language $(+, <, 0)$ and congruences of integers. This justifies the classification we obtain for trees and paths, where it is required that the length n of the path is congruent to 2 modulo 4.

The decidability of the existence of integer solutions of a linear system is a very well-known and long-studied problem in Computer Science. It was intensively studied in the field of mechanical theorem proving and it is most commonly known as (linear) integer-programming. It is actually one of the most important models in management science (capital budgeting, warehouse location, scheduling, etc) and there are many different efficient algorithms to address it.

As a consequence, given two trees, one can describe the linear system associated to them and use an algorithm to decide whether or not the system has an integer solution. If the answer is negative, that is, there is no integer solutions, then we can conclude that the corresponding groups are not commensurable. Furthermore if the answer to Question 1·4 (and Question 1·5) were positive, we could conclude that the commensurability problem between tree (and general) RAAGs is decidable as well as have a good understanding of its complexity, see [**CH17**] and references there.

2. *Systems of equations associated to tree RAAGs*

The main aim of this section is, given two RAAGs $\mathbb{G}(\Gamma_1)$ and $\mathbb{G}(\Gamma_2)$ defined by trees, to construct a linear system of equations and inequalities $S(\Gamma_1, \Gamma_2)$ such that if $\mathbb{G}(\Gamma_1)$ and $\mathbb{G}(\Gamma_2)$ are commensurable, then the system $S(\Gamma_1, \Gamma_2)$ has integer solutions, see Corollary 2·14.

In order to construct the system of equations, one needs to introduce several commensurability invariants, namely the (reduced) extension graph and the quotient graph. We assign certain labels to the quotient graph, deduce a system of linear equations and show that if two tree RAAGs are commensurable, then the exponents are positive integer solutions of the system of equations.

Since the quotient graph depends on the subgroup witnessing commensurability, so does the system of equations. In order to overcome this dependence, we introduce a new graph, the product graph, which only depends on the trees Γ_1 and Γ_2, we associate certain labels to the vertices and edges, describe a linear system of equations and inequalities and show again that if the groups are commensurable, the labels are integer solutions of the linear system.

This sections follows the ideas introduced in [**CKZ**]. For completeness, we recall the definitions and results needed in this paper.

2·1. *Reduced centraliser splitting*

Observe that tree RAAGs split as fundamental groups of graphs of groups, whose vertex groups are centralisers of vertex generators. We recall the notion of (reduced) centraliser splitting, as in [**CKZ**], which we will use in Section 5.

Definition 2·1 ((Reduced) Centraliser splitting). Let Δ be a tree and let $\mathbb{G}(\Delta)$ be the RAAG with the underlying graph Δ. The *centraliser splitting* of $\mathbb{G}(\Delta)$ is defined as follows. The graph of the splitting is isomorphic to Δ and the vertex group at every vertex is defined to be the centraliser of the corresponding vertex generator. Note that if v is some vertex of Δ, and $u_1, \ldots, u_s$ are all vertices of Δ adjacent to v, then $C(v) = \langle v, u_1, \ldots, u_s \rangle \cong \mathbb{Z} \times F_s$, where F_s is the free group of rank s, see [**CKZ**] for more details on centralizers in RAAGs. In particular, $C(v)$ is abelian if and only if v has degree 1, and in this case $C(v) \cong \mathbb{Z}^2$ is contained in the centraliser of the vertex adjacent to v. For an edge e connecting vertices u and v the edge group at e is $C(u) \cap C(v) = \langle u, v \rangle \cong \mathbb{Z}^2$.

Note that the centraliser splitting is neither reduced nor minimal, since for every vertex of degree 1 in Δ the vertex group is equal to the incident edge group. Thus it makes sense to consider the *reduced centraliser splitting* of $\mathbb{G}(\Delta)$ (for a tree Δ), which is obtained from the centraliser splitting by removing all vertices of degree 1. In this splitting all the vertex groups are non-abelian, and all the edge groups are isomorphic to $\mathbb{Z}^2$, in particular, this splitting is already minimal and reduced.

2·2. *Reduced extension graph and quotient graph*

In this section, we recall the notions of (reduced) extension and quotient graphs, see [**CKZ**] for further details.

Definition 2·2 (Extension graph, see [**KK13**]). Let $\mathbb{G}(\Gamma)$ be a RAAG with the underlying commutation graph Γ, then the *extension graph* Γ^e is defined as follows. The vertex set of Γ^e is the set of all elements of $\mathbb{G}(\Gamma)$ which are conjugate to the canonical generators (vertices of Γ). Two vertices are joined by an edge if and only if the corresponding group elements commute. The group $\mathbb{G}(\Gamma)$ acts on Γ^e by conjugation.

Definition 2·3 (Reduced extension graph). For a tree Γ, we define the *reduced extension graph* of Γ, and denote it by $\widetilde{\Gamma}^e$, to be the full subgraph of the extension graph Γ^e, whose vertex set is the set of all elements of $\mathbb{G}(\Gamma)$ which are conjugate to the canonical generators corresponding to vertices of Γ of degree more than 1 (which are exactly those which have non-abelian centralisers).

Suppose that Γ is a finite tree of diameter at least 3 and let $G = \mathbb{G}(\Gamma)$. Let $\widetilde{\Gamma}$ be the tree obtained from Γ by deleting all degree 1 vertices together with the incident edges. Then G

acts on Γ^e and on $\widetilde{\Gamma}^e$ by conjugation, so that $G\backslash\Gamma^e \cong \Gamma$ and $G\backslash\widetilde{\Gamma}^e \cong \widetilde{\Gamma}$; these actions can be thought of as the natural actions on the Bass–Serre trees of the centraliser splitting and the reduced centralizer splitting of G respectively, see [**CKZ**, lemma 3·4].

Suppose that H is a finite index subgroup of G. Let $\Psi(H) = H\backslash\widetilde{\Gamma}^e$, then $\Psi(H)$ is a finite graph, and there are natural graph morphisms $\gamma = \gamma_H : \widetilde{\Gamma}^e \to \Psi(H)$ and $\delta = \delta_H : \Psi(H) \to \Gamma$. Note that the image of δ lies in $\widetilde{\Gamma}$, so we can also think of δ as a morphism $\delta : \Psi(H) \to \widetilde{\Gamma}$.

2·3. *Labels in the reduced extension graph and the quotient graph*

Suppose H is a finite index subgroup of $G = \mathbb{G}(\Gamma)$. We can then associate some labels to the quotient graph as follows.

Definition 2·4 (Label of a vertex/edge). Let w be a vertex of $\widetilde{\Gamma}^e$, thus w is also an element of G. Define the *label of the vertex w*, denoted by $\overline{L}(w)$, to be the minimal positive integer k such that $w^k \in H$. Such number exists, since H has finite index in G.

For an edge f of $\widetilde{\Gamma}^e$ connecting vertices w_1 and w_2 define the *label of the edge f at the vertex w_1*, denoted by $\overline{l}_{w_1}(f)$, to be the minimal positive integer k such that there exists an integer l such that $w_1^k w_2^l \in H$. Without loss of generality, we can assume that l is non-negative. The label of f at w_2 is defined analogously. Note that, by definition, $\overline{L}(w_1) \geq \overline{l}_{w_1}(f)$, for all edges f.

Observe that the labels of vertices and edges are invariant under the action of H on $\widetilde{\Gamma}^e$ (by conjugation). Indeed, for $h \in H$ we have $w^k \in H$ iff $(w^h)^k \in H$, and $w_1^k w_2^l \in H$ iff $(w_1^h)^k (w_2^h)^l \in H$. This means that we can define labels for the quotient graph $\Psi(H)$. If v is a vertex of $\Psi(H)$, then define the *label of the vertex v*, denoted by $L(v)$, to be the label $\overline{L}(w)$, where w is some vertex of $\widetilde{\Gamma}^e$ such that $\gamma(w) = v$. Analogously, if p is an edge of $\Psi(H)$ connecting vertices v_1 and v_2, then define the *label of the edge p at the vertex v_1*, denoted by $l_{v_1}(p)$, to be the label $\overline{l}_{w_1}(f)$, where f is some edge of $\widetilde{\Gamma}^e$ such that $\gamma(f) = p$, and w_1 is the end of p such that $\gamma(w_1) = v_1$. It follows that these labels are well-defined. Note that the labels of vertices and edges of $\Psi(H)$ are positive integers.

2·4. *System of equations for the quotient graph*

Suppose now that Γ_1 and Γ_2 are two finite trees of diameters at least 3 such that $\mathbb{G}(\Gamma_1)$ and $\mathbb{G}(\Gamma_2)$ are commensurable, and $H_1 \leq G_1$, $H_2 \leq G_2$ are finite index isomorphic subgroups, and $\varphi : H_1 \to H_2$ is the isomorphism. All the definitions above apply to both H_1 in G_1 and H_2 in G_2. In [**CKZ**] we show that φ induces graph isomorphisms $\overline{\varphi} \colon \widetilde{\Gamma}_1^e \to \widetilde{\Gamma}_2^e$ and $\varphi_* : \Psi(H_1) \to \Psi(H_2)$, see [**CKZ**, lemma 3·6].

We associate a system of equations to the quotient graph and show that the edge and vertex labels of $\Psi(H_1)$ and $\Psi(H_2)$ are positive integer solutions of the system.

As above, we have surjective graph morphisms

$$\gamma_1 : \widetilde{\Gamma}_1^e \longrightarrow \Psi(H_1), \ \gamma_2 : \widetilde{\Gamma}_2^e \longrightarrow \Psi(H_2), \ \delta_1 : \Psi(H_1) \longrightarrow \widetilde{\Gamma}_1, \ \delta_2 : \Psi(H_2) \longrightarrow \widetilde{\Gamma}_2.$$

As in [**CKZ**], for an edge e of $\Psi(H_1)$ beginning in a vertex u we denote by $L(u)$ the vertex label of u in $\Psi(H_1)$, by $L'(u)$ the vertex label of $\varphi_*(u)$ in $\Psi(H_2)$, by $l_u(e)$ the edge label of e at the vertex u in $\Psi(H_1)$, and by $l'_u(e)$ the edge label of $\varphi_*(e)$ at the vertex $\varphi_*(u)$ in $\Psi(H_2)$. All these labels are positive integers by definition.

We summarise the equations obtained in [**CKZ**] in the following two lemmas.

LEMMA 2·5 (see [**CKZ**, lemma 3·16]). *Let e be an edge of $\Psi(H_1)$ connecting vertices v_1 and v_2. Then the following equations hold*:

$$\frac{L(v_1)}{l_{v_1}(e)} = \frac{L'(v_1)}{l'_{v_1}(e)} = \frac{L(v_2)}{l_{v_2}(e)} = \frac{L'(v_2)}{l'_{v_2}(e)} = q, \tag{2·1}$$

where q is some positive integer.

Let v be a vertex of $\Psi(H_1)$, and $u_1 = \delta_1(v) \in V(\widetilde{\Gamma}_1)$, $u_2 = \delta_2(\varphi_*(v)) \in V(\widetilde{\Gamma}_2)$. Let $p_1, \ldots, p_m$ be all vertices of $\widetilde{\Gamma}_1$ adjacent to u_1, and $q_1, \ldots, q_n$ be all vertices of $\widetilde{\Gamma}_2$ adjacent to u_2. Suppose that the edge e_i connects u_1 with p_i, $i = 1, \ldots, m$, and the edge f_j connects u_2 with q_j, $j = 1, \ldots, n$. Let $e_i^1, \ldots, e_i^{r_i}$ be all the edges of $\Psi(H_1)$ beginning in v which project into e_i under δ_1, for each $i = 1, \ldots, m$; note that $e_1^1, \ldots, e_1^{r_1}, e_2^1, \ldots, e_2^{r_2}, \ldots, e_m^1, e_m^2, \ldots, e_m^{r_m}$ are all the edges of $\Psi(H_1)$ beginning in v. Analogously, let $f_j^1, \ldots, f_j^{s_j}$ be all the edges of $\Psi(H_2)$ beginning in $\varphi_*(v)$ which project into f_j under δ_2, for each $j = 1, \ldots, n$; note that $f_1^1, \ldots, f_1^{s_1}, f_2^1, \ldots, f_2^{s_2}, \ldots, f_n^1, f_n^2, \ldots, f_n^{s_n}$ are all the edges of $\Psi(H_2)$ beginning in $\varphi_*(v)$, in particular, φ_* induces a bijection between them and the above edges in $\Psi(H_1)$ beginning in v.

Let also D_1 be the degree of u_1 in Γ_1 minus 1, and D_2 be the degree of u_2 in Γ_2 minus 1. Note that we take degrees in Γ_1, Γ_2, not in $\widetilde{\Gamma}_1, \widetilde{\Gamma}_2$. Below all the edge labels are taken at the end vertices, i.e., at the vertices which are not u_1 or u_2; we omit the lower index notation.

LEMMA 2·6. *For every vertex v of $\Psi(H_1)$, in the above notation the following equations hold*:

$$D_1 \sum_{i=1}^{r_1} l(e_1^i) = D_1 \sum_{i=1}^{r_2} l(e_2^i) = \cdots = D_1 \sum_{i=1}^{r_m} l(e_m^i)$$
$$= D_2 \sum_{j=1}^{s_1} l'(f_1^j) = D_2 \sum_{j=1}^{s_2} l'(f_2^j) = \cdots = D_2 \sum_{j=1}^{s_n} l'(f_n^j). \tag{2·2}$$

Proof. The statement follows immediately from [**CKZ**, lemmas 3·12 and 3·13].

2·5. *Product graph*

In this section, we introduce the product graph, which only depends on the graphs Γ_1 and Γ_2, and describe its relation with the quotient graph.

Recall that for two graphs Δ_1, Δ_2 their *direct product* (also called tensor product) is the graph $\Delta_1 \times \Delta_2$ defined as follows. The vertex set of $\Delta_1 \times \Delta_2$ is the (Cartesian) product of the vertex sets of Δ_1 and Δ_2. If u_1, v_1 are vertices of Δ_1, and u_2, v_2 are vertices of Δ_2, then we define two vertices (u_1, u_2) and (v_1, v_2) to be adjacent in $\Delta_1 \times \Delta_2$ if and only if u_1 is adjacent to v_1 in Δ_1 and u_2 is adjacent to v_2 in Δ_2. Note that this is a category-theoretic product, which means that there exist naturally defined projection morphisms

$$\pi_1^0 : \Delta_1 \times \Delta_2 \longrightarrow \Delta_1, \ \pi_1^0((u, v)) = u; \quad \pi_2^0 : \Delta_1 \times \Delta_2 \longrightarrow \Delta_2, \ \pi_2^0((u, v)) = v,$$

such that if Δ is a graph and $\delta_1^0 : \Delta \to \Delta_1$, $\delta_2^0 : \Delta \to \Delta_2$ are graph morphisms, then there exists a unique graph morphism $\delta^0 : \Delta \to \Delta_1 \times \Delta_2$ such that $\delta_1^0 = \pi_1^0 \delta^0$ and $\delta_2^0 = \pi_2^0 \delta^0$. Namely, $\delta^0(x) = (\delta_1^0(x), \delta_2^0(x))$ for every vertex x of Δ, and it is extended to the edges of Δ in a natural way.
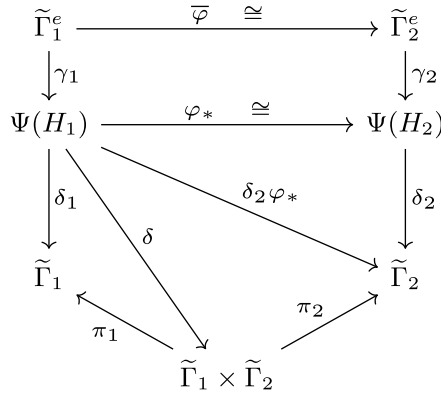
Fig. 1. Commutative diagram for the reduced extension graphs and quotient graphs of commensurable tree RAAGs

Definition 2·7 (Product graph and type morphism). Recall $\widetilde{\Gamma}_1$ is the subgraph obtained from Γ_1 by deleting all degree 1 vertices and incident edges, and similar for Γ_2. We define the *product graph* to be the direct product $\mathfrak{D} = \widetilde{\Gamma}_1 \times \widetilde{\Gamma}_2$ of the subgraphs $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$.

To abbreviate the notation, we will denote $\Psi(H_1)$ by Ψ. Let $\delta : \Psi \to \mathfrak{D}$ be the graph morphism defined by $\delta(x) = (\delta_1(x), \delta_2(\varphi_*(x)))$ for every vertex x of Ψ, extended to the edges in a natural way. Let $\pi_1 : \mathfrak{D} \to \widetilde{\Gamma}_1$, $\pi_2 : \mathfrak{D} \to \widetilde{\Gamma}_2$ be the projection morphisms as above. We call δ the *type morphism*. Recall that in [**CKZ**] we defined the pair of vertices $(\delta_1(u), \delta_2(\varphi_*(u)))$ to be the type of a vertex u of $\Psi(H_1)$. Thus, the type of a vertex u is now just $\delta(u)$, which is a vertex of $\mathfrak{D}$, see Figure 1.

We denote the image of Ψ in $\mathfrak{D}$ under δ by $\mathfrak{C}$. Thus the vertex set of $\mathfrak{C}$ is the set of all possible types of vertices of Ψ.

LEMMA 2·8. *In the above notation the following statements hold*:

(i) *the product graph $\mathfrak{D}$ has two connected components $\mathfrak{D}_1$ and $\mathfrak{D}_2$, and the graph $\mathfrak{C}$ is a connected subgraph of one of these components;*
(ii) *the restrictions of the projections $\pi_1 : \mathfrak{D} \to \widetilde{\Gamma}_1$ and $\pi_2 : \mathfrak{D} \to \widetilde{\Gamma}_2$ to $\mathfrak{C}$ are surjective;*
(iii) *moreover, the restrictions of π_1 and π_2 to $\mathfrak{C}$ are locally surjective, i.e. if $v \in V(\mathfrak{C})$, and $\pi_1(v) \in V(\widetilde{\Gamma}_1)$ is adjacent to some vertex $u \in V(\widetilde{\Gamma}_1)$, then there exists a vertex $w \in V(\mathfrak{C})$ adjacent to v in $\mathfrak{C}$ such that $\pi_1(w) = u$; the same is true for π_2.*

Proof. The first claim follows from the general graph-theoretic fact that the direct product of two connected graphs without cycles of odd length has two connected components [**We62**], in particular this is true for trees. Obviously, $\mathfrak{C}$ is connected, so it should lie inside one of the two connected components.

The second claim follows immediately from the fact that $\delta_1 = \pi_1\delta$ and $\delta_2 = \pi_2\delta\varphi_*^{-1}$ are surjective.

Note that $\delta_1\gamma_1$ is locally surjective in the sense above, so δ_1 is locally surjective as well, and then π_1 restricted to $\mathfrak{C}$ is also locally surjective; the same holds for π_2 restricted to $\mathfrak{C}$.

2·6. *Edge labels for the product graph*

As we did with the quotient graph, in this section we define certain labels and assign them to edges of the product graph.

We assume that all our graphs are directed, so that each edge e has initial vertex $\alpha(e)$, terminal vertex $\omega(e)$ and inverse edge e^{-1}. Note that the edge labels on Ψ defined above are defined for unoriented edges, i.e. they are the same for e and e^{-1} (but depend on the choice of the endpoint). We now define edge labels for the product graph $\mathfrak{D}$. Each oriented edge e will have 4 labels, denoted by $M_{11}(e)$, $M_{12}(e)$, $M_{21}(e)$ and $M_{22}(e)$.

Let e be an edge of $\mathfrak{D}$. If $e \notin E(\mathfrak{C})$, define $M_{11}(e) = M_{12}(e) = M_{21}(e) = M_{22}(e) = 0$. Otherwise, if $e \in E(\mathfrak{C})$, let $\delta^{-1}(e) = \{f_1, \ldots, f_k\}$ be the edges of Ψ which project into e under δ, $k \geq 1$. Let $\alpha(e) = u$, $\omega(e) = v$, and $\alpha(e_i) = u_i$, $\omega(e_i) = v_i$, for $i = 1, \ldots, k$, so that $\delta(u_i) = u$, $\delta(v_i) = v$, $i = 1, \ldots, k$. Now define the edge labels of e as follows:

$$M_{11}(e) = \sum_{i=1}^{k} L(u_i)l_{v_i}(e_i); \quad M_{12}(e) = \sum_{i=1}^{k} L(u_i)l'_{v_i}(e_i);$$

$$M_{21}(e) = \sum_{i=1}^{k} L'(u_i)l_{v_i}(e_i); \quad M_{22}(e) = \sum_{i=1}^{k} L'(u_i)l'_{v_i}(e_i). \tag{2·3}$$

LEMMA 2·9. *For every $e \in E(\mathfrak{D})$, all the labels of e are non-negative integers, and*

$$e \in E(\mathfrak{C}) \Longleftrightarrow M_{11}(e) > 0 \Longleftrightarrow M_{12}(e) > 0 \Longleftrightarrow M_{21}(e) > 0 \Longleftrightarrow M_{22}(e) > 0. \tag{2·4}$$

Proof. Follows immediately from the definition of labels in $\mathfrak{D}$ and the fact that all edge and vertex labels of Ψ are positive integers.

Remark 2·10. Note that the conditions on (local) surjectivity from Lemma 2·8 can be expressed as a union of linear equations and inequalities. Indeed, if $e' \in E(\widetilde{\Gamma}_j)$ and $e_1, \ldots, e_k \in E(\mathfrak{D})$ is the preimage $\pi_j^{-1}(e')$ in $\mathfrak{D}$, then, by Lemma 2·9, π_j is surjective on the edge e' if and only if

$$\sum_{i=1}^{k} M_{lm}(e_i) > 0, \quad j, l, m \in \{1, 2\}.$$

Similarly, if $v \in V(\mathfrak{C})$, $\pi_j(v) \in V(\widetilde{\Gamma}_j)$ is adjacent to some vertex $u \in V(\widetilde{\Gamma}_j)$, $e' = (\pi_j(v), u) \in E(\widetilde{\Gamma}_j)$ and $e_1, \ldots, e_k \in E(\mathfrak{D})$ are all the edges in the preimage $\pi_j^{-1}(e')$ in $\mathfrak{D}$ which begin in v, then $\sum_{i=1}^{k} M_{lm}(e_i) > 0$, $j, l, m \in \{1, 2\}$.

2·7. *Linear system for the product graph*

In this section, we define a linear system of equations and inequalities associated to the product graph and show that the labels of the edges are positive integer solutions of the system.

We first show that the following equations are satisfied for each edge.

LEMMA 2·11. *For every edge $e \in E(\mathfrak{D})$ the following holds*:

$$M_{11}(e) = M_{11}(e^{-1}), \ M_{12}(e) = M_{21}(e^{-1}), \ M_{21}(e) = M_{12}(e^{-1}), \ M_{22}(e) = M_{22}(e^{-1}). \tag{2·5}$$

Proof. If $e \notin E(\mathfrak{C})$, then all the labels of e and e^{-1} are 0, and so the claim holds automatically. Suppose now $e \in E(\mathfrak{C})$. Note that, in the above notation, by Lemma 2·5, we have that $L(v_i)l_{u_i}(e_i) = L(u_i)l_{v_i}(e_i)$, for all $i = 1, \ldots, k$, hence

$$M_{11}(e^{-1}) = \sum_{i=1}^{k} L(v_i) l_{u_i}(e_i) = \sum_{i=1}^{k} L(u_i) l_{v_i}(e_i) = M_{11}(e).$$

Analogously, by Lemma 2·5, we have that $L(v_i) l'_{u_i}(e_i) = L'(u_i) l_{v_i}(e_i)$, for all $i = 1, \ldots, k$ and so

$$M_{12}(e^{-1}) = \sum_{i=1}^{k} L(v_i) l'_{u_i}(e_i) = \sum_{i=1}^{k} L'(u_i) l_{v_i}(e_i) = M_{21}(e).$$

The proof of other two equalities in (2·5) is analogous.

