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F_σ GAMES AND REFLECTION IN $L(\mathbb{R})$

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Abstract. We characterize the determinacy of F_σ games of length ω^2 in terms of determinacy assertions for short games. Specifically, we show that F_σ games of length ω^2 are determined if, and only if, there is a transitive model of $\mathsf{KP} + \mathsf{AD}$ containing $\mathbb{R}$ and reflecting Π_1 facts about the next admissible set.

As a consequence, one obtains that, over the base theory $\mathsf{KP} + \mathsf{DC} +$ "$\mathbb{R}$ exists," determinacy for F_σ games of length ω^2 is stronger than AD, but weaker than $\mathsf{AD} + \Sigma_1$-separation.

§1. Introduction. We study the consistency strength of F_σ-determinacy for games of length ω^2. We see that the situation here is tied with that of short games. For example, over the base theory

$$\mathsf{KP} + \mathsf{DC} + \text{“}\mathbb{R} \text{ exists,”}$$

F_σ-determinacy for games of length ω^2 is much stronger than AD; however, it is much weaker than $\mathsf{AD} + \Sigma_1$-separation. Over ZFC, F_σ-determinacy for games of length ω^2 is stronger than the existence of a transitive model of $\mathsf{KP} + \mathsf{AD}$ containing all reals; yet weaker than the existence of a transitive model of $\mathsf{KP} + \mathsf{AD} + \Sigma_1$-separation containing all reals. The consistency strength of this theory is hard to describe in terms of large cardinals: determinacy assumptions for games of length ω^2 (over ZFC or over $\mathsf{KP} + \mathsf{DC} +$ "$\mathbb{R}$ exists") for all pointclasses between the clopen sets and the Borel sets lie strictly between the existence of all finite amounts of Woodin cardinals (as a schema) and the existence of infinitely many Woodin cardinals, in terms of consistency strength (see [3]).

Much like in the case of short F_σ games, long games are tied to reflection for set-theoretic formulae of a certain complexity, except that—rather than in L—one needs to consider reflection in the $L(\mathbb{R})$-hierarchy. Given an admissible set, let A^+ denote the *next admissible set*, in the sense of Barwise–Gandy–Moschovakis [7]; i.e., A^+ is the smallest admissible set containing A. By convention, all admissible sets are transitive.

Definition 1.1. An admissible set A is Π_1^+- *reflecting* if for every Π_1 formula ϕ with parameters in A and one free variable, if $A^+ \models \phi(A)$, then there is some $B \in A$ with all parameters in ϕ such that $B^+ \models \phi(B)$.

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Our main theorem is:

THEOREM 1.2. *The following are equivalent*:

1. $\mathbf{\Sigma}^0_2$*-determinacy for games of length* ω^2;
2. *There is a* Π^+_1*-reflecting model of* AD *containing* $\mathbb{R}$.

Precursors to this work include, on the side of long games, Blass' [8] theorem that determinacy for all games of length ω^2 is equivalent to $\mathsf{AD}_\mathbb{R}$, Neeman's [21] work on games of countable length, Trang's [24] work on analytic games of additively indecomposable length, as well as previous work on games of length ω^2 that are open (see Theorem 6.1 below), clopen [5], Borel [4], or projective [6]. On the side of F_σ games we mention Solovay's and Tanaka's [23] work in the contexts of subsystems of set theory and analysis, respectively, and Wolfe's [26] original proof of F_σ-determinacy.

§2. Preliminaries. In this section, we collect some preliminary definitions and results.

2.1. Games of transfinite length. We study two-player, perfect information games of length ω^2 which are F_σ, i.e., $\mathbf{\Sigma}^0_2$-definable. These are games in which, given $A \in \mathcal{P}(\mathbb{R}) \cap \mathbf{\Sigma}^0_2$, two players, Player I and Player II, alternate[1] ω^2-many turns playing natural numbers, thus producing a sequence $x \in \mathbb{N}^{\omega^2}$. Since the spaces $\mathbb{R} \setminus \mathbb{Q}$ and $\mathbb{N}^{\omega^2}$ are recursively homeomorphic, the sequence x may be identified with a(n irrational) real number. Player I wins if $x \in A$; otherwise, Player II wins.

As above, we will often identify sequences of natural numbers of length ω^2 with ω-sequences of reals and with single reals. Given reals $x_0, x_1, \ldots$, we will denote by

$$x = \langle x_0, x_1, \ldots \rangle$$

the *single* real number coding the sequence $(x_0, x_1, \ldots)$, via some fixed recursive coding. The precise coding used will be immaterial, except for the continuity property that the first n digits x should only depend on $x_0, x_1, \ldots, x_n$. Similarly, if s and t are infinite sequences, or reals coding them, we will denote by

$$s^\frown t$$

the single real number coding the result of concatenating s and t.

2.2. Reflection in $L(\mathbb{R})$. We recall the definition of the $L(\mathbb{R})$ hierarchy: $L_0(\mathbb{R})$ is defined to be $V_{\omega+1}$, the collection of all sets all of whose elements are hereditarily finite. $L_{\alpha+1}(\mathbb{R})$ is the set of all subsets of $L_\alpha(\mathbb{R})$ definable over $L_\alpha(\mathbb{R})$ with parameters. At limit stages, $L_\lambda(\mathbb{R})$ is the union of all $L_\alpha(\mathbb{R})$, for $\alpha < \lambda$. The function

$$\alpha \mapsto L_\alpha(\mathbb{R})$$

is Δ_1 with $\mathbb{R}$ as a parameter. The first step towards proving Theorem 1.2 is the observation that it suffices to restrict to admissible sets of the form $L_\alpha(\mathbb{R})$.

[1]Since the $\mathbf{\Sigma}^0_2$ sets are closed under recursive substitutions, the rules of the game may require either player to make the first move.

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We will use the following nonstandard notation:

Definition 2.1. Let α be an ordinal. We denote by

$$\alpha^+$$

the least ordinal β such that $\alpha < \beta$ and $L_\beta(\mathbb{R})$ is admissible.

Definition 2.2. An ordinal α is Π_1^+- *reflecting* if $L_\alpha(\mathbb{R})$ is Π_1^+-reflecting.

Lemma 2.3. *Suppose A is a Π_1^+-reflecting set and $\mathbb{R} \in A$. Let $\alpha = \mathsf{Ord} \cap A$; then, α is Π_1^+-reflecting.*

Proof. Let ϕ be a Π_1 formula with parameters in $L_\alpha(\mathbb{R})$ such that

$$\big(L_\alpha(\mathbb{R})\big)^+ \models \phi(L_\alpha(\mathbb{R})).$$

Clearly, $\big(L_\alpha(\mathbb{R})\big)^+ = L_{\alpha^+}(\mathbb{R})$. Moreover, A^+ is an admissible set containing $L_\alpha(\mathbb{R})$, so $L_{\alpha^+}(\mathbb{R}) \subset A^+$. Let ψ be the formula in the language of set theory with parameters in $A \cup \{A\}$ asserting that for all β, if no $\gamma \in (\alpha, \beta)$ is admissible, then $L_\beta(\mathbb{R}) \models \phi(L_\alpha(\mathbb{R}))$. Being admissible is expressible internally, α is Δ_1-definable from A, and—as remarked earlier— $L_\beta(\mathbb{R})$ is Δ_1-definable from β and $\mathbb{R}$ (which belong to $L_\alpha(\mathbb{R})$). Hence, ψ is Π_1. Moreover, $A^+ \models \psi(A)$, so, by reflection, there is $B \in A$ such that $B^+ \models \psi(B)$ and B contains all parameters in ψ. In particular, $\mathbb{R} \in B$. Let $\beta = \mathsf{Ord} \cap B$; since $B \in A$, $\beta \in A$, so, in particular, $\beta < \alpha$. $L_{\beta^+}(\mathbb{R}) \subset B^+$. By choice of ψ, we obtain that $L_{\beta^+}(\mathbb{R}) \models \phi(L_\beta(\mathbb{R}))$, as desired. ⊣

Corollary 2.4. *The following are equivalent*:

1. *There is a Π_1^+-reflecting model of* AD *containing* $\mathbb{R}$; *and*
2. *There is a Π_1^+-reflecting ordinal α such that $L_\alpha(\mathbb{R}) \models \mathsf{AD}$.*

Proof. Suppose that A is a Π_1^+-reflecting model of AD containing $\mathbb{R}$ and let $\alpha = \mathsf{Ord} \cap A$. By Lemma 2.3, $L_\alpha(\mathbb{R})$ is Π_1^+-reflecting. Moreover, it is a subset of A containing all reals, and thus all possible strategies for games; hence, $L_\alpha(\mathbb{R}) \models \mathsf{AD}$. The converse is immediate. ⊣

This motivates the following definition:

Definition 2.5. We denote by $\sigma_\mathbb{R}$ the least Π_1^+-reflecting ordinal.

$\sigma_\mathbb{R}$ is the analog of the ordinal σ_1^1 for $L(\mathbb{R})$. Unlike σ_1^1, the set-theoretic properties of $\sigma_\mathbb{R}$ are not decidable within ZF. For instance, under AD, $\sigma_\mathbb{R}$ is big—it is a limit of weakly inaccessible cardinals (this follows from Moschovakis [16, Theorem 5], by which, under AD, every $\mathbb{R}$-admissible ordinal is weakly inaccessible). In contrast, if $V = L$, then $\sigma_\mathbb{R} < \omega_2$.

