Combinatorics, Probability and Computing (2013) 22, 783–799. © Cambridge University Press 2013
doi:10.1017/S096354831300028X

Rainbow Perfect Matchings in Complete Bipartite Graphs: Existence and Counting

GUILLEM PERARNAU and ORIOL SERRA

Universitat Politècnica de Catalunya, Carrer Jordi Girona 1–3, 08034 Barcelona, Spain
(e-mail: {guillem.perarnau,oserra}@ma4.upc.edu)

Received 27 February 2012; revised 6 May 2013; first published online 9 July 2013

A perfect matching M in an edge-coloured complete bipartite graph $K_{n,n}$ is rainbow if no pair of edges in M have the same colour. We obtain asymptotic enumeration results for the number of rainbow perfect matchings in terms of the maximum number of occurrences of each colour. We also consider two natural models of random edge-colourings of $K_{n,n}$ and show that if the number of colours is at least n, then there is with high probability a rainbow perfect matching. This in particular shows that almost every square matrix of order n in which every entry appears n times has a Latin transversal.

2010 *Mathematics subject classification*: Primary 05A16
Secondary 05B15, 05C15, 05D40

1. Introduction

A subgraph H of an edge-coloured graph G is *rainbow* if no colour appears twice in $E(H)$. The study of rainbow subgraphs has a large literature: see, *e.g.*, [1, 6, 9, 10, 11]. In this paper we deal with *rainbow perfect matchings* of an edge-coloured complete bipartite graph $K_{n,n}$. Edge-coloured complete bipartite graphs $K_{n,n}$ are equivalent to integer matrices of size $n \times n$ (also called *n-squares*), and the problem of finding a rainbow perfect matching is equivalent to finding a Latin transversal of length n in the corresponding n-square (that is, a set of n pairwise distinct entries, no two in the same row or the same column). If an n-square contains exactly n copies of each entry, it is called an *equi-n-square*. In particular, proper edge-colourings of $K_{n,n}$ with n colours are equivalent to Latin squares, an interesting subclass of equi-n-squares. The following is a longstanding conjecture of Ryser [16] on the existence of Latin transversals in Latin squares.

Conjecture 1.1 (Ryser [16]). *Every Latin square of odd order admits a Latin transversal.*

The above conjecture is not true for even order Latin squares. For instance, the Latin square $A = (a_{ij})$, where $a_{ij} = i + j \pmod n$, contains no Latin transversals for even n.

Nevertheless, it was also conjectured by Brualdi that every Latin square has a partial Latin transversal of length $n-1$. This conjecture was extended by Stein [18] to equi-n-squares.

There are different approaches to these conjectures. For instance, Hatami and Shor [7] proved that every Latin square has a partial transversal of size $n - O(\ln^2 n)$. Snevily [17] conjectured that every subsquare of the addition table of an abelian group of odd order has a Latin transversal, a conjecture which was eventually proved by Arsovski [2]. Another approach was given by Erdős and Spencer [5]. They proved the following result.

Theorem 1.2 (Erdős and Spencer [5]). *Let A be an n-square. If every entry in A appears at most $\frac{n-1}{4e}$ times, then A has a Latin transversal.*

In order to get the above result the authors developed the *lopsided* version of the Lovász Local Lemma. The main idea of this version is to generalize the dependency graph through the so-called *lopsidependency graph*. In this graph, non-edges may no longer represent mutual independence, and the hypothesis of the Local Lemma is replaced by a weaker assumption.

In this paper we address two problems: first, the asymptotic enumeration of rainbow perfect matchings in a given edge-colouring of $K_{n,n}$, and second, the existence of rainbow perfect matchings in random edge-colourings of $K_{n,n}$. We consider not necessarily proper edge-colourings, but the asymptotic enumeration applies to proper ones as well.

Theorem 1.2 gives sufficient conditions on the existence of at least one Latin transversal. One of the goals of this article is to show that, under only slightly stronger assumptions, we can ensure the existence of many Latin transversals. Although there is no specific conjecture on the number of Latin transversals of a Latin square, Vardi [19] proposed the following conjecture for the particular class of addition tables of cyclic groups.

Conjecture 1.3 (Vardi [19]). *Let $z(n)$ be the number of Latin transversals in the table of the cyclic group of order n. There exist two constants $0 < c_1 < c_2 < 1$ such that*

$$c_1^n n! \leqslant z(n) \leqslant c_2^n n!,$$

for all odd n.

Recall that $z(n) = 0$ if n is even. In a more general setting, McKay, McLeod and Wanless [14] showed that $c_2 < 0.614$. Giving a lower bound on $z(n)$ is still an open problem. It is conjectured in [4] that the right asymptotic order of magnitude is around

$$z(n) \sim 0.39^n n!. \tag{1.1}$$

Here we provide, under the hypothesis of Theorem 1.2, upper and lower bounds for the number of rainbow perfect matchings in an edge-coloured $K_{n,n}$ that are asymptotically tight. The techniques used to derive these bounds are inspired by the framework devised by Lu and Székely [12] to obtain asymptotic enumeration results using the Lovász Local Lemma.

Our first result gives an asymptotic estimate of the expectation that a random matching is rainbow.

Theorem 1.4. *Consider an edge-colouring of $K_{n,n}$ such that no colour appears more than n/k times. Let $\mathcal{M}$ denote the family of pairs of non-incident edges that have the same colour and let M be a perfect matching of $K_{n,n}$ chosen uniformly at random. Denote by X_M the indicator variable that M is rainbow. Let $\mu = |\mathcal{M}|/n(n-1)$.*

If $k \geqslant 10.93$ then there exist constants $0 < c_1(k) < 1 < c_2(k)$ depending only on k such that

$$e^{-c_2(k)\mu} \leqslant \mathbb{P}(X_M = 1) \leqslant e^{-c_1(k)\mu}.$$

In the proof of Theorem 1.4 we obtain $c_1(k) = 1 - 2/k - 32/k^2 + o(1)$ and $c_2(k) = 1 + 16/k$. Therefore, if $k = \omega(1)$, then

$$\mathbb{P}(X_M = 1) = e^{-(1+o(1))\mu}.$$

Moreover, if $k = \omega(n^{1/2})$ then

$$\mathbb{P}(X_M = 1) = (1 + o(1))e^{-\mu}.$$

We note that the existence of a rainbow perfect matching in Theorem 1.2 is ensured with the slightly smaller value $k \geqslant 4e \approx 10.87$. We also observe that the bounds on the probability of a rainbow perfect matching in Theorem 1.4 depend only on the cardinality of $\mathcal{M}$, but not on the particular structure of the pairs of monochromatic non-incident edges composing $\mathcal{M}$. The dependence on $|\mathcal{M}|$ is natural, since a colouring in which all pairs of monochromatic edges are mutually incident ($|\mathcal{M}| = 0$) has $n!$ rainbow perfect matchings.

