A slowly converging sequence via Pythagoras

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1. *Introduction*

Some school textbooks contain end-of-chapter extension activities related to the topic that has just been covered. One that caught my eye recently was a simple problem in [1] concerning a so-called 'spiral of Pythagoras'. This spiral is constructed from right-angled triangles in a recursive manner, as follows. First, a right-angled triangle ABC is drawn with base $AB = 4$ units and height $BC = 1$ unit (noting that, with slight abuse of notation, we use AB, for example, to denote both the line segment AB and its length; the usage will be clear from the context however). Next, the hypotenuse AC of triangle AB serves as the base of a new triangle, ACS, for which the 'height' $CD = 1$ unit once again. The hypotenuse AD of triangle ACD then becomes the base of the following triangle, and this process continues until the spiral has come all the way round and started to overlap itself. The question posed was: How many of these triangles need to be constructed in order for an overlap to occur?

The answer to this question, 36 triangles, does not depend on the actual lengths of AB and BC, but rather on their ratio. For ease of notation in any calculations, therefore, we may set $AB = 1$ unit and $BC = x$ units for some $x > 0$. Note also that $CD = x$. A diagram containing the first two triangles in the construction for the case $x = \frac{1}{2}$ is given in Figure 1.

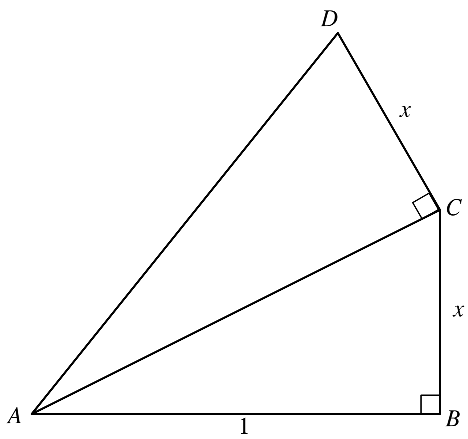

FIGURE 1: The start of a spiral of Pythagoras

We consider here a variation on the above problem, based on a spiral with an alternative structure. The difference is that rather than the 'height' remaining constant, each triangle is now similar to triangle ABC. The initial two triangles in this new construction, with $x = \frac{1}{2}$ once more, are shown in Figure 2. Rather than looking for an overlap, we are now interested only in

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situations whereby the spiral has gone through exactly 2π radians. The question is: What are we able to say about the area of the spiral as its constituent triangles become ever thinner? We adopt two different approaches to answering this question. As will be seen, the more direct approach of the two does in fact lead to a solution of greater complexity.

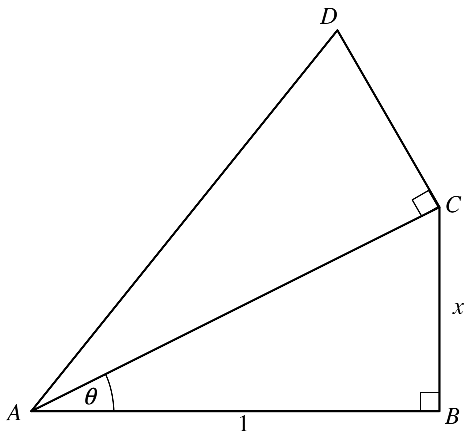

FIGURE 2: The start of an alternative spiral of Pythagoras

2. *Preliminary calculations*

We calculate here the area $A_n(x)$ of the spiral formed from n similar right-angled triangles for which the initial triangle has base 1 unit and height x units; in Figure 2 may be seen the spiral for $n = 2$. We will assume that the values of n and x are such that the spiral has gone through no more than 2π radians.

Let a_n and b_n denote the lengths of the base and the height, respectively, of the nth right-angled triangle in the construction, noting that $a_1 = 1$ and $b_1 = x$. We now show, by induction, that

$$a_n = (x^2 + 1)^{\frac{1}{2}(n-1)} \quad \text{and} \quad b_n = x(x^2 + 1)^{\frac{1}{2}(n-1)}$$

for all $n \geqslant 1$. First, by definition the above expressions for a_n and b_n are true when $n = 1$. Now assume that they are true for some $n \in \mathbb{N}$. Then, by Pythagoras' theorem,

$$\begin{aligned} a_{n+1}^2 &= a_n^2 + b_n^2 \\ &= (x^2 + 1)^{(n-1)} + x^2(x^2 + 1)^{(n-1)} \\ &= (x^2 + 1)^n, \end{aligned}$$

giving

$$a_{n+1} = (x^2 + 1)^{\frac{n}{2}}.$$

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Also, by similarity,

$$\frac{b_{n+1}}{a_{n+1}} = x,$$

which gives

$$b_{n+1} = x\left(x^2 + 1\right)^{\frac{n}{2}},$$

as required.

It then follows that

$$\begin{aligned} A_n(x) &= \frac{1}{2}\sum_{k=1}^{n} a_k b_k \\ &= \frac{1}{2}\sum_{k=1}^{n} x\left(x^2 + 1\right)^{(k-1)} \\ &= \frac{x}{2}\cdot\frac{\left(x^2 + 1\right)^n - 1}{\left(x^2 + 1\right) - 1} \\ &= \frac{\left(x^2 + 1\right)^n - 1}{2x}, \end{aligned} \tag{1}$$

where we have used the formula for the sum of a finite geometric progression [2].