We now describe the equations that we associate to each vertex of $\mathfrak{D}$. Let $w = (w_1, w_2) \in V(\mathfrak{D})$, where $w_1 = \pi_1(w) \in V(\widetilde{\Gamma}_1)$, $w_2 = \pi_2(w) \in V(\widetilde{\Gamma}_2)$. Let $p_1, \ldots, p_m$ be all vertices of $\widetilde{\Gamma}_1$ adjacent to w_1, and $q_1, \ldots, q_n$ be all vertices of $\widetilde{\Gamma}_2$ adjacent to w_2. Suppose that the edge e_i connects w_1 with p_i, $i = 1, \ldots, m$, and the edge f_j connects w_2 with q_j, $j = 1, \ldots, n$. Then $w_{ij} = (p_i, q_j)$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, are all the vertices of $\mathfrak{D}$ adjacent to w. Let e_{ij} be the (oriented) edge of $\mathfrak{D}$ beginning in w and ending in w_{ij}, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Thus, the edges $e_{i1}, e_{i2}, \ldots, e_{in}$ are all the edges of $\mathfrak{D}$ beginning in w which project into e_i under π_1, for $1 \leq i \leq m$ and the edges $e_{1j}, e_{2j}, \ldots, e_{mj}$ are all the edges of $\mathfrak{D}$ beginning in w which project into f_j under π_2, for $1 \leq j \leq n$.

Recall that D_1 is the degree of the vertex w_1 considered as a vertex of Γ_1 minus 1, and D_2 is the degree of the vertex w_2 considered as a vertex of Γ_2 minus 1. Note that we take degrees in Γ_1, Γ_2, not in $\widetilde{\Gamma}_1$, $\widetilde{\Gamma}_2$.

LEMMA 2·12. *For every vertex $w \in V(\mathfrak{D})$, in the above notation the following equations hold*:

$$D_1 \sum_{j=1}^{n} M_{11}(e_{1j}) = D_1 \sum_{j=1}^{n} M_{11}(e_{2j}) = \cdots = D_1 \sum_{j=1}^{n} M_{11}(e_{mj})$$
$$= D_2 \sum_{i=1}^{m} M_{12}(e_{i1}) = D_2 \sum_{i=1}^{m} M_{12}(e_{i2}) = \cdots = D_2 \sum_{i=1}^{m} M_{12}(e_{in}) \tag{2·6}$$

and

$$D_1 \sum_{j=1}^{n} M_{21}(e_{1j}) = D_1 \sum_{j=1}^{n} M_{21}(e_{2j}) = \cdots = D_1 \sum_{j=1}^{n} M_{21}(e_{mj})$$
$$= D_2 \sum_{i=1}^{m} M_{22}(e_{i1}) = D_2 \sum_{i=1}^{m} M_{22}(e_{i2}) = \cdots = D_2 \sum_{i=1}^{m} M_{22}(e_{in}). \tag{2·7}$$

Proof. If $w \notin V(\mathfrak{C})$, then the claim follows since in this case M_{ij}'s are all equal to 0. Hence, we can assume that $w \in V(\mathfrak{C})$. Let $\delta^{-1}(w) = \{w_1, \ldots, w_N\}$ be all the vertices of $\Psi(H_1)$ which project to w under δ. Now for each vertex w_k, $k = 1, \ldots, N$, write the equations (2·2) from Lemma 2·6, multiply each side by $L(w_k)$ and sum over $k = 1, \ldots, N$. Equation (2·6) follows now from the definitions of the labels on $\mathfrak{D}$. Analogously, for each vertex w_k, $k = 1, \ldots, N$, write the equations (2·2) from Lemma 2·6, multiply each side by $L'(w_k)$ and sum over $k = 1, \ldots, N$. Hence Equation (2·7).

In the above notation, for every vertex $w \in V(\mathfrak{D})$, define two labels:

$$R_1(w) = D_1 \sum_{j=1}^{n} M_{11}(e_{1j}), \quad R_2(w) = D_2 \sum_{i=1}^{m} M_{22}(e_{in}). \tag{2·8}$$

By Lemma 2·12, we can rewrite these labels in several different ways.

Definition 2·13 (Linear system $S(\Gamma_1, \Gamma_2)$). Let $\mathbb{G}(\Gamma_1)$ and $\mathbb{G}(\Gamma_2)$ be two RAAGs defined by trees Γ_1 and Γ_2 and let $\widetilde{\Gamma}_i$ be the induced subgraph of Γ_i defined by all non-leaf vertices, $i = 1, 2$. We denote by $S_i(\Gamma_1, \Gamma_2)$ the system of linear equations defined by the i-th connected component $\mathfrak{D}_i$ of the product graph $\mathfrak{D}$ of $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ in variables $M_{kl}(e)$, $k, l = 1, 2$, $e \in E(\mathfrak{D}_i)$, that is $S_i(\Gamma_1, \Gamma_2)$ is the union of Equations (2·5), (2·6) and (2·7) from Lemmas 2·11 and 2·12, for all edges and vertices of the ith connected component $\mathfrak{D}_i$.

Let P be the set of inequalities $M_{kl}(e) \geq 0$, for all $e \in E(\mathfrak{D})$. Let E be the following disjunction of equations and inequalities encoding surjectivity:

$$\sum_{i=1}^{k} M_{lm}(e_i) > 0,$$

for all $e' \in E(\widetilde{\Gamma}_j)$, where $e_1, \ldots, e_k \in E(\mathfrak{D})$ is the preimage $\pi_j^{-1}(e')$ in $\mathfrak{D}$, and $j, l, m \in \{1, 2\}$; and local surjectivity:

$$M_{kl}(e) > 0 \implies \sum_{i=1}^{k} M_{lm}(e_i) > 0,$$

or, equivalently,

$$M_{kl}(e) = 0 \ \vee \ \sum_{i=1}^{k} M_{lm}(e_i) > 0,$$

for all $v \in V(\mathfrak{D})$, for all $e = (v, v') \in E(\mathfrak{D})$, for all vertices $u \in V(\widetilde{\Gamma}_j)$ adjacent to $\pi_j(v) \in V(\widetilde{\Gamma}_j)$, $e' = (\pi_j(v), u) \in E(\widetilde{\Gamma}_j)$, where $e_1, \ldots, e_k \in E(\mathfrak{D})$ are all the edges in the preimage $\pi_j^{-1}(e')$ in $\mathfrak{D}$ which begin in v, and $j, l, m \in \{1, 2\}$.

Note that

$$(S_1(\Gamma_1, \Gamma_2) \ \vee \ S_2(\Gamma_1, \Gamma_2)) \ \wedge \ P \ \wedge \ E$$

is a disjunction of linear system of equations and inequalities in variables $M_{kl}(e)$, which we denote by $S(\Gamma_1, \Gamma_2)$.

COROLLARY 2·14. *In the above notation, if $\mathbb{G}(\Gamma_1)$ and $\mathbb{G}(\Gamma_2)$ are commensurable, then the (disjunction of) linear system of equations and inequalities $S(\Gamma_1, \Gamma_2)$ has an integer solution.*

Proof. It follows from Lemmas 2·11 and 2·12 that the labels assigned to vertices and edges of $\Psi(H)$ give rise to a solution of the linear system of equations S_i for some $i = 1, 2$. By Lemma 2·9 the solutions are non-negative and so satisfy the inequalities of the system P. Furthermore, by Lemma 2·8, the labels satisfy the disjunctions of inequalities encoding local surjectivity and so are a solution of the system E.
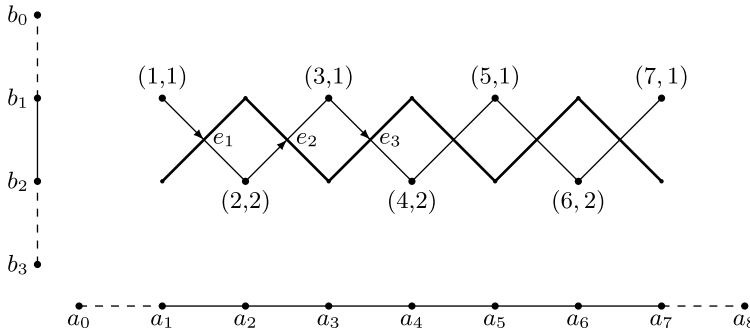
Fig. 2. Graph $\mathfrak{D}$ in the proof of Theorem 3·1 for $m = 8$. Edges of $\mathfrak{D}_1$ are thin and edges of $\mathfrak{D}_2$ are thick.

3. *The RAAGs $\mathbb{G}(P_3)$ and $\mathbb{G}(P_m)$ are not commensurable for $m \geq 5$*

In this section, we prove a special case of Theorem 1·2, namely we show that $\mathbb{G}(P_3)$ is not commensurable to $\mathbb{G}(P_n)$, for all $n > 4$. As we already mentioned, $\mathbb{G}(P_3)$ and $\mathbb{G}(P_4)$ are commensurable, see [**CKZ**] for more details on that.

The proof in this special case is easier and introduces the reader to the techniques and ideas behind the proof of Theorem 1·2. The proof of Theorem 1·2 is the most technically demanding in this paper and we will address it in the final section, Section 6.

THEOREM 3·1. $\mathbb{G}(P_3)$ *is not commensurable to $\mathbb{G}(P_m)$ for $m \geq 5$.*

Proof. Let $a_0, a_1, \ldots, a_m$ be the vertices of P_m, considered as canonical generators of $\mathbb{G}(P_m)$, and b_0, b_1, b_2, b_3 be the vertices of P_3, considered as canonical generators of $\mathbb{G}(P_3)$. Then, in the above notation, $\Gamma_1 = P_m$, $\Gamma_2 = P_3$, and $\widetilde{\Gamma}_1 = P_{m-2}$, with vertices $a_1, \ldots, a_{m-1}$, $\widetilde{\Gamma}_2 = P_1$, with vertices b_1, b_2.

Suppose that $\mathbb{G}(P_m)$ and $\mathbb{G}(P_3)$ are commensurable. Note that in our case $\mathfrak{D} = \widetilde{\Gamma}_1 \times \widetilde{\Gamma}_2 = P_{m-2} \times P_1$ is the following graph: its set of vertices is $\{(a_i, b_j), i = 1, \ldots, m-1; j = 1, \ldots, 2\}$, and two vertices (a_{i_1}, b_{j_1}) and (a_{i_2}, b_{j_2}) are connected by an edge in $\mathfrak{D}$ if and only if $|i_1 - i_2| = 1$ and $|j_1 - j_2| = 1$, for $i_1, i_2 = 1, \ldots, m-1$; $j_1, j_2 = 1, 2$. To abbreviate the notation, we will denote the vertex (a_i, b_j) of $\mathfrak{D}$ by (i, j), where $i = 1, \ldots, m-1$; $j = 1, 2$, see Figure 2.

Note that, as in Lemma 2·8, $\mathfrak{D}$ has two connected components, one of them, denoted by $\mathfrak{D}_1$, consisting of vertices (i, j), where $i + j$ is even, and the other one, denoted by $\mathfrak{D}_2$, where $i + j$ is odd, and $\mathfrak{C}$ lies in one of them. The automorphism of P_3 which reverses the order of its vertices (this also induces an automorphism of $\mathbb{G}(P_3)$) switches these components, which are isomorphic graphs, so, after applying this automorphism of $\mathbb{G}(P_3)$ if necessary, without loss of generality, we assume that $\mathfrak{C}$ lies in a particular component of $\mathfrak{D}$.

So we assume that $\mathfrak{C}$ lies in the component $\mathfrak{D}_1$ containing the vertex $(1, 1)$. Then $\mathfrak{D}_1$ contains vertices $(1, 1), (2, 2), (3, 1), (4, 2), \ldots, (m-1, n_0)$, where $n_0 = 1$ if m is even, and $n_0 = 2$ if m is odd, and all the connecting edges (as a graph, $\mathfrak{D}_1$ is isomorphic to the path of length $m - 2$). Note also that, by Lemma 2·8, $\mathfrak{C}$ should project surjectively on $\widetilde{\Gamma}_1$, so in fact $\mathfrak{C} = \mathfrak{D}_1$ in this case.

Denote by e_1 the (oriented) edge of $\mathfrak{D}$ beginning in $(1, 1)$ and ending in $(2, 2)$, by e_2 the edge beginning in $(2, 2)$ and ending in $(3, 1)$, and by e_3 the edge beginning in $(3, 1)$ and ending in $(4, 2)$ (note that such edges always exist, since $m \geq 5$, so $m - 1 \geq 4$), see Figure 2.

Note that we have the following equations on the labels of e_1, e_2, e_3: edge equations as in Lemma 2·11, and vertex equations which follow from Lemma 2·12 applied to the case under consideration.

From the $(1, 1)$ vertex we get

$$M_{11}(e_1) = M_{12}(e_1), \ M_{21}(e_1) = M_{22}(e_1);$$

from the $(2, 2)$ vertex we get

$$M_{11}(e_1^{-1}) = M_{11}(e_2) = M_{12}(e_1^{-1}) + M_{12}(e_2), \ M_{21}(e_1^{-1}) = M_{21}(e_2) = M_{22}(e_1^{-1}) + M_{22}(e_2);$$

and from the $(3, 1)$ vertex we get

$$M_{11}(e_2^{-1}) = M_{11}(e_3) = M_{12}(e_2^{-1}) + M_{12}(e_3), \ M_{21}(e_2^{-1}) = M_{21}(e_3) = M_{22}(e_2^{-1}) + M_{22}(e_3).$$

Using these equations, on the one hand we can write

$$M_{11}(e_1) = M_{12}(e_1) = M_{21}(e_1^{-1}) = M_{22}(e_1^{-1}) + M_{22}(e_2) = M_{22}(e_1) + M_{22}(e_2^{-1}), \quad (3\cdot1)$$

and on the other hand

$$\begin{aligned} M_{11}(e_1) = M_{11}(e_1^{-1}) = M_{12}(e_1^{-1}) + M_{12}(e_2) = M_{21}(e_1) + M_{21}(e_2^{-1}) = \\ = M_{22}(e_1) + M_{21}(e_2^{-1}) = M_{22}(e_1) + M_{22}(e_2^{-1}) + M_{22}(e_3). \end{aligned} \quad (3\cdot2)$$

Comparing (3·1) and (3·2), we see that $M_{22}(e_3) = 0$, so by (2·4) we have $e_3 \notin \mathfrak{C}$, which is a contradiction, since $\mathfrak{C} = \mathfrak{D}_1$ as mentioned above. This shows that $\mathbb{G}(P_3)$ is not commensurable to $\mathbb{G}(P_m)$ for $m \geq 5$.

COROLLARY 3·2. $\mathbb{G}(P_4)$ *is not commensurable to* $\mathbb{G}(P_m)$ *for* $m \geq 5$.

Proof. Follows immediately from Theorem 3·1 and the fact that $\mathbb{G}(P_3)$ and $\mathbb{G}(P_4)$ are commensurable, see [**CKZ**, proposition 4·4].

4. *Non-commensurability of RAAGs defined by paths and trees of diameter* 4

In this section, we address the commensurability relations between the classes of RAAGs defined by paths and trees of diameter 4. More precisely, we show that if $n > 5$ is not congruent to 2 modulo 4, then $\mathbb{G}(P_n)$ is not commensurable to any RAAG defined by a tree of diameter 4.

The general strategy to prove that two RAAGs defined by trees are not commensurable is common. First, we specialise the linear system of equations and inequalities given in Corollary 2·14 to the case under consideration. We then find a (local) pattern in the product graph and prove that the absence of certain set of edges S at a vertex implies the absence of other edges, see Lemma 4·3 and Figure 6. We then determine a vertex v for which the set of edges S is missing, see Lemma 4·5. This allows us to recursively remove edges using the identified local pattern and starting at v until we remove enough edges to contradict the local surjectivity at a vertex (assured by Lemma 2·8). This allows us to conclude that the system S does not have integer solutions and hence the RAAGs are not commensurable.

For our purposes, it will be convenient to encode finite trees of diameter four as follows. Let T be any finite tree of diameter four. Let f be a path (without backtracking) of length four from one leaf of T to another. By definition f contains 5 vertices and let $c_f \in V(T)$ be
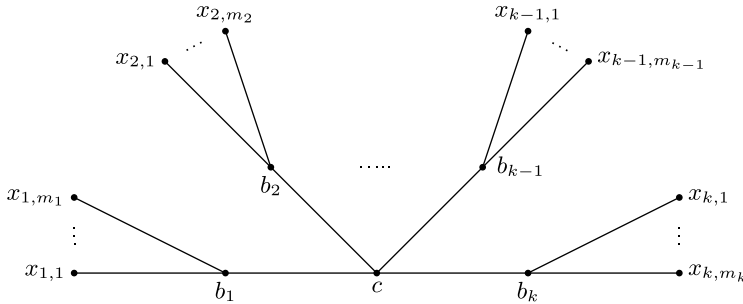
Fig. 3. Tree $T((m_k, 1), (m_{k-1}, 1), \ldots, (m_1, 1); 0)$ for $m_1 > m_2 > \cdots > m_{k-1} > m_k > 0$.

the middle vertex in f. It is immediate to see that the choice of the vertex $c = c_f$ does not depend on the choice of the path f of length four. We call c the *center* of T.

Any leaf of T connected to c by an edge is called a *hair vertex*. Vertices connected to c by an edge which are not hair are called *pivots*. Any finite tree T of diameter 4 is uniquely defined by the number q of hair vertices and by the number k_i of pivots of a given degree $d_i + 1$. Hence we encode any finite tree of diameter 4 as $T((d_1, k_1), \ldots, (d_l, k_l); q)$. Here all d_i and k_i and l are positive integers, $d_1 < d_2 < \cdots < d_l$, and q is a non-negative integer; moreover, either $l \geq 2$ or $l = 1$ and $k_1 \geq 2$, so that T indeed has diameter 4. See [**CKZ**] for more details.

THEOREM 4·1. *Let $m \geq 5$ and suppose that m is not 2 modulo 4. Then $\mathbb{G}(P_m)$ is not commensurable to any RAAG defined by a tree of diameter 4.*

Proof. Let T be a tree of diameter 4, and $m \geq 5$, m is not 2 modulo 4. We need to prove that $\mathbb{G}(P_m)$ is not commensurable to $\mathbb{G}(T)$. If $T = P_4$, then the claim follows from Corollary 3·2. Hence, without loss of generality, we can assume that $\mathbb{G}(T)$ is not commensurable to $\mathbb{G}(P_4)$. By [**CKZ**], this implies that $\mathbb{G}(T)$ is commensurable to $\mathbb{G}(T((m_k, 1), (m_{k-1}, 1), \ldots, (m_1, 1); 0))$ for some $m_1 > m_2 > \cdots > m_k \geq 1$. Here $T((m_k, 1), (m_{k-1}, 1), \ldots, (m_1, 1); 0)$ is the tree with central vertex c of degree k, connected to pivot vertices $b_1, \ldots, b_k$, and each b_i has degree $m_i + 1$ and is connected to m_i degree one vertices, see Figure 3.

Thus we can assume that $T = T((m_k, 1), (m_{k-1}, 1), \ldots, (m_1, 1); 0)$ for some $m_1 > m_2 > \cdots > m_k \geq 1$, $k \geq 2$. So, in the above notation, we have that $\Gamma_1 = P_m$, $m \geq 5$, with the vertices $a_0, a_1, \ldots, a_{m-1}, a_m$, and $\Gamma_2 = T$. It follows that $\widetilde{\Gamma}_1 = P_{m-2}$, with the vertices $a_1, \ldots, a_{m-1}$, and $\widetilde{\Gamma}_2 = \Delta$, where Δ is a tree of diameter 2 with central vertex c of degree k, connected to the degree one (in Δ) vertices $b_1, \ldots, b_k$. So $a_1, \ldots, a_{m-1}$ are those canonical generators of $\mathbb{G}(P_m)$ which have non-abelian centralisers, and $c, b_1, \ldots, b_k$ are those canonical generators of $\mathbb{G}(T)$ which have non-abelian centralisers.

We have that $\mathfrak{D} = P_{m-2} \times \Delta$ is a graph with vertices of the form (a_i, c) and (a_i, b_j), where $i = 1, \ldots, m-1$, $j = 1, \ldots, k$, and the following edges: for $i = 1, \ldots, m-2$ the vertex (a_i, c) is connected to the vertices (a_{i+1}, b_j), for all $j = 1, \ldots, k$, and for $i = 2, \ldots, m-1$ the vertex (a_i, c) is connected to the vertices (a_{i-1}, b_j), for all $j = 1, \ldots, k$. We will denote the vertex (a_i, c) by (i, c), and the vertex (a_i, b_j) by (i, j) for shortness, for $i = 1, \ldots, m-1$, $j = 1, \ldots, k$, see Figure 4.

By Lemma 2·8, the graph $\mathfrak{D}$ has two connected components, and $\mathfrak{C}$ is a connected subgraph of one of them. Denote the one which contains the vertex $(1, c)$ by $\mathfrak{D}_1$ and the other
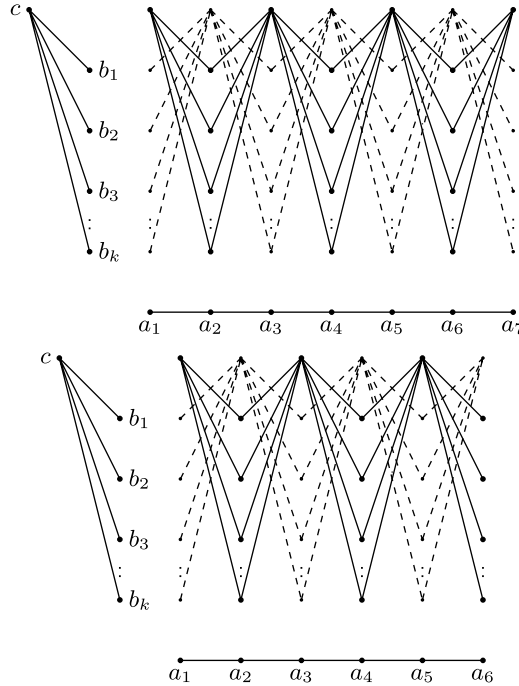
Fig. 4. Graph $\mathfrak{D}$ in the proof of Theorem 4·1 for $m = 8$ (on the left) and $m = 7$ (on the right). Edges of $\mathfrak{D}_1$ are solid and edges of $\mathfrak{D}_2$ are dashed.

one by $\mathfrak{D}_2$. Then $\mathfrak{D}_1$ contains all the vertices of the form (i, c), where i is odd, $1 \leq i \leq m - 1$, and (i', j), where i' is even, $1 \leq i' \leq m - 1$, $j = 1, \ldots, k$, as well as all the incident edges, and $\mathfrak{D}_2$ contains all the vertices of the form (i, c), where i is even, $1 \leq i \leq m - 1$, and (i', j), where i' is odd, $1 \leq i' \leq m - 1$, $j = 1, \ldots, k$, as well as all the incident edges.

Remark 4·2. Note that if m is odd, then the automorphism of P_m which reverses the order of its vertices (it also induces an automorphism of $\mathbb{G}(P_m)$) switches the components $\mathfrak{D}_1$ and $\mathfrak{D}_2$, which are in this case isomorphic graphs, so, after applying this automorphism of $\mathbb{G}(P_m)$ if necessary, without loss of generality, we can assume that $\mathfrak{C}$ lies in a particular component of $\mathfrak{D}$. However, if m is even (i.e., m is 0 modulo 4 in our case), then $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are not isomorphic graphs, and we should consider two cases, depending on whether $\mathfrak{C}$ lies in $\mathfrak{D}_1$ or $\mathfrak{D}_2$, see Figure 4.

Denote by $e_{l,c}^{i,j}$ the (oriented) edge of $\mathfrak{D}$ beginning in the vertex (l, c) and ending in the vertex (i, j), for all $i, l = 1, \ldots, m$, such that $|i - l| = 1$, and $j = 1, \ldots, k$. This is an edge of $\mathfrak{D}_1$ if l is odd and i is even, and of $\mathfrak{D}_2$ if l is even and i is odd. Denote also by $e_{i,j}^{l,c} = (e_{l,c}^{i,j})^{-1}$ the inverse edge, beginning in (i, j) and ending in (l, c).

Note that, in the notation of Lemma 2·12, we have $D_1 = 1$ for all vertices of $\mathfrak{D}$, and $D_2 = k - 1$ for the vertices (i, c) of $\mathfrak{D}$, $i = 1, \ldots, m - 1$, and $D_2 = m_j - 1$ for the vertices (i, j) of $\mathfrak{D}$, $i = 1, \ldots, m - 1$, $j = 1, \ldots, k$.