Definition 2.6. Let $A \subset \mathbb{R} \times \mathbb{R}$ and write $A_x = \{y \in \mathbb{R} : (x, y) \in \mathbb{R}\}$. We write $\Game^\mathbb{R} A$ for the set of all $x \in \mathbb{R}$ such that Player I has a winning strategy for the game on $\mathbb{R}$ with payoff A_x. We define

$$\Game^\mathbb{R} \boldsymbol{\Sigma}^0_2 = \{\Game^\mathbb{R} A : A \in \boldsymbol{\Sigma}^0_2\}.$$

(Classes such as $\Game^\mathbb{R} \boldsymbol{\Sigma}^0_1$ or $\Game^\mathbb{R} \boldsymbol{\Pi}^0_1$ are defined analogously.) To prove Theorem 1.2, it suffices to prove that determinacy for F_σ games of length ω^2 is equivalent to the fact

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that $L_{\sigma_{\mathbb{R}}}(\mathbb{R}) \models \mathsf{AD}$. Following previous proofs of determinacy for games of length ω^2, our first goal is to locate winning strategies for $\mathbf{\Sigma}^0_2$ games on $\mathbb{R}$. We shall show that

$$\mathcal{P}(\mathbb{R}) \cap \Sigma_1^{L_{\sigma_{\mathbb{R}}}(\mathbb{R})} = \Game^{\mathbb{R}}\mathbf{\Sigma}^0_2.$$

The main tool for this is the theory of inductive definitions on $\mathbb{R}$, the basics of which we recall next.

2.3. Inductive definitions. Suppose that $\phi : \mathcal{P}(\mathbb{R}^m) \to \mathcal{P}(\mathbb{R}^m)$ is an operator. We define sets $\phi^\lambda \subset \mathbb{R}^m$ inductively by

$$\begin{aligned} \phi^0 &= \varnothing, \\ \phi^{<\lambda} &= \bigcup_{\mu<\lambda} \phi(\phi^{<\mu}), \\ \phi^\lambda &= \phi^{<\lambda} \cup \phi(\phi^{<\lambda}), \\ \phi^\infty &= \bigcup_{\lambda \in \mathsf{Ord}} \phi^\lambda. \end{aligned}$$

The least κ such that $\phi^\infty = \phi^\kappa$ is called the *closure ordinal* of ϕ and denoted $|\phi|$. If Γ is a class of operators, we write

$$|\Gamma| = \sup\{|\phi| : \phi \in \Gamma\}.$$

Definition 2.7. Let Γ be a class of operators

$$\phi : \mathcal{P}(\mathbb{R}^m) \to \mathcal{P}(\mathbb{R}^m).$$

We say that $R \subset \mathbb{R}^n$ is Γ- *inductive* if there is $b \in \mathbb{R}^{m-n}$ such that for all $a \in \mathbb{R}^n$,

$$a \in R \text{ if, and only if, } (a,b) \in \phi^\infty.$$

An operator ϕ is defined by a formula $\psi(x,X)$ if, and only if,

$$\phi(X) = \{a \in \mathbb{R}^m : \psi(a,X)\}.$$

Thus, it is natural to consider classes of operators specified in terms of definability. Let Γ be a pointclass; we say that an operator ϕ is in Γ if it is defined by a formula in Γ with an additional predicate symbol X. We say that an operator is *positive* if this additional predicate symbol appears only positively, i.e., not in the scope of any negations (or in the antecedent of implications). An operator is *monotone* if $X \subset Y \subset \mathbb{R}$ implies $\phi(X) \subset \phi(Y)$. Every positive operator is monotone, of course. We may also speak of a formula defining an operator being *positive* under the obvious circumstances.

Definition 2.8. Let Γ be a pointclass.

1. Γ–IND is the pointclass of all Γ-inductive sets.
2. Γ^{pos}–IND is the pointclass of all (positive- Γ)-inductive sets.
3. Γ^{mon}–IND is the pointclass of all (monotone- Γ)-inductive sets.

Since every positive operator is monotone, we have

$$\Gamma^{pos}\text{–}\mathsf{IND} \subset \Gamma^{mon}\text{–}\mathsf{IND} \subset \Gamma\text{–}\mathsf{IND}.$$

DEFINITION 2.9. A subset of $\mathbb{R}$ is *inductive* if it is positive analytical inductive, i.e., letting IND be the class of all inductive sets, we have

$$\mathsf{IND} = \bigcup_{n\in\mathbb{N}} (\mathbf{\Sigma}^{1,pos}_n\text{–}\mathsf{IND}).$$

A subset of $\mathbb{R}$ is *coinductive* if its complement is inductive. We write coIND for the class of coinductive sets.

REMARK 2.10. One can actually show that the full analytical hierarchy is not necessary to construct all inductive sets, and indeed

$$\mathsf{IND} = \mathbf{\Sigma}^{1,pos}_n\text{–}\mathsf{IND}$$

whenever $2 \leq n$ (see Moschovakis [20, Exercise 7C.13]). For such an n, Theorem 2.14 below (due to Harrington and Kechris) implies that $\mathbf{\Sigma}^{1,mon}_n\text{–}\mathsf{IND} = \mathbf{\Sigma}^1_n\text{–}\mathsf{IND}$, and indeed one can show

$$\mathbf{\Sigma}^{1,pos}_n\text{–}\mathsf{IND} \subsetneq \mathbf{\Sigma}^{1,mon}_n\text{–}\mathsf{IND} = \mathbf{\Sigma}^1_n\text{–}\mathsf{IND},$$

which differs from the corresponding situation for (arithmetical) induction on $\mathbb{N}$. To verify the proper inclusion, it suffices to find an example of a $\mathbf{\Sigma}^1_n\text{–}\mathsf{IND}$ set which is not inductive.

We sketch the construction of such a set. First, fix a positive $\mathbf{\Sigma}^1_2$-inductive definition ψ constructing a prewellordering of a subset of $\mathbb{R}$ of length $\kappa^{\mathbb{R}}$, the closure ordinal of the inductive sets. Assume, without loss of generality, that ψ constructs this prewellordering by only using reals whose first digit is 0. $\phi(X)$ is defined as follows: as long as 2 does not belong to X, apply ψ to X, unless every real in the resulting set is already in X, in which case add 2 to X and add the pair (x, y) to X for every x whose first digit is 0 and every y whose first digit is 1. If 2 belongs to X, then apply to X the variant of ψ that constructs an isomorphic prewellordering, except that it only uses reals whose first digit is 1. This inductive definition produces a prewellordering of length $\kappa^{\mathbb{R}} + \kappa^{\mathbb{R}}$ and thus cannot be inductive. It can be seen to be $\mathbf{\Sigma}^1_n$ for some n (not 2), and not positive. Note the increase in complexity (from $\mathbf{\Sigma}^1_2$ to $\mathbf{\Sigma}^1_n$); because of this, this construction cannot be used to show that $\mathsf{coIND}^{pos}\text{–}\mathsf{IND}$ is a proper subset of $\mathsf{coIND}\text{–}\mathsf{IND}$. Indeed, it is not, by Theorem 2.16 below (due to Harrington and Moschovakis).

None of what has been said in this remark shall be used below, however.

Pointclasses of the form $\mathbf{\Sigma}^1_n$ (or of other forms) can be relativized by allowing sets of reals as parameters. This leads to a relativized version of the inductive sets:

DEFINITION 2.11. Let $X \subset \mathbb{R}$. A subset of $\mathbb{R}$ is *inductive on X* if it is positive analytical-on-X inductive, i.e., letting $\mathsf{IND}(X)$ be the class of all sets inductive on X, we have

$$\mathsf{IND}(X) = \bigcup_{n\in\mathbb{N}} (\mathbf{\Sigma}^{1,pos}_n(X)\text{–}\mathsf{IND}).$$

In the definition above, "positive" refers to the variable on which the induction is carried out, not to the parameter X.

Let κ be the closure ordinal of the inductive sets, i.e., the supremum of closure ordinals of positive analytical inductive definitions. Then $L_\kappa(\mathbb{R})$ is the smallest model of KP containing $\mathbb{R}$ and

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$$\mathsf{IND} = \mathcal{P}(\mathbb{R}) \cap \Sigma_1^{L_\kappa(\mathbb{R})},$$

where Σ_1-definability in the equation above allows for parameters. Subsets of $\mathbb{R}$ that are both inductive and coinductive are called *hyperprojective*. By Δ_1-separation, these are the sets of reals in $L_\kappa(\mathbb{R})$.

In general, let α be an ordinal and suppose that α is smaller than the least non-$\mathbb{R}$-projectible ordinal, i.e., that there is a surjection

$$\rho : \mathbb{R} \to L_\alpha(\mathbb{R}),$$

which is Σ_1-definable over $L_\alpha(\mathbb{R})$ (this restriction can be relaxed, but all ordinals relevant to this article will be of this form). Let X be a set of reals coding $L_\alpha(\mathbb{R})$ this way. Then, if κ_X is the closure ordinal of the X-inductive sets, we have that $L_{\kappa_X}(\mathbb{R})$ is the smallest model of KP containing $L_\alpha(\mathbb{R})$ and

$$\mathsf{IND}(X) = \mathcal{P}(\mathbb{R}) \cap \Sigma_1^{L_{\kappa_X}(\mathbb{R})}.$$

An alternate definition of the inductive sets is given by the following theorem:

THEOREM 2.12 (Moschovakis [18]). *The following are equivalent*:

1. *$A \subset \mathbb{R}$ is inductive,*
2. *there is a projective (or even analytic) set $B \subset \mathbb{R}$ such that for all $x \in \mathbb{R}$,*

$$x \in A \leftrightarrow \exists x_0 \forall x_1 \exists x_2 \ldots \exists n\, (x, \langle x_0, \ldots, x_n \rangle) \in B.$$

Moschovakis' theorem has the following easy consequence:

COROLLARY 2.13. $\mathsf{IND}(X) = \Game^{\mathbb{R}} \mathbf{\Sigma}^0_1(X)$.

