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SIMPLE GROUPS OF MORLEY RANK 5 ARE BAD

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Abstract. We show that any simple group of Morley rank 5 is a bad group all of whose proper definable connected subgroups are nilpotent of rank at most 2. The main result is then used to catalog the nonsoluble connected groups of Morley rank 5.

The groups of finite Morley rank form a model-theoretically natural and important class of groups that are equipped with a notion of dimension generalizing the usual Zariski dimension for affine algebraic groups. This class is known to contain many nonalgebraic examples, but nevertheless, there is an extremely tight connection with algebraic geometry witnessed by, among other things, the following conjecture of Cherlin and Zilber.

Algebraicity Conjecture. *An infinite simple group of finite Morley rank is isomorphic to an affine algebraic group over an algebraically closed field.*

This conjecture appears in Cherlin's early paper [7] analyzing groups of Morley rank at most 3, but even in that very small rank setting, a potential nonalgebraic simple group resisted all efforts to kill it until the recent work of Frécon [12] (see also [15]). The configuration Cherlin encountered is that of a so-called *bad group*, which is defined to be a nonsoluble group of finite Morley rank all of whose proper definable subgroups are nilpotent-by-finite. Despite the spectre of bad groups, a fair amount of progress has been made on the Algebraicity Conjecture. Most notable is the deep result of Altınel, Borovik, and Cherlin from 2008 that established the conjecture for those groups containing an infinite elementary abelian 2-group, [1]. The remaining cases to investigate turn out to be:

Odd type: when the group contains a copy of the Prüfer 2-group $\mathbb{Z}_{2^\infty}$ but no infinite elementary abelian 2-group, and

Degenerate type: when the group contains no involutions at all.

It is a theorem, involving ideas by Borovik, Corredor, Nesin, and Poizat, that bad groups are of degenerate type. As bad groups of rank 3 have now been dealt with by Frécon, some understanding of degenerate type groups is no longer such an unlikely dream. One may also hope to resolve the Algebraicity Conjecture for odd type groups, and here there exists a reasonably solid theory. But even in the presence of involutions, some extremely tight, nonalgebraic configurations have arisen. They

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first appeared in Cherlin and Jaligot's investigation of tame minimal simple groups of odd type [9] and persisted into the more recent and more general $N_\circ^\circ$-context studied by Jaligot and the first author in [11]. These configurations have been named **CiBo**$_1$, **CiBo**$_2$, and **CiBo**$_3$ and will be described in Section 1.1 below.

With the hope of shedding new light on the **CiBo** configurations, and hence on the Algebraicity Conjecture for odd type groups, we decided to study these pathologies in a small rank setting. Groups of rank 4 had already been addressed by the second author in [18], so we took up rank 5. Unlike in [18], we did not shy away from the heavy machinery. We were indeed successful in killing the **CiBo** configurations in rank 5, which is perhaps not surprising, but more importantly, our analysis hints at some general techniques: Moufang sets, and a closer study of involutions. Our main result, which may in the future be pushed further, is as follows.

Theorem. *A simple group of Morley rank* 5 *is a bad group all of whose proper definable connected subgroups are nilpotent of rank at most* 2.

Combining this theorem with several existing results, we obtain, with little effort, a classification of groups of rank 5 (up to bad groups of low rank). Regarding notation, $F^\circ(G)$ denotes the connected Fitting subgroup of G (see Section 2.1), G' denotes the commutator subgroup of G, and we write $G = A * B$ if A and B commute and generate G. Also, recall that a group is called *quasisimple* if it is perfect, and modulo its centre, it is simple.

Corollary. *If G is a nonsoluble connected group of Morley rank* 5, *then* $F := F^\circ(G)$ *has rank at most* 2, *and G is classified as follows.*

- *If* $\operatorname{rk} F = 2$, *then either*
  - $G/F \cong \mathrm{SL}_2(K)$ *acting naturally on* $F \cong K^2$ *with the extension split whenever* $\operatorname{char} K \neq 2$, *or*
  - $G = F * G''$ *with* $G'' \cong (\mathrm{P})\mathrm{SL}_2(K)$.
- *If* $\operatorname{rk} F = 1$, *then either*
  - $G = R * G''$ *with* $R/Z(R) \cong \mathrm{AGL}_1(L)$ *and* $G'' \cong (\mathrm{P})\mathrm{SL}_2(K)$,
  - $G = F * G'$ *with G' quasisimple and bad of rank* 4, *or*
  - *G is quasisimple and bad of rank* 5.
- *If* $\operatorname{rk} F = 0$, *then G' is quasisimple and bad of rank* 4 *or* 5.

Consequently, $\operatorname{rk} F \geq 1$ *whenever G contains an involution.*

We have thus far presented our work as simply a testing ground for the study of **CiBo**$_1$, **CiBo**$_2$, and **CiBo**$_3$, but it may also have applications to the current study of limits to the so-called degree of generic transitivity of a permutation group of finite Morley rank. This was indeed the case with the classification of groups of Morley rank 4, see [2 and 4].

§1. Background.