In our case the equations of Lemma 2·12 and Equation (2·8) have the following form. For a vertex $w = (1, j)$, where $j = 1, \ldots, k$, which has degree 1, we have

$$R_1(w) = M_{11}(e_{1,j}^{2,c}) = m_j M_{12}(e_{1,j}^{2,c}), \quad R_2(w) = M_{21}(e_{1,j}^{2,c}) = m_j M_{22}(e_{1,j}^{2,c}).$$
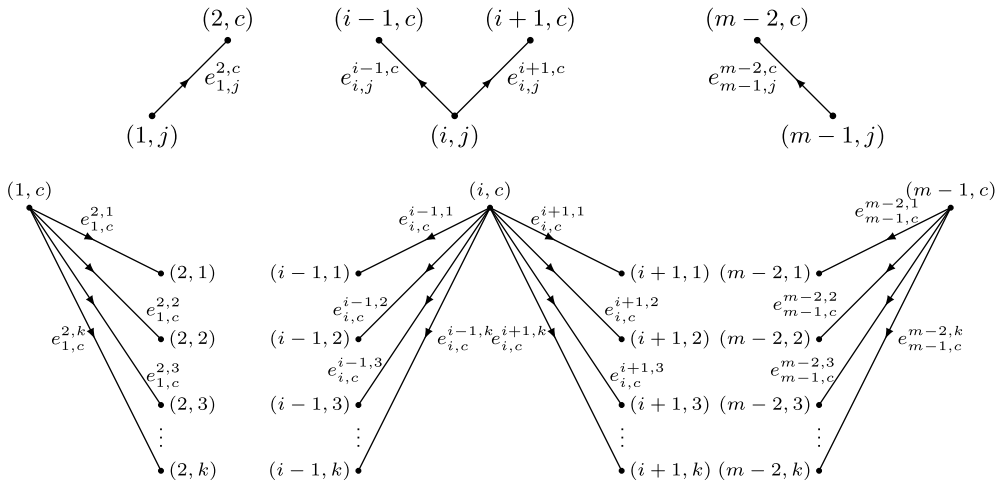
Fig. 5. Vertex stars of the graph $\mathfrak{D}$ in the proof of Theorem 4·1.

For a vertex $w = (i, j)$ of $\mathfrak{D}$, where $1 < i < m - 1$, $j = 1, \ldots, k$, which has degree 2, we have

$$R_1(w) = M_{11}(e_{i,j}^{i-1,c}) = M_{11}(e_{i,j}^{i+1,c}) = m_j(M_{12}(e_{i,j}^{i-1,c}) + M_{12}(e_{i,j}^{i+1,c})),$$
$$R_2(w) = M_{21}(e_{i,j}^{i-1,c}) = M_{21}(e_{i,j}^{i+1,c}) = m_j(M_{22}(e_{i,j}^{i-1,c}) + M_{22}(e_{i,j}^{i+1,c})).$$

For a vertex $w = (m - 1, j)$ of $\mathfrak{D}$, where $j = 1, \ldots, k$, which has degree 1, we have

$$R_1(w) = M_{11}(e_{m-1,j}^{m-2,c}) = m_j M_{12}(e_{m-1,j}^{m-2,c}), \quad R_2(w) = M_{21}(e_{m-1,j}^{m-2,c}) = m_j M_{22}(e_{m-1,j}^{m-2,c}).$$

We refer the reader to Figure 5 for notation.

For a vertex $w = (1, c)$ of $\mathfrak{D}$, which has degree k, we have

$$R_1(w) = \sum_{j=1}^{k} M_{11}(e_{1,c}^{2,j}) = (k-1)M_{12}(e_{1,c}^{2,j'}), \quad j' = 1, \ldots, k,$$
$$R_2(w) = \sum_{j=1}^{k} M_{21}(e_{1,c}^{2,j}) = (k-1)M_{22}(e_{1,c}^{2,j'}), \quad j' = 1, \ldots, k.$$

For a vertex $w = (i, c)$ of $\mathfrak{D}$, where $1 < i < m - 1$, which has degree $2k$, we have, for all $j' = 1, \ldots, k,$

$$R_1(w) = \sum_{j=1}^{k} M_{11}(e_{i,c}^{i-1,j}) = \sum_{j=1}^{k} M_{11}(e_{i,c}^{i+1,j}) = (k-1)(M_{12}(e_{i,c}^{i-1,j'}) + M_{12}(e_{i,c}^{i+1,j'})),$$

$$R_2(w) = \sum_{j=1}^{k} M_{21}(e_{i,c}^{i-1,j}) = \sum_{j=1}^{k} M_{21}(e_{i,c}^{i+1,j}) = (k-1)(M_{22}(e_{i,c}^{i-1,j'}) + M_{22}(e_{i,c}^{i+1,j'})).$$

Finally, for a vertex $w = (m - 1, c)$ of $\mathfrak{D}$, which has degree k, we have

$$R_1(w) = \sum_{j=1}^{k} M_{11}(e_{m-1,c}^{m-2,j}) = (k-1)M_{12}(e_{m-1,c}^{m-2,j'}), \quad j' = 1, \ldots, k,$$
$$R_2(w) = \sum_{j=1}^{k} M_{21}(e_{m-1,c}^{m-2,j}) = (k-1)M_{22}(e_{m-1,c}^{m-2,j'}), \quad j' = 1, \ldots, k.$$

Recall that $m_1 > m_2 > \cdots > m_k \geq 1$, $k \geq 2$ and $m \geq 5$. We will suppose that $s = 1, 2$ is such that $\mathfrak{C}$ lies in $\mathfrak{D}_s$.
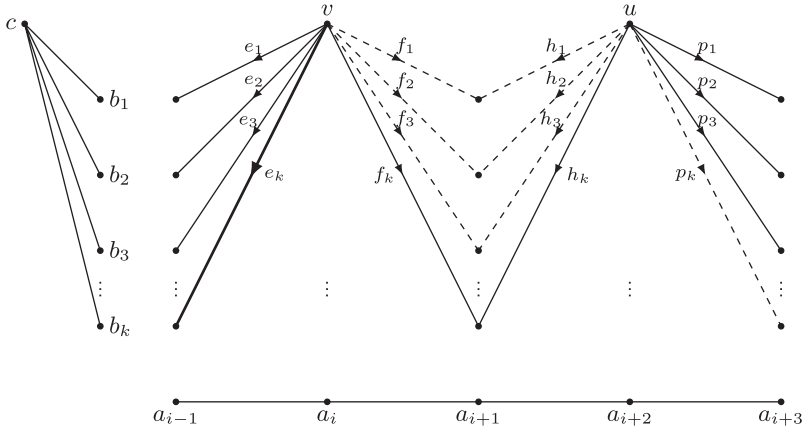
Fig. 6. Part of the graph $\mathfrak{D}$ in the proof of Lemma 4·3, in the case $2 \leq i \leq m - 4$. The thick edge is not in $\mathfrak{C}$ by assumptions of the lemma, and the dashed edges are claimed not to be in $\mathfrak{C}$ by the lemma.

LEMMA 4·3. *Let* $v = (i, c)$ *be a vertex of* $\mathfrak{D}_s$, *where* $1 \leq i \leq m - 2$. *Suppose that the edge* $e_{i,c}^{i-1,k}$ *is not in* $\mathfrak{C}$, *where* $i > 1$. *Then the edges* $e_{i,c}^{i+1,j'}$, $j' = 1, \ldots, k - 1$, *are not in* $\mathfrak{C}$.

If, in addition, $i \leq m - 3$, *then the edges* $e_{i+1,c}^{i+2,c}$ *are not in* $\mathfrak{C}$, *where* $j' = 1, \ldots, k - 1$. *Moreover, if* $i \leq m - 4$, *then also the edge* $e_{i+2,c}^{i+3,k}$ *is not in* $\mathfrak{C}$.

Proof. We will prove the lemma in the case $1 \leq i \leq m - 4$; the cases $i = m - 3$ and $i = m - 2$ are analogous. To abbreviate the notation, denote the edges $e_{i,c}^{i-1,j}$ by e_j (if $i \geq 2$), the edges $e_{i,c}^{i+1,j}$ by f_j, the edges $e_{i+2,c}^{i+2,j}$ by h_j, and the edges $e_{i+2,c}^{i+3,j}$ by p_j, for all $j = 1, \ldots, k$. So we know that $e_k \notin \mathfrak{C}$, see Figure 6.

By Lemmas 2·12 and 2·11 (applied to this case), we have

$$R_1(v) = \sum_{j=1}^k M_{11}(f_j) = \sum_{j=1}^k M_{11}(f_j^{-1}) = \sum_{j=1}^k m_j(M_{12}(f_j^{-1}) + M_{12}(h_j^{-1}))$$
$$= m_k \sum_{j=1}^k (M_{12}(f_j^{-1}) + M_{12}(h_j^{-1})) + D, \tag{4·1}$$

where

$$D = \sum_{j=1}^{k-1} (m_j - m_k)(M_{12}(f_j^{-1}) + M_{12}(h_j^{-1})).$$

Remark 4·4. Since $m_1 > m_2 > \cdots > m_{k-1} > m_k$, and M-labels are non-negative, we have that $D \geq 0$. Furthermore, $D = 0$ if and only if $M_{12}(f_j^{-1}) = M_{12}(h_j^{-1}) = 0$, for all $j = 1, \ldots, k - 1$, if and only if (by Equation (2·4)) $f_j, h_j \notin \mathfrak{C}$, for all $j = 1, \ldots, k - 1$.

Denote the vertex $(i + 2, c)$ by u. We continue Equation (4·1),

$$R_1(v) = m_k \sum_{j=1}^k (M_{12}(f_j^{-1}) + M_{12}(h_j^{-1})) + D$$
$$= m_k \sum_{j=1}^k M_{21}(f_j) + m_k \sum_{j=1}^k M_{21}(h_j) + D$$

$$= m_k(R_2(v) + R_2(u)) + D$$
$$= m_k(k-1)(M_{22}(f_k) + M_{22}(h_k) + M_{22}(p_k)) + D, \tag{4·2}$$

where the last equality holds since $R_2(u) = (k-1)(M_{22}(h_k) + M_{22}(p_k))$; if $i = 1$, then $R_2(v) = (k-1)M_{22}(f_k)$, and if $i \geq 2$, then still $R_2(v) = (k-1)(M_{22}(e_k) + M_{22}(f_k)) = (k-1)M_{22}(f_k)$, because $e_k \notin \mathfrak{C}$.

In the same way, we have $R_1(v) = (k-1)M_{12}(f_k)$. So

$$R_1(v) = (k-1)M_{12}(f_k) = (k-1)M_{21}(f_k^{-1}) = m_k(k-1)(M_{22}(f_k^{-1}) + M_{22}(h_k^{-1}))$$
$$= m_k(k-1)(M_{22}(f_k) + M_{22}(h_k)). \tag{4·3}$$

Comparing Equations (4·2) and (4·3), we obtain that $m_k(k-1)M_{22}(p_k) + D = 0$, but both $M_{22}(p_k)$ and D are non-negative, so $M_{22}(p_k) = D = 0$. By Equation (2·4), this implies that $p_k \notin \mathfrak{C}$. By Remark 4·4, it follows that $f_j, h_j \notin \mathfrak{C}$, for all $j = 1, \ldots, k-1$. This is exactly the claim of the lemma.

Suppose first that m is odd. By Remark 4·2, we can suppose that $\mathfrak{C}$ lies inside $\mathfrak{D}_1$, which is the connected component of $\mathfrak{D}$ containing $(1, c)$. Then the vertex $v = (1, c)$ satisfies the conditions of Lemma 4·3, so by this lemma the edge $e_{1,c}^{2,1}$ is not in $\mathfrak{C}$. This is a contradiction with Lemma 2·8, since local surjectivity at v of the projection π_2 does not hold. Thus, for odd m, the group $\mathbb{G}(P_m)$ is not commensurable with a RAAG defined by a tree of diameter 4.

So we can suppose that m is 0 modulo 4, in particular, $m \geq 8$. There are two cases — either $\mathfrak{C}$ lies inside $\mathfrak{D}_1$, or inside $\mathfrak{D}_2$. If $\mathfrak{C}$ lies inside $\mathfrak{D}_1$, then again applying Lemma 4·3 at the vertex $v = (1, c)$ results in a contradiction, as in the case of odd m. Thus, we can assume that $\mathfrak{C}$ lies inside $\mathfrak{D}_2$. Note that the vertices $(2, c)$ and $(m-2, c)$ belong to $\mathfrak{D}_2$.

LEMMA 4·5. *If $m \equiv 0 \pmod 4$ and $\mathfrak{C}$ lies inside $\mathfrak{D}_2$, then the edges $e_{2,c}^{1,j}$ and $e_{m-2,c}^{m-1,j}$, are not in $\mathfrak{C}$, where $j = 2, \ldots, k$.*

Proof. The proof is similar to that of Lemma 4·3. By symmetry, it suffices to prove that the edges $e_{2,c}^{1,j}$, for $j = 2, \ldots, k$, are not in $\mathfrak{C}$. To abbreviate the notation, denote the edges $e_{2,c}^{1,j}$ by e_j, the edges $e_{2,c}^{3,j}$ by f_j, and the edges $e_{4,c}^{3,j}$ by h_j, $j = 1, \ldots, k$. Denote the vertex $(2, c)$ by v, see Figure 7.

Note that, by Lemma 2·12, we have that $M_{11}(e_j^{-1}) = m_j M_{12}(e_j^{-1})$ and $M_{21}(e_j^{-1}) = m_j M_{22}(e_j^{-1})$ for all $j = 1, \ldots, k$. Then, we have

$$R_1(v) = \sum_{j=1}^{k} M_{11}(e_j) = \sum_{j=1}^{k} M_{11}(e_j^{-1}) = \sum_{j=1}^{k} m_j M_{12}(e_j^{-1}) = m_1 \sum_{j=1}^{k} M_{12}(e_j^{-1}) - C, \quad (4·4)$$

where

$$C = \sum_{j=2}^{k} (m_1 - m_j) M_{12}(e_j^{-1}).$$

Since $m_1 > m_2 > \cdots > m_{k-1} > m_k$, and M-labels are non-negative, we have $C \geq 0$, and $C = 0$ if and only if $M_{12}(e_j^{-1}) = 0$, for all $j = 2, \ldots, k$, if and only if (by Equation (2·4)) $e_j \notin \mathfrak{C}$, for all $j = 2, \ldots, k$.
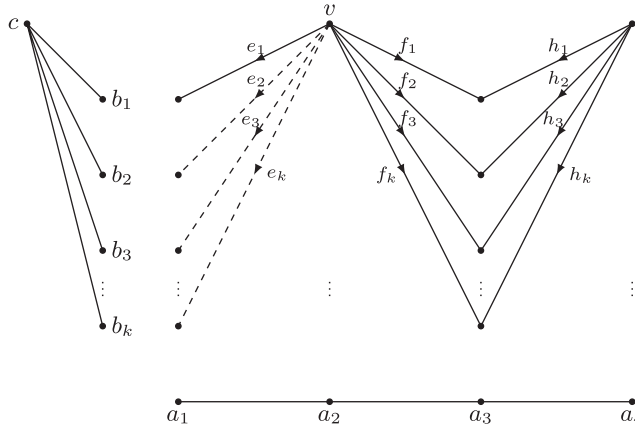
Fig. 7. Part of the graph $\mathfrak{D}_2$ in the proof of Lemma 4·5. The dashed edges are claimed not to be in $\mathfrak{C}$ by the lemma.
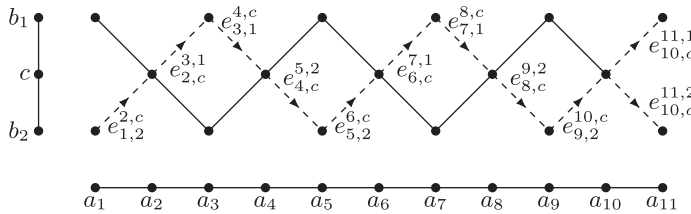

Fig. 8. The graph $\mathfrak{D}_2$ for $m = 12$ in the end of the proof of Theorem 4·1. The dashed edges are proved not to be in $\mathfrak{C}$, which leads to a contradiction.

We continue Equation (4·4),

$$R_1(v) = m_1 \sum_{j=1}^{k} M_{12}(e_j^{-1}) - C = m_1 \sum_{j=1}^{k} M_{21}(e_j) - C = m_1 R_2(v) - C \tag{4·5}$$
$$= m_1(k-1)(M_{22}(e_1) + M_{22}(f_1)) - C.$$

On the other hand, we have

$$R_1(v) = (k-1)(M_{12}(e_1) + M_{12}(f_1)) = (k-1)(M_{21}(e_1^{-1}) + M_{21}(f_1^{-1}))$$
$$= m_1(k-1)(M_{22}(e_1^{-1}) + M_{22}(f_1^{-1}) + M_{22}(h_1^{-1})) \tag{4·6}$$
$$= m_1(k-1)(M_{22}(e_1) + M_{22}(f_1) + M_{22}(h_1)),$$

since $M_{21}(f_1^{-1}) = M_{22}(f_1^{-1}) + M_{22}(h_1^{-1})$ by Lemma 2·12.

Comparing (4·5) and (4·6), we obtain that $C + m_1(k-1)M_{22}(h_1) = 0$, but both C and $M_{22}(h_1)$ are non-negative, so $C = M_{22}(h_1) = 0$. This means that $h_1 \notin \mathfrak{C}$, but also, by Remark 4·4, that $e_j \notin \mathfrak{C}$ for $j = 2, \ldots, k$. This is exactly the claim of the lemma.

We now turn to the proof of Theorem 4·1. As we already mentioned, we use Lemma 4·5 to deduce the absence of certain edges and Lemma 4·3 to recursively remove other edges. We combine these two lemmas until we assure that there is no local surjectivity contradicting Lemma 2·8, see Figure 8.
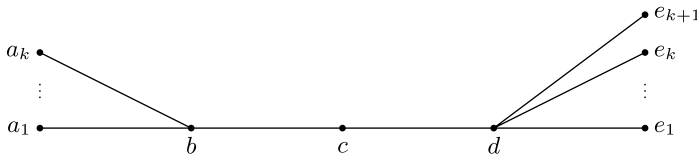
Fig. 9. The tree $T_{k,k+1}$.

By Lemma 4·5, we see that the edges $e_{2,c}^{1,j}$ are not in $\mathfrak{C}$ for $j = 2, \ldots, k$. This means that we can apply Lemma 4·3 to the vertex $v = (2, c)$ of $\mathfrak{D}_2$, and we conclude that the edges $e_{2,c}^{3,j'}$, for $j' = 1, \ldots, k-1$, are not in $\mathfrak{C}$. If $k \geq 3$, then we get a contradiction with Lemma 2·8: local surjectivity at v of the projection π_2 does not hold, since neither $e_{2,c}^{1,2}$ nor $e_{2,c}^{3,2}$ are in $\mathfrak{C}$. Thus we can suppose that $k = 2$.

We know that $e_{1,2}^{2,c} \notin \mathfrak{C}$, and by Lemma 4·3 the edges $e_{2,c}^{3,1}$, $e_{3,1}^{4,c}$ and $e_{4,c}^{5,2}$ are not in $\mathfrak{C}$. By Lemma 2·8 applied at the vertex $(5, 2)$, we also see that $e_{5,2}^{6,c} \notin \mathfrak{C}$. We claim that the edges $e_{4a-2,c}^{4a-1,1}$, $e_{4a-1,1}^{4a,c}$, $e_{4a,c}^{4a+1,2}$, $e_{4a+1,2}^{4a+2,c}$ are not in $\mathfrak{C}$, for $a = 1, \ldots, m/4 - 1$. We prove it by induction on a. Note that the claim holds for $a = 1$, as proved above. Suppose it holds for $a \leq a'$, $1 \leq a' \leq m/4 - 2$, and we prove it for $a = a' + 1$. Since $e_{4a'+2,c}^{4a'+1,2} \notin \mathfrak{C}$ by induction hypothesis for $a = a'$, by Lemma 4·3, applied at the vertex $(4a' + 2, c)$, we see that $e_{4a'+2,c}^{4a'+3,1}$, $e_{4a'+3,1}^{4a'+4,c}$, $e_{4a'+4,c}^{4a'+5,2}$, $\notin \mathfrak{C}$. Finally, by Lemma 2·8, we get $e_{4a'+5,2}^{4a'+6,c} \notin \mathfrak{C}$, and this proves the claim.

It follows from the claim with $a = m/4 - 1$ that $e_{m-3,2}^{m-2,c} \notin \mathfrak{C}$. By Lemma 4·3, applied at the vertex $(m - 2, c)$, we have that $e_{m-2,c}^{m-1,1} \notin \mathfrak{C}$. But by Lemma 4·5, $e_{m-2,c}^{m-1,2} \notin \mathfrak{C}$, and this is a contradiction with Lemma 2·8, see Figure 8.

Thus, if m is not 2 modulo 4, then $\mathbb{G}(P_m)$ is not commensurable to a tree of diameter 4. This proves the theorem.

Note that in this way we were able to show that for P_{4k+2} and $T_{k,k+1}$ the system has positive integer solutions and calculate these solutions, and this gave us a hint on how to construct the corresponding isomorphic finite index subgroups given in Section 5.

5. *Commensurability of RAAGs defined by paths and trees of diameter* 4

In this section, we characterise when a RAAG defined by a path P_n is commensurable to a RAAG defined by a tree of diameter 4. In Section 4, we have seen that a necessary condition for commensurability is that $n \equiv 2 \pmod 4$. In this section, we show that this is a sufficient condition.

Recall that by $T_{k,k+1}$ we denote a tree of diameter 4, with the central vertex of degree 2 and so that the two vertices adjacent to the central vertex have degrees $k + 1$ and $k + 2$ correspondingly, $T_{k,k+1}$ has $2k + 1$ leaves, see Figure 9.

THEOREM 5·1. *Let $k \geq 1$. Then $\mathbb{G}(P_{4k+2})$ is commensurable to $\mathbb{G}(T_{k,k+1})$.*

Note that Theorem 5·1 together with Theorem 4·1 immediately imply Theorem 1·3.

The remaining part of this section will be devoted to the proof of Theorem 5·1. In order to do so, we first define an abstract group as a fundamental group of a certain graph of groups X; we then exhibit finite index subgroups H and K of $\mathbb{G}(T_{k,k+1})$ and $\mathbb{G}(P_{4k+2})$ respectively and show that they are isomorphic to that abstract group. We define the subgroups by
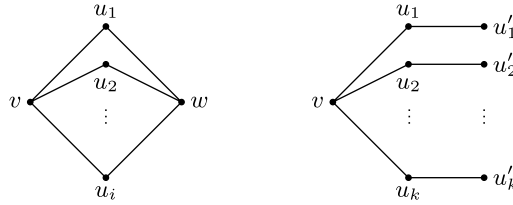
Fig. 10. The "diamond" graph D_i, $1 \leq i \leq k - 1$ (on the left) and the graph D_k (on the right).

describing them as fundamental groups of finite covers of the Salvetti complexes of the corresponding RAAGs. We divide the proof into five subsections. In Section 5·1 we construct X, in Section 5·2 we construct H, in Section 5·3 we prove that H is isomorphic to $\pi_1(X)$, in Section 5·4 we construct K, and in in Section 5·5 we prove that K is isomorphic to $\pi_1(X)$.

Throughout this section we always denote the conjugation as follows: $g^h = hgh^{-1}$. Also, in a group G, we denote the centraliser of an element g in G by $C(g)$, and the centraliser of g in a subgroup H of G by $C_H(g)$. We will occasionally use this notation even in the case when g is not in H (for some finite index subgroup H), and in this case, since centralizers in RAAGs are isolated (see [CKZ]), we have $C_H(g) = H \cap C(g) = C_H(g^n)$, where n is the minimal positive integer such that $g^n \in H$.

We will use basic facts from Bass-Serre theory, the reader is referred to [Serre] for details.

5·1. *Construction of the graph of groups X*

Recall that by $F(A)$ we mean the free group on A. We begin by defining a graph of groups X. The graph of groups X is built from some simpler pieces D_i, which are also graphs of groups. We begin by describing these pieces.

Let D_i, $i = 1, \ldots, k - 1$ be the "diamond" graph with $i + 2$ vertices, namely $v, w, u_1, \ldots, u_i$, where the vertices $u_1, \ldots, u_i$ have degree 2 and each of the vertices u_i is adjacent to the two vertices v and w of degree i, see Figure 10.

Let the vertex group at u_j be

$$V(u_j) = \langle r_j \rangle \times F(f_j, g_j).$$

Let the vertex groups at v and w be as follows,

$$V(v) = \langle s \rangle \times F(x_1, \ldots, x_{k^2}, y_1, \ldots, y_{k+1-i}, z_1, \ldots, z_i);$$
$$V(w) = \langle s' \rangle \times F(x'_1, \ldots, x'_{k^2+1}, y'_1, \ldots, y'_{k-i}, z'_1, \ldots, z'_i).$$

All edge groups are isomorphic to $\mathbb{Z}^2 = \langle p, q \rangle$.

