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The Hilbert Coefficients of the Fiber Cone and the *a*-Invariant of the Associated Graded Ring

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Abstract. Let $(A, \mathfrak{m})$ be a Noetherian local ring with infinite residue field and let I be an ideal in A and let $F(I) = \bigoplus_{n\geq 0} I^n/\mathfrak{m}I^n$ be the fiber cone of I. We prove certain relations among the Hilbert coefficients $f_0(I), f_1(I), f_2(I)$ of $F(I)$ when the a-invariant of the associated graded ring $G(I)$ is negative.

1 Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring with *infinite residue field* $k = A/\mathfrak{m}$. Let I be an ideal in A. The *fiber cone* of I is the standard graded k-algebra $F(I) = \bigoplus_{n\geq 0} I^n/\mathfrak{m}I^n$. Set $l(I) = \dim F(I)$, the *analytic spread* of I. The Hilbert polynomial of $F(I)$ is denoted by $f_I(z)$. Write $f_I(z) = \sum_{i=0}^{l-1}(-1)^i f_i(I)\binom{z+l-1-i}{l-1-i}$ where $l = l(I)$ We call $f_i(I)$ the i-th *fiber coefficient* of I.

Most recent results in the study of fiber cones involve the depth of the associated graded ring of I, $G(I) = \bigoplus_{n\geq 0} I^n/I^{n+1}$. When I is $\mathfrak{m}$-primary there has been some research relating $f_0(I)$ (the *multiplicity* of $F(I)$) with various other invariants of I (see [15, 4.1], [6, 4.3] and [4, 3.4]). In the case of $G(I)$ the relations among the Hilbert coefficients $e_0(I), e_1(I), e_2(I)$ are well known (see [28]). However there is no result relating $f_0(I), f_1(I)$, and $f_2(I)$. The reason for this is not difficult to find: any standard k-algebra can be thought of as a fiber cone of its graded maximal ideal. So any result involving the relation between $f_i(I)$ *would only hold in a restricted class of ideals*. Our paper explores the relation between $\mathbf{a}(I)$, *the a-invariant of* $G(I)$, and the Hilbert coefficients of $F(I)$. This is a new idea.

We first analyze when $l(I) = 2, 3$ as it throws light on the general result.

Theorem A *Let $(A, \mathfrak{m})$ be a Noetherian local ring with infinite residue field $k = A/\mathfrak{m}$. Let I be an ideal with $l(I) = 2$. If $\mathbf{a}(I) < 0$, then $f_1(I) \leq f_0(I) - 1$. Furthermore, equality holds if and only if $F(I^n)$ is Cohen–Macaulay for all $n \gg 0$. If* $\operatorname{grade}(I) = 2$, *then equality holds.*

This result should be compared with a result due to Northcott [18], which in our context states that $f_1(\mathfrak{m}) \geq f_0(\mathfrak{m}) - 1$ whenever A is Cohen–Macaulay. In Example 4.2, we exhibit a two-dimensional Noetherian local ring $(A, \mathfrak{m})$ with depth $A = 1$ but $f_1(I) < f_0(I) - 1$.

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To analyze the case when equality holds in Theorem A, we resolve $F(I^n)$ as an $F(J^{[n]}) = k[X_1^n, X_2^n]$-module and write it as:

$$0 \longrightarrow K_n \longrightarrow \bigoplus_{i=1}^{\beta_1^{[n]}} F(J^{[n]})(-1-\alpha_i^{[n]}) \longrightarrow F(J^{[n]})^{\beta_0^{[n]}} \longrightarrow F(I^n) \longrightarrow 0.$$

Here $\alpha_i^{[n]} \geq 0$ for all i. As depth $F(I^n) \geq 1$ for all $n \gg 0$, we get $K_n = 0$ for all $n \gg 0$. We show in Theorem 4.7 that if $\mathbf{a}(I) < 0$, then *for all* $n \gg 0$,

$$f_1(I) - f_0(I) + 1 = -\sum_{i=1}^{\beta_1^{[n]}} \alpha_i^{[n]} \quad \text{and}$$

$$\beta_1^{[n]} = 0 \text{ if and only if } \alpha_i^{[n]} = 0 \text{ for all } i\,.$$

Our second result, Theorem 5.5, has a noteworthy consequence when $G(I)$ is Cohen–Macaulay. Let $\operatorname{red}(I)$ denote the reduction number of I(see 1.1).