Here we simply collect a handful of results on "small" groups of finite Morley rank. For general reference on groups of finite Morley rank, we recommend [1, 5, 13]. We assume familiarity with basic definability and connectedness notions [5, Section 5.2], Zilber's Field Theorem [5, Theorem 9.1], and some standard involution-related techniques [5, Section 10] such as 2-Sylow theory and Brauer-Fowler estimates. We also use the fundamental notion of a decent torus [8].

For a group G, let $I(G)$ stand for the set of its involutions. The Prüfer 2-rank $\mathrm{pr}_2(G)$ is the maximal r such that $\bigoplus_r \mathbb{Z}_{2^\infty}$ is contained in a Sylow 2-subgroup of

G, $\mathbb{Z}_{2^\infty}$ being the Prüfer 2-group. The 2-rank $m_2(G)$ is simply the maximal rank (in the algebraic sense) of an elementary abelian 2-subgroup of G. As usual, H° stands for the connected component of a subgroup $H \leq G$.

1.1. $N_\circ^\circ$-groups and CiBo.

DEFINITION 1.1. A group of finite Morley rank G is called an $N_\circ^\circ$-*group* if for every nontrivial, definable, abelian, connected subgroup $A \leq G$, the connected component of its normalizer, $N^\circ(A)$, is soluble.

The condition for being an $N_\circ^\circ$-group is a local analog of minimal simplicity and implies a sense of smallness of the group. The only nonsoluble, *algebraic* $N_\circ^\circ$-groups are those of the form $(\mathrm{P})\mathrm{SL}_2$. The **CiBo** configurations are, at present, unavoidable complications in the study of $N_\circ^\circ$-groups for which the **C**entralizer of an **i**nvolution is a **Bo**rel subgroup. (A *Borel subgroup* is a maximal connected definable soluble subgroup.)

FACT 1.2 (Special case of [11]). *Let G be an infinite, connected, nonsoluble $N_\circ^\circ$-group of finite Morley rank with involutions, but without an infinite subgroup of exponent* 2. *Further assume that $C^\circ(i)$ is soluble for all $i \in I(G)$. Then the involutions of G are conjugate, and G is of one of the following types*:

CiBo$_1$: $\mathrm{pr}_2(G) = m_2(G) = 1$, *and $C(i)$ is a self-normalizing Borel subgroup of G*;
CiBo$_2$: $\mathrm{pr}_2(G) = 1$, $m_2(G) = 2$, *$C^\circ(i)$ is an abelian Borel subgroup of G inverted by any involution in $C(i) - \{i\}$, and* $\mathrm{rk}\, G = 3 \cdot \mathrm{rk}\, C(i)$;
CiBo$_3$: $\mathrm{pr}_2(G) = m_2(G) = 2$, *$C(i)$ is a self-normalizing Borel subgroup of G, and if $i \neq j \in I(G)$, then $C(i) \neq C(j)$*;
Algebraic: $G \cong \mathrm{PSL}_2(K)$.

1.2. Small groups and small actions. Here we briefly summarize the existing results about groups of small Morley rank.

FACT 1.3 ([14]). *If A is a connected group of rank* 1, *then A is either a divisible abelian group or an elementary abelian p-group for some prime p.*

FACT 1.4 ([7]). *If B is a connected group of rank* 2, *then B is soluble. If B is nilpotent and nonabelian, then B has exponent p or p^2 for some prime p, and if B is nonnilpotent, then $B/Z(B) \cong K_+ \rtimes K^\times$ for some algebraically closed field K.*

A nilpotent p-group of bounded exponent is called *p-unipotent*; such subgroups play an important role in the analysis of soluble groups and their intersections.

The main result for groups of rank 3 takes the following very satisfying form as a result of the aforementioned efforts of Frécon. Sadly, the situation in rank 4 is not yet so clear.

FACT 1.5 ([7 and 12]). *A nonsoluble, connected group of rank* 3 *is isomorphic to* $(\mathrm{P})\mathrm{SL}_2(K)$ *for some algebraically closed field K.*

FACT 1.6 ([18, Theorem A]). *A simple group of rank* 4 *is a bad group whose definable proper subgroups have rank at most* 1.

Finally, the ubiquity of the following result of Hrushovski in the study of groups of small rank is hard to overstate; despite nontrivial intersection with the above, we prefer to state it separately.

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FACT 1.7 (Hrushovski, see [5, Theorem 11.98]). *Let H be a group of finite Morley rank acting faithfully, transitively and definably on a definable set X of rank and degree 1. Then* $\operatorname{rk} H \leq 3$, *and if* $\operatorname{rk} H > 1$, *there is a definable algebraically closed field K such that the action of H on X is equivalent to* $\mathrm{AGL}_1(K) = K_+ \rtimes K^\times$ *acting on* $\mathbb{A}_1(K) = K$ *or* $\mathrm{PGL}_2(K)$ *acting on* $\mathbb{P}^1(K) = K \cup \{\infty\}$.

§2. Local analysis.

SETUP. Let G be a simple group of Morley rank 5.

The primary goal of the present section is to show that the Deloro-Jaligot analysis of Fact 1.2 applies to G; that is, we aim to establish the following proposition. (Note that, since G has rank 5, Fact 1.2 immediately rules out the possibility that G is of type **CiBo**$_2$.)