The embedding of $\mathbb{Z}^2$ of the edge (v, u_j) into $V(v)$, is defined by the map

$$p \longmapsto s, \ q \longmapsto z_j,$$

where $j = 1, \ldots, i$. The embedding of $\mathbb{Z}^2$ of the edge (v, u_j) into $V(u_j)$, is defined by the map

$$p \longmapsto f_j, \ q \longmapsto r_j,$$

where $j = 1, \ldots, i$. The embedding of $\mathbb{Z}^2$ of the edge (w, u_j) into $V(u_j)$, is defined by the map

$$p \longmapsto g_j, \ q \longmapsto r_j,$$

where $j = 1, \ldots, i$. The embedding of $\mathbb{Z}^2$ of the edge (w, u_j) into $V(w_j)$, is defined by the map

$$p \longmapsto s', \ q \longmapsto z'_j,$$

where $j = 1, \ldots, i$.

Let D_k be the graph with $2k + 1$ vertices defined as follows: it has 1 vertex v of degree k adjacent to k vertices $u_1, \ldots, u_k$ of degree 2; each vertex u_i is adjacent to a vertex u'_i of degree 1, $i = 1, \ldots, k$, see Figure 10.

The vertex group $V(v)$ at v is defined to be

$$V(v) = \langle s \rangle \times F(x_1, \ldots, x_{k^2}, z_1, \ldots, z_k, y_1).$$

Let the vertex groups at u_i and u'_i be as follows,

$$V(u_j) = \langle r_j \rangle \times F(f_j, g_j); \qquad V(u'_j) = \langle r'_j \rangle \times F(x'_{j1}, \ldots, x'_{j,k+1}, z'_j).$$

All edge groups are isomorphic to $\mathbb{Z}^2 = \langle p, q \rangle$. The embedding of $\mathbb{Z}^2$ of the edge (v, u_j) into $V(v)$, is defined by the map

$$p \longmapsto s, \ q \longmapsto z_j, \ \text{where } j = 1, \ldots, k.$$

The embedding of $\mathbb{Z}^2$ of the edge (v, u_j) into $V(u_j)$, is defined by the map

$$p \longmapsto f_j, \ q \longmapsto r_j, \ \text{where } j = 1, \ldots, k.$$

The embedding of $\mathbb{Z}^2$ of the edge (u_j, u'_j) into $V(u_j)$, is defined by the map

$$p \longmapsto r_j, \ q \longmapsto g_j, \ \text{where } j = 1, \ldots, k.$$

The embedding of $\mathbb{Z}^2$ of the edge (u_j, u'_j) into $V(u'_j)$, is defined by the map

$$p \longmapsto z'_j, \ q \longmapsto r'_j, \ \text{where } j = 1, \ldots, k.$$

We now consider the graph of groups X obtained by identifying the vertices v and w in the graphs of groups D_i in the following sequence

$$(D_k, D_1, D_{k-1}, D_2, \ldots, D_{k-1}, D_1, D_k), \tag{5·1}$$

where D_k and D_1 are identified along v, D_1 and D_{k-1} are identified along w, D_{k-1} and D_2 are identified along v etc, see Figure 11. This defines the graph of groups X.

5·2. *Construction of the finite index subgroup H of $\mathbb{G}(T_{k,k+1})$*

Recall that the tree $T_{k,k+1}$ has a central vertex c of degree 2, two adjacent vertices b and d of degree $k + 1$ and $k + 2$ respectively and leaves a_i adjacent to b, $i = 1, \ldots, k$ and e_j adjacent to d, $j = 1, \ldots, k + 1$, see Figure 9.

The subgroup H is defined as the full preimage under the natural epimorphism $\pi : \mathbb{G}(T_{k,k+1}) \to F(a_1, e_1)$ of a finite index subgroup H' of $F(a_1, e_1)$, that is $H := \pi^{-1}(H')$, where $H' <_{fi} F(a_1, e_1)$.

The subgroup H' is defined as the fundamental group of a finite cover S of degree $k(k + 1)$ of the bouquet of two circles. The aforementioned cover is defined as follows.
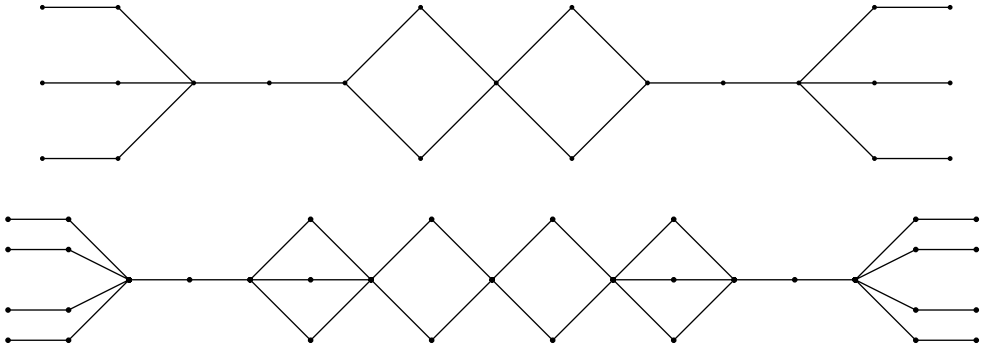
Fig. 11. The underlying graph for the graph of groups X in the cases $k = 3$ (above) and $k = 4$ (below).

Let P_i be $k + 1$-cycles labelled by a_1, $i = 1, \ldots, k$, and Q_i be k-cycles labelled by e_1, $i = 1, \ldots, k - 1$. We now glue these cycles according to the following pattern:

(i)   identify P_1 and Q_1 by 1 vertex, the basepoint;
(ii)   attach P_2 onto Q_1 by identifying $k - 1$ vertices;
(iii)   attach Q_2 onto P_2 (attached in the previous step) by identifying 2 (consecutive) vertices;
(iv)   attach P_3 onto Q_2 (attached in the previous step) by identifying $k - 2$ (consecutive) vertices;
(v)   …
(vi)   attach Q_{k-1} onto P_{k-1} (attached in the previous step) by identifying $k - 1$ vertices;
(vii)   attach P_k onto Q_{k-1} (attached in the previous step) by identifying 1 vertex.
(viii)   add loops labelled by e_1 at all the k vertices of P_1 which are not the basepoint (and so are not on Q_1), and similar for all the k vertices of P_k which are not on Q_{k-1}.

The attachments are always performed in such a way that the vertices in the intersection of Q_i and P_i appear in those cycles in the opposite order, and similar for the intersection of P_i and Q_{i-1}, for all $i = 2, \ldots, k - 1$. This defines S, and so also H.

By construction, the cycles P_l and P_m, $l \neq m$, Q_l and Q_m, $l \neq m$, P_l and Q_m, $l \neq m + 1$, $l \neq m$, do not share any vertices. We refer the reader to Figure 12 for the construction of the cover S in the cases $k = 3$ and $k = 4$.

Note that the vertices of S are in one-to-one correspondence with the right cosets of H' as follows: if g is the label of any path from the basepoint to a vertex v in S, then v corresponds to $H'g$. In other terms, S is the Schreier graph of H' in $F(a_1, e_1)$.

The group H' is a subgroup of $F(a_1, e_1)$ of index $k(k + 1)$, since it is defined by a finite cover of degree $k(k + 1)$. The group $\mathbb{G}(T_{k,k+1})$ retracts onto $F(a_1, e_1)$ and so the full preimage H of H' in $\mathbb{G}(T_{k,k+1})$ is a subgroup of index $k(k + 1)$.

5·3. *Isomorphism between H and $\pi_1(X)$*

The finite index subgroup H of $\mathbb{G}(T_{k,k+1})$ acts on the Bass–Serre tree T of the reduced centraliser splitting of $\mathbb{G}(T_{k,k+1})$ and so H has an induced graph of groups structure determined by the quotient of T by the action of H. In this subsection we prove the following.
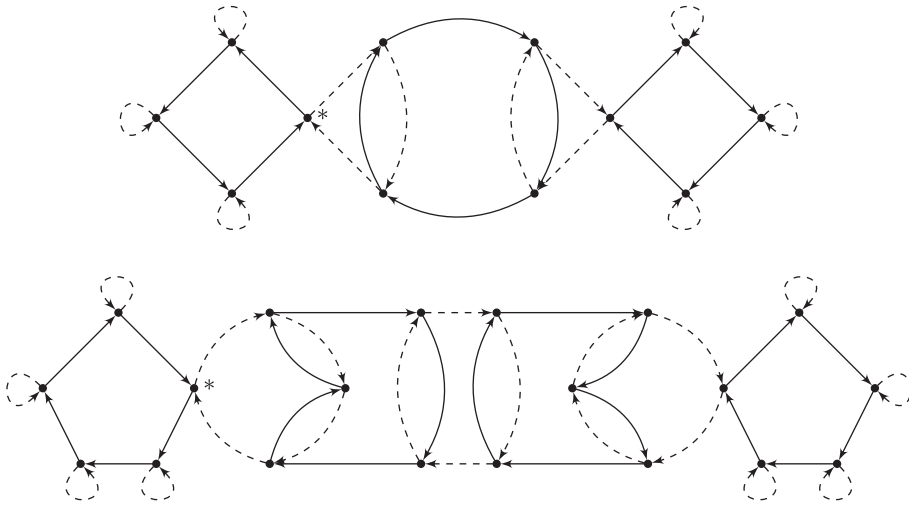
Fig. 12. The cover S defining the subgroup H' of $F(a_1, e_1)$ in the cases $k = 3$ (above) and $k = 4$ (below). Solid edges are labelled by a_1, and dashed edges are labelled by e_1. The basepoint is marked by a star.

PROPOSITION 5·2. *In the above notation, the subgroup H is isomorphic to the fundamental group of the graph of groups X. Namely, the induced splitting of H given by its action on T is X.*

Proof. By Bass–Serre theory the vertices of T correspond to the left cosets of the vertex groups of the reduced centraliser splitting, i.e. left cosets of centralisers of b, c and d in $\mathbb{G}(P_{k,k+1})$, and the action of $\mathbb{G}(P_{k,k+1})$ on T is by left multiplication.

We proceed by describing the fundamental domain of the action of H on T.

Let $1 \leq i \leq k - 1$. Let $\widetilde{D}_i$ be the graph with $2i + 1$ vertices described as follows: it has 1 vertex v of degree i adjacent to vertices $v_1, \ldots, v_i$ of degree 2; each v_j is adjacent to a vertex v'_j of degree 1, $j = 1, \ldots, i$.

We now consider the subtree of the Bass–Serre tree T which is isomorphic to $\widetilde{D}_i$, with vertex v labelled by $e_1(a_1 e_1)^{i-2} C(b)$, v_j labelled by $e_1(a_1 e_1)^{i-2} a_1^j C(c)$, v'_j labelled by $e_1(a_1 e_1)^{i-2} a_1^j C(d) = (e_1 a_1)^{i-1} a_1^{j-1} C(d)$, $j = 1, \ldots, i$. We now remove the vertices v'_j, $j = 2, \ldots, i$. The obtained subtree D'_i (without the removed vertices; no edges are removed) corresponds to a lift of D_i, where D_i is the $2i$th member of Sequence (5·1), for $1 \leq i \leq k - 1$. Note that, strictly speaking, D'_i is not a subtree, since for some of its edges one of the ends is not in D'_i, but, abusing the terminology, we will call it a subtree; the same observation applies for similar constructions below.

Similarly, consider a subtree of the Bass–Serre tree which is isomorphic to $\widetilde{D}_{k-i}$, with vertex v labelled by $(e_1 a_1)^{i-1} C(d)$, v_j labelled by $(e_1 a_1)^{i-1} e_1^j C(c)$, v'_j labelled by $(e_1 a_1)^{i-1} e_1^j C(b)$, $j = 1, \ldots, k - i$. We now remove the vertices v'_j, $j = 2, \ldots, k - i$. The obtained subtree D''_{k-i} (without the removed vertices; no edges are removed) corresponds to a lift of D_{k-i}, where D_{k-i} is the $(2i + 1)$th member of Sequence (5·1).

Let also D'_k be the subtree of the Bass-Serre tree T which is isomorphic to $\widetilde{D}_k$, with vertex v labelled by $C(b)$, v_j labelled by $a_1^j C(c)$, v'_j labelled by $a_1^j C(d)$, $j = 1, \ldots, k$. Note that no vertices are removed in this case. The subtree D'_k corresponds to a lift of D_k, which is the first member of Sequence (5·1).
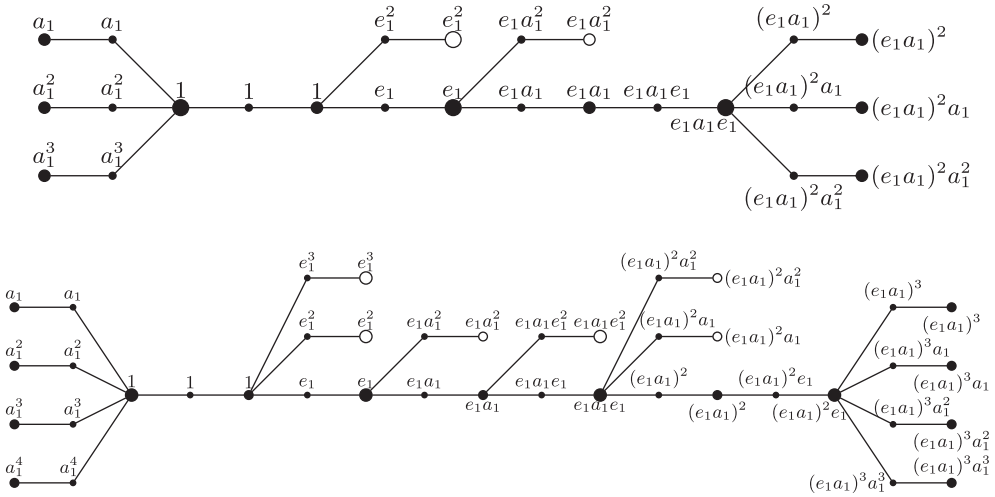
Fig. 13. The fundamental domain Y for the action of H on T in the case $k = 3$ (above) and $k = 4$ (below). Only the vertices that are denoted by disks belong to Y, while those denoted by circles do not. Small nodes correspond to the cosets of $C(c)$, huge nodes – to the cosets of $C(b)$, and middle-size nodes – to the cosets of $C(d)$. Representatives of the corresponding cosets are written next to the vertices.

Similarly, let D_k'' be the subtree of the Bass–Serre tree T which is isomorphic to $\widetilde{D}_k$, with vertex v labelled by $e_1(a_1e_1)^{k-2}C(b)$, v_j labelled by $e_1(a_1e_1)^{k-2}a_1^jC(c)$, v_j' labelled by $e_1(a_1e_1)^{k-2}a_1^jC(d) = (e_1a_1)^{k-1}a_1^{j-1}C(d)$, $j = 1, \ldots, k$. Note that no vertices are removed in this case. The subtree D_k'' corresponds to a lift of D_k, which is the last member of Sequence (5·1).

One can readily check that the union Y of all the subtrees D_i' and D_{k-i}'' of T, for $1 \le i \le k-1$, together with D_k' and D_k'', is connected. See Figure 13 for Y in the cases $k = 3$ and $k = 4$.

We need a few additional lemmas.

LEMMA 5·3. *There is a one-to-one correspondence θ_c between the vertices of Y which are the left cosets of $C(c)$ and the vertices in the cover S defining H', i.e. all the right cosets of H'.*

Under θ_c a coset $gC(c)$ representing a vertex of Y is mapped to the vertex of S, where one gets after reading the word $\pi(g)$ starting at the basepoint, i.e. to the vertex representing the right coset $H'\pi(g)$.

Proof. Note that θ_c is a well-defined map, since $C(c) = \langle b, c, d \rangle$ is in the kernel of π.

Now, by the definition of Y, the set of left cosets of $C(c)$ which are the vertices of Y has the following set of representatives (one for each coset), for $1 \le i \le k-1$:

$$e_1(a_1e_1)^{i-2}a_1^j, \ 1 \le j \le i; \quad (e_1a_1)^{i-1}e_1^j, \ 1 \le j \le k-i; \quad a_1^j, \ e_1(a_1e_1)^{k-2}a_1^j, \ 1 \le j \le k. \tag{5·2}$$

Then it follows from the definition of H' that the elements from (5·2) form the set of right coset representatives for H', or, in other words, for each vertex v in S there is exactly one of the elements in (5·2) which labels a path from the basepoint to v in S. Indeed, the elements a_1^j, $1 \le j \le i$, correspond to the vertices in P_1 but not Q_1, the elements $(e_1a_1)^{i-1}e_1^j$, $1 \le j \le k-i$, correspond to the vertices in $Q_i \cap P_{i+1}$ for all $1 \le i \le$

$k-1$, the elements $e_1(a_1e_1)^{i-2}a_1^j$, $1 \le j \le i$, correspond to the vertices in $P_i \cap Q_i$ for all $1 \le i \le k-1$, and the elements $e_1(a_1e_1)^{k-2}a_1^j$, $1 \le j \le k$, correspond to the vertices in P_k but not Q_{k-1}.

This proves that θ_c is indeed a bijection.

LEMMA 5·4. *There is a one-to-one correspondence θ_b between the vertices of Y which are the left cosets of $C(b)$ and the cycles labelled by a_1 in the cover S defining H'. Under this correspondence, a coset $gC(b)$ representing a vertex of Y is mapped to the a_1-cycle passing through the vertex of S where one gets after reading the word $\pi(g)$ starting at the basepoint (i.e. through the vertex representing the coset $H'\pi(g)$).*

Similarly, there is a one-to-one correspondence θ_d between the vertices of Y which are the left cosets of $C(d)$ and all the cycles labelled by e_1 in the cover S defining H'. Under this correspondence, a coset $gC(d)$ representing a vertex of Y is mapped to the e_1-cycle passing through the vertex of S where one gets after reading the word $\pi(g)$ starting at the basepoint (i.e., through the vertex representing the coset $H'\pi(g)$).

Proof. Note that θ_b is a well-defined map, since $C(b)$ does not contain e_1 and so taking another representative g' from the coset $gC(b)$ would result in a vertex on the same a_1-cycle.

Now, by the definition of Y, the set of left cosets of $C(b)$ which are the vertices of Y has the following set of representatives (one for each coset):

$$e_1(a_1e_1)^{i-2}, \ 1 \le i \le k.$$

It follows from the definition of H' that each word $e_1(a_1e_1)^{i-2}$ labels a path in S from the basepoint to a vertex on the cycle P_i, for all $1 \le i \le k$. Since $P_1, \ldots, P_k$ are all the cycles in S labelled by a_1, this proves the first claim.

Similarly, θ_d is a well-defined map, and the set of left cosets of $C(d)$ which are vertices of Y has the following set of representatives (one for each coset):

$$(e_1a_1)^{i-1}, \ 1 \le i \le k-1, \ a_1^j, \ (e_1a_1)^{k-1}a_1^{j-1}, \ 1 \le j \le k.$$

It follows from the definition of H' that each word $(e_1a_1)^{i-1}$ labels a path in S from the basepoint to a vertex on the cycle Q_i, for all $1 \le i \le k-1$, and together with a_1^j, $(e_1a_1)^{k-1}a_1^{j-1}$, which label paths from the basepoint to e_1-loops in S, this gives all the e_1-cycles, and hence the desired result.

The correspondences from Lemmas 5·3 and 5·4 can be easily traced on Figures 12 and 13 for $k=3$ and $k=4$ (notice that they respect the colours).

LEMMA 5·5. *Y is a fundamental domain of the action of H on T, i.e.,*

(i) *no 2 vertices (edges) of Y belong to the same H-orbit;*

(ii) *any vertex (edge) of the tree T can be brought to one of the vertices (edges) of Y by the action of H.*

Proof. Claim (i) follows from Lemmas 5·3 and 5·4. Indeed, suppose that two vertices $v = gC(c)$ and $v' = g'C(c)$ of Y are equivalent under the action of H, i.e., $g'C(c) = hgC(c)$ for some $h \in H$. Then $\pi(h) \in H'$ and so $\theta_c(g'C(c)) = \theta_c(gC(c))$, since both correspond to the coset $H'\pi(g) = H'\pi(h)\pi(g)$. Then $v = v'$, since θ_c is injective.

Similarly, if $v = gC(b)$ and $v' = g'C(b)$, and $g'C(b) = hgC(b)$ for some $h \in H$, then $\theta_b(g'C(b)) = \theta_b(gC(b))$, since both correspond to the a_1-cycle passing through the vertex $H'\pi(g) = H'\pi(h)\pi(g)$, so $v = v'$, since θ_b is injective. In the same way, if $v = gC(d)$ and $v' = g'C(d)$, and $g'C(d) = hgC(d)$, then $v = v'$. Since no two vertices of Y are in the same H-orbit, the same is true for edges of Y. This proves the first claim.

We now prove claim (ii). Let v be a vertex of Y, labelled by $gC(\alpha)$, where α is b, c or d, and let $St(v)$ be the star at v in the Bass–Serre tree T. Note that the stabiliser of v under the action of H is $C_H(\alpha^g)$. We show that, modulo $C_H(\alpha^g)$, any vertex $w \in St(v) \smallsetminus \{v\}$ belongs to the orbit of a vertex labelled by:

    (i) $gC(b), gC(d)$ if $\alpha = c$;

    (ii) $ge_1^j C(c)$, if $\alpha = d$, where $j = 0, \ldots, k - 1$;

    (iii) $ga_1^j C(c)$, if $\alpha = b$, where $j = 0, \ldots, k$.

Indeed, suppose first $v = gC(c) \in Y$ and let $e = (gC(c), gxC(b)) \notin Y$. Then $x \in C(c) = \langle b, c, d \rangle$ and without loss of generality we can assume that $x \in \langle b, d \rangle$. It suffices to find $h \in H \cap C(c^g)$ such that $h \cdot gxC(b) = gC(b)$. Hence, $h = gx^{-1}g^{-1}$ satisfies the requirements. The case when $e = (gC(c), gxC(d)) \notin Y$ is similar.

Let now $v = gC(d) \in Y$, and let $e = (gC(d), gxC(c)) \notin Y$. Then $x \in C(d) = \langle c, d, e_i, i = 1, \ldots, k + 1 \rangle$, and without loss of generality we can assume that $x \in \langle c, e_i, i = 1, \ldots, k + 1 \rangle$. We now find $h \in H \cap C(d^g)$ such that $h \cdot gxC(c) = ge_1^j C(c)$, for some $j = 0, \ldots, k - 1$. Set $h = ge_1^j x^{-1} g^{-1} \in H$, where j is the sum of exponents of e_1 in x modulo k. Note that the choice of j guarantees that $e_1^j x^{-1} \in C(d)$ is a loop in the graph.

The argument for the case $\alpha = b$ is identical.

We now show that all the edges incident to a vertex in Y can be taken to Y by elements of H.

For the edges that are incident to a vertex of type $gC(c)$ in Y it follows directly from the above claim: any edge connecting $gC(c)$ to $w = g'C(b)$ can be brought to an edge connecting $gC(c)$ to $gC(b)$, which is in Y, and similar for d.

We now consider the case of edges incident to a vertex $v = gC(b)$ of Y. According to the above claim, it suffices to prove that any edge connecting v to $ga_1^j C(c)$ for $j = 0, \ldots, k$, can be brought to an edge of Y. If $v = C(b)$ or $v = e_1(a_1e_1)^{k-2}C(b)$, then all these edges are already in Y, and there is nothing to prove. Otherwise, we have $v = e_1(a_1e_1)^{i-2}C(b)$ for some $1 \leq i \leq k - 1$, and then the edges $(gC(b), ga_1^j C(c))$ are already in Y for $j = 1, \ldots, i$, but not for $j = i + 1, \ldots, k$. Thus it suffices to show that the edges $(gC(b), ga_1^j C(c))$ can be brought to Y for $j = i + 1, \ldots, k$. Indeed, let $h_j = ge_1^{k+1-j} a_1^{-j} g^{-1}$. It follows from the construction of S that $h_j \in H$. We then have

$$h_j \cdot ga_1^j C(c) = ge_1^{k+1-j} a_1^{-j} g^{-1} ga_1^j C(c) = ge_1^{k+1-j} C(c) \quad \text{and}$$

$$h_j \cdot gC(b) = ge_1^{k+1-j} a_1^{-j} C(b) = ge_1^{k+1-j} C(b).$$

It follows that the edge $(gC(b), ga_1^j C(c))$ is mapped by h_j to the edge $(ge_1^{k+1-j} C(b), ge_1^{k+1-j} C(c))$, which belongs to Y since $1 \leq k + 1 - j \leq k - i$.