PROOF. Assume $X = \varnothing$ for simplicity. To see that

$$\mathsf{IND} \subset \Game^{\mathbb{R}} \mathbf{\Sigma}^0_1,$$

it suffices to notice that given a projective B, statements of the form

$$\exists x_0 \forall x_1 \exists x_2 \ldots \exists n\, (x, \langle x_0, \ldots, x_n \rangle) \in B$$

can be decided by an open (in our sense) game on $\mathbb{R}$ (the details can be verified e.g., as in [2]). Conversely, let $U \in \mathbf{\Sigma}^0_1 \cap \mathcal{P}(\mathbb{R}^2)$. Write

$$U = \bigcup_{i \in \mathbb{N}} U_i$$

as a countable union of basic clopen sets. Thus,

$$x \in \Game^{\mathbb{R}} U \text{ if, and only if, } \exists x_0 \in \mathbb{R} \forall x_1 \in \mathbb{R} \ldots \exists n \in \mathbb{N}\, (x, \langle x_0, x_1, \ldots \rangle) \in U_n.$$

Since U_n is basic open, it is of the form

$$O(s,t) = \{(x,y) \in \mathbb{R} : (s,t) \sqsubset (x,y)\},$$

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where $\sqsubset$ denotes end-extension. Let k_n be the length of s and t. By our choice of coding of infinite sequences of real numbers by real numbers, the first k_n digits of a sequence $\langle x_0, x_1, \ldots\rangle$ depend only on $x_0, \ldots, x_{k_n}$. Clearly, the set

$$U^*_n = \big\{(x, \langle k, x_0, \ldots, x_k\rangle) : k \text{ is greater than or equal to the length of the unique finite sequences } s \text{ and } t \text{ such that } U_n = O(s,t), \text{ and } \exists x_{k+1}\, \exists x_{k+2} \ldots (x, \langle x_0, x_1, \ldots\rangle) \in U_n)\big\};$$

is projective. Let

$$A = \big\{x \in \mathbb{R} : \exists x_0 \in \mathbb{R}\, \forall x_1 \in \mathbb{R} \ldots \exists n, m \in \mathbb{N}\, (x, \langle n, x_0, x_1, \ldots, x_n\rangle) \in U^*_m\big\}.$$

We claim that A is inductive. To see this, first notice that the collection of all pairs $(x, \langle n, x_0, x_1, \ldots, x_n\rangle)$ belonging to some U^*_m is projective. Moreover, the real $\langle n, x_0, x_1, \ldots, x_{\langle n,m\rangle}\rangle$ coding the tuple $(n, x_0, x_1, \ldots, x_{\langle n,m\rangle})$ agrees on the first n digits with the real $\langle n, x_0, x_1, \ldots, x_n\rangle$ coding the tuple $(n, x_0, x_1, \ldots, x_n)$. Thus, it follows from the definition of U^*_m that

$$(x, \langle n, x_0, x_1, \ldots, x_n\rangle) \in U^*_m$$

implies

$$(x, \langle\langle n,m\rangle, x_0, x_1, \ldots, x_{\langle n,m\rangle}\rangle) \in U^*_m.$$

Hence, A is inductive. Finally, writing k_n for the length of the finite sequences s and t such that $U_n = O(s,t)$,

$$\begin{aligned} x \in A \text{ iff } & \exists x_0 \in \mathbb{R}\, \forall x_1 \in \mathbb{R} \ldots \exists n, m \in \mathbb{N}\, (x, \langle n, x_0, x_1, \ldots, x_n\rangle) \in U^*_m \\ \text{iff } & \exists x_0 \in \mathbb{R}\, \forall x_1 \in \mathbb{R} \ldots \exists n, m \in \mathbb{N}\; k_m \leq n \wedge \\ & \qquad \exists y_{n+1}\, \exists y_{n+2} \ldots (x, \langle x_0, x_1, \ldots, x_n, y_{n+1}, \ldots\rangle) \in U_m \\ \text{iff } & \exists x_0 \in \mathbb{R}\, \forall x_1 \in \mathbb{R} \ldots \exists m \in \mathbb{N}\, (x, \langle x_0, x_1, \ldots\rangle) \in U_m. \end{aligned}$$

Here, the first and second equivalences hold by definition. The third equivalence holds because whether $(x, \langle x_0, x_1, \ldots\rangle)$ belongs to U_m depends only on the first k_m digits of $\langle x_0, x_1, \ldots\rangle$ (by definition of k_m), which in turn depend only on $x_0, x_1, \ldots, x_{k_m}$. ⊣

We mention some results on the relation between inductive definitions, monotone inductive definitions, and positive inductive definitions.[2] For a collection Γ of sets, we denote by $\breve{\Gamma}$ the *dual* class consisting of complements of sets in Γ. Similarly for a collection of operators.

THEOREM 2.14 (Harrington–Kechris [11, 10]). *Let $\mathcal{F}$ be a collection of operators containing Π^1_1 and closed under $\wedge, \vee, \exists^{\mathbb{R}}$ and recursive substitutions. Suppose that*

1. *$WF \in \breve{\mathcal{F}}$, and*
2. *$\breve{\mathcal{F}} \subset \mathcal{F}^{mon}$–IND.*

Then $\mathcal{F}^{mon}$–IND $= \mathcal{F}$–IND.

[2] We would like to thank A. S. Kechris for sharing the notes [10] with us.

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We state the following theorem without defining all notions involved, and afterwards state the instance in which we will be interested:

THEOREM 2.15 (Harrington–Moschovakis [12]). *Let Q be a quantifier on $\mathbb{R}$. Let Γ be the pointclass of all sets which are positive Q-inductive and let $\breve{\Gamma}$ be the pointclass dual to Γ. Then,*

$$\breve{\Gamma}\text{–IND} = \breve{\Gamma}^{pos}\text{–IND}.$$

In particular, letting Q be the quantifier $\exists^{\mathbb{R}}$, we have $\Gamma = \mathsf{IND}$ and $\breve{\Gamma} = \mathsf{coIND}$, and thus:

THEOREM 2.16 (Harrington–Moschovakis).

$$\mathsf{coIND}\text{–IND} = \mathsf{coIND}^{pos}\text{–IND}.$$

We mention that Harrington and Moschovakis' theorem as stated in [12] is more general and e.g., implies the following classical result:

THEOREM 2.17 (Grilliot). $\Sigma^1_1\text{–IND} = \Sigma^{1,pos}_1\text{–IND}$.

We finish by recalling the definition of a Spector class (on $\mathbb{R}$).

DEFINITION 2.18. A pointclass Γ is called[3] a *Spector class on* $\mathbb{R}$ if the following conditions hold:

1. Γ is closed under $\wedge, \vee, \exists^{\mathbb{R}}, \forall^{\mathbb{R}}$;
2. Γ contains all analytical relations;
3. Γ is parametrized by $\mathbb{R}$;
4. Γ has the prewellordering property.

2.4. Short F_σ games. In order to locate winning strategies for $\mathbf{\Sigma}^0_2$ games on $\mathbb{R}$ within the $L(\mathbb{R})$-hierarchy, it might be helpful to recall the location of winning strategies for Σ^0_2 games on $\mathbb{N}$ in the L-hierarchy. This is a result of Solovay and is obtained by combining three theorems in recursion theory.

The first one of them is also due to Solovay:

THEOREM 2.19 (Solovay). $\Sigma^{1,pos}_1\text{–IND} = \Game\Sigma^0_2$.

A proof of Solovay's theorem can be found in [13] or in [20]. It will adapt to prove the analog for games on $\mathbb{R}$ later, as well as the fact that $\Game^{\mathbb{R}}\mathbf{\Sigma}^0_2$-determinacy implies determinacy for $\mathbf{\Sigma}^0_2$ games of length ω^2. The second theorem is Grilliot's Theorem 2.17 which, together with Theorem 2.19 implies that

$$\Sigma^1_1\text{–IND} = \Game\Sigma^0_2.$$

The final theorem is due to Aczel and Richter. Recall that σ^1_1 denotes the least Σ^1_1-reflecting ordinal.

[3]This differs from the notion of "Spector *pointclass*" in Moschovakis [20].

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THEOREM 2.20 (Aczel–Richter [1]). $|\Sigma^1_1| = \sigma^1_1$, *where* $|\Sigma^1_1|$ *refers to closure under inductive operators on* $\mathbb{N}$.

Combining everything, one sees that, in order to know which player wins a (lightface) Σ^0_2 game on $\mathbb{N}$, one needs not search beyond $L_{\sigma^1_1}$ for a winning strategy. Our plan for locating Player I's winning strategies for $\mathbf{\Sigma}^0_2$ games on $\mathbb{R}$ will be to follow the same steps as in the argument for games on $\mathbb{N}$.

§3. Coinductive operators and games.

THEOREM 3.1. $\Game^{\mathbb{R}}\mathbf{\Sigma}^0_2 = \mathsf{coIND}^{pos}\text{–}\mathsf{IND}$.