PROPOSITION 2.1. *The group G is an $N_\circ^\circ$-group that either has no involutions or is of type* **CiBo**$_1$ *or* **CiBo**$_3$.

First notice that by Fact 1.7, G has no definable subgroups of rank 4. Consequently, if $A \neq 1$ is a connnected soluble subgroup of G, then $N^\circ(A)/A$ has rank at most 2, so $N^\circ(A)$ is soluble by Fact 1.4. Combining this with the classification of even and mixed type simple groups from [1], we easily arrive at the following lemma.

LEMMA 2.2. *The group G is an $N_\circ^\circ$-group of odd or degenerate type.*

To apply the Deloro-Jaligot analysis to G, it remains to prove solubility of centralisers of involutions—this, however, requires knowledge of intersections of Borel subgroups, which is where we begin. However, after solubility is obtained in Section 2.1, we shall in Section 2.2 study intersections a bit more for later use.

2.1. Solubility of centralisers.

Since G is an $N_\circ^\circ$-group, G enjoys certain general so-called "uniqueness principles" [11, Fact 8]. However, the small rank context allows for a much stronger version with the effect of completely short-cutting Burdges' elaborate unipotence theory, on which we shall therefore not dwell.

Recall that the connected Fitting subgroup $F^\circ(H)$ of a group H is its characteristic subgroup generated by all definable, connected, normal nilpotent subgroups of H; when H has finite Morley rank, $F^\circ(H)$ is definable and nilpotent [5, Theorem 7.3]. Also, for soluble connected H of finite Morley rank, one has $H' \leq F^\circ(H)$ [5, Corollary 9.9], and $C_H^\circ(F^\circ(H)) \leq F^\circ(H)$ [5, Proposition 7.4]; finally, $H/F^\circ(H)$ is a divisible abelian group [5, Theorem 9.21]. Borel subgroups were defined at the beginning of Section 1.1.

LEMMA 2.3. *If B is a rank 3 Borel subgroup of G, then* $\operatorname{rk} F^\circ(B) \geq 2$.

PROOF. By Zilber's Field Theorem and the aforementioned fact that $C_H^\circ(F^\circ(H)) \leq F^\circ(H)$ for connected soluble H of finite Morley rank, one has $\operatorname{rk} F^\circ(B) > 1$. ⊣

LEMMA 2.4. *For $i \in I(G)$, $C^\circ(i)$ is soluble.*

PROOF. First observe that by torality [6, Theorem 3], $C_i = C^\circ(i)$ contains i. If C_i is not soluble, then it has rank 3; as it contains i in its centre, Fact 1.5 yields $C_i \simeq \mathrm{SL}_2(K)$ in characteristic not 2. In particular, the Prüfer rank of G is 1. Torality again and the conjugacy of Sylow 2-subgroups (alternatively, of maximal decent tori) imply that involutions are conjugate in G and Sylow 2-subgroups of G are like that of $\mathrm{SL}_2(K)$.

Let $j \in I(G)$ be a generic conjugate of i, and set $A := (C_i \cap C_j)^\circ > 1$. Since i is the only involution in C_i, the group A contains no involutions: it is therefore a unipotent (in the algebraic sense) subgroup of C_i, contained in the Borel subgroup $B_i = N_{C_i}(A)$ of C_i (respectively $B_j = N_{C_j}(A)$ in C_j). Of course $i \in B_i$.

Since $i \neq j$, one has $B_i \neq B_j$, so $N := N^\circ(A) \geq \langle B_i, B_j \rangle$ has rank at least 3. Now, N is soluble since G is an $N_\circ^\circ$-group, so $\operatorname{rk} N = 3$. However N is nonnilpotent since otherwise $i \in Z(N)$ by the so-called rigidity of tori [5, Theorem 6.16] (or [5, Section 9.2, Exercise 3]) which would imply that $j = i$ by the structure of the Sylow 2-subgroup. So, by Lemma 2.3, the Fitting subgroup $F := F^\circ(N)$ has rank exactly 2. If F contains some involution ℓ, then by the rigidity of tori again ℓ is central in N, so using the structure of the Sylow 2-subgroup, $i = \ell = j$, which is a contradiction.

Now let k be an involution generic over i and j. If $A_k := (C_k \cap N)^\circ$, which is nontrivial by rank considerations, contains an involution, it can only be k, which then normalises A against genericity. Hence, A_k is again a unipotent subgroup of C_k; in particular it has no involutions. By a standard "torsion-lifting" argument [5, Section 5.5, Exercise 11], $A_k F$ has no involutions, so it is proper in N. Since F has rank 2, one gets $A_k \leq F$.

But also, $A \leq F$ and $A_k \neq A$ since otherwise k normalises A, against genericity. Thus, the rank 2, nilpotent group F contains two distinct infinite abelian subgroups—it is a classical consequence of the normaliser condition [5, Lemma 6.3] (or of [5, Section 6.1, Exercise 5]) that F itself is abelian. In particular $F \leq N_k := N^\circ(A_k)$.