3. *An initial approach*

We work directly here with the exact expression for the area of the spiral, and consider the behaviour of $A_n(x)$ for the special case in which the spiral has gone through exactly 2π radians. Note that, with $\theta = \angle BAC$, it is the case, by similarity, that $\theta = \angle CAD$, and so on. Then, since $x = \tan\theta$, we have

$$x^2 = \tan^2\theta = \sec^2\theta - 1,$$

from which it follows, using (1), that

$$A_n(x) = A_n\left(\tan\theta\right) = \frac{\sec^{2n}\theta - 1}{2\tan\theta}.$$

If we now set $n\theta = 2\pi$, where $n \in \mathbb{N}$ such that $n \geqslant 5$, we have

$$A_n\left(\tan\left(\frac{2\pi}{n}\right)\right) = \frac{\sec^{2n}\left(\frac{2\pi}{n}\right) - 1}{2\tan\left(\frac{2\pi}{n}\right)}. \tag{2}$$

This gives the area of a spiral that has gone through precisely 2π radians.

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The Laurent series for $\cot\theta$ [3] about 0 is given by

$$\cot\theta = \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!}\theta^{2k-1} = \frac{1}{\theta} - \frac{1}{3}\theta - \frac{1}{45}\theta^3 - \dots ,$$

where B_k is the kth Bernoulli number, while the Maclaurin series for $\sec\theta$ [4] is

$$\sec\theta = \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k}}{(2k)!}\theta^{2k} = 1 + \frac{1}{2}\theta^2 + \frac{5}{24}\theta^4 + \dots ,$$

where E_k is the kth Euler number. Both of these series converge for any fixed $\theta \in \mathbb{R}$ such that $0 < \theta < \frac{\pi}{2}$ (in fact, the series for $\cot\theta$ and $\sec\theta$ are also valid when $\theta = \frac{\pi}{2}$ and $\theta = 0$, respectively). We thus have, for fixed $n \geqslant 5$, that

$$\begin{aligned} A_n\left(\tan\left(\frac{2\pi}{n}\right)\right) &= \frac{1}{2}\cot\left(\frac{2\pi}{n}\right)\left(\sec^{2n}\left(\frac{2\pi}{n}\right) - 1\right) \\ &= \pi + \frac{2(3n-1)}{3n^2}\pi^3 + \frac{8(5n^2-1)}{15n^4}\pi^5 + \dots \\ &= \pi + \sum_{k=1}^{\infty}\frac{f_k(n)}{g_{2k}(n)}\pi^{2k+1}, \qquad (3) \end{aligned}$$

where $f_k(n)$ and $g_{2k}(n)$ are polynomials of degrees k and $2k$, respectively.

Since the series given by (3) converges, we know in particular that the sum

$$\sum_{k=1}^{\infty}\frac{f_k(n)}{g_{2k}(n)}\pi^{2k+1}$$

is finite for any $n \geqslant 5$. From this it follows that

$$\lim_{n\to\infty}\left(\sum_{k=1}^{\infty}\frac{f_k(n)}{g_{2k}(n)}\pi^{2k+1}\right) = 0.$$

It is thus the case that

$$\lim_{n\to\infty} A_n\left(\tan\left(\frac{2\pi}{n}\right)\right) = \pi,$$

showing that the area of the spiral tends to the area of a circle of radius 1 unit as the number of constituent triangles increases without limit.

One possibly has to be a little guarded against stating that such a result is 'obvious'. After all, although increasing the number of triangles that make up the spiral leads to, for fixed k, a decrease in the ratio of the lengths of the hypotenuses between the kth and the $(k + 1)$th triangles, this will, to some extent at least, be countered by the fact that there are a greater number of iterations taking place; at least some degree of analysis is therefore required.

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The sequence of areas with nth term given by (2) converges to π very slowly indeed. Using a well-known computer algebra package, we found that (2) does not give π correct to one decimal place until $n = 7389$. In fact, it does not achieve an accuracy to five decimal places until n is well in excess of twenty-five million.

4. *An alternative approach*

We can also answer the question posed in the Introduction without having to derive an expression for the exact area of the spiral, and we outline such an approach here. Suppose that a spiral of n triangles has gone through exactly 2π radians. With $x = \tan\theta$ and $\theta = 2\pi / n$, the length of the final hypotenuse constructed can be shown to be

$$a_{n+1} = \sec^n\left(\frac{2\pi}{n+1}\right).$$

On referring to Figure 2, it follows that the spiral encloses a circle of radius 1 unit with centre at A, while a circle of radius a_{n+1} units with centre at A encloses the spiral. Therefore, in order to demonstrate that the area of the spiral tends to π units^2, we just need to show that

$$\lim_{n\to\infty} \sec^{2n}\left(\frac{2\pi}{n+1}\right) = 1$$

and then apply the 'squeeze rule'. Not surprisingly perhaps, as in the previous section, convergence is very slow.

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