In particular, observe that any proper edge-colouring where each colour appears exactly n/k times, satisfies $|\mathcal{M}| \sim n^3/2k$. This implies the following corollary of Theorem 1.4, which has the same form as Conjecture 1.3.

Corollary 1.5. *The number $r(n,k)$ of rainbow perfect matchings in a proper edge-colouring of $K_{n,n}$ in which each colour appears exactly n/k times, $k \geqslant 10.93$, satisfies*

$$\gamma_1(k)^n n! \leqslant r(n,k) \leqslant \gamma_2(k)^n n!$$

for some constants $0 < \gamma_1(k) < \gamma_2(k) < 1$ which depend only on k.

One interesting question is how far k can be pushed down and still have at least $c^n n!$ rainbow perfect matchings in a proper edge-colouring of $K_{n,n}$, for some $0 < c < 1$. Wanless [20, Section 3] defines the function $f(n)$ to be the minimum number of Latin transversals among all the Latin squares of order n (case $k = 1$). Notice that $f(2n) = 0$ and Ryser's conjecture states that $f(2n+1) > 0$ for any $n \geqslant 0$. As far as we know, this function has not yet been studied.

The results in Theorem 1.4 require the condition $k \geqslant 10.93$, which is close to the one given by Erdős and Spencer [5] for the existence of rainbow perfect matchings. It seems difficult to drop this condition, at least by using a probabilistic approach. This prompts us to ask what we can say about most edge-colourings of $K_{n,n}$ in the more general setting when $k \geqslant 1$ (we cannot use less than n colours). Thus we study the existence of rainbow perfect matchings in *random* edge-colourings. We restrict ourselves to colourings with a fixed number $s = kn$ of colours. We define two natural random models that fit with this condition.

In the *uniform* random model, $\mathcal{C}_u(n,s)$, each edge gets one of the s colours independently and uniformly at random. In this model, every possible edge-colouring with at most s colours appears with the same probability. In the *regular* random model, $\mathcal{C}_r(n,s)$, we choose an edge-colouring uniformly at random among all the equitable edge-colourings. Recall that a colouring is called *equitable* if each colour class has the same size. Although the models have the same expected behaviour, we consider both interesting to analyse. Results analogous to the one in Theorem 1.4 can be proved for these random models.

Proposition 1.6. *Let C be a random edge-colouring of $K_{n,n}$ in the model $\mathcal{C}_u(n,s)$ (or $\mathcal{C}_r(n,s)$) with $s = kn$ colours ($k > 1$). Then,*

$$\mathbb{P}(X_M = 1) = e^{-(c(k)+o(1))\mu},$$

where

$$\mu \sim \frac{n^2}{2s} \quad \text{and} \quad c(k) = 2k\left((k-1)\ln\left(\frac{k-1}{k}\right)+1\right).$$

For $k = 1$, we have $\mathbb{P}(X_M = 1) = e^{-(2+o(1))\mu}$.

Obviously, we have that $c_2(k) < c(k) < c_1(k)$, where $c_1(k)$ and $c_2(k)$ are the constants appearing in Theorem 1.4. It is worth noting that, for any $k \geqslant 1$, we have $c(k) > 1$. Observe also that when $s = n$, the number of rainbow perfect matchings is w.h.p. around $(e^{-1})^n n!$. If (1.1) holds, an edge-colouring induced by a cyclic group of odd order would contain more rainbow perfect matchings than a typical edge-colouring of the complete bipartite graph.

Since the colourings are random, we have a stronger concentration of the rainbow perfect matching probability than in the case of arbitrary edge-colourings. We say that a property holds *with high probability* (w.h.p.) in $\mathcal{C}_u(n,s)$ (or $\mathcal{C}_r(n,s)$) if the probability that the property is satisfied by an edge-colouring chosen uniformly at random from $\mathcal{C}_u(n,s)$ (or $\mathcal{C}_r(n,s)$) tends to one as $n \to +\infty$.

By using the random model $\mathcal{C}_u(n,s)$ we can show that for any $s \geqslant n$, almost all edge-colourings have a rainbow perfect matching.

Theorem 1.7. *A random edge-colouring of $K_{n,n}$ in the $\mathcal{C}_u(n,s)$ model ($s \geqslant n$) contains a rainbow perfect matching w.h.p.*

To prove Theorem 1.7 we use the second moment method on the random variable that counts the number of rainbow perfect matchings in the $\mathcal{C}_u(n,s)$ model. This result can be proved for the $\mathcal{C}_r(n,s)$ model using the same ideas. In particular, this implies that the Ryser conjecture is true w.h.p. for equi-n-squares.

The paper is organized as follows. In Section 2 we provide a proof for Theorem 1.4. The random colouring models are defined in Section 3, where we also prove Proposition 1.6. Theorem 1.7 is proved in Section 4. Finally, in Section 5, we discuss some open problems on rainbow perfect matchings that arise from the paper.

2. Asymptotic enumeration

In this section we prove Theorem 1.4. The theorem provides exponential upper and lower bounds for the probability that a random perfect matching in an edge-coloured complete bipartite graph is rainbow.

2.1. Lower bound

For a given perfect matching, the property of being rainbow can be expressed in terms of the non-occurrence of certain partial matchings. One of the standard tools to give a lower bound for the probability of the existence of a structure that avoids some given bad events is the Lovász Local Lemma. As shown in [5], it is convenient in our current setting to use its *lopsided* version.

Given a set of events $\mathcal{A} = \{A_1,\dots,A_m\}$, a graph H with vertex set $V(H) = [m]$ is a *lopsidependency graph* for the events in $\mathcal{A}$ if, for each i and each subset $S \subseteq \{j \mid ij \notin E(H),\, j \neq i\}$, we have

$$\mathbb{P}(A_i \mid \cap_{j\in S}\overline{A_j}) \leqslant \mathbb{P}(A_i).$$

Thus, the lopsided version weakens the condition on the dependency graph used. Instead of having each event independent of every subset of its non-neighbours, we need the event to be negatively correlated to this subset.