By conjugacy of involutions in G and of unipotent subgroups in C_i, the groups A and A_k are however G-conjugate, so N and N_k are as well. The latter is therefore a rank 3 Borel subgroup with no involutions in $F_k := F^\circ(N_k)$. Lifting torsion again, FF_k has no involutions, it must be that $F = F_k$. Taking normalisers, $N = N_k$, so k normalises N, against genericity again. ⊣

Proof of Proposition 2.1. Fact 1.2 now applies to G, and as rank considerations rule out the possibility that G is of the form PSL_2 or $\mathbf{CiBo}_2$, Proposition 2.1 is proven. ⊣

2.2. More on uniqueness principles. We now continue the local analysis of G (without using Fact 1.2) and give two auxiliary results that build on Lemma 2.3.

Lemma 2.5. *If $B_1 \neq B_2$ are two Borel subgroups of G, then $(F^\circ(B_1) \cap F^\circ(B_2))^\circ = 1$.*

Proof. Let $X = (F^\circ(B_1) \cap F^\circ(B_2))^\circ$, and suppose that $X \neq 1$.

First notice that X cannot be normal in both B_1 and B_2 since otherwise $B_1 = N^\circ(X) = B_2$, a contradiction.

We now show that X cannot be normal in either. If say $X \trianglelefteq B_1$, then X cannot be normal in B_2, so $X < F^\circ(B_2)$. By the normaliser condition, $K_2 := N^\circ_{F^\circ(B_2)}(X) > X$. Of course $K_2 \leq N^\circ(X) = B_1$. If $K_2 \trianglelefteq B_1$ then $K_2 \leq F^\circ(B_1)$, against the definition of X. In particular B_1 has rank 3 and is nonnilpotent. By Lemma 2.3, $F^\circ(B_1)$ has rank 2 and so does K_2. Also, X has rank 1. Now the rank 1 group X being

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normal in the nilpotent groups $F^\circ(B_1)$ and K_2 is central in each, hence also in $B_1 = \langle K_2, F^\circ(B_1)\rangle$. Since B_1 is not nilpotent, the rank 2 factor group $\beta = B_1/X$ is not either, so by Fact 1.4, its structure is known: modulo a finite centre it is isomorphic to some $K_+ \rtimes K^\times$. Since $K_2 \not\leq F^\circ(B_1)$, K_2 covers $\beta/F(\beta)$, so K_2 contains an infinite divisible torsion subgroup. Lifting torsion, there is at most one definable infinite subgroup of K_2 containing no divisible torsion, and if it exists, it must be definably characteristic in K_2.

Now, still assuming $X \trianglelefteq B_1$, recall that X cannot be normal in B_2. However, whether B_2 is nilpotent or not, one has $K_2 \trianglelefteq B_2$, so X cannot be definably characteristic in K_2. Thus, as observed above, X must contain divisible torsion. Since $\operatorname{rk} X = 1$, X is a decent torus so is contained in the (unique) maximal decent torus of $F^\circ(B_2)$. By the rigidity of tori, X is central in B_2. This is a contradiction, and we conclude that X is normal in neither B_1 nor B_2.

Consider $N := N^\circ(X)$. As above $K_2 > X$; likewise $K_1 := N^\circ_{F^\circ(B_1)}(X) > X$, so $N = \langle K_1, K_2\rangle$ is a rank 3 Borel subgroup. But now $X = (F^\circ(B_1) \cap F^\circ(N))^\circ$ is normal in N, against the preceding analysis. ⊣

It follows that no Borel subgroup of rank 3 (if there is any) can be nilpotent as it would intersect its conjugates non-trivially.

Corollary 2.6. *If $B_1 \neq B_2$ are two rank 3 Borel subgroups of G, then either* $\operatorname{rk}(B_1 \cap B_2) = 1$, *or there exist commuting involutions i and j for which $B_1 = C^\circ(i)$ and $B_2 = C_2^\circ(j)$.*

Proof. As G has rank 5, $\operatorname{rk}(B_1 \cap B_2) \geq 1$. Now, assume that $H := (B_1 \cap B_2)^\circ$ has rank 2. Consider the canonical map from H to $B_1/F^\circ(B_1) \times B_2/F^\circ(B_2)$; by Lemma 2.5 it has a finite kernel. Thus, the connected group H' is finite, so H is abelian. Also, $B_1 = H \cdot F^\circ(B_1)$ since otherwise $H = F^\circ(B_1)$ implying that the rank 3 group B_2 contains the rank 2 subgroups $F^\circ(B_1)$ and $F^\circ(B_2)$, which by Lemma 2.5 have a finite intersection.

Set $A_1 := (H \cap F^\circ(B_1))^\circ$ and $A_2 := (H \cap F^\circ(B_2))^\circ$. By Lemma 2.5, $(A_1 \cap A_2)^\circ = 1$. Since $F^\circ(B_1)$ has rank 2, A_1 must be central in $F^\circ(B_1)$, and as H is abelian, we find that A_1 is central in $B_1 = H \cdot F^\circ(B_1)$. Moreover, B_1/A_1 must be of the form $K_+ \rtimes K^\times$ modulo a finite centre, so $H/A_1 \cong K^\times$ for some algebraically closed field K of characteristic not 2 (using Lemma 2.2). But, $H = A_1 \cdot A_2$, so A_2 contains some involution j, and $B_2 = C^\circ(j)$. An analogous argument shows that $B_1 = C^\circ(i)$ for some involution $i \in A_1$, which commutes with j, so we are done. ⊣

§3. Using the $N_\circ^\circ$-analysis.