Finally, consider the case of edges incident to a vertex $v = gC(d)$ of Y. According to the above claim, it suffices to prove that any edge connecting v to $ge_1^j C(c)$ for $j = 0, \ldots, k - 1$ can be brought to an edge of Y. Suppose first $v = a_1^l C(d)$ for $1 \leq l \leq k$. Then the edge $(a_1^l C(d), a_1^l e_1^j C(c))$ can be brought to the edge $(a_1^l C(d), a_1^l C(c)) \in Y$, by the element $a_1^l e_1^{-j} a_1^{-l}$, which belongs to H by the construction of S. The case when

$v = (e_1a_1)^{k-1}a_1^{l-1}C(d)$ is similar. Otherwise, $v = (e_1a_1)^{i-1}C(d)$ for some $1 \le i \le k-1$, and the proof in this case is similar to the one above for the cosets of $C(b)$.

This shows that indeed all the edges incident to a vertex in Y can be taken to Y by elements of H.

To finish the proof of claim (ii) we are left to consider the case when $v \in Y$, $(v, w) \in Y$ but w is not in Y. By definition of Y, this is possible only in the following setting. Let (u_j, v_j, w, u_1, v_1) be a path of length 4 in the Bass–Serre tree T, where $g = (e_1a_1)^{i-1}$ for some $1 \le i \le k-1$ and

$$u_j = ge_1^jC(b), \quad v_j = ge_1^jC(c), \quad w = gC(d), \quad \text{where } j = 1, \ldots, k-i.$$

We show that u_1 and u_j are in the same H-orbit, that is there exist $h_j \in H$ so that $h_j \cdot ge_1^jC(b) = ge_1C(b)$, for all $j = 1, \ldots, k-i$. Indeed, one can take $h_j = ge_1a_1^{1-j}e_1^{-j}g^{-1}$, and it follows from the construction of S that $h_j \in H$.

The case when $w = gC(b)$ is identical and is left to the reader.

Now it is standard that Y is the fundamental domain. Indeed, one can see by induction on the distance between an edge e of T and the closest to e vertex of Y that any edge of T can be taken to Y by an element of H, and similar for vertices.

This proves Lemma 5·5.

It follows that the quotient of the action of T by H is a graph isomorphic to the one associated to X. We need one more lemma about the structure of centralisers in H.

LEMMA 5·6. *The following formulas hold for centralizers in H:*

$$C_H(c) = \langle c \rangle \times F(b, d),$$

$$C_H(d) = \langle d \rangle \times F(e_i^{e_1^j}, e_1^k, c^{e_1^j} \mid j = 0, \ldots, k-1, \ i = 2, \ldots, k+1),$$

$$C_H(b) = \langle b \rangle \times F(a_i^{a_1^j}, a_1^{k+1}, c^{a_1^j} \mid j = 0, \ldots, k, \ i = 2, \ldots, k).$$

More generally, the following holds:

(i) $C_H(c^g) = \langle c^g \rangle \times F(b^g, d^g) \cong \mathbb{Z} \times F_2$ *for every* $g \in G$;

(ii) $C_H(b^g) = \langle b^g \rangle \times F(a_i^{ga_1^j}, (a_1^{k+1})^g, c^{ga_1^j} \mid j = 0, \ldots, k, \ i = 2, \ldots, k) \cong \mathbb{Z} \times F_{k^2+k+1}$ *for every* $g \in G$;

(iii) *if* $g \in G$ *is such that* $\pi(g)$ *labels a path in S from the basepoint to a vertex on one of the cycles* $Q_1, \ldots, Q_{k-1}$, *then*

$$C_H(d^g) = \langle d^g \rangle \times F(e_i^{ge_1^j}, (e_1^k)^g, c^{ge_1^j} \mid j = 0, \ldots, k-1, \ i = 2, \ldots, k+1)$$
$$\cong \mathbb{Z} \times F_{k^2+k+1};$$

(iv) *if* $g \in G$ *is such that* $\pi(g)$ *labels a path in S from the basepoint to a vertex on the cycle* P_1 *but not* Q_1, *or* P_k *but not* Q_{k-1}, *then*

$$C_H(d^g) = \langle d^g \rangle \times F(e_i^g, c^g \mid i = 1, \ldots, k+1) \cong \mathbb{Z} \times F_{k+2}.$$

Proof. We just prove the first statement, the others follow from the construction of the cover S in a similar way, since $C(\alpha^g) = (C(\alpha))^g$, where $C(\alpha)$ is a subgroup of $H^{g^{-1}} = \pi^{-1}((H')^{\pi(g)^{-1}})$ for $\alpha = b, c, d$, and the cover defining $(H')^{\pi(g)^{-1}}$ can be obtained from S just by moving the basepoint along a path with the label $\pi(g)$.

Note that

$$C(b) = \langle a_1, \ldots, a_k, b, c \rangle \cong \mathbb{Z} \times F_{k+1}, \quad C(c) = \langle b, c, d \rangle \cong \mathbb{Z} \times F_2,$$
$$C(d) = \langle c, d, e_1, \ldots, e_{k+1} \rangle \cong \mathbb{Z} \times F_{k+2}.$$

It is immediate that $C_H(c) = \langle b, c, d \rangle$. Now, we have

$$C_H(b) = H \cap C(b) = H \cap \langle a_1, \ldots, a_k, b, c \rangle = \langle b \rangle \times (H \cap \langle a_1, \ldots, a_k, c \rangle),$$

and $H_0 = H \cap \langle a_1, \ldots, a_k, c \rangle$ is a subgroup of the free group $F(a_1, \ldots, a_k, c)$ which is defined by the following cover: take the cycle P_1 (labelled by a_1, of length $k + 1$) and add loops labelled by $a_2, \ldots, a_k, c$ at every vertex. It follows that H_0 has a basis $\{a_i^{a_1^j}, a_1^{k+1}, c^{a_1^j}, \ j = 0, \ldots, k, \ i = 2, \ldots, k\}$, and so $C_H(b)$ has the desired form. The proof for $C_H(d)$ is similar.

We now show that H is isomorphic to the fundamental group of the graph of groups X. To see this we need to check that vertex groups, edge groups and the embeddings are the same. The vertex groups are simply centralisers (in H) of conjugates of generators, which we computed in Lemma 5·6. Edge groups are clearly free abelian groups of rank two and the embeddings are mapping generators of the edge groups to the corresponding generators of the vertex groups. Now it follows directly from the definition of X, Lemma 5·5 and Lemma 5·6 that H is isomorphic to the fundamental group of the graph of groups X. This proves Proposition 5·2.

5·4. *Construction of the finite index subgroup K in $\mathbb{G}(P_{4k+2})$*

Let now P_{4k+2} be the path of length $4k + 2$. Let

$$\{A, D_1, C_1, B_1, C_1', D_2, C_2, B_2, C_2', \ldots, D_k, C_k, B_k, C_k', D_{k+1}, E\}$$

be the ordered list of vertices of P_{4k+2}. From now on denote by $G = \mathbb{G}(P_{4k+2})$ the corresponding RAAG. Let $F = F(C_1, \ldots, C_k, C_1', \ldots, C_k')$ be the free group on the indicated set of generators. We construct a finite index subgroup K' of F. The group K' corresponds to the degree $k(k + 1)$ cover Z of the bouquet of $2k$ circles defined as follows:

(i) there are exactly i cycles spanned by edges labelled by C_i', $i = 1, \ldots, k$. In particular, there is only one cycle spanned by C_1', its length is $k(k + 1)$;

(ii) edges labelled by C_k' span k cycles of length $k + 1$ each;

(iii) edges labelled by C_i' span i cycles, $i = 1, \ldots, k - 1$: one cycle of length $k(k + 1) - (i - 1)$ and $i - 1$ loops;

(iv) there are $k + 1 - i$ cycles spanned by C_i, $i = 1, \ldots, k$. In particular, there is one cycle spanned by C_k, its length is $k(k + 1)$;

(v) edges labelled by C_1 span k cycles of length $k + 1$ each;

(vi) edges labelled by C_i span $k + 1 - i$ cycles, $i = 2, \ldots, k$: one cycle of length $k(k + 1) - (k - i)$ and $k - i$ loops;

(vii) there are no loops at the basepoint;

(viii) the graph spanned by the edges labelled by C_i and C_i' is connected, for every $i = 1, \ldots, k$;

(xi) the graph spanned by the edges labelled by C_{i-1}' and C_i is connected, for every $i = 2, \ldots, k$.
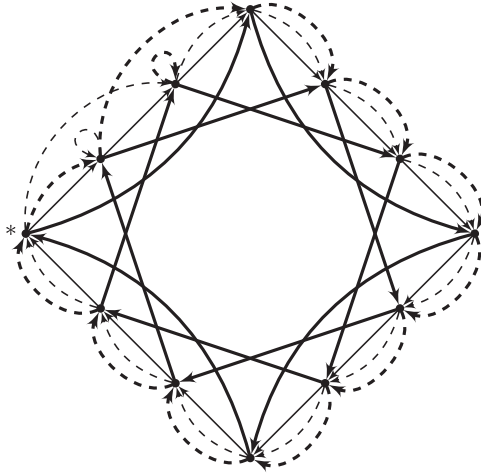
Fig. 14. A cover Z defining the subgroup K' in F in the case $k = 3$. The basepoint is marked by a star. Each solid thin edge corresponds to two edges, one labelled by C'_1 and the other by C_3; each solid thick edge corresponds to two edges, one labelled by C_1 and the other by C'_3; dashed thin edges are labelled by C_2 and dashed thick edges are labelled by C'_2. Here we have $\alpha_{2,1} = 1$, $\beta_{2,1} = 1$.

It is easy to see that such a cover always exists. For instance, one could have at most one loop at each vertex, it is possible since there are $(k - 1)(k - 2)$ loops altogether and $k(k + 1)$ vertices, and in this case the last two conditions follow from the previous ones. Such a cover is not unique, but we can choose any cover satisfying the above assumptions, and this will give isomorphic subgroups, as we will see below. See Figures 14 and 15 for covers Z in the cases $k = 3$ and $k = 4$ respectively.

Without loss of generality, we assume that the vertices of Z which can be reached from the basepoint by reading the words $C_1'^{j}$, $j = 0, \ldots, k - 1$, all belong to different cycles labelled by C_1, and similarly the vertices of Z which can be reached from the basepoint by reading the words C_k^{j}, $j = 0, \ldots, k - 1$, all belong to different cycles labelled by C'_k.

We let K to be the full preimage of K' in G under the natural epimorphism $\pi : G \to F$. By definition, K has index $k(k + 1)$ in G.

Let $2 \leq i \leq k - 1$. By construction, the graph spanned by the edges labelled by C'_{i-1} and C_i is connected, and the graph spanned by the edges labelled only by C_i has $k + 1 - i$ connected components, all but one being loops, and so these loops are incident to the vertices $v_1, \ldots, v_{k-i}$, which belong to the cycle labelled by C'_{i-1} that goes through the basepoint (of length $k(k + 1) - (i - 1)$). Let $\alpha_{i,1} < \alpha_{i,2} < \cdots < \alpha_{i,k-i}$ be the lengths of the shortest oriented paths labelled by C'_{i-1} starting at the basepoint and ending at the vertices $v_1, \ldots, v_{k-i}$ (i.e, those which have loops labelled by C_i), see Figures 14, 15. In the same way, exchanging the roles of C'_{i-1} and C_i, one can define $\alpha'_{i,1} < \alpha'_{i,2} < \cdots < \alpha'_{i,i-2}$ to be the lengths of the shortest oriented paths labelled by C_i starting at the basepoint and ending at the vertices with loops labelled by C'_{i-1}, one for each of $i - 2$ such loops.

Similarly, for $2 \leq i \leq k - 1$, the graph spanned by the edges labelled by C_i and C'_i is connected, and the graph spanned by the edges labelled only by C'_i has i connected components, all but one being loops, and so these loops are incident to the vertices $w_1, \ldots, w_{i-1}$, which belong to the cycle labelled by C_i that goes through the basepoint (of length $k(k + 1) - (k - i)$). Let $\beta_{i,1} < \beta_{i,2} < \cdots < \beta_{i,i-1}$ be the lengths of the shortest oriented paths labelled by C_i starting at the basepoint and ending in the vertices $w_1, \ldots, w_{i-1}$ (i.e, those
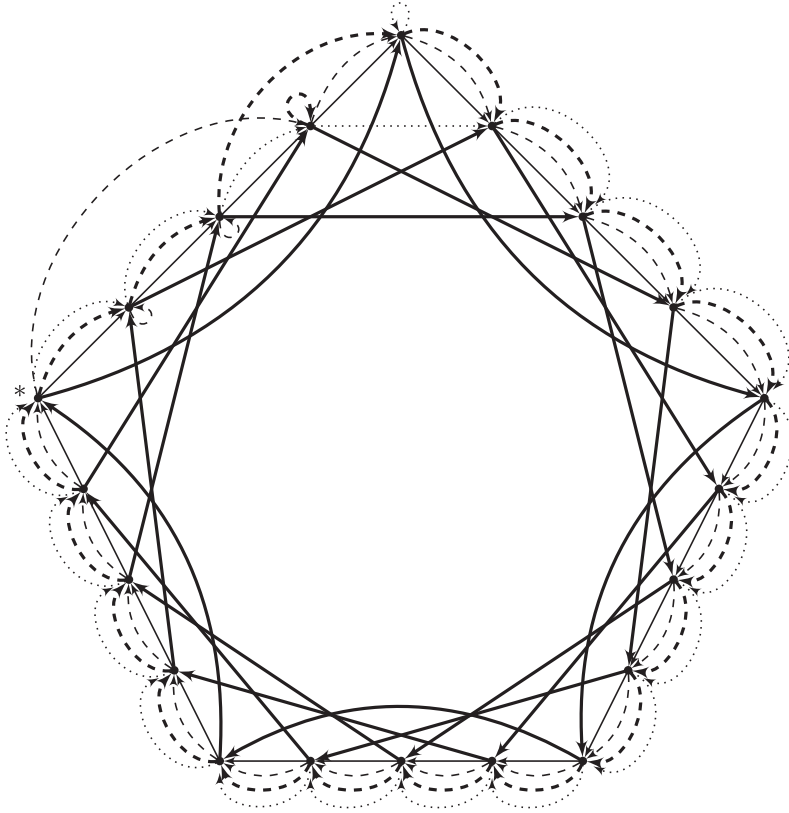
Fig. 15. A cover Z defining the subgroup K' in F in the case $k = 4$. The basepoint is marked by a star. Each solid thin edge corresponds to two edges, one labelled by C'_1 and the other by C_4; each solid thick edge corresponds to two edges, one labelled by C_1 and the other by C'_4; each dashed thin edge corresponds to two edges, one labelled by C_2 and the other by C'_3; dashed thick edges are labelled by C'_2 and dotted edges are labelled by C_3. Here we have $\alpha_{2,1} = 1$, $\alpha_{2,2} = 2$, $\alpha_{3,1} = 3$, $\beta_{2,1} = 1$, $\beta_{3,1} = 1$, $\beta_{3,2} = 2$.

which have loops labelled by C'_i), see Figures 14, 15. In the same way, exchanging the roles of C_i and C'_i, one can define $\beta'_{i,1} < \beta'_{i,2} < \cdots < \beta'_{i,k-i}$ to be the lengths of the shortest oriented paths labelled by C'_i starting at the basepoint and ending at the vertices with loops labelled by C_i, one for each of $k - i$ such loops.

We additionally define $\alpha_{i,0} = \beta_{i,0} = 0$ for $2 \leq i \leq k - 1$.

5·5. *Isomorphism between K and $\pi_1(X)$*

From now on we will denote by T the Bass–Serre tree of the reduced centraliser splitting of $\mathbb{G}(P_{4k+2})$. The finite index subgroup K of $\mathbb{G}(P_{4k+2})$ acts on T, and so K has an induced graph of groups structure determined by the quotient of T by the action of K. In this subsection we prove the following.

PROPOSITION 5·7. *In the above notation, the subgroup K is isomorphic to the fundamental group of the graph of groups X. Namely, the induced splitting of K given by its action on T gives precisely X.*

Proof. Consider the full subgraph (subtree) Y_0 of the Bass–Serre tree T spanned by the following vertices:
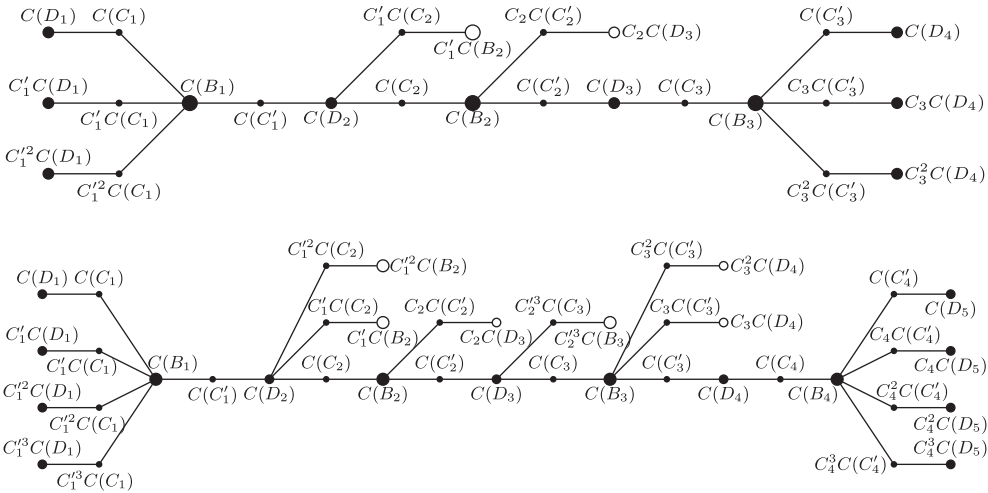
Fig. 16. The fundamental domain Y for the action of K on T in the case $k = 3$ (above) and $k = 4$ (below), where K is defined via K' as on Figures 14, 15. Only the vertices that are filled belong to Y, while those with blank interior do not. Small nodes correspond to the cosets of $C(C_i)$, $C(C_i')$, huge nodes – to the cosets of $C(B_i)$, and middle-size nodes – to the cosets of $C(D_i)$.

(i) $C(D_i)$, $C(C_i)$, $C(B_i)$, $C(C_i')$, $i = 1, \ldots, k$, $C(D_{k+1})$;

(ii) $C_1'^{j} C(C_1)$, $C_1'^{j} C(D_1)$, $j = 1, \ldots, k-1$;

(iii) $C_k^{j} C(C_k')$, $C_k^{j} C(D_{k+1})$, $j = 1, \ldots, k-1$;

(iv) $C_{i-1}'^{\alpha_{i,j}} C(C_i)$, $C_{i-1}'^{\alpha_{i,j}} C(B_i)$, $i = 2, \ldots, k-1$, $j = 1, \ldots, k-i$;

(v) $C_i^{\beta_{i,j}} C(C_i')$, $C_i^{\beta_{i,j}} C(D_{i+1})$, $i = 2, \ldots, k-1$, $j = 1, \ldots, i-1$.

Let Y be obtained from Y_0 by deleting the following vertices (without deleting any edges):

(i) $C_{i-1}'^{\alpha_{i,j}} C(B_i)$, $i = 2, \ldots, k-1$, $j = 1, \ldots, k-i$;

(ii) $C_i^{\beta_{i,j}} C(D_{i+1})$, $i = 2, \ldots, k-1$, $j = 1, \ldots, i-1$.

The subtree Y is shown on Figure 16 for $k = 3$ and $k = 4$.

LEMMA 5·8. *Y is a fundamental domain of the action of K on T.*

Proof. We first show that no 2 vertices in Y belong to the same K-orbit. This is immediate for the cosets of the centralisers of C_1', C_k, B_1, B_2, $\ldots$, B_k and D_2, $\ldots$, D_k, since for them there is only one vertex in Y even in each G-orbit.

Suppose that there exists $h \in K$ such that for some $2 \leq i \leq k-1$ and $0 \leq j < l \leq k-i$ we have $h \cdot C_{i-1}'^{\alpha_{ij}} C(C_i) = C_{i-1}'^{\alpha_{il}} C(C_i)$. By definition, $h = C_{i-1}'^{\alpha_{il}} x C_{i-1}'^{-\alpha_{ij}}$, where $x \in C(C_i) = \langle B_i, C_i, D_i \rangle$. Then $\pi(h) = C_{i-1}'^{\alpha_{il}} w C_{i-1}'^{-\alpha_{ij}} \in K'$, where $w = \pi(x)$ is a power of C_i. This means that the vertices in Z obtained from the basepoint after reading $C_{i-1}'^{\alpha_{il}}$ and $C_{i-1}'^{\alpha_{ij}}$ are on the same cycle labelled by C_i, which contradicts the definition of α's. Similar argument shows that there is no $h \in K$ such that for some $2 \leq i \leq k-1$ and $0 \leq j < l \leq k-i$ we have $hC_i^{\beta_{i,j}} C(C_i') = C_i^{\beta_{i,l}} C(C_i')$.

In the same way, if there is $h \in K$ such that $hC_1'^{j} C(C_1) = C_1'^{l} C(C_1)$ for some $0 \leq j < l \leq k-1$, then $C_1'^{j} x C_1'^{-l} \in K$ for some $x \in C(C_1) = \langle D_1, C_1, B_1 \rangle$, and so $C_1'^{j} w C_1'^{-l} \in K'$, where $w = \pi(x)$ is a power of C_1, so the vertices in Z obtained from the basepoint after reading $C_1'^{j}$ and $C_1'^{l}$ are on the same cycle labelled by C_1, which contradicts our choice of Z.

Also, if there is $h \in K$ such that $hC_1'^{j}C(D_1) = C_1'^{l}C(D_1)$ for some $0 \le j < l \le k-1$, then $C_1'^{j}xC_1'^{-l} \in K$ for some $x \in C(D_1) = \langle A, D_1, C_1 \rangle$, so $C_1'^{j}wC_1'^{-l} \in K'$, where $w = \pi(x)$ is a power of C_1, which is again a contradiction as above. Similar arguments apply to the cosets of C_k' and D_{k+1}. Thus no 2 vertices of Y are in the same K-orbit.

We now show that all the vertices $C_{i-1}'^{\alpha_{ij}}C(B_i)$, $i = 2, \ldots, k-1$, $j = 1, \ldots, k-i$, can be mapped to $C(B_i)$ and all the vertices $C_i'^{\beta_{ij}}C(D_{i+1})$, $i = 2, \ldots, k-1$, $j = 1, \ldots, i-1$, can be mapped to $C(D_{i+1})$ by some elements of K.

Indeed, we show that there exists $h \in K$ such that $h \cdot C(B_i) = C_{i-1}'^{\alpha_{ij}}C(B_i)$. It suffices to take $h = C_{i-1}'^{\alpha_{ij}}w_j \in K$, where $w_j \in C(B_i) = \langle C_i, B_i, C_i' \rangle$. Let v be the vertex of Z where one gets after reading the label $C_{i-1}'^{\alpha_{ij}}$ from the basevertex. It now suffices to choose $w_j = w_j(C_i, C_i')$ to be the label of a path joining v to the basepoint inside the graph spanned by the edges labelled by C_i and C_i', and such a path indeed exists since this graph is connected.

The proof for cosets of $C(D_i)$ is analogous and is left to the reader.

We are left to show that all the edges e in the Bass–Serre tree T which share a vertex with Y can be mapped into Y by an element of K.

Suppose first that $e = (xC(C_i), xgC(B_i))$, where $x = 1$ and $1 \le i \le k$, or $x = C_{i-1}'^{\alpha_{i,j}}$, $i = 2, \ldots, k-1$, $j = 1, \ldots, k-i$, or $x = C_1'^{j}$, $j = 1, \ldots, k-1$ with $i = 1$. In particular, $xC(C_i)$ is in Y. Then $g \in C(C_i) = \langle B_i, C_i, D_i \rangle$, so, without loss of generality, we can assume that $g = w(D_i, B_i)$, and so $g' = xg^{-1}x^{-1} \in K$ and g' takes e to $(xC(C_i), xC(B_i))$ which is in Y by construction.

Suppose next that $e = (C(B_i), gC(C_i'))$, $i = 1, \ldots, k$. Then $g \in C(B_i)$, and so, without loss of generality, we can assume that $g = w(C_i, C_i')$. We show that for all such g there exists $h \in C_K(B_i)$ and $j = 0, \ldots, i-1$ such that $hgC(C_i') = C_i'^{\beta_{ij}}C(C_i')$. Indeed, let v be the vertex of Z where one gets after reading the word w from the basepoint of Z. By definition, one can choose j such that the path labelled by $C_i'^{\beta_{ij}}$ and starting at the basepoint of Z ends in the C_i'-cycle that passes through v, and then $h = C_i'^{\beta_{ij}}C_i'^{l}g^{-1} \in K$ for some l, and $h \in C(B_i)$, as desired. Then h takes the edge e into Y.