Following Lu and Székely [12], we adopt the more explanatory term *negative dependency graph* for this kind of dependency graph. We next recall the statement of the Lopsided Lovász Local Lemma we will use. It includes an intermediate step, that appears in its proof, which will also be used later on.

Lemma 2.1 (LLLL). *Let $\mathcal{A} = \{A_1,\dots,A_m\}$ be a set of events and let H be a negative dependency graph for $\mathcal{A}$.*

If there exist $x_1,\dots,x_m \in (0,1)$ such that, for each i,

$$\mathbb{P}(A_i) \leqslant x_i \prod_{ij\in E(H)} (1-x_j), \tag{2.1}$$

then, for each $T \subset [m]$ we have

$$\mathbb{P}(A_i \mid \cap_{j\in T}\overline{A_j}) \leqslant x_i.$$

In particular, for each pair $S,T \subset [m]$ of disjoint sets we have

$$\mathbb{P}(\cap_{i\in S}\overline{A_i} \mid \cap_{j\in T}\overline{A_j}) \geqslant \prod_{i\in S}(1-x_i), \tag{2.2}$$

and

$$\mathbb{P}(\cap_{i=1}^m \overline{A_i}) \geqslant \prod_{i=1}^m (1-x_i).$$

Recall that $\mathcal{M}$ denotes the family of pairs of non-incident edges that have the same colour in a given edge-colouring of $K_{n,n}$. For each such pair $\{e,f\} \in \mathcal{M}$, let $A_{e,f}$ denote the event that the pair $\{e,f\}$ belongs to the random perfect matching M. We define $\mathcal{A}_{\mathcal{M}}$ to be the set of events $A_{e,f}$ for any $\{e,f\} \in \mathcal{M}$. Consider the following dependency graph.

Definition. The *rainbow dependency graph* H has the family $\mathcal{M}$ as its vertex set. Two elements in $\mathcal{M}$ are adjacent in H if they contain at least two incident edges in $K_{n,n}$.

It is shown by Erdős and Spencer [5] that the graph H defined above is a negative dependency graph. The following lower bound can be obtained in a similar way to Lu and Székely [12, Lemma 2]. Recall that we consider edge-colourings of $K_{n,n}$ in which each colour appears at most n/k times.

Lemma 2.2. *With the above notations, if* $k \geqslant 10.93$ *and* $n > 200$ *then*

$$\mathbb{P}(\cap_{\{e,f\}\in\mathcal{M}} \overline{A_{e,f}}) \geqslant e^{-(1+16/k)\mu},$$

where

$$\mu = \sum_{\{e,f\}\in\mathcal{M}} \mathbb{P}(A_{e,f}).$$

Proof. Set $\mathcal{A}_{\mathcal{M}} = \{A_1, \ldots, A_m\}$, where $A_i = A_{e,f}$ for some $\{e,f\} \in \mathcal{M}$. The size of $\mathcal{M}$ depends on the configuration of the colours in $E(K_{n,n})$. Let us give an upper bound on $|\mathcal{M}|$. Since each colour appears at most n/k times, in the worst case, we have a proper edge-colouring with kn colours. Thus,

$$|\mathcal{M}| \leqslant kn\binom{n/k}{2} \leqslant \frac{n^2(n-1)}{2k}.$$

The probability that a given pair of non-incident edges belongs to a random perfect matching is

$$p = \mathbb{P}(A_i) = \frac{1}{n(n-1)}.$$

Therefore,

$$\mu = \frac{|\mathcal{M}|}{n(n-1)} \leqslant \frac{n}{2k}. \tag{2.3}$$

Since $n > 200$, we have that $0 < p < 10^{-4}/4$. For any $k \geqslant 10.93$, set $t = 4/k$. It can be checked that for any such t and p we have

$$pe^{(1+4t)t} < 1 - e^{-(1+4t)p}.$$

https://doi.org/10.1017/S096354831300028X Published online by Cambridge University Press

Thus, choose $x_i \in (pe^{(1+4t)t}, 1 - e^{-(1+4t)p})$. Observe that the maximum degree in H is less than $4n(n-1)/k$: given a pair $\{e, f\} \in \mathcal{M}$, there are at most $4n$ possibilities for selecting an edge e' incident to either e or f, and at most $n/k - 1$ choices for a second edge f' with the same colour as e'. Hence, for any $1 \leqslant i \leqslant m$, we have that

$$\sum_{ij \in E(H)} \mathbb{P}(A_j) \leqslant \frac{4n(n-1)}{k} \cdot \frac{1}{n(n-1)} = t.$$

Using the previous inequality, for any $1 \leqslant i \leqslant m$ we have

$$\mathbb{P}(A_i) = p < x_i e^{-(1+4t)t} < x_i \prod_{ij \in E(H)} e^{-(1+4t)\mathbb{P}(A_j)} < x_i \prod_{ij \in E(H)} (1 - x_j). \tag{2.4}$$

Thus, by Lemma 2.1,

$$\mathbb{P}(\cap_{i=1}^m \overline{A_i}) \geqslant \prod_{i=1}^m (1 - x_i) \geqslant e^{-(1+16/k)\mu}.$$

This proves the lemma. □

2.2. Upper bound

Lu and Székely [12] propose a new enumeration tool using the Lovász Local Lemma. Their objective is to find an upper bound for the non-occurrence of rare events. In order to adapt the Local Lemma, they define a new type of parametrized dependency graph: the *ε-near-positive dependency graph.*

Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a set of events. A graph H with vertex set $[m]$ is an ε-near-positive dependency graph if:

(i) $\mathbb{P}(A_i \cap A_j) = 0$ for each $ij \in E(H)$, and
(ii) for any set $S \subseteq \{j : ij \notin E(H), j \neq i\}$ we find that $\mathbb{P}(A_i \mid \cap_{j \in S} \overline{A_j}) \geqslant (1 - \varepsilon)\mathbb{P}(A_i)$.

Condition (i) implies that only incompatible events can be connected. Condition (ii) says that the non-occurrence of any set of non-neighbours cannot shrink the probability of A_i too much. An ϵ-near-positive dependency graph provides an upper bound for the probability that no event in $\mathcal{A}$ occurs.

Theorem 2.3 (Lu and Székely [12]). *Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be a set of events with an ε-near-positive dependency graph H. Then,*

$$\mathbb{P}(\cap_{i=1}^m \overline{A_i}) \leqslant \prod_{i=1}^m (1 - (1 - \varepsilon)\mathbb{P}(A_i)).$$

Observe that this upper bound implies an exponential upper bound of the form $e^{(1-\varepsilon)\mu}$, where $\mu = \sum_{i=1}^m \mathbb{P}(A_i)$.