By Proposition 2.1, the simple rank 5 group G has no involutions or is of type $\mathbf{CiBo}_k$ for $k \in \{1, 3\}$. Different cases beg for distinct methods.

3.0. The bad case.

Let us quickly address the case without involutions.

Proposition 3.1. *If G has no involutions, then G is a bad group all of whose proper definable connected subgroups are nilpotent of rank at most* 2.

Proof. Suppose G has a subgroup H of rank 3 and divide into cases.

First, assume that H is nonsoluble. Then having no involutions, H is a bad group of rank 3: hence for $g \notin N(H)$, the intersection $H \cap H^g$ has rank 1. So, the orbit of H^g under conjugation by H has rank 2, and as $X := \{H^g : g \in G\}$ has rank 2

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and degree 1, we find that the action of G on X is generically 2-transitive, i.e., that there is a unique generic orbit on $X \times X$. Lifting torsion, this creates an involution in G, a contradiction. (Of course there is Frécon's Theorem as well.)

Now suppose that H is soluble. To avoid the same contradiction as above, for generic g, the intersection $H \cap H^g$ has rank 2, but this contradicts Corollary 2.6.

So G has no definable, connected, proper subgroup of rank 3. Having no involutions, its rank 2 subgroups are nilpotent; G is a bad group. ⊣

3.1. Killing $\mathbf{CiBo}_1$: Moufang sets. We target the following proposition.

Proposition 3.2. *The group G cannot have type* $\mathbf{CiBo}_1$.

Setup. Assume G has type $\mathbf{CiBo}_1$ (see Fact 1.2).

The two main ingredients of our analysis in this case are (a small fragment of) the theory of Moufang sets and the Brauer-Fowler computation. The latter will be used later; for the moment we focus on the former. Moufang sets encode the essence of so-called split BN-pairs of (Tits) rank 1, and as such, they sit at the low end of an important geometric framework. (So low, in fact, there is no honest geometry to speak of, at least not in the sense of Tits.)

Definition 3.3. For a set X with $|X| \geq 3$ and a collection of groups $\{U_x : x \in X\}$ with each $U_x \leq \mathrm{Sym}(X)$, we say that $(X, \{U_x : x \in X\})$ is a *Moufang set* if for $P := \langle U_x : x \in X\rangle$ the following conditions hold:

1. each U_x fixes x and acts regularly on $X \setminus \{x\}$,
2. $\{U_x : x \in X\}$ is a conjugacy class of subgroups in P.

We call P the *little projective group* of the Moufang set, and each U_x is called a *root group*. The 2-point stabilisers in P are called the *Hua subgroups*.

The result we require is the following, though likely we could do with less if we were willing to work a little harder.

Fact 3.4 (see [17, Theorem A]). *Let $(X, \{U_x : x \in X\})$ be a Moufang set of finite Morley rank with abelian Hua subgroups and infinite root groups that contain no involutions. If the little projective group P of the Moufang set has odd type, then $P \cong \mathrm{PSL}_2(K)$ for some algebraically closed field K.*

Lemma 3.5. *There are no rank* 3 *Borel subgroups of G.*

Proof. Assume that B is a rank 3 Borel subgroup of G, and let X be the set of G-conjugates of B. We now consider the permutation group (X, G), and show that it is associated to a Moufang set. For $\beta \in X$, let $U_\beta := F^\circ(\beta)$. Of course the stabiliser of β is $N(\beta)$ and $N^\circ(\beta) = \beta$. As a consequence of Corollary 2.6 in type $\mathbf{CiBo}_1$ (and the fact that X has degree 1), we find that G acts 2-transitively on X with each 2-point stabiliser of rank 1. We now claim that U_β acts regularly on $X \setminus \{\beta\}$.

First, we show transitivity. As X has rank 2 and degree 1, it suffices to show that U_β has no orbits of rank 0 or 1 other than $\{\beta\}$. By connectedness of U_β, a rank 0 orbit is a fixed point of U_β, but U_β is too large to be contained in a 2-point stabiliser. Next, a rank 1 orbit gives rise to a rank 1 subgroup $A_\beta := U_\beta \cap N(\gamma)$ for some $\gamma \neq \beta$. By 2-transitivity, $A_\gamma := U_\gamma \cap N(\beta)$ also has rank 1, and by Lemma 2.5,

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$A^\circ_\beta \neq A^\circ_\gamma$. Thus, the 2-point stabiliser $G_{\beta,\gamma}$ has rank 2, which is a contradiction. So U_β acts transitively on $X \setminus \{\beta\}$.

We now show that the action of U_β on $X \setminus \{\beta\}$ is free. Suppose $u \in U_\beta$ fixes γ for $\gamma \neq \beta$, so $u \in U_\beta \cap N(\gamma)$. First, if U_β is abelian, then the transitivity of U_β on $X \setminus \{\beta\}$ forces u to be in the kernel of the action of G on X, and simplicity of G implies that $u = 1$. Thus, we may assume that U_β is a nilpotent nonabelian group of rank 2. Thus, U_β is p-unipotent (see Section 1.2 for the definition), and u is a p-element. Of course, U_γ is also p-unipotent, so by [5, Section 6.4], u has an infinite centraliser in $Z^\circ(U_\gamma)$, which has rank exactly 1. Hence u centralises $Z^\circ(U_\gamma)$. We now have that $C^\circ(u) \geq \langle Z^\circ(U_\beta), Z^\circ(U_\gamma)\rangle$.