PROOF. We begin by quoting Wolfe's proof of Σ^0_2-determinacy and Solovay's proof of Theorem 2.19 to prove that

$$\Game^{\mathbb{R}}\mathbf{\Sigma}^0_2 \subset \mathsf{coIND}^{pos}\text{–}\mathsf{IND}. \tag{1}$$

Although the argument for this inclusion is very similar to the one for games on $\mathbb{N}$, a variation of it will be used below in the proof of Lemma 5.1 below, so we prefer to include all the details.

Let $A \subset \mathbb{R}^2$ be a Σ^0_2 set (where we assume no parameter is needed; the general result follows by relativization). Say A is given by

$$(x,y) \in A \leftrightarrow \exists n \forall m\, (n, x \upharpoonright m, y \upharpoonright m) \in P,$$

where P is recursive. The proof proceeds by showing that having a winning strategy in a certain game $G^*(\langle\rangle)$ equivalent to the Gale–Stewart game on reals with payoff A_x is equivalent to the membership of x in a set in $\mathsf{coIND}^{pos}\text{–}\mathsf{IND}$. For s a finite sequence of real numbers of even length, define $G^*(s)$ to be the following game:

1. Players I and II alternate turns playing real numbers α_n.
2. After infinitely many rounds have taken place, Player I wins if, letting

$$t(m) = s^\frown\langle \alpha_0, \alpha_1, \dots, \alpha_m\rangle$$

(i.e., letting t be the real coding the result of concatenating s with the first m moves of the play $(\alpha_0, \alpha_1, \dots)$), we have

$$\exists n \forall m \forall m' < m\, (n, x \upharpoonright m', t(m) \upharpoonright m') \in P.$$

Note that by our conventions on coding sequences of reals by single reals, we have

$$s^\frown\langle \alpha_0, \alpha_1, \dots\rangle \upharpoonright m = t(m) \upharpoonright m.$$

This implies that Player I has a winning strategy for the game on A if, and only if, she has one for the game on $G^*(\langle\rangle)$. The game $G^*(s)$ is like $G^*(\langle\rangle)$, except that we assume that s has already been played. We shall show that Player I having a winning strategy is in $\mathsf{coIND}^{pos}\text{–}\mathsf{IND}$.

Let s be a finite sequence of real numbers of even length and X be a set of reals. Consider the following game, $G(X, s)$:

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1. Players I and II alternate turns playing real numbers α_n.
2. After infinitely many rounds have taken place, Player I wins if, and only if, letting

$$t(2m) = s^\frown\langle\alpha_0, \dots, \alpha_{2m}\rangle,$$

one of the following holds for each $m \in \mathbb{N}$:
(a) there is $n \leq \text{LTH}(s)$ such that we have

$$\forall m' < m\,(n, x \restriction m', t(2m) \restriction m') \in P.$$

(b) $t(2m) \in X$.

The formula

$$\phi(s, X) = \text{“}s \text{ has even length and Player I has a winning strategy in } G(X, s)\text{”} \quad (2)$$

is clearly in $\Game^{\mathbb{R}}\Pi^0_1$ with an additional predicate for X and X appears positively in it. Since Π^0_1 games on $\mathbb{R}$ are determined by the Gale–Stewart theorem [9], the dual pointclass of $\Game^{\mathbb{R}}\Pi^0_1$ is $\Game^{\mathbb{R}}\Sigma^0_1$. By Corollary 2.13, (2) is in coIND^{pos}. The claim is now that

$$s \in \phi^\infty \leftrightarrow \text{“Player I has a winning strategy in } G^*(s).\text{”}$$

As in Solovay's proof, it is first shown that if $s \in \phi^\xi$, then Player I has a winning strategy in $G^*(s)$, by induction on ξ. Suppose that this holds for $\phi^{<\xi}$ and that $s \in \phi^\xi$, so that Player I has a winning strategy in $G(\phi^{<\xi}, s)$, say, σ. A winning strategy for $G^*(s)$ is obtained as follows: Player I begins by playing $G^*(s)$ by σ so long as the first winning condition of $G(s, X)$ is satisfied, i.e., so long as after round m, letting $t(m) = s^\frown\langle\alpha_0, \alpha_1, \dots, \alpha_m\rangle$, we have

$$\exists n \leq \text{LTH}(s)\,\forall m' < m\,(n, x \restriction m', t(2m) \restriction m') \in P.$$

If after some round this condition is not satisfied, then (since σ is a winning strategy), we must have

$$\exists \zeta < \xi \left(s^\frown\langle\alpha_0, \dots, \alpha_{2m}\rangle \in \phi^\zeta\right),$$

in which case the induction hypothesis yields a winning strategy for the game $G^*(s^\frown\langle\alpha_0, \dots, \alpha_{2m}\rangle)$ which should now be followed.

Conversely, if Player I has a winning strategy σ in $G^*(s)$ then s must belong to ϕ^∞, for otherwise (by monotonicity of ϕ and determinacy of closed games) Player II has a winning strategy τ in $G(s, \phi^\infty)$. If so, we could face off the strategies σ and τ against each other. Since τ is winning for Player II, after finitely many rounds, one will reach a partial play $t(2m_1) = s^\frown\langle\alpha_0, \dots, \alpha_{2m_1}\rangle$ such that the following hold:

1. $\forall n \leq \text{LTH}(s)\,\exists m' < m_1\,(n, x \restriction m', t(2m_1) \restriction m') \notin P$; and
2. $t(2m_1) \notin \phi^\infty$.

The second condition yields a new strategy τ_1 for Player II that can now be played against σ in $G(t(2m_1), \phi^\infty)$ until some stage m_2 at which the two conditions above hold again, etc. Continuing this process infinitely often yields a play

$$t = s^\frown\langle\alpha_0, \alpha_1, \dots\rangle$$

such that for each k,

$$\forall n \leq \textsc{lth}(t(2m_k))\, \exists m' < m_k\, (n, x \upharpoonright m', t(2m_{k+1}) \upharpoonright m') \notin P,$$

so that

$$\forall n\, \exists m' (n, x \upharpoonright m', t \upharpoonright m') \notin P,$$

contradicting the fact that σ was a winning strategy for Player I in $G(s)$. This completes the proof of (1).

The second step consists in showing that

$$\mathsf{coIND}^{pos}\text{–}\mathsf{IND} \subset \Game^{\mathbb{R}}\boldsymbol{\Sigma}^0_2. \tag{3}$$

Let $\phi(s, X)$ be a positive coIND operator.

By a theorem of Kechris and Moschovakis (see [13, Theorem 2.18]) and since $\boldsymbol{\Sigma}^0_2$ is parametrized by $\mathbb{R}$ and has the prewellordering property, $\Game^{\mathbb{R}}\boldsymbol{\Sigma}^0_2$ is a Spector class on $\mathbb{R}$. By the "Main Lemma" of Moschovakis [19] (see also [13, Theorem 1.7]), in order to see that $\phi^\infty \in \Game^{\mathbb{R}}\boldsymbol{\Sigma}^0_2$, it suffices to show that $\Game^{\mathbb{R}}\boldsymbol{\Sigma}^0_2$ is closed under ϕ, i.e., that for all $A \in \Game^{\mathbb{R}}\boldsymbol{\Sigma}^0_2$, the set

$$A_\phi = \{(x,z) : \phi(x, \{y : (y,z) \in A\})\}$$

is in $\Game^{\mathbb{R}}\boldsymbol{\Sigma}^0_2$. Let us put

$$A_z = \{y \in \mathbb{R} : (y,z) \in A\}$$

and verify that

$$\{(x,z) : \phi(x, A_z)\} \in \Game^{\mathbb{R}}\boldsymbol{\Sigma}^0_2.$$

Since ϕ is in coIND, it follows from Moschovakis [18] that there is a $\boldsymbol{\Pi}^1_1$ formula ψ such that

$$\phi(x, X) \leftrightarrow \exists y_0 \forall y_1 \ldots \forall n\, \psi(x, \langle y_0, \ldots, y_n \rangle, X),$$

say,

$$\phi(x, X) \leftrightarrow \exists y_0 \forall y_1 \ldots \forall n \forall w\, \psi_0(w, x, \langle y_0, \ldots, y_n \rangle, X),$$

for some arithmetical ψ_0.

To verify that A_ϕ is in $\Game^{\mathbb{R}}\boldsymbol{\Sigma}^0_2$, fix x and z; we play the natural game given by the equivalence above:

1. Players I and II begin by playing real numbers $y_0, y_1, \ldots$ until Player II decides to move on to the next stage after turn n. If this never happens, Player I wins.
2. Player II plays $w \in \mathbb{R}$. Let $\theta_0 = \psi_0(w, x, \langle y_0, \ldots, y_n \rangle, A_z)$ and assume without loss of generality that θ_0 has been rewritten without implications and without negations whose scope is not a single atomic formula.
3. If θ_k has been defined and is a formula in the language of second-order arithmetic with an additional predicate A_z, we proceed by cases:
   (a) If the outermost logical connective of θ_k is a disjunction, then Player I chooses one of the disjuncts; we set θ_{k+1} equal to this choice.
   (b) If the outermost logical connective of θ_k is a conjunction, then Player II chooses one of the conjuncts; we set θ_{k+1} equal to this choice.

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(c) If θ_k is atomic then either it is an atomic formula not involving X, in which case the game ends and Player I wins if, and only if, θ_k holds; or it is of the form $a \in A_z$ (since ϕ is positive). In the latter case, the game continues.
(d) Let $B \in \mathbf{\Sigma}^0_2$ be such that $A_z = \Game^{\mathbb{R}} B$. Players I and II alternate infinitely many turns playing reals $w_0, w_1, w_2, \dots$. At the end, Player I wins if, and only if, $(\langle w_0, w_1, \dots \rangle, w, a, z) \in B$.