Lu and Székely [12] also showed that an ε-near-positive dependence H can be constructed using a family of matchings $\mathcal{M}$. Unfortunately the conditions of [12, Theorem 4] that would provide the upper bound in our case do not apply to our family $\mathcal{M}$ of matchings. Instead we give a direct proof for the upper bound which is inspired by their approach.

Lemma 2.4. *With the hypothesis of Lemma 2.2, the rainbow dependency graph H is an ε-near-positive dependency graph with*

$$\varepsilon = 1 - e^{-(2/k + 32/k^2 + o(1))}.$$

Proof. Set $\mathcal{A}_{\mathcal{M}} = \{A_1, \ldots, A_m\}$, where $A_i = A_{e,f}$ for some $\{e,f\} \in \mathcal{M}$. The rainbow dependency graph H for $\mathcal{A}_{\mathcal{M}}$ clearly satisfies condition (i) in the definition of ε-near-positive dependency graph, since two adjacent events have incident edges and a matching is composed of a set of non-incident edges. For condition (ii) we want to show that, for each i and each $T \subseteq \{j \mid ij \notin E(H),\ j \neq i\}$, we have the inequality

$$\mathbb{P}(A_i|B) \geqslant (1-\epsilon)\mathbb{P}(A_i),$$

where $B = \cap_{j\in T}\overline{A_j}$. This is equivalent to showing

$$\mathbb{P}(B|A_i) \geqslant (1-\epsilon)\mathbb{P}(B).$$

Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be the vertices of the two stable sets of $K_{n,n}$. By symmetry, we may assume that A_i consists of the event related to the 2-matching $\{a_{n-1}b_{n-1}, a_nb_n\}$. Then $\{A_j : j \in T\}$ consists of a set of events related to the 2-matchings in $K_{n,n} - \{a_{n-1}a_n, b_{n-1}b_n\}$. This graph is just the complete bipartite graph $K_{n',n'}$, $n' = n-2$, with an edge-colouring in which each colour appears at most n'/k' times, where $k' = k(1-2/n)$. Let us call B' the event B viewed in $K_{n',n'}$ (primes indicate changing the probability space from random perfect matchings in $K_{n,n}$ to random perfect matchings in $K_{n',n'}$), so that

$$\mathbb{P}(B|A_i) = \mathbb{P}(B'). \tag{2.5}$$

For any $r, s \in [n]$, $r \neq s$, let $C_{r,s}$ denote the event related to the 2-matching $\{a_{n-1}b_r, a_nb_s\}$. Define $T_{r,s} \subset T$ as the maximal subset of indices of the 2-matchings which meet none of the two vertices b_r, b_s. Set $B_{r,s} = \cap_{j\in T_{r,s}}\overline{A_j}$. Let us show that

$$\mathbb{P}(B) = \frac{1}{n(n-1)} \sum_{r\neq s} \mathbb{P}(B'_{r,s}), \tag{2.6}$$

where, as before, $B'_{r,s}$ denotes the event $B_{r,s}$ in the probability space of random perfect matchings in $K_{n',n'}$.

For convenience, we may split the event B corresponding to $\cap_{j\in T}\overline{A_j}$ in several events depending on the perfect matching containing $\{a_{n-1}b_r, a_nb_s\}$. We note that, by the definition of $B_{r,s}$, we have $B \cap C_{r,s} = B_{r,s} \cap C_{r,s}$. Thus,

$$\mathbb{P}(B) = \sum_{r\neq s} \mathbb{P}(B \cap C_{r,s}) = \sum_{r\neq s} \mathbb{P}(B_{r,s} \cap C_{r,s}).$$

Note that, since by the definition of T none of the perfect matchings involved in B meets vertices in $\{a_{n-1}a_n, b_{n-1}b_n\}$, we have, for all r, s, $r \neq s$,

$$\mathbb{P}(B_{r,s}|C_{r,s}) = \mathbb{P}(B_{r,s}|C_{n-1,n}).$$

https://doi.org/10.1017/S096354831300028X Published online by Cambridge University Press

Moreover, we observe that $\mathbb{P}(B_{r,s}|C_{n-1,n}) = \mathbb{P}(B'_{r,s})$. Therefore

$$\mathbb{P}(B) = \sum_{r\neq s} \mathbb{P}(B_{r,s}|C_{r,s})\mathbb{P}(C_{r,s}) = \frac{1}{n(n-1)} \sum_{r\neq s} \mathbb{P}(B_{r,s}|C_{n-1,n}) = \frac{1}{n(n-1)} \sum_{r\neq s} \mathbb{P}(B'_{r,s}),$$

giving equation (2.6).

Set $x_i = 1 - e^{-(1+16/k')p'}$. By inequality (2.4) and the choice of x_i, the hypothesis (2.1) of Lemma 2.1 is satisfied. Thus, we can now use the intermediate inequality (2.2) of Lemma 2.1 with $S = T \setminus T_{r,s}$ to obtain

$$\mathbb{P}(B') = \mathbb{P}(B'_{r,s})\mathbb{P}(\cap_{j\in S}\overline{A_j} \mid B'_{r,s}) \geqslant \mathbb{P}(B'_{r,s}) \prod_{j\in S}(1-x_j), \tag{2.7}$$

for any r, s, $r \neq s$. By combining (2.5) with (2.7) we get

$$n(n-1)\mathbb{P}(B|A_i) \geqslant \sum_{r\neq s} \mathbb{P}(B_{r,s}) \prod_{j\in S}(1-x_j). \tag{2.8}$$

Now we give a uniform bound on $\prod_{j\in S}(1-x_j)$. Recall that $S = T \setminus T_{r,s}$ is the set of indices of 2-matchings in $\mathcal{M}'$ that are incident to b_r or b_s. The size of this set can be bounded independently of r and s by

$$|S| \leqslant 2n'\left(\frac{n'}{k'} - 1\right) \leqslant 2\frac{n^2}{k}.$$

Observe that

$$x_i = 1 - e^{-(1+16/k')p'} \leqslant 1 - e^{-(1+16/k+o(1))p}$$

(where $p = 1/n(n-1)$), and we have

$$\prod_{j\in S}(1-x_j) \geqslant e^{-(1+16/k+o(1))p|S|} \geqslant e^{-(2/k+32/k^2+o(1))}. \tag{2.9}$$

By using (2.8) with (2.9) and (2.6) we get

$$\mathbb{P}(B|A_i) \geqslant e^{-(2/k+32/k^2+o(1))} \frac{1}{n(n-1)} \sum_{r\neq s} \mathbb{P}(B_{r,s}) \geqslant e^{-(2/k+32/k^2+o(1))}\mathbb{P}(B). \tag{2.10}$$

Therefore,

$$\varepsilon = 1 - e^{-(2/k+32/k^2+o(1))}$$

satisfies the conclusion of the lemma. □

Now we are able to prove Theorem 1.4.