Theorem B *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d = 3$. Let I be an $\mathfrak{m}$-primary ideal with $G(I)$ Cohen–Macaulay and* $\operatorname{red}(I) = 2$. *Then*

$$f_2(I) \geq f_1(I) - f_0(I) + 1. \tag{1.1}$$

We extend our results to higher analytic spread using Rees-superficial sequences (see the Appendix for details), under some mild assumptions on grade(I). We state some of our noteworthy results. The first one (see Theorem 6.6) states that if $l(I) \geq 2$, grade$(I) \geq l(I) - 2$, and $\operatorname{red}(I) \leq 1$, then $f_1(I) \leq f_0(I) - 1$ with equality if grade$(I) = l(I)$. An immediate consequence (see Corollary 6.7) is that if $(A, \mathfrak{m})$ is Cohen–Macaulay with $\dim A \geq 2$, I an $\mathfrak{m}$-primary ideal and the second Hilbert-Samuel coefficient $e_2(I) = 0$, then $f_1(I) = f_0(I) - 1$.

Finally, we show that if A is a Cohen–Macaulay ring of dimension at least three and if I is an $\mathfrak{m}$-primary ideal of reduction number two whose associated graded ring is Cohen–Macaulay, then $f_2(I) \geq f_1(I) - f_0(I) + 1$ (see Theorem 6.8).

Here is an overview of the contents of the paper. In Section 2 we introduce some notations and necessary preliminary facts. In Section 3 we introduce two complexes which will be used in the subsequent sections. In Section 4 we prove Theorem A. In Section 5 we prove our second main theorem and as a consequence obtain Theorem B. In Section 6 we obtain results on the coefficients of the fiber cone for any analytic spread. In the appendix we recall some basic facts regarding minimal reductions and filter-regular elements and prove an elementary result that is useful in Section 4.

2 Preliminaries

From now on, $(A, \mathfrak{m})$ is a Noetherian local ring of dimension d, with infinite residue field. All modules are assumed to be finitely generated. For a finitely generated module M, we denote its length by $\ell(M)$.

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Let $J = (x_1, \dots, x_l)$ be a minimal reduction of I. We denote by

$$\operatorname{red}_J(I) := \min\{n \mid JI^n = I^{n+1}\}$$

the *reduction number of I with respect to J*. Let

$$\operatorname{red}(I) = \min\{\operatorname{red}_J(I) \mid J \text{ is a reduction of } I\}$$

be the *reduction number of I*.

As a reference for local cohomology we use [1] (see especially Chapter 18 for relations between local cohomology and reductions). We take local cohomology of $G(I)$ with respect to $G(I)_+ = \bigoplus_{n\geq 1} I^n/I^{n+1}$. Set $G = G(I)$ and $G_+ = G(I)_+$. For each $i \geq 0$ the local cohomology modules $H^i_{G_+}(G)$ are graded G-modules. Furthermore $H^i_{G_+}(G)_n = 0$ for all $n \gg 0$. For each $i \geq 0$ set $a_i(I) = \max\{n \mid H^i_{G_+}(G)_n \neq 0\}$.

Set $l = l(I)$. Then $H^l_{G_+}(G) \neq 0$ and $H^i_{G_+}(G) = 0$ for all $i > l$ (see [10, 2.3]). We call $\mathbf{a}(I) = a_l(I)$ the *a-invariant* of $G(I)$. The *(Castelnuovo–Mumford) regularity* of $G(I)$ is $\operatorname{reg}(G(I)) = \max\{a_i(G) + i \mid 0 \leq i \leq l\}$. The *regularity of $G(I)$ at and above level r*, denoted by $\operatorname{reg}^r(G(I))$, is $\operatorname{reg}^r(G(I)) = \max\{a_i(G) + i \mid r \leq i \leq l\}$.

Observation 2.1 Let $x \in I \setminus I^2$ be a I-superficial element of I. Also assume x is A-regular. For all $r \geq 1$ and $s \geq 0$ it is easy to show

$$\operatorname{reg}^r(G(I)) \leq s \implies \operatorname{reg}^r(G(I/(x)) \leq s.$$

We will use the following beautiful result due to Hoa.

Theorem 2.2 **(Hoa[10, 2.6])** *There exist non-negative integers $n_0, \mathfrak{r}(I)$, such that for all $n \geq n_0$ and every minimal reduction J of I^n we have* $\operatorname{red}_J(I^n) = \mathfrak{r}(I)$. *Furthermore,*

$$\mathfrak{r}(I) = \begin{cases} l(I) - 1 & \textit{if } \mathbf{a}(I) < 0, \\ l(I) & \textit{if } \mathbf{a}(I) \geq 0. \end{cases}$$

For the definition and basic properties of superficial sequences see [20, pp. 86–87].