We still aim at proving freeness of U_β on $X \setminus \{\beta\}$. Briefly suppose $C^\circ(u)$ to be soluble. Then every p-unipotent subgroup of $C^\circ(u)$ lies in $F^\circ(C^\circ(u))$ (since $H/F^\circ(H)$ is divisible for soluble connected H, see the beginning of Section 2.1); if $C^\circ(u)$ is a Borel subgroup, this contradicts Lemma 2.5. Hence $\operatorname{rk} C^\circ(u) = 2$; as it is generated by two distinct rank 1, bounded exponent groups, it is abelian. Now $Z^\circ(U_\beta) \leq N^\circ(Z^\circ(U_\gamma)) = \gamma$, against Lemma 2.5 again. So, $C^\circ(u)$ is a nonsoluble group of rank 3. By the work of Frécon (or the fact that since $\beta \cap \gamma$ normalizes $\langle Z^\circ(U_\beta), Z^\circ(U_\gamma)\rangle = C^\circ(u)$, $Z^\circ(U_\beta) \cdot (\beta \cap \gamma)^\circ$ is a rank 2 subgroup of $C^\circ(u)$), we find that $C^\circ(u) \cong (\mathrm{P})\mathrm{SL}_2(K)$, and as this cannot happen inside of a $\mathbf{CiBo}_1$ group (by looking at the structure of the Sylow 2-subgroup), we conclude that the action of U_β on $X \setminus \{\beta\}$ is free.

This shows that $(X, \{U_\beta : \beta \in X\})$ is a Moufang set, and by connectedness results for Moufang sets (see [16, Proposition 2.3]), each Hua subgroup is a connected rank 1, hence abelian, group. Moreover, we claim that U_β contains no involutions. Indeed, if U_β contained an involution, it would contain a maximal 2-torus T, the definable hull $d(T)$ of which would be central in β by rigidity. But then, $\beta/d(T)$ would be a nonnilpotent group of rank 2 without involutions. This is a contradiction, so the root groups of the Moufang set have no involutions. By Fact 3.4, we have a contradiction. ⊣

Notice that since we are in type $\mathbf{CiBo}_1$, using the work of Frécon one could now even claim that G has no definable subgroups of rank 3 at all.

Proof of Proposition 3.2. The proof combines Brauer-Fowler computations and genericity arguments, the latter being mostly [6, Theorem 1] which asserts that the definable hull of a generic element contains a maximal decent torus (in our case a nontrivial 2-torus will suffice). Let $i, j \in G$ be independent involutions and set $x = ij$. This strongly real element is no involution.

First we argue that $B_i = C^\circ(i)$ has rank 2; of course, by Lemma 3.5, the rank is at most 2. Consider $C = C^\circ(x)$, which is normalised by i; by the structure of the Sylow 2-subgroup, C contains no involutions. If C were not soluble it would be of the form $\mathrm{PSL}_2(K)$ or a bad group; the first case contradicts the structure of the Sylow 2-subgroup and the second contradicts the solubility of B_i as bad groups have no involutive automorphisms [5, Proposition 13.4] (besides not existing in rank 3). Thus, C is soluble, so by Lemma 3.5 again, $\operatorname{rk} C \leq 2$.

Consider the map $\mu : I(G) \times I(G) \to G$ mapping (i, j) to x. Notice that the image is the set of strongly real elements. By the structure of the Sylow 2-subgroup and the genericity result we quoted in the first paragraph, the generic element is not strongly real, so the image set of μ has rank at most 4. On the other hand, the fibre over x is in definable bijection with the set of involutions inverting x, i.e., of involutions in $C^{\pm}(x) \setminus C(x)$, where $C^{\pm}(x)$ is the group of elements centralising or inverting x. Clearly the fibre has rank $f \leq \operatorname{rk} C \leq 2$. The Brauer-Fowler estimate is simply that, by additivity,

$$2 \operatorname{rk} I(G) - f \leq \operatorname{rk} \left(\mu(I(G) \times I(G))\right);$$

or put otherwise, since the map μ has nongeneric image, $5 < 2 \operatorname{rk} B_i + f$.

In particular one must have $\operatorname{rk} B_i = f = \operatorname{rk} C = 2$. Now there is a whole coset of C consisting of involutions inverting x, so C is abelian, inverted by i. From this we derive a highly nongeneric property. Suppose C (or of course any conjugate) contains a generic element of G. As the latter contains in its definable hull a maximal 2-torus, the structure of the Sylow 2-subgroup forces $i \in C$, which is impossible. So $\bigcup_G C^g$ contains no generic element and we derive the contradiction as follows.