Proof of Theorem 1.4. Set $\mathcal{A}_{\mathcal{M}} = \{A_1, \dots, A_m\}$, where $A_i = A_{e,f}$ for some $\{e, f\} \in \mathcal{M}$.

By Lemma 2.4, the graph H is an ε-near-positive dependency graph with

$$\varepsilon = 1 - e^{-(2/k+32/k^2+o(1))}.$$

https://doi.org/10.1017/S096354831300028X Published online by Cambridge University Press

It follows from Theorem 2.3 that the probability of having a rainbow perfect matching is upper-bounded by

$$\mathbb{P}(\cap_{i=1}^m \overline{A_i}) \leqslant \prod_{i=1}^m (1-(1-\varepsilon)\mathbb{P}(A_i)) \leqslant e^{-(1-\epsilon)\mu}.$$

By plugging in our value of ε and by using

$$e^{-(2/k+32/k^2+o(1))} \geqslant 1-\frac{2}{k}-\frac{32}{k^2}+o(1),$$

we obtain

$$\mathbb{P}(\cap_{i=1}^m \overline{A_i}) \leqslant e^{-(1-2/k-32/k^2+o(1))\mu}.$$

Combining this upper bound with the lower bound obtained in Lemma 2.2, we obtain

$$\exp\left\{-\left(1+\frac{16}{k}\right)\mu\right\} \leqslant \mathbb{P}(\cap_{i=1}^m \overline{A_i}) \leqslant \exp\left\{-\left(1-\frac{2}{k}-\frac{32}{k^2}+o(1)\right)\mu\right\}.$$

This proves the theorem. □

In particular, if $k=\omega(\sqrt{n})$,

$$\mathbb{P}(\cap_{i=1}^m \overline{A_i}) = (1+o(1))e^{-\mu},$$

as a corollary of Theorem 1.4. This means that when k is sufficiently large with n, the asymptotic estimate coincides with the one obtained by assuming that the bad events A_i are mutually independent.

3. Random colourings

In this section we will analyse the existence of rainbow perfect matchings in random edge-colourings of $K_{n,n}$.

Recall that, in the uniform random colouring model $\mathcal{C}_u(n,s)$, each edge of $K_{n,n}$ is given a colour uniformly and independently chosen from a set of s colours, *i.e.*, every possible colouring with at most s colours appears with the same probability.

In the regular random colouring model $\mathcal{C}_r(n,s)$, a colouring is chosen uniformly at random among all colourings of $E(K_{n,n})$ with equitable colour classes of size n^2/s. Let us give a set-up for this model. Consider two sets A and B, each with n^2 points. Partition A into s cells $C_1,\dots,C_s$, each with n^2/s elements, representing the different colours. Let B represent the edges of $K_{n,n}$. A perfect matching between the points of A and B induces an equitable edge-colouring of the entire graph. The probability space $\mathcal{C}_r(n,s)$ of colourings is settled by choosing such a perfect matching uniformly at random. Let us show that $\mathcal{C}_r(n,s)$ is a uniform model for the set of equitable edge-colourings of $K_{n,n}$.

Lemma 3.1. *Every equitable edge-colouring with s colours has the same probability in the $\mathcal{C}_r(n,s)$ model.*

https://doi.org/10.1017/S096354831300028X Published online by Cambridge University Press

Proof. We show that every equitable edge-colouring arises from the same number of perfect matchings from A to B. Let C be an equitable edge-colouring of $K_{n,n}$. Let $E_i \subset B$ be the set of edges that have colour i under C. We have $|E_i| = n^2/s$ and there are $(n^2/s)!$ perfect matchings from C_i to E_i assigning colour i to the edges in E_i. Therefore, there are exactly $((n^2/s)!)^s$ perfect matchings from A to B giving rise to the edge-colouring C. This number does not depend on C. □

We consider these two models since they simulate the worst situation among the colourings admitted in Theorem 1.4: the probability that a perfect matching is rainbow only depends on the size of $\mathcal{M}$, and this set has its largest cardinality when there are few colours and the number of occurrences of each of them is maximized. This means that there are exactly $s = nk$ colours with n/k occurrences each. Observe that in both random models, the expected size of each colour class is also n/k. In this sense, they are consistent with the hypothesis of Theorem 1.4.

Proof of Proposition 1.6. Consider an edge-colouring obtained using the $\mathcal{C}_u(n,s)$ model and let M denote a fixed perfect matching of $K_{n,n}$. If X_M is the random variable indicating that M is rainbow, then

$$\begin{aligned}\mathbb{P}(X_M = 1) &= \frac{s}{s} \cdot \frac{s-1}{s} \cdot \frac{s-2}{s} \cdot \dots \cdot \frac{s-(n-1)}{s} \\ &= \prod_{i=0}^{n-1}\left(1 - \frac{i}{s}\right). \end{aligned} \tag{3.1}$$

For $s = n$ we can get directly from (3.1)

$$\mathbb{P}(X_M = 1) = \frac{n!}{n^n} = e^{-(2+o(1))\mu}.$$

Assume $s > n$. By writing $(1-x) = \exp\{\ln(1-x)\}$ for $0 < x < 1$, we have

$$\begin{aligned}\mathbb{P}(X_M = 1) &= \prod_{i=0}^{n-1} \exp\left\{\ln\left(1-\frac{i}{s}\right)\right\} \\ &= \exp\left\{\sum_{i=0}^{n-1} \ln\left(1-\frac{i}{s}\right)\right\} \\ &= \exp\left\{\int_0^n \ln\left(1-\frac{x}{s}\right)dx \; + e_1(n,s)\right\},\end{aligned}$$

where $e_1(n,s)$ is the error term obtained by replacing the sum for the integral.