Clearly the winning condition for the game is $\mathbf{\Sigma}^0_2$. It is easy to verify that Player I has a winning strategy in this game if, and only if $(x,z) \in A_\phi$. This proves (3). ⊣

Together with Harrington and Moschovakis's Theorem 2.16, the previous result yields:

Corollary 3.2. coIND–$\mathsf{IND} = \Game^{\mathbb{R}} \mathbf{\Sigma}^0_2$.

§4. Coinductive operators and reflection. Let $\preceq$ be a binary relation. As a convention, we will use $\prec$ to denote the strict part of $\preceq$ (i.e., $x \prec y$ whenever $x \preceq y$ and $y \not\preceq x$) and $\equiv$ to denote $\preceq$-equivalence (i.e., $x \equiv y$ whenever $x \preceq y$ and $y \preceq x$). Similar conventions shall apply to variations of $\preceq$, e.g., by subscripts. Notice that any two of $\preceq$, $\prec$, and $\equiv$ determine the other one.

Given a binary relation $\preceq$ and an equivalence relation E with the same field, $\preceq$ is said to commute with E if $x \preceq y$, xEa, and yEb imply $a \preceq b$, and similarly for the strict part $\prec$. A binary relation $\preceq$ *commutes with itself* if it commutes with $\equiv$; this follows from transitivity. An induction on the complexity of formulae shows that if $\preceq$ is a binary relation that commutes with itself, then any first-order formula in the vocabulary $\{\prec, =\}$ holds of some tuple $(x_1, \dots, x_n)$ of objects in field$(\preceq)$ if, and only if, it holds of any tuple $(y_1, \dots, y_n)$ such that x_i is $\equiv$-equivalent to y_i for each i, so long as equality is interpreted as $\equiv$.

We define in the natural way the notion of a preorder on $\mathbb{R}$ coding an initial segment of $L(\mathbb{R})$:

Definition 4.1. Let α be an ordinal. A *coding* of $L_\alpha(\mathbb{R})$ is a preorder $\preceq$ on a subset of $\mathbb{R}$ such that

$$(L_\alpha(\mathbb{R}), \in, =) \cong (\text{field}(\preceq), \prec, \equiv)/\equiv.$$

The right-hand side of the displayed equation is the quotient structure of the structure (field$(\preceq), \prec, \equiv)$ by $\equiv$. The relation $\equiv$ becomes equality in this quotient. If $\preceq$ is a coding of some $L_\alpha(\mathbb{R})$, then $\prec$ induces a wellfounded transitive relation on $\equiv$-equivalence classes and there is a surjection ρ from field $(\preceq)$ to $L_\alpha(\mathbb{R})$ such that $x \prec y$ if, and only if, $\rho(x) \in \rho(y)$. If so then *in this section only*, let us denote $\rho(x)$ by $|x|_\preceq$.

Theorem 4.2. $|\mathsf{coIND}| = \sigma_{\mathbb{R}}$.

The proof of Theorem 4.2 will be divided into two lemmata:

Lemma 4.3. $|\mathsf{coIND}|$ *is* Π^+_1*-reflecting.*

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Proof. The lemma will be proved by an argument similar to the one of [1, Theorem 10.7]. We have not checked whether the results of [1] (e.g., the lemma proved in the appendix) hold in full in the context of recursion on $\mathbb{R}$ but, since we are only interested in proving Lemma 4.3 (and are not analyzing, e.g., ordinals of the form $|\Sigma^1_n|$), we do not need to be very careful with issues of definability, which greatly simplifies the situation.

Given an inductive definition ϕ, there is a natural way of associating to it a prewellordering on $\mathbb{R}$, namely, letting

$$w(x) = \begin{cases} \text{least } \xi \text{ such that } x \in \phi^{\xi+1}, & \text{if it exists,} \\ \infty, & \text{otherwise;} \end{cases}$$

we put

$$x \preceq_\phi y \text{ if, and only if, } w(x) \leq x(y).$$

Let ϕ be a universal coinductive operator, i.e., a coinductive operator such that whenever ψ is a coinductive operator, we have

$$\psi(X) = \{x \in \mathbb{R} : (x, a) \in \phi(X)\}$$

for some $a \in \mathbb{R}$. (See Moschovakis [19] for a proof of the existence of universal coinductive sets; they can be defined uniformly in X.) Below, we will define a coinductive operator Θ which simultaneously applies ϕ, defines the prewellordering associated with ϕ, and codes initial segments of $L_\alpha(\mathbb{R})$ along the prewellordering given by ϕ. Given $X \subset \mathbb{R}$, write

$$(i, X) = \{(i, x) : x \in X\}.$$

Conversely, we write

$$X_i = \{x \in \mathbb{R} : (i, x) \in X\}.$$

Let $X \subset \mathbb{R}$ and $\preceq$ be a binary relation on X. Suppose moreover that U is a set of real numbers and that X consists only of tuples of real numbers none of whose first coordinate belongs to U. We define a set $\mathrm{Def}(X, \preceq)$ and a binary relation $\preceq^+$ such that if $\preceq$ has field X and codes $L_\alpha(\mathbb{R})$, then $\preceq^+$ is a relation on $X \oplus \mathrm{Def}(X, \preceq)$ which codes $L_{\alpha+1}(\mathbb{R})$. The reason for mentioning U at all will become clear soon; roughly, it contains indicators which we will use to identify which reals belong to $\mathrm{Def}(X, \preceq)$ and which do not.

$\mathrm{Def}(X, \preceq)$ is the set of all tuples $(x_0, \varphi, \vec{b})$, where $x_0 \in U$, φ is a formula of arity $\mathrm{LTH}(\vec{b}) + 1$ in the vocabulary $\{=, \prec\}$ and $\vec{b}$ is a finite tuple of elements of X. For $x, y \in X \oplus \mathrm{Def}(X, \preceq)$, we put $x \prec^* y$ if, and only if, one of the following holds:

1. $x, y \in X$ and $x \prec y$;
2. $y \in \mathrm{Def}(X, \preceq)$ is of the form $(x_0, \varphi, \vec{b})$ and $(X, \prec, \equiv) \models \varphi(\vec{b}, x)$.

Afterwards, let xEy if, and only if x and y have exactly the same $\prec^*$-predecessors, and put $x \preceq^+ y$ if, and only if, xEy or there are aEx and bEy such that $a \prec^* b$. Hence, the relation $\prec^*$ is a first approximation to $\prec^+$. It extends $\prec$ by specifying which of the elements of X are smaller than which of the new elements. It may thus be that an element x of $\mathrm{Def}(X, \preceq)$ turns out to have exactly the same $\prec^*$-predecessors as some other element y of $\mathrm{Def}(X, \preceq)$ or of X. We would then like to force these x and y to

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be $\equiv^+$-equivalent, which is what we do. Note that this definition depended on U, so when in need of precision, we may write $\mathrm{Def}(X, \preceq, U)$ for $\mathrm{Def}(X, \preceq)$ and $\preceq^{+U}$ for $\preceq^+$. By inspecting the construction, one sees that both $\mathrm{Def}(X, \preceq, U)$ and $\preceq^+$ are, say, hyperprojective (in fact, much simpler) in X, U, and $\preceq$. Another feature of this definition is that if $\{\preceq_\iota\}_\iota$ is an increasing family of relations obtained this way, and $\preceq_0$ is a coding of some $L_\alpha(\mathbb{R})$ then $\bigcup_\iota \preceq_\iota$ is a coding of some $L_\beta(\mathbb{R})$ (this is where the set U in the definition is used).

We now define the operator Θ. Given $X \subset \mathbb{R}$, Θ will map X to a set Y with four parts, Y_0, Y_1, Y_2, and Y_3 defined in terms of X_0, X_1, X_2, and X_3. If X is of the right form, Y_1 will be a prewellordering of some length $\alpha + 1$ and Y_3 will be a coding of $L_{\alpha+1}(\mathbb{R})$ with field Y_2.

1. $Y_0 = \phi(X_0)$;
2. $Y_1 = X_1 \cup \left\{(x,y) : x \in Y_0 \cup \mathrm{field}(X_1) \text{ and } y \in Y_0 \setminus \mathrm{field}(X_1)\right\}$;
3. $Y_2 = X_2 \cup \mathrm{Def}(X_2, X_3, U)$, where $U = Y_0 \setminus \mathrm{field}(X_1)$;
4. $Y_3 = X_3^{+U}$, where $U = Y_0 \setminus \mathrm{field}(X_1)$.

The class of coinductive operators is closed under Boolean connectives and real quantifiers, so Θ is coinductive. The operator generates the inductive definition ϕ^∞ on its first component, while simultaneously coding $\preceq_\phi$ in its second component. At each stage ξ, all new elements added to the field of $\preceq_\phi$, i.e., those of $\preceq_\phi$-rank $\xi + 1$, are used to define the new elements of the set $(\Theta^\xi)_2$ which acts as the field of a relation $(\Theta^\xi)_3$ that codes $L_{\xi+1}(\mathbb{R})$.