Let now $e = (C_1'^{j}C(D_1), C_1'^{j}gC(C_1))$, $j = 0, \ldots, k-1$. Then $g \in C(D_1)$, and without loss of generality we can assume that $g = w(A, C_1)$. Hence, $\pi(g) = C_1^{l}$ for some l, and so $\pi(C_1^{l}g^{-1}) = 1$. Therefore, $h = C_1'^{j}C_1^{l}g^{-1}C_1'^{-j} \in K$. Now h takes e into $(C_1'^{j}C(D_1), C_1'^{j}C(C_1))$, which is in Y.

All the other cases are similar to the above and left to the reader.

Now it is routine to see that Z is a fundamental domain for the action of K on T. This proves Lemma 5·8.

We established that the quotient of the action of T by K is a graph isomorphic to the one associated to X. We need one more lemma about centralisers in K.

LEMMA 5·9. *The following formulas hold for centralizers in K:*

(i) $C(C_i^{g}) = (K \cap \langle C_i^{g} \rangle) \times F(D_i^{g}, B_i^{g}) \cong \mathbb{Z} \times F_2$ *for all* $g \in G$, $i = 1, \ldots, k$;

(ii) $C(C_i'^{g}) = (K \cap \langle C_i'^{g} \rangle) \times F(B_i^{g}, D_{i+1}^{g}) \cong \mathbb{Z} \times F_2$ *for all* $g \in G$, $i = 1, \ldots, k$;

(iii) $C(B_i) = \langle B_i \rangle \times L_i \cong \mathbb{Z} \times F_{k^2+k+1}$ *for* $i = 2, \ldots, k-1$, *where* L_i *is a subgroup of index* $k(k+1)$ *in* $F(C_i, C_i')$ *with basis*

$$\left\{ C_i^{k(k+1)-(k-i)}, C_i^{(C_i'^{\beta_{i,j}'})}, C_i'^{k(k+1)-(i-1)}, \; \middle| \; j = 1, \ldots, k-i, \, l = 1, \ldots, i-1 \right\},$$
$$\left. C_i'^{(C_i^{\beta_{i,l}})}, U_{i,1}, \ldots, U_{i,k^2} \right.$$

where each of $U_{i,1}, \ldots, U_{i,k^2}$ is not conjugate to a power of C_i or C_i';

(iv) $C(D_i) = \langle D_i \rangle \times L_i' \cong \mathbb{Z} \times F_{k^2+k+1}$ for $i = 2, \ldots, k$, where L_i' is a subgroup of index $k(k+1)$ in $F(C_i, C_{i-1}')$ with the basis

$$
\left\{
\begin{array}{l}
C_i^{k(k+1)-(k-i)}, C_i^{(C_{i-1}'^{\alpha_{i,j}})}, C_{i-1}'^{k(k+1)-(i-2)}, \\
C_{i-1}'^{(C_i^{\alpha_{i,l}'})}, U_{i,1}', \ldots, U_{i,k^2+1}'
\end{array}
\; \middle| \; j = 1, \ldots, k-i, \; l = 1, \ldots, i-2
\right\},
$$

where each of $U_{i,1}', \ldots, U_{i,k^2+1}'$ is not conjugate to a power of C_i or C_{i-1}';

(v) $C(B_1) = \langle B_1 \rangle \times L_1 \cong \mathbb{Z} \times F_{k^2+k+1}$, where L_1 is a subgroup of index $k(k+1)$ in $F(C_1, C_1')$ with the basis $\{C_1'^{k(k+1)}, (C_1^{k+1})^{(C_1'^j)}, U_{1,1}, \ldots, U_{1,k^2}, \; j = 0, \ldots, k-1\}$, where each of $U_{1,1}, \ldots, U_{1,k^2}$ is not conjugate to a power of C_1 or C_1';

(vi) $C(B_k) = \langle B_k \rangle \times L_k \cong \mathbb{Z} \times F_{k^2+k+1}$, where L_k is a subgroup of index $k(k+1)$ in $F(C_k, C_k')$ with the basis $\{C_k^{k(k+1)}, (C_k'^{k+1})^{(C_k^j)}, U_{k,1}, \ldots, U_{k,k^2}, \; j = 0, \ldots, k-1\}$, where each of $U_{k,1}, \ldots, U_{k,k^2}$ is not conjugate to a power of C_k or C_k';

(vii) $C(D_1^{C_1'^j}) = \langle D_1^{C_1'^j} \rangle \times F((C_1^{k+1})^{C_1'^j}, A^{C_1'^j}, A^{C_1'^j C_1}, \ldots, A^{C_1'^j C_1^k}) \cong \mathbb{Z} \times F_{k+2}$, where $j = 0, \ldots, k-1$;

(viii) $C(D_{k+1}^{C_k^j}) = \langle D_{k+1}^{C_k^j} \rangle \times F((C_k'^{k+1})^{C_k^j}, E^{C_k^j}, E^{C_k^j C_k'}, \ldots, E^{C_k^j C_k'^k}) \cong \mathbb{Z} \times F_{k+2}$, where $j = 0, \ldots, k-1$.

Proof. Recall that $C(C_i) = \langle C_i \rangle \times F(B_i, D_i)$, $C(C_i') = \langle C_i' \rangle \times F(B_i, D_{i+1})$, $C(B_i) = \langle B_i \rangle \times F(C_i, C_i')$ for $i = 1, \ldots, k$, and $C(D_i) = \langle D_i \rangle \times F(C_{i-1}', C_i)$ for $i = 1, \ldots, k-1$, $C(D_1) = \langle D_1 \rangle \times F(A, C_1)$, $C(D_{k+1}) = \langle D_{k+1} \rangle \times F(C_k', E)$.

The first two claims of the lemma are immediate. We now prove the third one. Let Z_i be the (connected) graph spanned by the edges labelled by C_i and C_i' in Z. By definition, Z_i is a cover of the bouquet of two circles, labelled by C_i and C_i'. Let $L_i < F(C_i, C_i')$ be the free group corresponding to the cover Z_i. Since the index of L_i in $F(C_i, C_i')$ is $k(k+1)$, its rank is $k(k+1)+1$. Now it is easy to see that L_i has the desired basis by first choosing a basis corresponding to a maximal subtree in Z_i with $k(k+1) - (k-i) - 1$ edges labelled by C_i and $k - i$ edges labelled by C_i', and then applying appropriate Nielsen transformations.

Similarly, to prove the fifth claim, we let Z_1 to be the graph spanned by C_1 and C_1' and L_1 be the corresponding subgroup of index $k(k+1)$ in $F(C_1, C_1')$. Again, one can see that L_1 has the desired basis by first choosing a basis corresponding to a maximal subtree in Z_1 which includes k edges from each cycle labelled by C_1 in Z_1, as well as a path from the basepoint of length $k - 1$ with all the edges labelled by C_1', and then applying appropriate Nielsen transformations.

To prove claim (vii) for $j = 0$, we define L_0 to be the index $k + 1$ subgroup of $F(A, C_1)$ given by the cover Z_0 obtained from a cycle of length $k + 1$ labelled by C_1 by adding loops labelled by A at every vertex. The desired basis for L_0 then corresponds to a maximal subtree in Z_0. The case $j > 0$ is similar.

The proofs of all the other claims are similar and left to the reader.

Finally, we show that K is isomorphic to the fundamental group of the graph of groups X. To see this we need to check that vertex groups, edge groups and the embeddings are the same. The vertex groups are simply centralisers (in K) of conjugates of generators, which we computed in Lemma 5·9. Edge groups are clearly free abelian groups of rank two and the embeddings are mapping generators of the edge groups to the corresponding generators
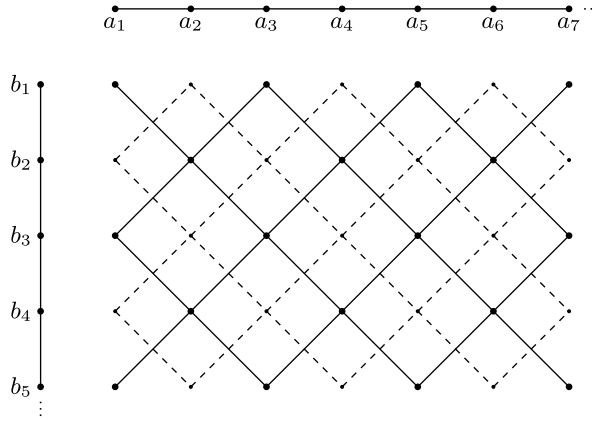
Fig. 17. Structure of the product graph $\mathfrak{D} = P_{m-2} \times P_{n-2}$. The solid edges are in $\mathfrak{D}_1$, and the dashed edges are in $\mathfrak{D}_2$.

of the vertex groups. Now Proposition 5·7 follows directly from the definition of X, Lemma 5·8 and Lemma 5·9.

Propositions 5·2 and 5·7 together imply that H and K are isomorphic, and so $\mathbb{G}(T_{k,k+1})$ and $\mathbb{G}(P_{4k+2})$ are indeed commensurable. This finishes the proof of Theorem 5·1.

6. *Path RAAGs are not commensurable*

In this section we address the proof of Theorem 1·2 which states that $\mathbb{G}(P_m)$ and $\mathbb{G}(P_n)$ are not commensurable, with the only exception of $\mathbb{G}(P_3)$ and $\mathbb{G}(P_4)$.

By Corollary 2·14, it suffices to show that the linear system of equations and inequalities associated to the product graph does not have integer solutions. The key tool is Lemma 6·3, which allows, given a local pattern in the graph, to deduce that some edges do not exist, see Figure 21. Applying this lemma recursively, the structure of the graph is significantly simplified and a case-by-case analysis on the parities of n and m allows us to conclude that the system does not have integer solutions.

We can suppose that $m, n \geq 3$, since $\mathbb{G}(P_0) \cong \mathbb{Z}$, $\mathbb{G}(P_1) \cong \mathbb{Z}^2$, $\mathbb{G}(P_2) \cong F_2 \times \mathbb{Z}$ and $\mathbb{G}(P_n)$ for some fixed $n \geq 3$ are pairwise not commensurable (not even quasi-isometric, see [**BN08**]). Furthermore, we can suppose that $m > n \geq 5$, since other cases are already covered by Theorem 3·1.

6·1. *Product graph for two paths*

We fix some $m > n \geq 3$. Let $a_0, a_1, \ldots, a_m$ be the vertices of P_m, considered as canonical generators of $\mathbb{G}(P_m)$, and $b_0, b_1, \ldots, b_n$ be the vertices of P_n, considered as canonical generators of $\mathbb{G}(P_n)$. Then, in the above notation, $\Gamma_1 = P_m$, $\Gamma_2 = P_n$, and $\widetilde{\Gamma}_1 = P_{m-2}$, with vertices $a_1, \ldots, a_{m-1}$, $\widetilde{\Gamma}_2 = P_{n-2}$, with vertices $b_1, \ldots, b_{n-1}$.

Suppose that $\mathbb{G}(P_m)$ and $\mathbb{G}(P_n)$ are commensurable. Note that in our case $\mathfrak{D} = \widetilde{\Gamma}_1 \times \widetilde{\Gamma}_2 = P_{m-2} \times P_{n-2}$ is the following graph: its set of vertices is $\{(a_i, b_j), i = 1, \ldots, m-1; j = 1, \ldots, n-1\}$, and two vertices (a_{i_1}, b_{j_1}) and (a_{i_2}, b_{j_2}) are connected by an edge in $\mathfrak{D}$ if and only if $|i_1 - i_2| = 1$ and $|j_1 - j_2| = 1$, for $i_1, i_2 = 1, \ldots, m-1$; $j_1, j_2 = 1, \ldots, n-1$. To abbreviate the notation, we will denote the vertex (a_i, b_j) of $\mathfrak{D}$ by (i, j), for $i = 1, \ldots, m-1$; $j = 1, \ldots, n-1$, see Figure 17.

Note that, by Lemma 2·8, $\mathfrak{D}$ has two connected components, one of them, denoted by $\mathfrak{D}_1$, consisting of vertices (i, j) with $i + j$ even, and the other one, denoted by $\mathfrak{D}_2$ with $i + j$ odd, and $\mathfrak{C}$ lies in one of them.

Remark 6·1. If m is odd, then the automorphism of P_m which reverses the order of its vertices (it also induces an automorphism of $\mathbb{G}(P_m)$) switches these components, which are in this case isomorphic graphs, so, after applying this automorphism of $\mathbb{G}(P_m)$ if necessary, without loss of generality we can assume that $\mathfrak{C}$ lies in a particular component of $\mathfrak{D}$. The same is true if n is odd. However, if both m and n are even, then the two connected components of $\mathfrak{D}$ are not isomorphic, and we should consider two cases, depending on whether $\mathfrak{C}$ lies in one or the other connected component of $\mathfrak{D}$.

Note that Lemmas 2·11 and 2·12 apply and provide us with a system of equations on the edge labels of $\mathfrak{D}$. The equations of Lemma 2·12 get simplified in our case, in particular, in the notation of this lemma we always have $D_1 = D_2 = 1$. We will now show that this system of equations has no solutions, provided that restrictions on $\mathfrak{C}$ given by Lemmas 2·8 and 2·9 hold, and this derives a contradiction.

6·2. *Notation*

We now introduce some auxiliary notation used in the proof.

Note that $\mathfrak{D}$ is a planar graph, so we can think of $\mathfrak{D}$ as a graph on the plane, and use "compass notation", with the first coordinate increasing from west to east, and the second coordinate increasing from north to south. So we have the vertex $(1, 1)$ in the top left (NW) corner, vertex $(1, n - 1)$ in bottom left corner (SW), vertex $(m - 1, 1)$ in top right corner (NE), and $(m - 1, n - 1)$ in bottom right (SE) corner. Every vertex (i, j) in $\mathfrak{D}$ has some of the following incident edges (with a minimum of one): the NW edge, going to $(i - 1, j - 1)$; the SW edge, going to $(i - 1, j + 1)$; the NE edge, going to $(i + 1, j - 1)$; and the SE edge, going to $(i + 1, j + 1)$. Thus, the vertices of $\mathfrak{D}$ can be subdivided into inner vertices, which have degree 4 — those which are of the form (i, j) with $1 < i < m - 1$, $1 < j < n - 1$, and boundary vertices — all the rest. Among boundary vertices there are four corner vertices, which have degree 1, and all the rest, which have degree 2, see Figure 17.

Fix some $s = 1, 2$, and recall that $\mathfrak{D}_s$ is one of the connected components of $\mathfrak{D}$. Consider the following auxiliary graph $\mathfrak{D}'_s$: the set of vertices of $\mathfrak{D}'_s$ coincides with the set of vertices of $\mathfrak{D}_s$, and the set of edges of $\mathfrak{D}'_s$ is equal to the union of the set of edges of $\mathfrak{D}_s$ and the set of new edges called boundary, which connect the vertices $(1, k_1)$ with $(1, k_1 + 2)$ (west boundary); $(m - 1, k_2)$ with $(m - 1, k_2 + 2)$ (east boundary); $(k_3, 1)$ with $(k_3 + 2, 1)$ (north boundary); $(k_4, n - 1)$ with $(k_4 + 2, n - 1)$ (south boundary) for all such natural k_1, k_2, k_3, k_4 that the vertices above belong to $\mathfrak{D}_s$. Note that $\mathfrak{D}'_s$ contains $\mathfrak{D}_s$ as a subgraph.

Obviously, $\mathfrak{D}'_s$ is also a planar graph, so we can speak about faces of $\mathfrak{D}'_s$ — the set of all regions bounded by edges, here we do not consider the unbounded region. Let $\mathfrak{F}$ be the set of all such bounded faces of $\mathfrak{D}'_s$ (we omit the index s which is fixed). Abusing the terminology, we will also call them faces of D_s. Note that there are two types of faces in $\mathfrak{F}$ — square faces, which are bounded by four edges, all belonging to $\mathfrak{D}_s$, and triangle (boundary) faces, which are bounded by three edges, one of them boundary and the other two belonging to $\mathfrak{D}_s$. Boundary faces can be further subdivided into west, east, north and south boundary faces, depending on their boundary edge. Each square face has four sides, which are all edges of $\mathfrak{D}_s$ – the NW side, the SW side, the NE side, and the SE side; for a triangle face only two of
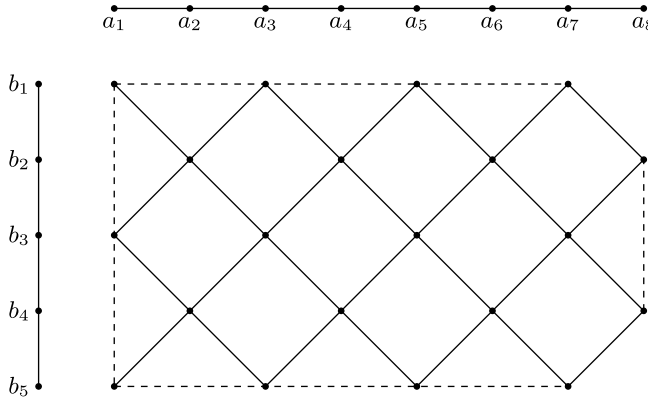
Fig. 18. The graphs $\mathfrak{D}_1$ (solid edges) and $\mathfrak{D}'_1$ (solid and dashed edges) for $m = 9$ and $n = 6$.
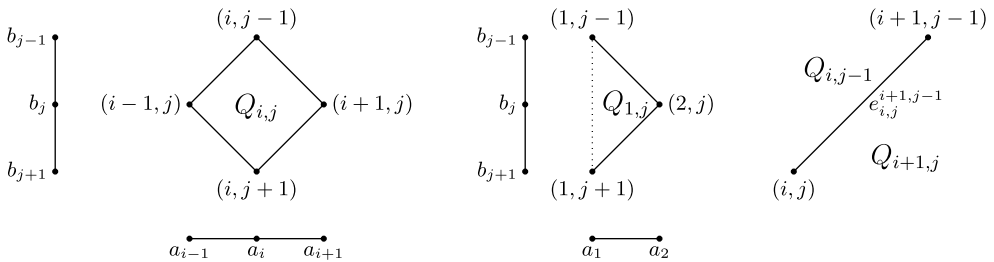

Fig. 19. Notation for edges and faces of $\mathfrak{D}_1$ or $\mathfrak{D}_2$.

the sides are defined. Each square face has four corners, which are all vertices of $\mathfrak{D}_s$ — the north, south, east and west corner; for a triangle face only three of the corners are defined. Two faces are adjacent if they have a common side, see Figure 18.

Recall that $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are the connected components of $\mathfrak{D}$, such that $\mathfrak{D}_1$ contains vertices (i, j) with $i + j$ even, and $\mathfrak{D}_2$ with $i + j$ odd. Note that for every face of $\mathfrak{D}_2$ there exists exactly one vertex of $\mathfrak{D}_1$ inside this face (when considered on the plane), and this vertex is not a corner vertex; and vice versa, each vertex of $\mathfrak{D}_1$ which is not a corner vertex belongs to exactly one face of $\mathfrak{D}_2$. This means that there is a bijection between the faces of $\mathfrak{D}_2$ and the vertices of $\mathfrak{D}_1$ which are not corner vertices. Analogous statement holds with the roles of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ interchanged. We denote the face of $\mathfrak{D}_2$ (or $\mathfrak{D}_1$) corresponding to the non-corner vertex (i, j) of $\mathfrak{D}_1$ (or of $\mathfrak{D}_2$, respectively) by $Q_{i,j}$. This means that if (i, j) is an inner vertex of $\mathfrak{D}$ (i.e., $1 < i < m - 1$, $1 < j < n - 1$), then $Q_{i,j}$ is a face of $\mathfrak{D}_1$ if $i + j$ is odd, and a face of $\mathfrak{D}_2$ if $i + j$ is even, and in both cases it is a square face with corners $(i - 1, j)$ (west), $(i, j - 1)$ (north), $(i + 1, j)$ (east) and $(i, j + 1)$ (south). For a vertex $(1, j)$, $1 < j < n - 1$, $Q_{1,j}$ is a face of $\mathfrak{D}_1$, if j is even, and a face of $\mathfrak{D}_2$, if j is odd, and in both cases it is a west boundary triangle face, with the corners $(1, j - 1)$ (north), $(2, j)$ (east) and $(1, j + 1)$ (south); analogously for other boundaries, see Figure 19.

We also denote by $e_{i,j}^{k,l}$ (or $e_{k,l}^{i,j}$) the (non-oriented) edge of $\mathfrak{D}$ which connects the vertices (i, j) and (k, l), for all possible i, j, k, l. This means that $e_{i,j}^{k,l}$ is also a side of faces $Q_{i,l}$ and $Q_{k,j}$ (if these faces exist, which is always true except when (i, l) or (k, j) is a corner vertex), see Figure 19.

6·3. *System of equations for the product graph of two paths*

Note that, in our case and in the above notation, Lemma 2·12 and Equation (2·8) have the following form. If w is an inner vertex of $\mathfrak{D}$, and e_1, e_2, e_3, e_4 are the NW, NE, SE, SW edges of $\mathfrak{D}$ incident to w respectively, all oriented from w, then

$$R_1(w) = M_{11}(e_1) + M_{11}(e_4) = M_{11}(e_2) + M_{11}(e_3)$$
$$= M_{12}(e_1) + M_{12}(e_2) = M_{12}(e_4) + M_{12}(e_3),$$
$$R_2(w) = M_{21}(e_1) + M_{21}(e_4) = M_{21}(e_2) + M_{21}(e_3)$$
$$= M_{22}(e_1) + M_{22}(e_2) = M_{22}(e_4) + M_{22}(e_3).$$

Also it follows from Lemma 2·8 (claim iii, local surjectivity) that if such w is in $\mathfrak{C} \subseteq \mathfrak{D}_s$, then at least one of each pair of the edges (e_1, e_2), (e_2, e_3), (e_3, e_4), (e_4, e_1) is in $\mathfrak{C}$. For the boundary vertices, we have similar equations. For example, if w is the NW corner with the SE edge e_3 beginning in w, we have

$$R_1(w) = M_{11}(e_3) = M_{12}(e_3); \quad R_2(w) = M_{21}(e_3) = M_{22}(e_3).$$

If w is on the west boundary, but not in a corner, and e_2, e_3 are the NE, SE edges beginning in w respectively, then

$$R_1(w) = M_{11}(e_2) + M_{11}(e_3) = M_{12}(e_2) = M_{12}(e_3),$$

$$R_2(w) = M_{21}(e_2) + M_{21}(e_3) = M_{22}(e_2) = M_{22}(e_3),$$

and analogous equations hold for the other boundary vertices.

It follows from Lemma 2·8 (local surjectivity) that if w is a boundary vertex (of degree 1 or 2) which is in $\mathfrak{C} \subseteq \mathfrak{D}_s$, then all the edges of $\mathfrak{D}_s$ incident to w are also in $\mathfrak{C}$.

6·4. *Face labels*

For every face F in $\mathfrak{F}$ define two labels as follows. If F is a square face, and w_1, w_2, w_3, w_4 are its west, north, east and south corners respectively, then let

$$R_1(F) = R_1(w_2) + R_1(w_4), \quad R_2(F) = R_2(w_1) + R_2(w_3). \tag{6·1}$$

If F is a triangle face, then exactly one of w_1, w_2, w_3, w_4 above will be missing, say w_1 (so F is west boundary face), and then define

$$R_1(F) = R_1(w_2) + R_1(w_4), \quad R_2(F) = R_2(w_3); \tag{6·2}$$

the other cases are analogous (just think of the missing vertex as having labels 0).

LEMMA 6·2. *In the above notation, for every face F in $\mathfrak{F}$ we have $R_1(F) = R_2(F)$.*

Proof. Suppose first that F is a square face. Let w_1, w_2, w_3, w_4 be the corners of F as above.

Let e_1, e_2, e_3, e_4 be oriented edges which are sides of F, such that e_1 goes from w_1 to w_2 (NW side), e_2 goes from w_2 to w_3 (NE side), e_3 goes from w_3 to w_4 (SE side), and e_4 goes from w_4 to w_1 (SW side), see Figure 20.