Since $\ln\left(1-\frac{i}{s}\right)$ is decreasing, we have

$$\sum_{i=1}^{n} \ln\left(1-\frac{i}{s}\right) \leqslant \int_0^n \ln\left(1-\frac{x}{s}\right)dx \leqslant \sum_{i=0}^{n-1} \ln\left(1-\frac{i}{s}\right).$$

https://doi.org/10.1017/S096354831300028X Published online by Cambridge University Press

Thus, the error term can be bounded by

$$e_1(n,s) \leqslant \left| \sum_{i=0}^{n-1} \ln\left(1-\frac{i}{s}\right) - \int_0^n \ln\left(1-\frac{x}{s}\right) dx \right| \tag{3.2}$$

$$\leqslant \left| \sum_{i=0}^{n-1} \ln\left(1-\frac{i}{s}\right) - \sum_{i=1}^{n} \ln\left(1-\frac{i}{s}\right) \right|$$

$$= \left| \ln\left(1-\frac{n}{s}\right) \right|$$

$$= \ln\left(\frac{k}{k-1}\right) = O(1),$$

where $k = s/n$.

Also,

$$\int_0^n \ln\left(1-\frac{x}{s}\right) dx = -(s-n)\ln\left(\frac{s-n}{s}\right) - n. \tag{3.3}$$

Using $\mu \sim \frac{n}{2k}$, we get

$$\mathbb{P}(X_M = 1) = \exp\left\{ -\left((k-1)\ln\left(\frac{k-1}{k}\right) + 1 \right) n + e_1(n,s) \right\}$$

$$= \exp\left\{ -2k\left((k-1)\ln\left(\frac{k-1}{k}\right) + 1 + o(1) \right)\mu \right\},$$

proving the first part of the proposition for the $\mathcal{C}_u(n,s)$ model.

Now we study the probability that a fixed perfect matching M is rainbow in the $\mathcal{C}_r(n,s)$ model. According to the construction of the $\mathcal{C}_r(n,s)$ model, the probability of M being rainbow is

$$\mathbb{P}(X_M = 1) = \frac{n^2}{n^2} \cdot \frac{n^2 - \frac{n^2}{s}}{n^2 - 1} \cdot \frac{n^2 - 2\frac{n^2}{s}}{n^2 - 2} \cdot \cdots \cdot \frac{n^2 - (n-1)\frac{n^2}{s}}{n^2 - (n-1)}$$

$$= \prod_{i=0}^{n-1} \left(1 - \frac{i(n^2 - s)}{s(n^2 - i)} \right)$$

$$= \exp\left\{ \sum_{i=0}^{n-1} \ln\left(1 - \frac{i(n^2 - s)}{s(n^2 - i)} \right) \right\}$$

$$= \exp\left\{ \int_0^n \ln\left(1 - \frac{x(n^2 - s)}{s(n^2 - x)} \right) dx + e_2(n,s) \right\},$$

where $e_2(n,s)$ is the error term obtained by replacing the sum for the integral.

If $s = n$ we have

$$\int_0^n \ln\left(1 - \frac{x(n-1)}{(n^2 - x)}\right) dx = -n(n-1)\ln\left(\frac{n}{n-1}\right),$$

which, by using the Taylor expansion of the logarithm, gives

$$\mathbb{P}(X_M = 1) = e^{-(2+o(1))\mu}.$$

https://doi.org/10.1017/S096354831300028X Published online by Cambridge University Press

By arguments analogous to those in (3.2), we can bound the error $e_2(n,s) = O(1)$. In the case where $s > n$, and using $k = s/n$, we have

$$\int_0^n \ln\left(1 - \frac{x(n^2-s)}{s(n^2-x)}\right)dx = -\left((k-1)\ln\left(\frac{k-1}{k}\right) - (n-1)\ln\left(\frac{n-1}{n}\right)\right)n$$
$$= -\left((k-1)\ln\left(\frac{k-1}{k}\right) + 1 + o(1)\right)n.$$

Hence

$$\mathbb{P}(X_M = 1) = \exp\left\{-2k\left((k-1)\ln\left(\frac{k-1}{k}\right) + 1 + o(1)\right)\mu\right\}.$$

□

Note that, for both models of random edge-colourings, the probability that a fixed perfect matching is rainbow is asymptotically the same. Observe that for the two random models we obtain the exact asymptotic value of the probability, while bounds provided by Theorem 1.4 (when the size $|\mathcal{M}|$ of the set of bad events is maximum) are probably not sharp, though consistent with the values for the random models.

We finally observe that for both models, when $k = 1$ we have $\mathbb{P}(X_M = 1) = e^{-(2+o(1))\mu}$, while if $k \to +\infty$ then $\mathbb{P}(X_M = 1) \to e^{-\mu}$, since

$$2k\left(1 - (k-1)\ln\left(\frac{k}{k-1}\right)\right) = 1 + O\left(\frac{1}{k}\right).$$

This reflects the fact that, when k is large, the number of bad events decreases and the model behaves as though they were independent.

4. Existence of rainbow perfect matchings

The aim of this section is to prove that there exists w.h.p. a rainbow perfect matching for any edge-colouring of $E(K_{n,n})$ with $s \geqslant n$ colours. We only consider the $\mathcal{C}_u(n,s)$ model, but the results can be adapted to the $\mathcal{C}_r(n,s)$ model as well. The number of rainbow perfect matchings is counted by $X = \sum_M X_M$, which, according to Proposition 1.6, has expected value

$$\mu = \mathbb{E}(X) = \mathbb{P}(X_M = 1)n! = \exp\left\{-2k\left((k-1)\ln\left(\frac{k-1}{k}\right) + 1 + o(1)\right)\mu\right\}n!.$$

In order to have a rainbow perfect matching we just need to show that $X \neq 0$.

Proof of Theorem 1.7. To show that there exists some rainbow perfect matching w.h.p. we will use the second moment method. By the Chebyshev inequality, we have

$$\mathbb{P}(X = 0) \leqslant \mathbb{P}(|X - \mu| > \mu) \leqslant \frac{\sigma^2}{\mu^2}, \tag{4.1}$$

where σ^2 denotes the variance of X. Observe that $X = 0$ is equivalent to the non-existence of rainbow perfect matchings. Therefore, we need to compute σ^2 and show that it is asymptotically smaller than μ^2. Note that

$$\mathbb{E}(X^2) = \sum_{(M,N)} \mathbb{E}(X_M X_N).$$

Let M and N denote two perfect matchings of $K_{n,n}$ with $|M \cap N| = z$. Then

$$\mathbb{E}(X_M X_N) = \mathbb{P}(X_M = 1)\mathbb{P}(X_N = 1 \mid X_M = 1).$$

If $X_M = 1$, we know that the edges of $M \cap N$ are rainbow. In the remaining $n - z$ edges to colour, we must avoid the z colours that appear in $M \cap N$. Thus,

$$\mathbb{P}(X_N = 1 \mid X_M = 1) = \prod_{i=z}^{n-1} \left(1 - \frac{i}{s}\right) \sim \exp\left\{\frac{\alpha(z)z^2}{2s}\right\}\mathbb{P}(X_M = 1). \tag{4.2}$$

where $1 \leqslant \alpha(z) \leqslant 2$, as can be derived from (3.3). Observe that the events $X_M = 1$ and $X_N = 1$ are positively correlated.