Any fixed point of Θ will contain a fixed point of ϕ in its first component. Because ϕ is universal coinductive, an argument as in [1, Theorem 8.5] (but using continuous reducibility in place of many-one reducibility[4]) shows that

$$|\Theta| = |\phi| = |\mathsf{coIND}|.$$

Hence, we need to show that $|\Theta|$ is Π^+_1-reflecting (although, to show that there is a Π^+_1-reflecting ordinal $\leq |\mathsf{coIND}|$—which is really the point of this lemma, we do not need the argument from [1, Theorem 8.5]). Recall that, given a coding $\preceq$ of some $L_\alpha(\mathbb{R})$, in this section we write $|x|_\preceq$ for the element of $L_\alpha(\mathbb{R})$ coded by x, if any.

SUBLEMMA 4.4. *Let ψ be a Π_1 formula in the language of set theory. Then, there is a coinductive operator Ψ, such that whenever $\lambda \leq |\Theta|$,*

1. *for all $c_1, \ldots, c_l \in (\Theta^\lambda)_2$,*
$$L_{\lambda^+}(\mathbb{R}) \models \psi\big(|c_1|_{(\Theta^\lambda)_3}, \ldots, |c_l|_{(\Theta^\lambda)_3}\big) \leftrightarrow (c_1, \ldots, c_l) \in \Psi(\Theta^\lambda).$$
2. *for all $c_1, \ldots, c_l \in \mathbb{R}$, if there is some $i < l+1$ such that $c_i \notin (\Theta^\lambda)_2$, then*
$$(c_1, \ldots, c_l) \notin \Psi(\Theta^\lambda).$$

PROOF. The main observation is that by Barwise–Gandy–Moschovakis [7, Lemma 2.9], for every admissible set A, a relation P on A is coinductive on A with parameters if, and only if, it is Π_1 over A^+, with parameters in $A \cup \{A\}$. Moreover,

[4]A proof of the Kleene's Recursion Theorem in this context can be found e.g., in [15, Theorem 3.1].

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this correspondence is uniform, in that the definition of the coinductive relation depends only on the Π_1 formula and its parameters, and not on the admissible set A (as long as it contains the parameters),[5] and vice-versa.

Hence, given a Π_1-formula ψ, the set of all $a_1, \dots, a_l \in L_\lambda(\mathbb{R})$ such that

$$L_{\lambda^+}(\mathbb{R}) \models \psi(a_1, \dots, a_l)$$

is a coinductive subset of $L_\lambda(\mathbb{R})$ (and thus of $L_{\lambda+1}(\mathbb{R})$). Since $(\Theta^\lambda)_3$ is a coding of $L_{\lambda+1}(\mathbb{R})$, there is a surjection

$$\rho : (\Theta^\lambda)_2 \to L_{\lambda+1}(\mathbb{R}),$$

such that a pair (x, y) belongs to the strict part of $(\Theta^\lambda)_3$ if, and only if, $\rho(x) \in \rho(y)$. It follows that the preimage of a coinductive subset of $L_{\lambda+1}(\mathbb{R})$ under ρ is coinductive on $\mathbb{R}$ from the parameters $(\Theta^\lambda)_2$ and $(\Theta^\lambda)_3$. Hence, there is a coinductive operator Ψ' such that

1. for all $c_1, \dots, c_l \in (\Theta^\lambda)_2$,

$$L_{\lambda^+}(\mathbb{R}) \models \psi(|c_1|_{(\Theta^\lambda)_3}, \dots, |c_l|_{(\Theta^\lambda)_3}) \leftrightarrow (c_1, \dots, c_l) \in \Psi'((\Theta^\lambda)_2 \oplus (\Theta^\lambda)_3).$$

2. for all $c_1, \dots, c_l \in \mathbb{R}$, if there is some $i < l+1$ such that $c_i \notin (\Theta^\lambda)_2$, then

$$(c_1, \dots, c_l) \notin \Psi'((\Theta^\lambda)_2 \oplus (\Theta^\lambda)_3).$$

Now, both $(\Theta^\lambda)_2$ and $(\Theta^\lambda)_3$ reduce to Θ^λ continuously, and this reduction is uniform in λ, so there is an operator Ψ as desired. ⊣

Let ψ be a Π_1 sentence and let Ψ be given by the sublemma, so that for all $\lambda \leq |\Theta|$ and all $c_1, \dots, c_l \in (\Theta^\lambda)_2$, we have

$$L_{\lambda^+}(\mathbb{R}) \models \psi(|c_1|_{(\Theta^\lambda)_3}, \dots, |c_l|_{(\Theta^\lambda)_3}) \leftrightarrow (c_1, \dots, c_l) \in \Psi(\Theta^\lambda).$$

Since ϕ was chosen to be a universal coinductive operator, $\Theta(X)$ is complete coinductive for all X, so there is a continuous function

$$g : \mathbb{R}^l \mapsto \mathbb{R}$$

such that for all $c_1, \dots, c_l \in \mathbb{R}$ and all $X \subset \mathbb{R}$ do we have

$$(c_1, \dots, c_l) \in \Psi(X) \leftrightarrow g(c_1, \dots, c_l) \in \Theta(X).$$

Hence,

$$L_{\lambda^+}(\mathbb{R}) \models \psi(|c_1|_{(\Theta^\lambda)_3}, \dots, |c_l|_{(\Theta^\lambda)_3}) \leftrightarrow g(c_1, \dots, c_l) \in \Theta(\Theta^\lambda). \tag{4}$$

[5]This can be shown directly for admissible sets of the form $L_\alpha(\mathbb{R})$: given such a set, one defines an operator, elementary in $L_\alpha(\mathbb{R})$, that successively outputs extensions of $L_\alpha(\mathbb{R})$, like Θ does. To ensure that the operator is elementary, one can have it e.g., add only Σ_1-definable sets at each stage, instead of all first-order–definable sets. One can ask the operator to also check at each limit stage whether the Π_1 fact with parameters in $L_\alpha(\mathbb{R})$ holds of the structure it outputs. Like we did with Θ, this operator can be ensured to have closure ordinal α^+ by asking it to generate a universal $\Sigma_1(L_\alpha(\mathbb{R}))$ inductive definition in parallel. The operator for that can be obtained e.g., from the universal $\Sigma_1(P)$ subset of $\mathbb{R}$, where P is a subset of $\mathbb{R}$ coding $L_\alpha(\mathbb{R})$. The set P can be obtained uniformly in a $\Sigma_1(L_\alpha(\mathbb{R}))$ way (for multiplicatively indecomposable α; see Steel [22, Lemma 1.4]).

Suppose then that there are elements $\gamma_1,\dots,\gamma_l$ of $L_\lambda(\mathbb{R})$ such that

$$L_{|\Theta|^+}(\mathbb{R}) \models \psi\big(\gamma_1,\dots,\gamma_l\big).$$

By construction, $(\Theta^{<\lambda})_3$ is a coding of $L_\lambda(\mathbb{R})$ for each λ, so $(\Theta^\infty)_3$ is a coding of $L_{|\Theta|}(\mathbb{R})$ with field $(\Theta^\infty)_2$. Thus, there is a surjection

$$\rho : (\Theta^\infty)_2 \to L_{|\Theta|}(\mathbb{R})$$

such that the pair (x,y) belongs to the strict part of $(\Theta^\infty)_3$ if, and only if, $\rho(x) \in \rho(y)$. Pick $c_1,\dots,c_l$ such that for each $0 < i < l+1$, $\rho(c_i) = \gamma_i$. Then,

$$L_{|\Theta|^+}(\mathbb{R}) \models \psi\big(|c_1|_{(\Theta^\infty)_3},\dots,|c_l|_{(\Theta^\infty)_3}\big).$$

By (4),

$$g(c_1,\dots,c_l) \in \Theta(\Theta^\infty) \subset \Theta^\infty.$$

$|\Theta|$ is a limit ordinal, so there is some $\lambda < |\Theta|$ such that

$$g(c_1,\dots,c_l) \in \Theta(\Theta^\lambda),$$

so that, by (4),

$$L_{\lambda^+}(\mathbb{R}) \models \psi\big(|c_1|_{(\Theta^\lambda)_3},\dots,|c_l|_{(\Theta^\lambda)_3}\big).$$

Notice that the sequence of sets $\{(\Theta^\lambda)_2 : \lambda < |\Theta|\}$, as well as the sequence of isomorphisms witnessing that each $(\Theta^\lambda)_3$ is a coding of $L_{\lambda+1}(\mathbb{R})$ are strictly increasing, and this implies that

$$|c_i|_{(\Theta^\lambda)_3} = |c_i|_{(\Theta^\infty)_3}$$

for each $0 < i < l+1$, which yields the result. This proves the lemma. ⊣

Lemma 4.5. *If κ is Π_1^+-reflecting, then* $|\mathsf{coIND}| \leq \kappa$.

Proof. This proof is like that of [1, Lemma 10.1]. Suppose κ is Π_1^+-reflecting and let ϕ be a coinductive operator. Since κ is Π_1^+-reflecting, it is $\mathbb{R}$-recursively inaccessible. Because coinductive relations are universal relations on the next admissible set, it follows that if $A \in \mathcal{P}(\mathbb{R}) \cap L_\alpha(\mathbb{R})$, then $L_{\alpha^+ +1}(\mathbb{R})$ can compute coinductive relations on A. Now, $L_\kappa(\mathbb{R})$ can compute the sequence $\{\phi^\lambda : \lambda < \kappa\}$ in a Π_1 way (with parameters): x is the λth element of this sequence if every transitive set of the form $L_\alpha(\mathbb{R})$ containing an increasing $(\lambda+1)$-sequence of $\mathbb{R}$-admissible sets believes that x is the λth element of the sequence. Moreover, this definition is uniform on $\mathbb{R}$-recursively inaccessible ordinals. Thus, letting ψ be a Π_1 formula such that

$$a \in \phi^{<\kappa} \leftrightarrow L_\kappa(\mathbb{R}) \models \exists\lambda\, \psi(a,\lambda),$$

the question of whether a belongs to $\phi(\phi^{<\kappa})$ is coinductive over $\phi^{<\kappa}$, hence over $L_\kappa(\mathbb{R})$, hence universal over $L_{\kappa^+}(\mathbb{R})$, so, by Π_1^+-reflection, $\phi(\phi^{<\kappa}) \subset \phi^{<\kappa}$, as desired. ⊣

This concludes the proof of Theorem 4.2. The main consequence of interest to us is:

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Corollary 4.6. $\mathcal{P}(\mathbb{R}) \cap \Sigma_1^{L_{\sigma_\mathbb{R}}(\mathbb{R})} = \Game^\mathbb{R}\mathbf{\Sigma}_2^0$.