By the $N_\circ^\circ$-property together with Lemma 3.5, C must be almost self-normalising. Moreover if there is a nontrivial $c \in C \cap C^g$ for some $g \notin N(C)$ then $C^\circ(c) \geq \langle C, C^g \rangle$. As this centraliser is generated by two groups of rank 2 that are each without 2-torsion, it can be no group of rank 3: hence $C^\circ(c) = G$. But, G is simple, so $C \cap C^g = \{1\}$. Thus, C is disjoint from its proper conjugates, and a standard computation now gives

$$\operatorname{rk} \bigcup_{g \in G} C^g = \operatorname{rk} \left(G/N_G(C)\right) + \operatorname{rk}(C) = \operatorname{rk} G.$$

This shows that $\bigcup_G C^g$ contains a generic element: a contradiction. ⊣

3.2. Killing $\mathbf{CiBo}_3$: conjugacy methods. To conclude the proof of our main theorem, it remains to prove the following proposition.

PROPOSITION 3.6. *The group G cannot have type* $\mathbf{CiBo}_3$.

SETUP. Assume G has type $\mathbf{CiBo}_3$ (see Fact 1.2).

We begin with some general observations.

REMARK 3.7. Let $i \in I(G)$. As distinct involutions give rise to distinct centralisers by Fact 1.2, no maximal 2-torus of $B_i := C(i)$ is contained in $F^\circ(B_i)$; in particular B_i is not nilpotent. Since its Prüfer 2-rank is 2, by Zilber's Field Theorem one cannot have $\operatorname{rk} B_i \leq 2$ (see also Fact 1.4). Hence $\operatorname{rk} B_i = 3$, and by Lemma 2.3, $F^\circ(B_i)$ has rank 2. Returning to maximal 2-tori and Zilber's Field Theorem, one sees that $B_i/F^\circ(B_i)$ and $F^\circ(B_i)$ both have Prüfer 2-rank 1. Consequently, $Z^\circ(B_i)$ is a rank 1 group containing divisible torsion, which in turn implies that $F^\circ(B_i)$ is abelian.

LEMMA 3.8. *If i, j are noncommuting involutions, then $C(i, j)$ has rank* 1 *and contains a unique involution.*

PROOF. Let $H := C(i, j)$. Since i and j do not commute, $B_i := C(i)$ and $B_j := C(j)$ differ. Consequently, $\operatorname{rk} H = 1$ by Corollary 2.6, and H° is abelian (Fact 1.3). Now if $\operatorname{pr}_2(H^\circ) = 2$, then H° contains i and j, which implies that i and j commute, a contradiction. Therefore $\operatorname{pr}_2(H^\circ) \leq 1$. If $\operatorname{pr}_2(H^\circ) = 0$, then $H^\circ \leq F^\circ(B_i)$ and $H^\circ \leq F^\circ(B_j)$, against Lemma 2.5. Hence $\operatorname{pr}_2(H^\circ) = 1$, so H° contains a unique

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involution k. Now if $\ell \in I(H)$, then ℓ commutes with both i and k, which commute themselves; by the structure of the Sylow 2-subgroup, $\ell \in \{i, k, ik\}$. The same shows that $\ell \in \{j, k, jk\}$, and therefore $\ell = k$. So $I(H) = \{k\}$, which is what we wanted. ⊣

Proof of Proposition 3.6. Let $X = \{(i, j) \in I(G)^2 : ij \neq ji\}$, which G definably permutes by conjugation. Notice that X has rank 4. By Lemma 3.8, the stabiliser $H = C(i, j)$ of any pair in X has rank 1, so every orbit of G is generic in X. Thus, G is transitive on X. Fix $(i, j) \in X$. Let k be the unique involution of $C(i, j)$, so $C(i, j) = C(i, j, k) \leq C(i, jk)$. Notice that i and jk do not commute, so by transitivity, there is a $g \in G$ such that $i^g = i$ and $j^g = jk$. Now, $C(i, j) \leq C(i, jk)$ being conjugate, equality holds. Thus, g normalises this group, so it centralises k. Finally, $j^{g^2} = j$, so $g^2 \in C(j, k)$. However, since j and k commute, $C(j, k)$ contains a Sylow 2-subgroup of G; this forces $g \in C(j, k)$, a contradiction. ⊣

§4. Proof of the main corollary.

We now take up the classification of nonsoluble connected groups of Morley rank 5. Facts 1.4 and 1.5 as well as the following corollary to Fact 1.6 will be used frequently in our analysis.

Fact 4.1 ([18, Corollary A] together with [12]). *If G is a nonsoluble connected group of Morley rank 4, then for $F := F^\circ(G)$, $\operatorname{rk} F \leq 1$, and G is classified as follows.*

1. $\operatorname{rk} F = 1$ *and* $G = F * Q$ *with* $Q \cong (\mathrm{P})\mathrm{SL}_2(K)$, *or*
2. $\operatorname{rk} F = 0$ *and G is a quasisimple bad group.*

Setup. Let G be a nonsoluble connected group of rank 5; set $F := F^\circ(G)$.

Since connected groups of rank 2 are soluble, we have that $\operatorname{rk} F \leq 2$ and that $\operatorname{rk} G' \geq 3$.