For any perfect matching M and any integer z, such that $0 \leqslant z \leqslant n$, we claim that there exist at most $\binom{n}{z}(e^{-1}(n-z)! + 1)$ perfect matchings N such that $|M \cap N| = z$. We can assume without loss of generality that M is given by the identity and N by a permutation $\pi \in \mathcal{S}_n$. There are $\binom{n}{z}$ ways to choose which edges of M will be shared by N, *i.e.*, the set $\mathcal{I} = \{i : \pi(i) = i\}$. In order for π to correspond to a matching N with exactly z common edges with M, its restriction to $[n] \setminus \mathcal{I}$ must be a derangement. It is well known that the proportion of derangements among all the permutations of length $n - z$ is

$$\sum_{i=0}^{n-z} \frac{(-1)^i}{i!} \leqslant e^{-1} + \frac{1}{(n-z)!}.$$

Therefore there are at most $e^{-1}(n-z)! + 1$ ways to complete the perfect matching, concluding our claim. Hence,

$$\mathbb{E}(X^2) = n! \sum_{z=0}^{n} \binom{n}{z} \big(e^{-1}(n-z)! + 1\big)\mathbb{P}(X_M = 1)\mathbb{P}(X_N = 1 \mid X_M = 1).$$

Since $\sigma^2 = \mathbb{E}(X^2) - \mathbb{E}(X)^2$,

$$\begin{aligned}\frac{\sigma^2(X)}{\mathbb{E}(X)^2} &= \frac{n!\sum_{z=0}^{n}\binom{n}{z}\big(e^{-1}(n-z)!+1\big)\mathbb{P}(X_M = 1)\mathbb{P}(X_N = 1 \mid X_M = 1)}{(n!\mathbb{P}(X_M = 1))^2} - 1 \\ &= e^{-1}\sum_{z=0}^{n} \frac{1}{z!}\left(1 + \frac{e}{(n-z)!}\right)\frac{\mathbb{P}(X_N = 1 \mid X_M = 1)}{\mathbb{P}(X_M = 1)} - 1.\end{aligned}$$

For the sake of simplicity, let us define

$$f(s) = \sum_{z=0}^{n} \frac{1}{z!}\left(1 + \frac{e}{(n-z)!}\right)\frac{\mathbb{P}(X_N = 1 \mid X_M = 1)}{\mathbb{P}(X_M = 1)}.$$

https://doi.org/10.1017/S096354831300028X Published online by Cambridge University Press

Then, using (4.2),

$$\begin{aligned} f(s) &\leqslant \sum_{z=0}^{n} \frac{1}{z!}\left(1+\frac{e}{(n-z)!}\right)\exp\left\{\frac{\alpha(z)z^2}{2s}\right\} \\ &\leqslant \sum_{z=0}^{\infty} \frac{1}{z!}\exp\left\{\frac{\alpha(z)z^2}{2s}\right\} + \frac{e}{n!}\sum_{z=0}^{n}\binom{n}{z}\exp\left\{\frac{\alpha(z)z^2}{2s}\right\} \\ &= \sum_{z=0}^{\infty}\frac{1}{z!}\sum_{t=0}^{\infty}\frac{1}{t!}\left(\frac{\alpha(z)z^2}{2s}\right)^t + \frac{e}{n!}\sum_{z=0}^{n}\binom{n}{z}\sum_{t=0}^{\infty}\frac{1}{t!}\left(\frac{\alpha(z)z^2}{2s}\right)^t \\ &= \sum_{t=0}^{\infty} a_t s^{-t}, \end{aligned}$$

where

$$a_t = \frac{1}{2^t t!}\left(\sum_{z=0}^{\infty}\frac{(\alpha(z)z^2)^t}{z!} + \frac{e}{n!}\sum_{z=0}^{n}\binom{n}{z}(\alpha(z)z^2)^t\right).$$

Observe that $a_0 = e\left(1+\frac{2^n}{n!}\right)$. Since $s \leqslant n^2$,

$$f(s) \leqslant e + O(s^{-1}).$$

Observe that $s \geqslant n$; otherwise, $\mathbb{P}(X_M = 1) = 0$ in equation (3.1). Hence,

$$\frac{\sigma^2}{\mu^2} = e^{-1}f(s) - 1 = O(s^{-1}) \to 0.$$

This concludes the proof. □

Corollary 4.1. *For any* $\varepsilon > 0$*, an equitable edge-colouring of* $K_{n,n}$ *with* s *colours,* $s \geqslant n$*, contains more than* $(1-\varepsilon)c(k)^n n!$ *rainbow perfect matchings with probability at least* $1 - O(\varepsilon^{-2}s^{-1})$.

Proof. It follows from the proof of Theorem 1.7 that

$$\mathbb{P}(X > (1-\varepsilon)\mu) \leqslant \frac{\sigma^2}{\varepsilon^2\mu^2} = O(\varepsilon^{-2}s^{-1}) \to 0.$$ □

5. Final remarks and some open problems

Theorem 1.4 provides upper and lower bounds for the number of rainbow perfect matchings of a given edge-colouring of $K_{n,n}$ such that the number of occurrences of each colour is at most n/k and $k \geqslant 10.93$. It is probably not true that each such edge-colouring contains $e^{-(1+o(1))c(k)\mu}n!$ rainbow perfect matchings, where $c(k)$ is defined in Proposition 1.6. It would be interesting to see how tight these upper and lower bounds $c_1(k)$ and $c_2(k)$ provided in Theorem 1.4 are, by showing extremal examples.

We believe that every edge-colouring with colour classes of cardinality n/k still has an exponential fraction of perfect matchings that are rainbow for any $k > 1$. Determining smaller values of k with this property may shed some additional light on the open

https://doi.org/10.1017/S096354831300028X Published online by Cambridge University Press

conjectures on Latin transversals. Corollary 4.1 shows that almost all equitable edge-colourings with n colours contain an exponential fraction of perfect matchings that are rainbow.