Proof. By Theorems 3.1 and 4.2, we need to show that

$$\mathcal{P}(\mathbb{R}) \cap \Sigma_1^{L_{|\mathsf{coIND}|}(\mathbb{R})} = \mathsf{coIND–IND}.$$

That the class on the right-hand side is contained in the one on the left is clear. The converse follows from the proof of Lemma 4.3: one can use a coinductive operator to inductively generate codes for initial segments of $L(\mathbb{R})$ below $|\mathsf{coIND}|$, so an existential statement about $L_{|\mathsf{coIND}|}(\mathbb{R})$ can be rephrased in terms of membership in the least fixed point of a coinductive operator. ⊣

§5. F_σ-determinacy.

Lemma 5.1. *Suppose that all sets in $\Game^\mathbb{R}\mathbf{\Sigma}_2^0$ are determined. Then, all $\mathbf{\Sigma}_2^0$ games of length ω^2 are determined.*

Proof. Note that $\mathbf{\Sigma}_2^0$ games of length ω with moves in $\mathbb{R}$ are determined. Suppose that all sets in $\Game^\mathbb{R}\mathbf{\Sigma}_2^0$ are determined and let $A \in \mathbf{\Sigma}_2^0$. We show that the game of length ω^2 with payoff A is determined. Let us refer to this game as G. The proof begins very much like that of Theorem 3.1: we begin by replacing G with a game $G^*(\langle\rangle)$. Without loss of generality we shall assume that there is a recursive set P such that A is given by

$$x \in A \leftrightarrow \exists n \forall m\, (n, x \upharpoonright m) \in P,$$

For s a finite sequence of real numbers of even length, define $G^*(s)$ to be the following game:

1. Players I and II alternate ω^2-many turns playing natural numbers to produce a countable sequence $\{\alpha_n : n \in \mathbb{N}\}$ of real numbers.
2. After ω^2-many rounds have taken place, Player I wins if, letting

   $$t(m) = s^\frown\langle \alpha_0, \alpha_1, \dots, \alpha_m\rangle,$$

   we have

   $$\exists n \forall m \forall m' < m\, (n, t(m) \upharpoonright m') \in P.$$

As before, Player I has a winning strategy for G if, and only if, she has one for $G^*(\langle\rangle)$. Moreover, Player II has a winning strategy for G if, and only if, she has one for $G^*(\langle\rangle)$. The game $G^*(s)$ is like $G^*(\langle\rangle)$, except that we assume that s has already been played.

Let n_s be a natural number, s be a sequence of natural numbers of even length $\omega \cdot n_s$ and X be a set of sequences of natural numbers of length $<\omega^2$. Consider the following game, $G(X, s)$:

1. Players I and II alternate ω^2-many turns playing natural numbers to produce digits corresponding to a countable sequence $\{\alpha_n : n \in \mathbb{N}\}$ of real numbers.
2. After ω^2-many rounds have taken place, Player I wins if, and only if, letting

   $$t(m) = s^\frown\langle \alpha_0, \dots, \alpha_m\rangle,$$

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one of the following holds for each $m \in \mathbb{N}$:

(a) there is $n \leq n_s$ such that we have

$$\forall m' < m\,(n, t(m) \upharpoonright m') \in P.$$

(b) $t(m) \in X$.

Thus $G(X,s)$ is a game on $\mathbb{N}$ of length ω^2. Write

$$\phi(s,X) = \text{"Player I has a winning strategy in } G(X,s)\text{."} \tag{5}$$

The argument from Theorem 3.1 shows that if $s \in \phi^\infty$, then Player I has a winning strategy in $G^*(s)$. To complete the proof, it remains to show that if $s \notin \phi^\infty$, then Player II has a winning strategy in $G^*(s)$.

Notice that $G(X,s)$ is a game of length ω^2 with moves in $\mathbb{N}$ and payoff in Π^0_1 (using X as a second-order oracle). We consider the following auxiliary game, which we shall denote by $H(X,s)$:

$$\begin{array}{c|ccccc} \text{I} & & \sigma_0 & & \sigma_1 & \ldots \\ \text{II} & \tau_0 & & \tau_1 & & \ldots \end{array}$$

Here, σ_i and τ_i are real numbers coding strategies for games on $\mathbb{N}$ of length ω. Player I wins the game if, and only if, the sequence

$$(s, \sigma_0 * \tau_0 *, \sigma_1 * \tau_1, \ldots)$$

satisfies the winning condition of $G(X,s)$. Here, $\sigma * \tau$ denotes the real number obtained when the strategies σ and τ are faced off against each other. Thus, $H(X,s)$ is a game of length ω with moves in $\mathbb{R}$ and payoff in $\Pi^0_1(X)$. We will need a sublemma:

Sublemma 5.2. *Suppose that all subsets of $\mathbb{R}$ in $\Game^{\mathbb{R}}\boldsymbol{\Sigma}^0_1(X)$ are determined. Then, the following are equivalent*:

1. *Player* I *has a winning strategy in $H(X,s)$; and*
2. *Player* I *has a winning strategy in $G(X,s)$.*

Moreover, the games are determined.

Proof. The equivalence is proved by arguing as in the proof of [5, Theorem 1.1]. This is possible because the game $H(s,X)$ is $\Pi^0_1(X)$. The proof of [5, Theorem 1.1] uses determinacy of sets in $\Game^{\mathbb{R}}\boldsymbol{\Sigma}^0_1$; in this case, we need determinacy of sets in $\Game^{\mathbb{R}}\boldsymbol{\Sigma}^0_1(X)$, which we are assuming. $H(X,s)$ is closed and thus determined by the Gale–Stewart theorem. The proof of [5, Theorem 1.1] also shows that if Player II has a winning strategy in $H(X,s)$, then she has one in $G(X,s)$, so that this game is also determined. ⊣

Remark 5.3. Sublemma 5.2 essentially states that Lemma 5.1 holds if one replaces $\boldsymbol{\Sigma}^0_2$ by $\boldsymbol{\Pi}^0_1$. The part of the proof of [5, Theorem 1.1] used in the proof of Sublemma 5.2 goes through for $\boldsymbol{\Pi}^0_1$ games. It would also go through for $\boldsymbol{\Sigma}^0_1$ games if one switched the order in which the players move in the definition of $H(X,s)$. The proof, however, does not go through for more complicated games. By different arguments (involving the theory of scales in $L(\mathbb{R})$), one can prove analogues of Lemma 5.1 for more complicated pointclasses, up to $\boldsymbol{\Delta}^1_1$ (see [4]). The result of replacing $\boldsymbol{\Sigma}^0_2$ by $\boldsymbol{\Pi}^1_1$ in the statement of Lemma 5.1 is not provable in ZFC.

It follows from Corollary 2.13 that if $X \in L_\alpha(\mathbb{R})$, then all winning strategies for games on $\mathbb{R}$ with payoff in $\mathbf{\Sigma}^0_1(X)$ are definable over $L_{\alpha^+}(\mathbb{R})$, so $\Game^{\mathbb{R}}\mathbf{\Sigma}^0_1(X)$ belongs e.g., to $L_{\alpha^{++}}(\mathbb{R})$.

SUBLEMMA 5.4. *$\Game^{\mathbb{R}}\mathbf{\Sigma}^0_2$ is closed under the open-game-on- $\mathbb{R}$ quantifier; i.e., if $X \in \Game^{\mathbb{R}}\mathbf{\Sigma}^0_2$, then $\Game^{\mathbb{R}}\mathbf{\Sigma}^0_1(X) \subset \Game^{\mathbb{R}}\mathbf{\Sigma}^0_2$.*

PROOF. This can be proved directly by an argument like the one of (3) in Theorem 3.1. Alternatively, one can also appeal to the "Main Lemma" of Moschovakis [19] (as in the proof of Theorem 3.1): since operators definable from X by an open game quantifier are X-inductive (by Corollary 2.13) and $\Game^{\mathbb{R}}\mathbf{\Sigma}^0_2$ is a Spector class, it suffices to show that $\Game^{\mathbb{R}}\mathbf{\Sigma}^0_2$ is closed under the inductive step. But the inductive step is positive analytical on X and a real parameter, so clearly $\Game^{\mathbb{R}}\mathbf{\Sigma}^0_2$ is closed under it. ⊣

Write

$$\phi'(s,X) = \text{“Player I has a winning strategy in } H(X,s).\text{”}$$

Clearly, $\phi'(s,X)$ is an operator in $\Game^{\mathbb{R}}\mathbf{\Pi}^0_1$ and thus in coIND (by Corollary 2.13). By Theorem 4.2, there is some $\eta \leq \sigma_{\mathbb{R}}$ such that $\phi'^{\infty} = \phi'^{\eta}$. By Corollary 4.6 and the hypothesis of the lemma,

$$L_{\sigma_{\mathbb{R}}}(\mathbb{R}) \models \mathsf{AD}.$$

Since $\sigma_{\mathbb{R}}$ is recursively $\mathbb{R}$-inaccessible, i.e., $\mathbb{R}$-admissible and a limit of $\mathbb{R}$-admissibles, it follows that for every $\zeta < \eta$, every subset of $\mathbb{R}$ in $\Game^{\mathbb{R}}\mathbf{\Sigma}^0_1(\phi'^{\zeta})$ is determined. Hence, an induction along ζ shows that

$$\phi'\zeta = \phi^{\zeta},$$

for all $\zeta < \eta$. It follows that

$$\phi^{\infty} = \phi'\infty \in \Sigma_1^{L_{\sigma_{\mathbb{R}}}(\mathbb{R})} = \Game^{\mathbb{R}}\mathbf{\Sigma}^0_2.$$

(In case $\eta = \sigma^{\mathbb{R}}$, the last step of the induction follows from Sublemma 5.4.)