Lemma 4.2. *If G has a definable, connected, normal subgroup Q of rank 3, then $G = H * Q$ with $Q \cong (\mathrm{P})\mathrm{SL}_2(K)$ and H connected and soluble. Consequently, in this case we have $1 \leq \operatorname{rk} F \leq 2$ and $Q = G''$.*

Proof. Since G is nonsoluble, the same is true of Q, so (invoking [12]) we find that $Q \cong (\mathrm{P})\mathrm{SL}_2(K)$ for some algebraically closed field K. By [1, II, Corollary 2.26], $G = Q * C(Q)$. As $\operatorname{rk} C(Q) = 2$, $C^\circ(Q)$ is soluble. ⊣

Lemma 4.3. *If $\operatorname{rk} F = 0$, then G' is quasisimple and bad of rank 4 or 5.*

Proof. Assume $\operatorname{rk} F = 0$. By Lemma 4.2, G' has rank 4 or 5, and in either case, $F^\circ(G') = 1$. Now, if Q is a nontrivial proper definable connected normal subgroup of G', then it must be that Q has rank 3 and $Q = G''$. Consequently, Lemma 4.2 implies that F is nontrivial, a contradiction, so no such Q exists. Thus, if N is any proper normal subgroup of G', then $[N, G']$, which is definable and connected, must be trivial, so every proper normal subgroup of G' is central in G'.

We now have that G' is quasisimple with a finite centre, so combining our main theorem with Fact 4.1, we find that G' modulo its finite centre is a bad group. Hence G' is bad. ⊣

Lemma 4.4. *If $\operatorname{rk} F = 1$ and G is quasisimple, then G is a bad group.*

Proof. By Fact 4.1, we find that G/F is a bad group, and since $F = Z^\circ(G)$, it is easy to see that G must also be bad. ⊣

Lemma 4.5. *If* rk $F = 1$ *and* G *is not quasisimple, then either* $G = H * G''$ *with* $H/Z(H) \cong \mathrm{AGL}_1(L)$ *and* $G'' \cong (\mathrm{P})\mathrm{SL}_2(K)$, *or* $G = F * G'$ *with* G' *a rank* 4 *quasisimple bad group.*

Proof. Assume rk $F = 1$. By considering the generalized Fitting subgroup of G, we see that G contains some proper definable normal quasisimple subgroup Q (see [1, I, Section 7]). If Q has rank 3, then, by Lemma 4.2, $G = H * Q$ with H connected and soluble and $Q \cong (\mathrm{P})\mathrm{SL}_2(K)$ for some algebraically closed field K. Since rk $F = 1$, H is nonilpotent, so $H/Z(H) \cong \mathrm{AGL}_1(L)$ for some algebraically closed field L. Clearly, in this case, $Q = G''$. It remains to consider the case when Q has rank 4. By Fact 4.1, we find that Q is bad and that F intersects Q in a finite set. Thus, $G = F * Q$, and it must be that $Q = G'$. ⊣

Lemma 4.6. *If* rk $F = 2$ *and* $F \leq Z(G)$, *then* $G = F * Q$ *with* $Q \cong (\mathrm{P})\mathrm{SL}_2(K)$.

Proof. Again by considering the generalized Fitting subgroup of G, we see that G contains some definable normal quasisimple subgroup Q. Note that the theory of central extension prohibits $G = Q$ (see [1, II, Proposition 3.1]. If Q has rank 3, we are done, so assume that Q has rank 4. Here, $Q \cap F$ has rank 1, so by Fact 4.1, we find that $Q = F(Q) * R$ with R quasisimple. But this contradicts the fact that Q is quasisimple. ⊣

Lemma 4.7. *If* rk $F = 2$, *then either* $G = F * G''$ *with* $G'' \cong (\mathrm{P})\mathrm{SL}_2(K)$ *or* F *is* G*-minimal.*

Proof. Assume that F has a definable G-normal subgroup A of rank 1. Then, by Fact 4.1, $G/A = F/A * R/A$ for some definable connected subgroup R of G containing A with R/A quasisimple. As R has rank 4, we find that $R = A * Q$ with $Q \cong (\mathrm{P})\mathrm{SL}_2(K)$. Certainly, $Q = G''$. ⊣

So, it remains to treat the case where F is G-minimal (hence abelian) and noncentral. The key is, of course, [10].

Lemma 4.8. *If* F *is an abelian noncentral rank* 2 *subgroup of* G, *then there is an algebraically closed field* K *for which* $F \cong K^2$ *and* $G/F \cong \mathrm{SL}_2(K)$ *acting naturally on* F. *Moreover, if* char$(K) \neq 2$, *then the extension splits as* $G = F \rtimes C(i)$ *for* i *an involution of* G.

Proof. By [10, Theorem A], there is an algebraically closed field K for which $F \cong K^2$ and $G/C(F) \cong \mathrm{SL}_2(K)$ in its natural action. Since $C(F)$ is a finite extension of F, the theory of central extensions can be applied to G/F to see that $C(F) = F$. Now, assume char$(K) \neq 2$, and let i be an involution of G. Set $H := C(i)$. The image of i in G/F inverts F, so as F has no involutions, we find that $G = F \rtimes C(i)$, see [3, Lemma 9.3]. ⊣

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on **CiBo**$_3$. Incidentally, there was a casualty: the geometry of involutions. But geometry is not dead and will return in due time.

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