By Lemma 2·12, we have

$$R_1(w_2) = M_{12}(e_1^{-1}) + M_{12}(e_2), \quad R_1(w_4) = M_{12}(e_3^{-1}) + M_{12}(e_4).$$
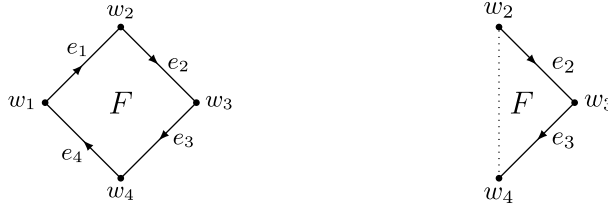
Fig. 20. Face labels: Lemma 6·2 claims that $R_1(w_2) + R_1(w_4) = R_2(w_1) + R_2(w_3)$ for the square face on the left, and $R_1(w_2) + R_1(w_4) = R_2(w_3)$ for the triangle face on the right, and similar for other triangle faces.

Together with Equation (6·1) and Lemma 2·11, this means that

$$
\begin{aligned}
R_1(F) = R_1(w_2) + R_1(w_4) &= M_{12}(e_1^{-1}) + M_{12}(e_2) + M_{12}(e_3^{-1}) + M_{12}(e_4) \\
&= M_{21}(e_1) + M_{21}(e_2^{-1}) + M_{21}(e_3) + M_{21}(e_4^{-1}).
\end{aligned}
\tag{6·3}
$$

On the other hand, again by Lemma 2·12, we have

$$
R_2(w_1) = M_{21}(e_1) + M_{21}(e_4^{-1}), \quad R_2(w_3) = M_{21}(e_2^{-1}) + M_{21}(e_3),
$$

so, by Equation (6·1), this gives us

$$
R_2(F) = R_2(w_1) + R_2(w_3) = M_{21}(e_1) + M_{21}(e_4^{-1}) + M_{21}(e_2^{-1}) + M_{21}(e_3),
$$

which is the same as the right-hand side of Equation (6·3), so $R_1(F) = R_2(F)$.

If F is a triangle face, then the proof is similar, with some summands missing in the argument above. For example, if F is a west boundary face, w_2, w_3, w_4 are its corners as above, and e_2 goes from w_2 to w_3 (NE side), e_3 goes from w_3 to w_4 (SE side), see Figure 20, then, by Lemma 2·12, Lemma 2·11 and Equation (6·2), we have

$$
\begin{aligned}
R_1(F) = R_1(w_2) + R_1(w_4) &= M_{12}(e_2) + M_{12}(e_3^{-1}) = \\
&= M_{21}(e_2^{-1}) + M_{21}(e_3) = R_2(w_3) = R_2(F).
\end{aligned}
$$

Other cases are analogous.

We fix s equal to 1 or 2 such that $\mathfrak{C}$ is a subgraph of $\mathfrak{D}_s$, as above.

6·5. *Key lemma*

The following lemma is key in this proof. It provides us with a way of applying consequently the equations from above to prove that some edges of $\mathfrak{D}_s$ do not belong to $\mathfrak{C}$, until we obtain a contradiction with Lemma 2·8.

LEMMA 6·3. *In the above notation, suppose that Q_1, Q_2 are two adjacent faces in $\mathfrak{F}$, such that Q_1 is either square or west boundary face, Q_2 is either square or north boundary face, and NE side of Q_1 coincides with SW side of Q_2. Suppose, in addition, that if Q_1 is square, then the west corner of Q_1 does not have a NW edge in $\mathfrak{C}$, and, if Q_2 is square, then the north corner of Q_2 does not have a NW edge in $\mathfrak{C}$.*

Then the south corner of Q_1 does not have a SE edge in $\mathfrak{C}$, and the east corner of Q_2 does not have a SE edge in $\mathfrak{C}$.

Analogous three statements hold with all the directions above rotated by $\pi/2$, π and $3\pi/2$.
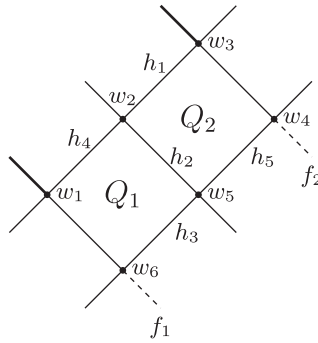
Fig. 21. Part of the graph $\mathfrak{D}$ in the proof of Lemma 6·3, consisting of two adjacent faces in the case when both faces are square. Thick edges are not in $\mathfrak{C}$ by the conditions of the lemma, and dashed edges are claimed not to be in $\mathfrak{C}$ by the lemma.

In the statement of Lemma 6·3, when we say that the west corner of Q_1 does not have a NW edge in $\mathfrak{C}$, we mean that there is either no such edge in $\mathfrak{D}$ (this will be the case when the west corner of Q_1 is on the left boundary, i.e. is of the form $(1, k)$), or there is such an edge in $\mathfrak{D}$, but it does not belong to $\mathfrak{C}$, which is equivalent to saying that all (or just one) of its labels are 0, by Equation (2·4).

By the expression "all the directions rotated by $3\pi/2$", we mean that in the statement north is changed to east, east – to south, south – to west, west – to north, and NE – to SE, SE – to SW, SW – to NW, NW – to NE; other rotations are defined analogously in a natural way.

Proof. Note that it suffices to prove the original statement above, the proofs of all statements with rotated directions are similar, up to corresponding change of directions.

Suppose first that Q_1 and Q_2 are both square faces.

Let w_1 be the west vertex of Q_1, w_2 be the north vertex of Q_1 (which is also the west vertex of Q_2), w_3 be the north vertex of Q_2, w_4 be the east vertex of Q_2, w_5 be the south vertex of Q_2 (which is also the east vertex of Q_1), and w_6 be the south vertex of Q_1. Let h_1 be the edge going from w_2 to w_3 (NW side of Q_2), h_2 be the edge going from w_2 to w_5 (NE side of Q_1 and SW side of Q_2), h_3 be the edge going from w_5 to w_6 (SE side of Q_1), h_4 be the edge going from w_1 to w_2 (NW side of Q_1), and h_5 be the edge going from w_4 to w_5 (SE side of Q_2). Since we will use only M_{11} and M_{22} labels below, the orientation of edges is not important here, see Figure 21.

We can suppose that there are SE edges in $\mathfrak{D}_s$ both from w_6 and w_4, otherwise the proof is similar. Let f_1 be the SE edge at w_6, and f_2 be the SE edge at w_4. By Lemma 6·2, we have $R_1(Q_1) = R_2(Q_1)$. Note that $R_1(Q_1) = R_1(w_2) + R_1(w_6)$ by Equation (6·1), and $R_1(w_2) = M_{11}(h_1) + M_{11}(h_2)$, $R_1(w_6) = M_{11}(h_3) + M_{11}(f_1)$ by Equation (2·8). Also we have $R_2(Q_1) = R_2(w_1) + R_2(w_5)$ by Equation (6·1), and $R_2(w_1) = M_{22}(h_4)$, $R_2(w_5) = M_{22}(h_2) + M_{22}(h_5)$ by Equation (2·8) and since, by the conditions of this lemma, w_1 has no NW edge in $\mathfrak{C}$ (and so this edge, even if it exists in $\mathfrak{D}_s$, has M_{22} label equal to 0 by Equation (2·4)). Thus, we have

$$
\begin{aligned}
R_1(Q_1) &= M_{11}(h_1) + M_{11}(h_2) + M_{11}(h_3) + M_{11}(f_1) \\
&= R_2(Q_1) = M_{22}(h_4) + M_{22}(h_2) + M_{22}(h_5).
\end{aligned}
\tag{6·4}
$$

In the same way, by Lemma 6·2, we have $R_1(Q_2) = R_2(Q_2)$. Note that $R_1(Q_2) = R_1(w_3) + R_1(w_5)$, by Equation (6·1), and $R_1(w_3) = M_{11}(h_1)$, $R_1(w_5) = M_{11}(h_2) + M_{11}(h_3)$ by Equation (2·8) and since, by the conditions of this lemma, w_3 has no NW edge in $\mathfrak{C}$ (and so this edge, even if it exists in $\mathfrak{D}_s$, has M_{11} label equal to 0 by Equation (2·4)). Also, we have $R_2(Q_2) = R_2(w_2) + R_2(w_4)$ by Equation (6·1), and $R_2(w_2) = M_{22}(h_4) + M_{22}(h_2)$, $R_2(w_4) = M_{22}(h_5) + M_{22}(f_2)$ by Equation (2·8). Thus, we have

$$
\begin{aligned}
R_1(Q_2) &= M_{11}(h_1) + M_{11}(h_2) + M_{11}(h_3) \\
&= R_2(Q_2) = M_{22}(h_4) + M_{22}(h_2) + M_{22}(h_5) + M_{22}(f_2).
\end{aligned}
\tag{6·5}
$$

Substracting (6·5) from (6·4), we obtain

$$
R_1(Q_1) - R_1(Q_2) = M_{11}(f_1) = R_2(Q_1) - R_2(Q_2) = -M_{22}(f_2),
\tag{6·6}
$$

but $M_{11}(f_1) \geq 0$, $-M_{22}(f_2) \leq 0$, so (6·6) implies that $M_{11}(f_1) = M_{22}(f_2) = 0$, and so $f_1, f_2 \notin \mathfrak{C}$ by Equation (2·4), as required.

The other cases, when Q_1 is the west boundary, or Q_2 is the north boundary, or both, are analogous, with the only difference in the proof being that some summands do not appear (we can think of them as being equal to 0). For example, if Q_1 is a west boundary, and Q_2 is a square, then all the notation and equalities are almost the same as above, except that w_1 and h_4 do not exist, so $R_2(Q_1) = R_2(w_5) = M_{22}(h_2) + M_{22}(h_5)$, and $R_2(w_2) = M_{22}(h_2)$, so $R_2(Q_2) = M_{22}(h_2) + M_{22}(h_5) + M_{22}(f_2)$. The same argument as above applies, hence the lemma.

6·6. *Case of odd m or n*

We now continue with the proof of the theorem. Suppose first that at least one of m and n is odd. Then, by Remark 6·1, we can always choose one of the connected components of $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of $\mathfrak{D}$, and suppose that $\mathfrak{C}$ is a (connected) subgraph of this component. We will choose now $\mathfrak{D}_2$, and so we can suppose that $\mathfrak{C} \subseteq \mathfrak{D}_2$.

Note that all faces $Q_{i,j}$ of $\mathfrak{D}_2$ have even $i + j \geq 4$. For an even k such that $4 \leq k \leq n + 1$ denote by S_k the set of all faces $Q_{i,j}$ of $\mathfrak{D}_2$ such that $i + j = k$. Then for a given $k \leq n - 1$ all faces in S_k form a "SW-NE diagonal":

$$
S_k = \{Q_{1,k-1}, \; Q_{2,k-2}, \ldots, \; Q_{k-2,2}, \; Q_{k-1,1}\},
$$

with the NE side of $Q_{i,k-i}$ coinciding with the SW side of $Q_{i+1,k-i-1}$ for $i = 1, \ldots, k - 2$; for $k = n$ or $k = n + 1$ (the one which is even) S_k is as above, but with the first face missing, and if $k = n + 1$ and $m = n + 1$, also the last face missing.

Let also E_k, for an even k such that $4 \leq k \leq n + 1$, be the set of all edges of $\mathfrak{D}_2$ which go from NW to SE and are the sides of some faces in S_k. This means that, if $k \leq n$, then

$$
E_k = \{e_{1,k-2}^{2,k-1}, \; e_{2,k-3}^{3,k-2}, \ldots, \; e_{k-3,2}^{k-2,3}, \; e_{k-2,1}^{k-1,2}\};
$$

for $k = n + 1$ (in the case it is even) E_k is as above, but with the first edge missing. Note also that each edge set E_k, for $4 \leq k \leq n + 1$, separates our graph $\mathfrak{D}_2$ in two, which means that, after deleting all the edges of E_k from $\mathfrak{D}_2$, the remaining graph will become disconnected, namely it will have two connected components.

The idea is to proceed from NW to SE, showing that the edges in E_k are not in $\mathfrak{C}$, with increasing k, until we get a contradiction when $k = n$ or $k = n + 1$.
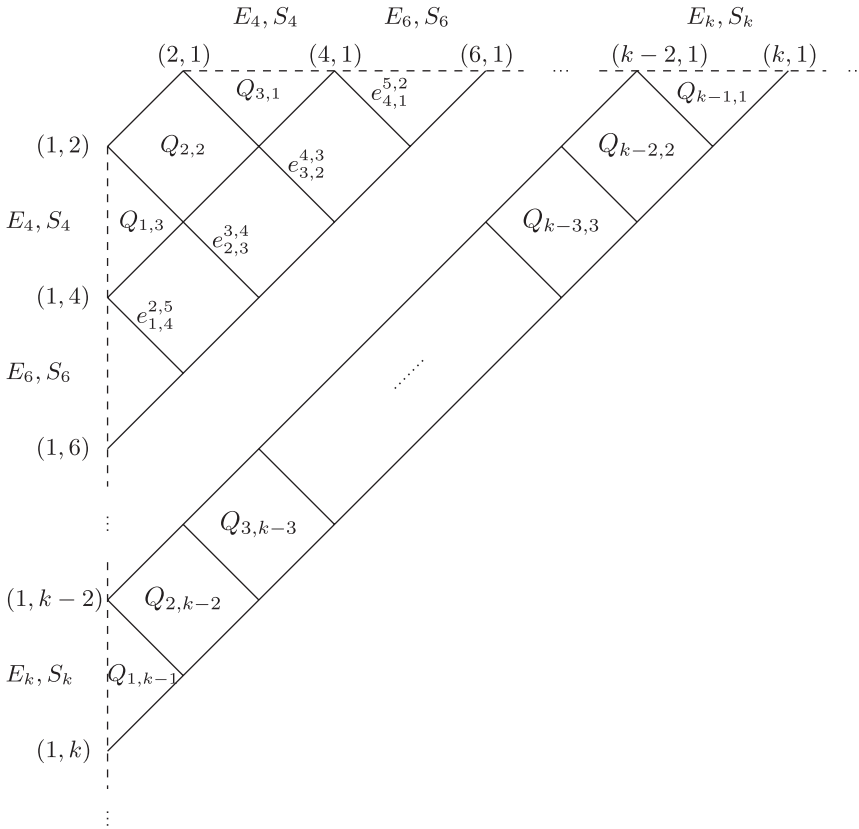
Fig. 22. Part of the graph $\mathfrak{D}_2$ in the case of odd m or n. The solid edges are in $\mathfrak{D}_2$, and the dashed edges are the boundary edges of $\mathfrak{D}'_2$, which are not in $\mathfrak{D}_2$.

First apply Lemma 6·3 to $Q_{1,3}$ and $Q_{2,2}$; the conditions of the lemma are satisfied, since NE side of $Q_{1,3}$ coincides with the SW side of $Q_{2,2}$, $Q_{1,3}$ is west boundary face, and the north vertex of $Q_{2,2}$ is on the north boundary of $\mathfrak{D}$, in particular it has no NW edge. By Lemma 6·3, the south corner of $Q_{1,3}$ does not have a SE edge in $\mathfrak{C}$, which, in our terms, means that if $n \geq 6$, then $e_{1,4}^{2,5} \notin \mathfrak{C}$ (if $n = 5$, then the south corner of $Q_{1,3}$ is on the south boundary, so this condition is vacuous); and the east corner of $Q_{2,2}$ does not have a SE edge in $\mathfrak{C}$, which in our terms means that $e_{3,2}^{4,3} \notin \mathfrak{C}$, see Figure 22.

Analogously, apply Lemma 6·3 to $Q_{2,2}$ and $Q_{3,1}$; the conditions are again satisfied. We obtain that the south corner of $Q_{2,2}$ does not have a SE edge in $\mathfrak{C}$, which means that $e_{2,3}^{3,4} \notin \mathfrak{C}$, and the east corner of $Q_{3,1}$ does not have a SE edge in $\mathfrak{C}$, which means that $e_{4,1}^{5,2} \notin \mathfrak{C}$ (recall that $m \geq 6$). Thus, all the edges of E_6 are not in $\mathfrak{C}$, but $\mathfrak{C}$ is connected, so it should be contained in one of the connected components of the graph obtained by deleting the edges of E_6 from $\mathfrak{D}_2$. It cannot be the "NW component" (i.e., the one containing the vertex $(2, 1)$), since in this component there are no vertices of the form $(5, i)$ for some i, so it does not project surjectively to $\widetilde{\Gamma}_1$, but $\mathfrak{C}$ should, by Lemma 2·8. This means that $\mathfrak{C}$ is contained in the other component. In particular, we see that none of the edges of E_4, as well as E_6, are in $\mathfrak{C}$.

We claim that none of the edges of E_k, for even k such that $4 \leq k \leq n + 1$, are in $\mathfrak{C}$. (For $n = 5$ and $n = 6$ this is already proved). We prove this by induction, for E_4 and E_6 it is

proved above, so suppose the claim is proved for all even k such that $4 \leq k \leq k_0$, where $6 \leq k_0 \leq n-1$, k_0 is even, and let us prove it for $k = k_0 + 2$.

Consider all the pairs of adjacent faces in S_{k_0}: Q_{1,k_0-1} and Q_{2,k_0-2}, Q_{2,k_0-2} and Q_{3,k_0-3}, ..., $Q_{k_0-2,2}$ and $Q_{k_0-1,1}$. By induction hypothesis, all these pairs satisfy the conditions of Lemma 6·3. Indeed, all the existing in $\mathfrak{D}_2$ NW edges at the north and west corners of the faces from S_{k_0} belong to E_{k_0-2} by definition, and so they do not belong to $\mathfrak{C}$. Applying Lemma 6·3 to all the pairs above, we obtain that all the SE edges at the south and east corners of the faces from S_{k_0}, which are exactly all the edges from E_{k_0+2}, are not in $\mathfrak{C}$, and the claim is proved.

Thus, in particular, if n is even, then no edges from E_n are in $\mathfrak{C}$, and if $n+1$ is even, then no edges from E_{n+1} are in $\mathfrak{C}$. Suppose first that n is even. Then deleting all the edges of E_n from $\mathfrak{D}_2$ results in a graph with two connected components, none of which projects surjectively to $\widetilde{\Gamma}_1$ (namely, one of them does not contain vertices with the first coordinate equal to 1, and the other one does not contain vertices with the first coordinate equal to $n-1$). This is impossible by Lemma 2·8. In the same way, if n is odd, then again deleting all the edges of E_{n+1} from $\mathfrak{D}_2$ results in a graph with two connected components, none of which projects surjectively to $\widetilde{\Gamma}_1$ (namely, one of them does not contain vertices with the first coordinate equal to 1, and the other one does not contain vertices with the first coordinate equal to n), and this is impossible by Lemma 2·8.

Thus, $\mathbb{G}(P_m)$ and $\mathbb{G}(P_n)$ are not commensurable if $m > n \geq 5$ and at least one of m and n is odd.

6·7. *Case of even m and n*

It remains to consider the case when both m and n are even and $m > n \geq 6$; in particular, $m \geq n+2$. In this case $\mathfrak{D}_1$ (the connected component of $\mathfrak{D}$ containing $(1, 1)$) contains all four corners of $\mathfrak{D}$, and $\mathfrak{D}_2$ contains no corners. We now have to consider two subcases, depending on whether $\mathfrak{C}$ lies inside $\mathfrak{D}_1$ or $\mathfrak{D}_2$.

Suppose first that $\mathfrak{C}$ lies inside $\mathfrak{D}_2$. Then similarly to the case when n and m are odd we can derive a contradiction. Indeed, in the above notation and in the same way as above we can prove by induction that none of the edges of E_k, for even k such that $4 \leq k \leq n$, are in $\mathfrak{C}$. In particular, no edges from E_n are in $\mathfrak{C}$, and deleting all the edges of E_n from $\mathfrak{D}_2$ results in a graph with two connected components, none of which projects surjectively to $\widetilde{\Gamma}_1$ (namely, one of them does not contain vertices with the first coordinate equal to 1, and the other one does not contain vertices with the first coordinate equal to $n-1$). This is impossible by Lemma 2·8.

So it remains to consider the case when $\mathfrak{C}$ lies inside $\mathfrak{D}_1$, which is more subtle.

For a vertex (i, j) of $\mathfrak{D}_1$ denote by $A_{i,j}$ the set of all edges on the "NW-SE diagonal" of $\mathfrak{D}_1$ passing through (i, j), i.e., $A_{i,j}$ consists of all edges connecting vertices (i', j') of $\mathfrak{D}_1$ with $i' - j' = i - j$. These are the edges of the form $e_{i+k,j+k}^{i+k+1,j+k+1}$, where $\max\{1 - i, 1 - j\} \leq k \leq \min\{m - 2 - i, n - 2 - j\}$. Note that every $A_{i,j}$ is equal to $A_{i',j'}$ with $i' = 1$ or $j' = 1$ (west or north boundary), and to $A_{i'',j''}$ with $i'' = m - 1$ or $j'' = n - 1$ (east or south boundary).

Analogously, for a vertex (i, j) of $\mathfrak{D}_2$ denote by $B_{i,j}$ the set of all edges on the "SW-NE diagonal" of $\mathfrak{D}_2$ passing through (i, j), i.e., $B_{i,j}$ consists of all edges connecting vertices (i', j') of $\mathfrak{D}_1$ with $i' + j' = i + j$. These are the edges of the form $e_{i+k,j-k}^{i+k+1,j-k-1}$, where $\max\{1 - i, j + 1 - n\} \leq k \leq \min\{m - 2 - i, j - 2\}$. Note that every $B_{i,j}$ is equal to $B_{i',j'}$ with $i' = 1$ or $j' = n - 1$ (west or south boundary), and to $B_{i'',j''}$ with $i'' = m - 1$ or $j'' = 1$ (east or north boundary).

Recall that $\mathfrak{D}_1$ consists of all vertices (i, j) of $\mathfrak{D}$ such that $i + j$ is even (or, equivalently, $i - j$ is even). Successive application of Lemma 6·3 allows us to prove that, informally speaking, every second diagonal of $\mathfrak{D}_1$ is not in $\mathfrak{C}$, which is the content of the following lemma.

LEMMA 6·4. *In the above notation, with even m and n, suppose that (i, j) is a vertex of $\mathfrak{D}_1$ (i.e. $i + j$ and $i - j$ are even, $1 \leq i \leq m - 1$, $1 \leq j \leq n - 1$). If one of the following conditions holds*:

  (i) *$i - j$ is 2 modulo 4;*
  (ii) *$(m - i) - (n - j)$ is 2 modulo 4, or, equivalently, $i - j + (n - m)$ is 2 modulo 4;*

then all the edges of $A_{i,j}$ are not in $\mathfrak{C}$. If one of the following conditions holds:

  (iii) *$i + j - m$ is 2 modulo 4;*
  (iv) *$i + j - n$ is 2 modulo 4.*

then all the edges of $B_{i,j}$ are not in $\mathfrak{C}$.

Proof. We first prove the first claim, so we suppose that $i - j$ is 2 modulo 4, and we need to prove that all the edges of $A_{i,j}$ are not in $\mathfrak{C}$. According to the remarks above, it suffices to prove the claim when $i = 1$ or $j = 1$.

We start by proving the claim for $A_{1,3}$ and $A_{3,1}$. The idea is to proceed diagonally from NW to SE successively applying Lemma 6·3. Note that

$$A_{1,3} = \{e_{1,3}^{2,4}, e_{2,4}^{3,5}, \ldots, e_{n-4,n-2}^{n-3,n-1}\}, \quad A_{3,1} = \{e_{3,1}^{4,2}, e_{4,2}^{5,3}, \ldots, e_{n,n-2}^{n+1,n-1}\}.$$

Note that the conditions of Lemma 6·3 are satisfied for $Q_{1,2}$ and $Q_{2,1}$ (since $Q_{1,2}$ is west boundary face, and $Q_{2,1}$ is north boundary face), so applying this lemma we deduce that $e_{1,3}^{2,4}$, $e_{3,1}^{4,2} \notin \mathfrak{C}$. Furthermore, $Q_{2,3}$ and $Q_{3,2}$ satisfy conditions of Lemma 6·3 (since the west corner of $Q_{2,3}$ is on the west boundary of $\mathfrak{D}$, and the north corner of $Q_{3,2}$ is on the north boundary), so applying this lemma we deduce that $e_{2,4}^{3,5}$, $e_{4,2}^{5,3} \notin \mathfrak{C}$.