Now, suppose that $s \notin \phi^{\infty}$. By definition, Player I does not have a winning strategy in $G(\phi^{\infty}, s)$. By the two sublemmata, $G(\phi^{\infty}, s)$ is determined. Thus, Player II has a winning strategy in $G(\phi^{\infty}, s)$. But then, an argument as in the proof of Theorem 3.1 shows how to turn this into a winning strategy for $G^*(s)$. This concludes the proof of the lemma. ⊣

We are now led to the main theorem of the article.

THEOREM 5.5. *The following are equivalent:*

1. $\mathbf{\Sigma}^0_2$*-determinacy for games of length* ω^2;
2. *There is a* Π^+_1*-reflecting model of* AD *containing* $\mathbb{R}$.

PROOF. If $\mathbf{\Sigma}^0_2$ games of length ω^2 are determined, then clearly sets in $\Game^{\mathbb{R}}\mathbf{\Sigma}^0_2$ are determined, so that $L_{\sigma_{\mathbb{R}}}(\mathbb{R}) \models \mathsf{AD}$, by Corollary 4.6.

Conversely, suppose that there is a Π^+_1-reflecting model of AD containing $\mathbb{R}$. By Corollary 2.4, $L_{\sigma_{\mathbb{R}}}(\mathbb{R}) \models \mathsf{AD}$. Since $\sigma_{\mathbb{R}}$ is $\mathbb{R}$-admissible,

$$\mathcal{P}(\mathbb{R}) \cap L_{\sigma_{\mathbb{R}}}(\mathbb{R}) = \mathcal{P}(\mathbb{R}) \cap \Delta_1^{L_{\sigma_{\mathbb{R}}}(\mathbb{R})}.$$

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Let Γ be the pointclass of all sets of reals which are Σ_1-definable over $L_{\sigma_\mathbb{R}}(\mathbb{R})$ with parameters. Using that $L_{\sigma_\mathbb{R}}(\mathbb{R}) \models \mathsf{AD}$, Steel [22, Theorem 2.1] implies that Γ has the scale property. Since $\sigma_\mathbb{R}$ is $\mathbb{R}$-admissible, Γ is closed under existential and universal real quantification and thus by Moschovakis [17], it has the uniformization property. Hence, the hypotheses for the Kechris–Woodin determinacy transfer theorem [14] are satisfied, and we may apply it to conclude that all sets in $\Sigma_1^{L_{\sigma_\mathbb{R}}(\mathbb{R})}$ are determined. By Corollary 4.6, all sets in $\Game^\mathbb{R}\boldsymbol{\Sigma}^0_2$ are determined. By Lemma 5.1, $\boldsymbol{\Sigma}^0_2$-games of length ω^2 are determined. ⊣

§6. Concluding remarks. Let us remark that the proof of Theorem 5.5 also yields the following result, which was pointed out in [2]:

THEOREM 6.1. *The following are equivalent over* ZFC:

1. $\boldsymbol{\Sigma}^0_1$-*determinacy for games of length* ω^2,
2. *There is a transitive model of* $\mathsf{KP}+\mathsf{AD}$ *containing the reals.*

To prove Theorem 6.1, repeat the proof of Theorem 5.5, using the least κ such that $L_\kappa(\mathbb{R})$ is admissible in place of $\sigma_\mathbb{R}$, Corollary 2.13 in place of Corollary 4.6, and Sublemma 5.2 in place of Lemma 5.1.

A classical theorem of Kripke states that, in L, the least α such that L_α is a model of $\mathsf{KP}+\Sigma_1$-separation is the union of a chain of Σ_1-elementary substructures. The situation is analogous in $L(\mathbb{R})$. This and Theorem 1.2 imply the consistency results mentioned in the introduction.

THEOREM 6.2. *The following theories are strictly increasing in consistency strength*:

1. $\mathsf{KP}+\mathsf{DC}+$ "$\mathbb{R}$ *exists*" $+\ \mathsf{AD}$;
2. $\mathsf{KP}+\mathsf{DC}+$ "$\mathbb{R}$ *exists*" $+\ \boldsymbol{\Sigma}^0_2$-*determinacy for games of length* ω^2;
3. $\mathsf{KP}+\mathsf{DC}+$ "$\mathbb{R}$ *exists*" $+\ \mathsf{AD}+\Sigma_1$-*separation*.

PROOF. Working in $\mathsf{KP}+\mathsf{DC}+$ "$\mathbb{R}$ exists," assume that $\boldsymbol{\Sigma}^0_2$ games of length ω^2 are determined. We show that there is a model of

$$\mathsf{KP}+\mathsf{DC}+\text{“}\mathbb{R}\text{ exists”}+\mathsf{AD}.$$

Consider the following game of length ω^2:

1. Player I begins by choosing a $\boldsymbol{\Sigma}^0_1$ game of length ω^2;
2. Player II decides which role she wants to have in the game;
3. Players I and II play the game chosen by Player I, taking the roles specified by Player II.

This game is perhaps not $\boldsymbol{\Sigma}^0_1$, but it is certainly, say, Δ^0_2. It clearly cannot be won by Player I, and any winning strategy for Player II easily reduces continuously to a winning strategy for some player for any prescribed $\boldsymbol{\Sigma}^0_1$ game of length ω^2. Since $\mathbb{R}$ exists, one may use collection to conclude that there is a set A containing winning strategies for all $\boldsymbol{\Sigma}^0_1$ games of length ω^2. From A and $\mathbb{R}$ one may use Σ_0-separation to conclude that there is a universal $\Game^\mathbb{R}\boldsymbol{\Sigma}^0_1$ set. Hence, there is a universal IND set, from which one can easily compute a prewellordering of $\mathbb{R}$ whose length is

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an $\mathbb{R}$-admissible ordinal κ. By collection, this ordinal exists, so $L_\kappa(\mathbb{R}) \models \mathsf{KP} + \mathsf{DC}$ and

$$\mathsf{IND} = \mathcal{P}(\mathbb{R}) \cap \Sigma_1^{L_\kappa(\mathbb{R})}.$$

Since all $\mathbf{\Sigma}^0_2$ games of length ω^2 are determined, all $\Game^{\mathbb{R}}\mathbf{\Sigma}^0_1$ games of length ω are determined, and so

$$L_\kappa(\mathbb{R}) \models \mathsf{AD}.$$

For the remaining implication, suppose that AD holds and Σ_1-separation holds. It follows that

$$L(\mathbb{R}) \models \mathsf{AD} + \Sigma_1 - \text{Separation}.$$

To see this, we assume without loss of generality that for no ordinal η do we have

$$L_\eta(\mathbb{R}) \models \mathsf{AD} + \Sigma_1 - \text{Separation},$$

i.e., that the least non- $\mathbb{R}$-projectible ordinal does not exist. Thus, it suffices to show that every subset of $\mathbb{R}$ which is Σ_1-definable (with parameters) over $L(\mathbb{R})$ belongs to $L(\mathbb{R})$. If not, then by Steel [22, Lemma 1.14], there is a partial surjection from $\mathbb{R}$ onto $L(\mathbb{R})$ which is Σ_1-definable over $L(\mathbb{R})$ with parameters. However, $L(\mathbb{R})$ is a Σ_1-definable class and Σ_1-separation holds (in V), so this surjection is actually a set. But this is impossible, for its restriction to the ordinals is also a surjection. Thus,

$$L(\mathbb{R}) \models \mathsf{AD} + \Sigma_1 - \text{Separation}$$

as claimed. Working in $L(\mathbb{R})$, there are arbitrarily large $\mathbb{R}$-stable ordinals. But it is immediate from the definition that if

$$L_\alpha(\mathbb{R}) \prec_1 L_{\alpha^+ + 1}(\mathbb{R}),$$

(recall Definition 2.1) then $\sigma_{\mathbb{R}} < \alpha$, so there is a (set) model of KP in which all games of length ω^2 are determined (and AD holds in this model, additionally). ⊣

By a slightly more elaborate argument involving Inner Model Theory (one can e.g., use the results in Part II of Müller [25]), one can replace KP by ZFC in the statement of Theorem 6.2(2) (and, in fact, by much stronger theories). Similarly, and using Theorem 6.1, one can replace Theorem 6.2(1) by the theory ZFC + $\mathbf{\Sigma}^0_1$-determinacy for games of length ω^2. We omit the details.

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