For $s = n$, equation (4.1) provides an upper bound estimate for the probability p that a random colouring has no rainbow perfect matchings of the type

$$p = O(n^{-1}). \tag{5.1}$$

The proportion of Latin squares among the set of square matrices with n symbols is of the order of $e^{-(1+o(1))2n^2}$, so this estimate falls short of proving an asymptotic version of Ryser's original conjecture. We have provided a probabilistic approach to the problem by showing that every equi-n-square admits a Latin transversal with high probability. Even though there are some almost sure results on Latin squares (see *e.g.*, [13, 3]), and some results on generating random Latin squares [15, 8], to our knowledge there are no random models for Latin squares that could pave the way to an asymptotic version of the conjectures of Ryser or Brualdi on the existence of Latin transversals in Latin squares.

On the other hand the following example shows how to construct exponentially many Latin squares (in general edge-colourings of $K_{2k,2k}$ with $2k$ colours) which have no rainbow perfect matchings. Let k be odd. Choose two arbitrary colourings α_1, α_2 of $K_{k,k}$ with colours $\{a_1, \dots, a_k\}$ and two arbitrary colourings β_1, β_2 of $K_{k,k}$ with colours $\{b_1, \dots, b_k\}$.

Let $\{A_1 \cup A_2, B_1 \cup B_2\}$ be the stable sets of $K = K_{2k,2k}$ with $|A_i| = |B_i| = k$, and use α_i for the edges connecting A_i with B_i, $i = 1, 2$, and β_i for the edges connecting A_i with B_j, $i \neq j$. Suppose that the resulting edge-coloured graph has a rainbow perfect matching M. Since M must use the k colours $a_1, \dots, a_k$, we may assume that it uses at least $(k+1)/2$ of these colours from the subgraph $K[A_1, B_1]$ induced by $A_1 \cup B_1$. But then each of the subgraphs $K[A_1, B_2]$ and $K[A_2, B_1]$ can only use $(k-1)/2$ colours $b_1, \dots, b_k$ and some colour b_i cannot be used in the perfect matching, contradicting that M is rainbow.

It is easy to see that there are about n^{n^2} equitable edge-colourings of $K_{n,n}$. By the construction displayed above, if $n \equiv 2 \bmod 4$, we can get $(k^{k^2})^4 = 2^{-n^2} n^{n^2}$ equitable edge-colourings that do not contain rainbow perfect matchings. Thus, for any colouring of $C \in \mathcal{C}_r(n, s)$,

$$\begin{aligned} \mathbb{P}(C \text{ does not contain a rainbow matching}) \geqslant 2^{-n^2} &\gg e^{-(1+o(1))2n^2} \\ &\approx \mathbb{P}(C \text{ is a proper colouring}), \end{aligned}$$

and there is no chance of proving Ryser's conjecture w.h.p. by improving the upper bound in (5.1).

Acknowledgement

This work was partially supported by the Catalan Research Council under grant 2009SGR01387 and the Spanish Council under project MTM2011-28800-C02-01. The first author is grateful to the FPU grant from the Ministerio de Educación de España. Finally, we would also like to thank anonymous referees for their valuable comments and suggestions.

References

[1] Alon, N., Jiang, T., Miller, Z. and Pritikin, D. (2003) Properly colored subgraphs and rainbow subgraphs in edge-colorings with local constraints. *Random Struct. Alg.* **23** 409–433.

[2] Arsovski, B. (2011) A proof of Snevily's conjecture. *Israel J. Math.* **182** 505–508.

[3] Cavenagh, N. J., Greenhill, C. and Wanless, I. M. (2008) The cycle structure of two rows in a random Latin square. *Random Struct. Alg.* **33** 286–309.

[4] Cooper, C., Gilchrist, R., Kovalenko, I. N. and Novakovic, D. (1999) Deriving the number of 'good' permutations, with applications to cryptography. *Kibernet. Sistem. Anal.* **5** 10–16.

[5] Erdős, P. and Spencer, J. (1991) Lopsided Lovász local lemma and Latin transversals. *Discrete Appl. Math.* **30** 151–154.

[6] Frieze, A. and Krivelevich, M. (2008) On rainbow trees and cycles. *Electron. J. Combin.* **15** #R59.

[7] Hatami, P. and Shor, P. W. (2008) A lower bound for the length of a partial transversal in a Latin square. *J. Combin. Theory Ser. A* **115** 1103–1113.

[8] Jacobson, M. T. and Matthews, P. (1996) Generating uniformly distributed random Latin squares. *J. Combin. Designs* **4** 405–437.

[9] Janson, S. and Wormald, N. (2007) Rainbow Hamilton cycles in random regular graphs. *Random Struct. Alg.* **30** 35–49.

[10] Kano, M. and Li, X. (2008) Monochromatic and heterochromatic subgraphs in edge-colored graphs: A survey. *Graphs Combin.* **24** 237–263.

[11] LeSaulnier, T. D., Stocker, C., Wenger, P. S. and West, D. B. (2010) Rainbow matching in edge-colored graphs. *Electron. J. Combin.* **17** #N26.

[12] Lu, L. and Szekely, L. A. (2009) A new asymptotic enumeration technique: the Lovasz Local Lemma. ArXiv e-prints.

[13] McKay, B. D. and Wanless, I. M. (1999) Most Latin squares have many subsquares. *J. Combin. Theory Ser. A* **86** 322–347.

[14] McKay, B. D., McLeod, J. C. and Wanless, I. M. (2006) The number of transversals in a Latin square. *Des. Codes Cryptogr.* **40** 269–284.

[15] McKay, B. D. and Wormald, N. C. (1991) Uniform generation of random Latin rectangles. *J. Combin. Math. Combin. Comput.* **9** 179–186.

[16] Ryser, H. J. (1967) Neuere Probleme der Kombinatorik. In *Vorträge über Kombinatorik*, Mathematisches Forschungsinstitut Oberwolfach, pp. 24–29.

[17] Snevily, H. S. (1999) Unsolved problems: The Cayley addition table of Z_n. *Amer. Math. Monthly* **106** 584–585.

[18] Stein, S. K. (1975) Transversals of Latin squares and their generalizations. *Pacific J. Math.* **59** 567–575.

[19] Vardi, I. (1991) *Computational Recreations in Mathematica*, Addison-Wesley.

[20] Wanless, I. M. (2007) Transversals in Latin squares. *Quasigroups Related Systems* **15** 169–190.