Proceeding by induction, we prove that $e_{k-2,k}^{k-1,k+1}$, $e_{k,k-2}^{k+1,k-1} \notin \mathfrak{C}$ for $3 \leq k \leq n - 2$. Indeed, suppose that this is true for $3 \leq k \leq k_0$, for some $4 \leq k_0 \leq n - 3$ (which is the case for $k_0 = 3, 4$ as shown above), and prove it for $k = k_0 + 1$. The faces Q_{k_0-1,k_0} and Q_{k_0,k_0-1} satisfy the conditions of Lemma 6·3 (since the NW edge at the west corner of Q_{k_0-1,k_0} is $e_{k_0-3,k_0-1}^{k_0-2,k_0} \notin \mathfrak{C}$ by the induction hypothesis for $k = k_0 - 1$, and the NW edge at the north corner of Q_{k_0,k_0-1} is $e_{k_0-1,k_0-3}^{k_0,k_0-2} \notin \mathfrak{C}$ by the induction hypothesis for $k = k_0 - 1$). So, applying Lemma 6·3, we get that $e_{k_0-1,k_0+1}^{k_0,k_0+2}$, $e_{k_0+1,k_0-1}^{k_0+2,k_0} \notin \mathfrak{C}$, which is exactly what we wanted.

This already shows that all the edges in $A_{1,3}$ are not in $\mathfrak{C}$, and just two more edges from $A_{3,1}$ remain. Applying Lemma 6·3 to the faces $Q_{n-3,n-2}$ and $Q_{n-2,n-3}$ (which is possible since we proved above that $e_{n-5,n-3}^{n-4,n-2}$, $e_{n-3,n-5}^{n-2,n-4} \notin \mathfrak{C}$), we obtain that $e_{n-1,n-3}^{n,n-2} \notin \mathfrak{C}$. Applying Lemma 6·3 to the faces $Q_{n-2,n-1}$ and $Q_{n-1,n-2}$ (which is possible since we proved above that $e_{n-4,n-2}^{n-3,n-1}$, $e_{n-2,n-4}^{n-1,n-3} \notin \mathfrak{C}$), we obtain that $e_{n,n-2}^{n+1,n-1} \notin \mathfrak{C}$. We conclude that all the edges of $A_{1,3}$ and $A_{3,1}$ are not in $\mathfrak{C}$, see Figure 23.

We now prove that all the edges in $A_{1,j}$ are not in $\mathfrak{C}$ for all $3 \leq j \leq n - 1$, j is 3 modulo 4 (so that $1 - j$ is 2 modulo 4). For $j = 3$ this is proved above. We proceed by induction. Let $3 \leq j_0 \leq n - 5$, j_0 is 3 modulo 4, and suppose this is true for all j which are 3 modulo 4, $3 \leq j \leq j_0$, we will prove this is also true for $j = j_0 + 4$. Note that if $j_0 + 4 = n - 1$,
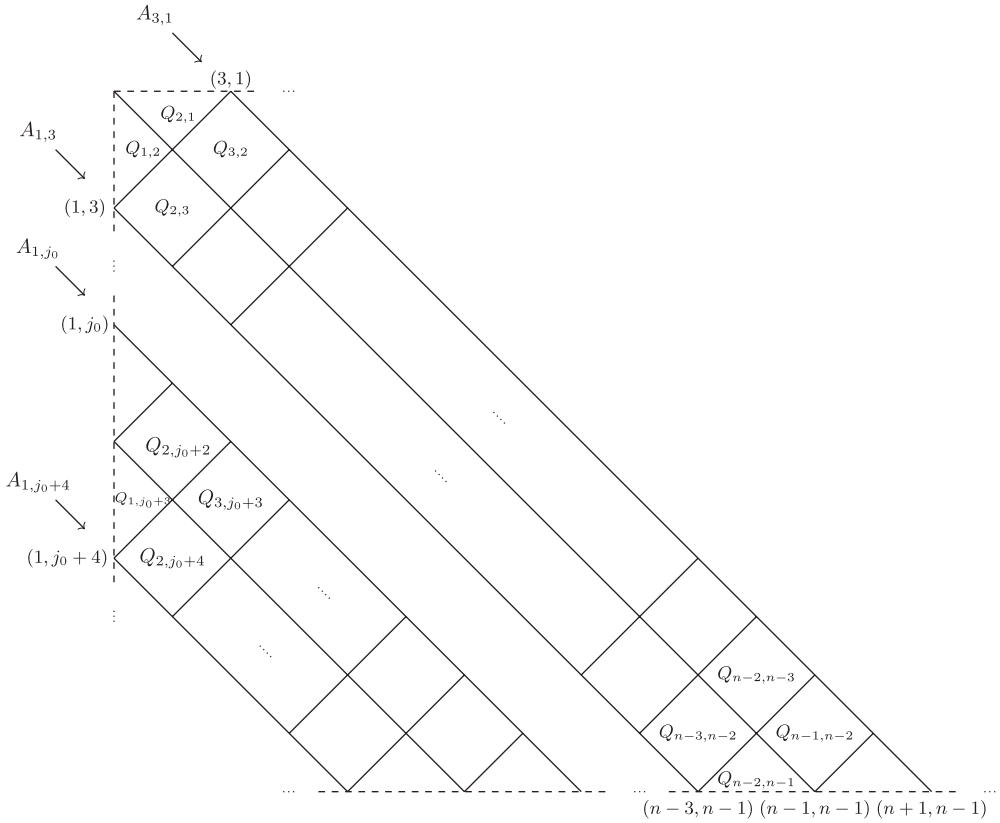
Fig. 23. Part of the graph $\mathfrak{D}_1$ in the case of even m and n, as in the proof of Lemma 6·4. The solid edges are in $\mathfrak{D}_1$, and the dashed edges are the boundary edges of $\mathfrak{D}_1'$, which are not in $\mathfrak{D}_1$.

then the claim is vacuous, since A_{1,j_0+4} is empty, so we can suppose that $j_0 + 4 \leq n - 3$, or $j_0 \leq n - 7$.

Note that

$$A_{1,j_0} = \{e_{1,j_0}^{2,j_0+1}, e_{2,j_0+1}^{3,j_0+3}, \ldots, e_{n-j_0-1,n-2}^{n-j_0,n-1}\}, \quad A_{1,j_0+4} = \{e_{1,j_0+4}^{2,j_0+5}, e_{2,j_0+5}^{3,j_0+6}, \ldots, e_{n-j_0-5,n-2}^{n-j_0-4,n-1}\}.$$

We know by induction hypothesis that all the edges in A_{1,j_0} are not in $\mathfrak{C}$, and need to prove the same for A_{1,j_0+4}. Applying Lemma 6·3 to Q_{1,j_0+3} and Q_{2,j_0+2} (this is possible, since Q_{1,j_0+3} is a west boundary face, and the NW edge at the north corner of Q_{2,j_0+2} is $e_{1,j_0}^{2,j_0+1} \in A_{1,j_0}$, so it is not in $\mathfrak{C}$), we obtain that $e_{1,j_0+4}^{2,j_0+5} \notin \mathfrak{C}$ (since it is the SE edge at the south vertex of Q_{1,j_0+3}).

Now apply Lemma 6·3 to Q_{2,j_0+4} and Q_{3,j_0+3} (this is possible, since the west corner of Q_{2,j_0+4} is on the west boundary of $\mathfrak{D}$, and the NW edge at the north corner of Q_{3,j_0+3} is $e_{2,j_0+1}^{3,j_0+2} \in A_{1,j_0}$, so it is not in $\mathfrak{C}$), we obtain that $e_{2,j_0+5}^{3,j_0+6} \notin \mathfrak{C}$ (since it is the SE edge at the south vertex of Q_{2,j_0+4}).

If $j_0 = n - 7$, so $j_0 + 4 = n - 3$ and A_{1,j_0+4} contains only two edges, then we are done; otherwise, $j_0 \leq n - 9$, and we proceed by (local) induction (inside the main induction) to show that $e_{1+k,j_0+4+k}^{2+k,j_0+5+k} \notin \mathfrak{C}$ for $0 \leq k \leq n - j_0 - 6$. Suppose this is true for all $0 \leq k \leq k_0$, for some $2 \leq k_0 \leq n - j_0 - 7$ (which is the case for $k_0 = 0$ and $k_0 = 1$, as shown above), and we need to prove it for $k = k_0 + 1$. Indeed, we can apply Lemma 6·3 to Q_{k_0+2,j_0+k_0+4} and

Q_{k_0+3, j_0+k_0+3} (this is possible, since the NW edge at the west corner of Q_{k_0+2, j_0+k_0+4} is $e_{k_0, j_0+k_0+3}^{k_0+1, j_0+k_0+4}$, which is not in $\mathfrak{C}$ by the (local) induction hypothesis for $k = k_0 - 1$, and the NW edge at the north corner of Q_{k_0+3, j_0+k_0+3} is $e_{k_0+2, j_0+k_0+1}^{k_0+3, j_0+k_0+2}$, which is in A_{1, j_0}, and so also not in $\mathfrak{C}$). So, we get that the SE edge at the south vertex of Q_{k_0+2, j_0+k_0+4}, which is $e_{k_0+2, j_0+k_0+5}^{k_0+3, j_0+k_0+6}$, is not in $\mathfrak{C}$, and this is exactly what we need (in the local induction).

This shows that all the edges of A_{1, j_0+4} are not in $\mathfrak{C}$, and this is what we need in the main induction. Thus all the edges in $A_{1, j}$ for $3 \leq j \leq n - 1$, j is 3 modulo 4, are not in $\mathfrak{C}$, see Figure 23.

To prove the lemma, it remains to show that all the edges in $A_{i, 1}$ for $3 \leq i \leq m - 1$, i is 3 modulo 4, are not in $\mathfrak{C}$. The proof is similar to the one for edges $A_{1, j}$, but formally we need to consider two cases, depending on whether $A_{i, 1}$ finishes on the south or east boundary of $\mathfrak{D}$: when $3 \leq i \leq m - n + 1$, and when $m - n + 1 < i \leq m - 1$.

Suppose first that $3 \leq i \leq m - n + 1$. If $i = 3$, then the claim is already proved above. We proceed by induction. Let $3 \leq i_0 \leq m - n - 3$, and suppose all the edges of $A_{i, 1}$ are not in $\mathfrak{C}$ for all i which are 3 modulo 4, $3 \leq i \leq i_0$, we need to prove that all the edges of $A_{i_0+4, 1}$ are also not in $\mathfrak{C}$. This can be done by Lemma 6·3, applied successively (by induction, as above) to the pairs of faces $(Q_{i_0+2, 2}, Q_{i_0+4, 1})$, $(Q_{i_0+3, 3}, Q_{i_0+5, 2})$, ..., $(Q_{i_0+n-1, n-1}, Q_{i_0+n, n-2})$.

Finally, the proof that all the edges in $A_{i, 1}$ for $m - n + 1 < i \leq m - 1$, i is 3 modulo 4, are not in $\mathfrak{C}$, is analogous to the proofs above; we omit the details. This proves the first claim of the lemma.

Now note that the second, third and fourth claims of the lemma can be obtained from the first claim by rotating all the directions by π, $\pi/2$ and $3\pi/2$ respectively. Rotation by $\pi/2$ means replacing i by $m - i$, leaving j unchanged, and changing A's to B's; rotation by π means replacing i by $m - i$ and j by $n - j$, without changing the A's; and rotation by $3\pi/2$ means leaving i unchanged, replacing j by $n - j$ and changing A's to B's. So the proof is similar to the one above, with application of the corresponding claims of Lemma 6·3. This proves the lemma.

Recall that both m and n are even. Suppose first that one of m and n is 0 modulo 4, and the other is 2 modulo 4. Then every vertex (i, j) of $\mathfrak{D}_1$ satisfies one of the first two conditions and one of the last two conditions of Lemma 6·4. Hence, all the edges of $A_{i, j}$ and $B_{i, j}$ (for all vertices (i, j) of $\mathfrak{D}_1$), which are all the edges of $\mathfrak{D}_1$, are not in $\mathfrak{C}$, and this is a contradiction, so in this case $\mathbb{G}(P_m)$ and $\mathbb{G}(P_n)$ are not commensurable.

Suppose now that both m and n are 0 modulo 4. In particular, $n \geq 8$, $m \geq n + 4 \geq 12$. By Lemma 6·4, for a vertex (i, j) of $\mathfrak{D}_1$ we have that, if $i - j$ is 2 modulo 4, then all the edges of $A_{i, j}$ are not in $\mathfrak{C}$, and if $i + j$ is 2 modulo 4, then all the edges of $B_{i, j}$ are not in $\mathfrak{C}$. In other words, we have

$$e_{i, j}^{i+1, j+1} \notin \mathfrak{C}, \text{ if } i - j = 2 \mod 4, \quad e_{i', j'}^{i'+1, j'-1} \notin \mathfrak{C}, \text{ if } i' + j' = 2 \mod 4. \tag{6·7}$$

Let $m_0 = m/2$, $n_0 = n/2$, so that, when considered on the plane, the vertices of the form (i, n_0) of $\mathfrak{D}_1$ are on the horizontal axis of symmetry of $\mathfrak{D}_1$, the vertices of the form (m_0, j) are on the vertical axis of symmetry of $\mathfrak{D}_1$, and (m_0, n_0) is the "central" vertex of $\mathfrak{D}_1$.

Let V be the set of vertices (i, j) of $\mathfrak{D}_1$ such that either $j \leq n_0$ and $j < i < m - j$, or $j \geq n_0$ and $n - j < i < m - n + j$. Let also E be the set of edges of $\mathfrak{D}_1$ which have at least one of the end vertices in V. Note that all the vertices of $\mathfrak{D}_1$ of the form (m_0, j), for all $j = 1, \ldots, n - 1$, are in V, so all the edges adjacent to them are in E, and this means that
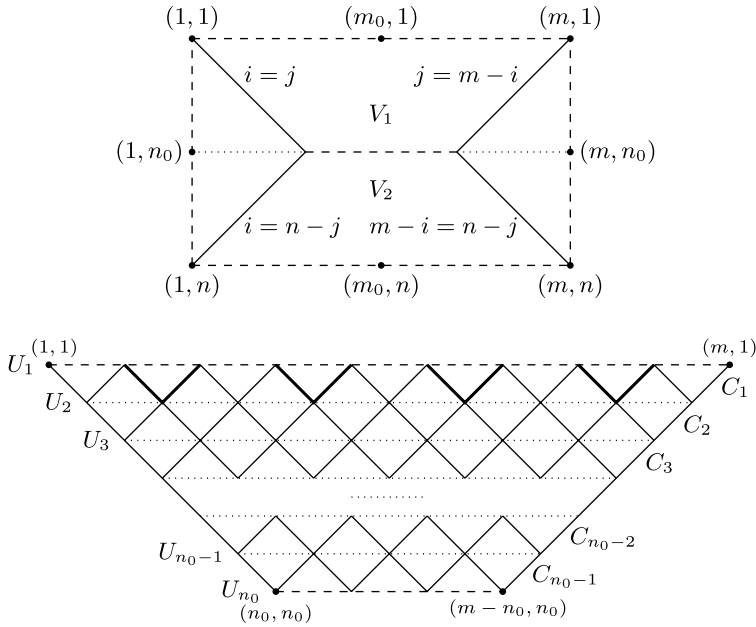
Fig. 24. Above: schematic figure depicting the graph $\mathfrak{D}_1$ in the case when both m and n are 0 modulo 4, with the regions containing the vertices in V_1 and V_2 marked. Below: the region with vertices in V_1, with all solid edges in $\mathfrak{D}_1$ and thick edges not in $\mathfrak{C}$ by Lemma 6·4.

the graph obtained from $\mathfrak{D}_1$ by deleting all the edges from E does not project surjectively to $\widetilde{\Gamma}_1$. We claim that all the edges of E are not in $\mathfrak{C}$. This will immediately imply a contradiction by Lemma 2·8.

Let $V = V_1 \cup V_2$, where V_1 are all vertices of V of the form (i, j) with $j \leq n_0$, and V_2 are the vertices of V of the form (i, j) with $j \geq n_0$. Let also $E = E_1 \cup E_2$, where E_1 are all edges in E with both end vertices in V_1, and E_2 are all edges in E with both end vertices in V_2. It suffices to prove that all the edges in E_1 are not in $\mathfrak{C}$. The proof for E_2 is similar.

Let $V_1 = U_1 \cup U_2 \cup \cdots \cup U_{n_0}$, where U_j, for $1 \leq j \leq n_0$, are all the vertices of V_1 of the form (i, j), for all admissible i, namely $j < i < m - j$. Let also $E_1 = C_1 \cup C_2 \cup \cdots \cup C_{n_0-1}$, where C_k, for $1 \leq k \leq n_0 - 1$, consists of all edges of E_1 connecting vertices with second coordinate equal to k to vertices of E_1 with second coordinate equal to $k + 1$. This means that

$$C_k = \{e_{k+1,k+1}^{k+2,k}, \ e_{k+2,k}^{k+3,k+1}, \ e_{k+3,k+1}^{k+4,k}, \ldots, \ e_{m-k-3,k+1}^{m-k-2,k}, \ e_{m-k-2,k}^{m-k-1,k+1}\},$$
$$U_k = \{(k + 2, k), (k + 4, k), \ldots, (m - k - 2, k)\}.$$

In particular, C_k contains $m - 2k - 2$ edges, see Figure 24.

We now prove by induction on k that all the edges in C_k are not in $\mathfrak{C}$, for all $1 \leq k \leq n_0 - 1$. First consider the case $k = 1$. We have

$$C_1 = \{e_{2,2}^{3,1}, \ e_{3,1}^{4,2}, \ e_{4,2}^{5,1}, \ldots, \ e_{m-5,1}^{m-4,2}, \ e_{m-4,2}^{m-3,1}, \ e_{m-3,1}^{m-2,2}\},$$
$$U_1 = \{(3, 1), (5, 1), \ldots, (m - 3, 1)\}.$$

By (6·7), we know that all the edges from C_1 of the from $e_{4i-1,1}^{4i,2}$ and $e_{4i,2}^{4i+1,1}$, where $i = 1, \ldots, m/4 - 1$, are not in $\mathfrak{C}$. But, since $e_{4i-1,1}^{4i,2} \notin \mathfrak{C}$, the vertex $(4i - 1, 1)$ of U_1 is not in

$\mathfrak{C}$ by Lemma 2·8 (local surjectivity), so $e_{4i-2,2}^{4i-1,1} \notin \mathfrak{C}$, for all $i = 1, 2, \ldots, m/4 - 1$. In the same way, since $e_{4i,2}^{4i+1,1} \notin \mathfrak{C}$, also $(4i + 1, 1) \notin \mathfrak{C}$ by Lemma 2·8, and so $e_{4i+1,1}^{4i+2,2} \notin \mathfrak{C}$, for all $i = 1, 2, \ldots, m/4 - 1$. This shows that indeed all the edges of C_1 are not in $\mathfrak{C}$.

Now suppose that for some $1 \le k \le n_0 - 2$ all the edges in C_k are not in $\mathfrak{C}$, and we need to prove that all the edges in C_{k+1} are also not in $\mathfrak{C}$. We have C_k and U_k as above, so

$$C_{k+1} = \{e_{k+2,k+2}^{k+3,k+1}, \ e_{k+3,k+1}^{k+4,k+2}, \ e_{k+4,k+2}^{k+5,k+1}, \ldots, e_{m-k-4,k+2}^{m-k-3,k+1}, \ e_{m-k-3,k+1}^{m-k-2,k+2}\},$$
$$U_{k+1} = \{(k + 3, k + 1), (k + 5, k + 1), \ldots, (m - k - 3, k + 1)\}.$$

By the induction hypothesis, all the NE and NW edges at all the vertices of U_{k+1} are not in $\mathfrak{C}$, since they belong to C_k. By Lemma 2·8 (local surjectivity), all the vertices of U_{k+1} are also not in $\mathfrak{C}$, and so all the SE and SW edges at all the vertices of U_{k+1} are not in $\mathfrak{C}$, and these are exactly all the edges in C_{k+1}.

Thus, all the edges in C_k, for $1 \le k \le n_0 - 1$, which are all the edges of E_1, are not in $\mathfrak{C}$. This means that $\mathbb{G}(P_m)$ and $\mathbb{G}(P_n)$ are not commensurable if both m and n are 0 modulo 4.

It remains to consider the case when both m and n are 2 modulo 4. According to Theorem 4·1 in this case, if $m = 4k + 2$ and $n = 4l + 2$, for some $k \ne l$, $k, l \ge 1$, $\mathbb{G}(P_m)$ is commensurable to $\mathbb{G}(T_{k,k+1})$ and $\mathbb{G}(P_n)$ is commensurable to $\mathbb{G}(T_{l,l+1})$, but $\mathbb{G}(T_{k,k+1})$ and $\mathbb{G}(T_{l,l+1})$ are not commensurable, according to [**CKZ**, theorem 4·5], so $\mathbb{G}(P_m)$ and $\mathbb{G}(P_n)$ are also not commensurable. Note that this can also be proved directly using the above equations, but it is not completely straightforward, and we omit the argument.

This completes the proof of Theorem 1·2.

REFERENCES

[A13]   I. AGOL. The virtual Haken conjecture. *Doc. Math.* **18**: 1045–1087.
[B72]   H. BASS. The degree of polynomial growth of finitely generated nilpotent groups. *Proc. London Math. Soc.* **3**, 4 (1972), 603–614.
[BJN09]   J. BEHRSTOCK, T. JANUSZKIEWICZ and W. NEUMANN. Commensurability and QI classification of free products of finitely generated abelian groups. *Proc. Amer. Math. Soc.* **137**, 3 (2009), 811–813.
[BN08]   J. BEHRSTOCK and W. NEUMANN. Quasi-isometric classification of graph manifold groups. *Duke Math. J.* **141**, 2 (2008), 217–240.
[BM00]   M. BURGER and S. MOZES. Lattices in product of trees. *Inst. Hautes Études Sci. Publ. Math.* **92** (2000), 151–194.
[CKZ]   M. CASALS-RUIZ, I. KAZACHKOV and A. ZAKHAROV. On commensurability of right-angled Artin groups I: RAAGs defined by trees of diameter 4, *Revista Matemática Iberoamericana* **35**:2 (2019), 521–560.
[CH17]   D. CHISTIKOV and C. HAASE. On the complexity of quantified integer programming. In ICALP'17, vol. 80 of LIPIcs (2017), pp 94:1–94:13.
[DW93]   P. DELIGNE and G. D. MOSTOW. Commensurabilities among lattices in $PU(1, n)$. Annals of Math. Stud, **132** (Princeton University Press, 1993).
[Ga16]   A. GARRIDO. Abstract commensurability and the Gupta–Sidki group, *Groups Geom. Dyn.* **10** (2016), 523–543.
[GrW03]   R. GRIGORCHUK and J. WILSON, A structural property concerning abstract commensurability of subgroups. *J. London Math. Soc.* **68**, 3 (2003), 671–682.

[Gr81]   M. GROMOV. Groups of polynomial growth and expanding maps. *Inst. Hautes Études Sci. Publ. Math.*, **53** (1981), 53–78.

[Gr93]   M. GROMOV. Geometric group theory, vol. 2: asymptotic invariants of infinite groups. *London Math. Soc. Lecture Notes* **182** (1993), 1–295.

[H16]   J. HUANG. Commensurability of groups quasi-isometric to RAAGs. *Invent. Math.* **213**, 3 (2018), 1179–1247.

[KPS73]   A. KARRASS, A. PIETROWSKI and D. SOLITAR. Finite and infinite cyclic extensions of free groups, *J. Australian Math. Soc* **16**, 04 (1973), 458–466.

[KK13]   S. KIM and T. KOBERDA. Embedability between right-angled Artin groups, *Geom. Topol.* **17** (2013), 493–530.

[KK14]   S. KIM and T. KOBERDA. The geometry of the curve graph of a right-angled Artin group, *Int. J. Algebra Comput.* **24** (2014), 121–169.

[Mar73]   G. A. MARGULIS. Arithmeticity of nonuniform lattices, *Funkcional. Anal. i Prilozen.* **7**, 3 (1973), 88–89.

[Sch95]   R. E. SCHWARTZ. The quasi-isometry classification of rank one lattices, *Publ. Math. Inst. Hautes Etudes Sci.* **82** (1995), 133–168.

[Serre]   J.-P. SERRE. *Trees*, (Springer-Verlag, 1980).

[Si43]   C. L. SIEGEL. Symplectic geometry, *Amer. J. Math.* **65** (1943), 1–86.

[St68]   J. R. STALLINGS. On torsion-free groups with infinitely many ends, *Ann. Math.* (1968), 312-334.

[We62]   P. WEICHSEL, The Kronecker product of graphs. *Proc. Amer. Math. Soc.* **13** (1962), 47–52.

[Wh10]   K. WHYTE. Coarse bundles, arXiv:1006.3347.

[Wi96]   D. WISE. *Non-Positively Curved Squared Complexes, Aperiodic Tilings and Non-Residually Finite Groups* (Princeton University, 1996).