If I is $\mathfrak{m}$-primary, then let $p_I(z)$ be the *Hilbert–Samuel polynomial* of A with respect to I (so $\ell(A/I^{n+1}) = p_I(n)\ \forall n \gg 0$). Write $p_I(z) = \sum_{i=0}^{d} (-1)^i e_i(I) \binom{z+d-i}{d-i}$. For $i \geq 0$, we call $e_i(I)$ the i-th *Hilbert coefficient* of I.

The Hilbert series of $F(I)$, $G(I)$ is denoted by $H(F(I), z)$, $H(G(I), z)$ respectively, *i.e.*,

$$H(F(I), z) = \sum_{n\geq 0} \ell\Big(\frac{I^n}{\mathfrak{m}I^n}\Big) z^n \quad \text{and} \quad H(G(I), z) = \sum_{n\geq 0} \ell\Big(\frac{I^n}{I^{n+1}}\Big) z^n$$

If $x \in I^j \setminus \mathfrak{m}I^j$, then we denote by x° its image in $F(I)_j = I^j/\mathfrak{m}I^j$.

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3 Two Complexes

Our results are based on analyzing two complexes which we describe in this section. Both the complexes are defined *using the maps in the Koszul complex.* Throughout, I is an ideal in $(A,\mathfrak{m})$ with a minimal reduction $J=(x_1,\dots,x_l)$ where $l=l(I)$. Note that $x_1,\dots,x_l$ are analytically independent [19], [2, 4.6.9]). Using analyticity to detect exactness (at some stage) of a complex has been studied first in [21].

The first complex is when $l=2$:

$$\mathcal{C}_\bullet(I)\colon 0\to\frac{A}{\mathfrak{m}}\xrightarrow{\alpha_2}\left(\frac{I}{\mathfrak{m}I}\right)^2\xrightarrow{\alpha_1}\frac{IJ}{\mathfrak{m}IJ}\to 0, \tag{3.1}$$

where

$$\alpha_2(a+\mathfrak{m})=\begin{pmatrix}-x_2a+\mathfrak{m}I\\ x_1a+\mathfrak{m}I\end{pmatrix}\quad\text{and}\quad \alpha_1\begin{pmatrix}a+\mathfrak{m}I\\ b+\mathfrak{m}I\end{pmatrix}=x_1a+x_2b+\mathfrak{m}IJ.$$

Observation 3.1 (i) $H_2(\mathcal{C}_\bullet(I))=0$ since x_1,x_2 are analytically independent.
(ii) If x_1,x_2 is a regular sequence in A, then clearly $H_1(\mathcal{C}_\bullet(I))=0$.
(iii) Clearly α_1 is surjective. So $H_0(\mathcal{C}_\bullet(I))=0$.

The second complex, $\mathcal{D}_\bullet(I)$, is when $l=3$.

$$0\to\frac{A}{\mathfrak{m}}\xrightarrow{\alpha_3}\left(\frac{I}{\mathfrak{m}I}\right)^3\xrightarrow{\alpha_2}\left(\frac{I^2}{\mathfrak{m}I^2}\right)^3\xrightarrow{\alpha_1}\frac{I^2J}{\mathfrak{m}I^2J}\to 0, \tag{3.2}$$

where

$$\alpha_3(a+\mathfrak{m})=\begin{pmatrix}x_3a+\mathfrak{m}I\\ -x_2a+\mathfrak{m}I\\ x_1a+\mathfrak{m}I\end{pmatrix},\quad \alpha_1\begin{pmatrix}a+\mathfrak{m}I^2\\ b+\mathfrak{m}I^2\\ c+\mathfrak{m}I^2\end{pmatrix}=x_1a+x_2b+x_3c+\mathfrak{m}I^2J,$$

$$\alpha_2\begin{pmatrix}a+\mathfrak{m}I\\ b+\mathfrak{m}I\\ c+\mathfrak{m}I\end{pmatrix}=\begin{pmatrix}-x_2a-x_3b+\mathfrak{m}I^2\\ x_1a-x_3c+\mathfrak{m}I^2\\ x_1b+x_2c+\mathfrak{m}I^2\end{pmatrix}.$$

Observation 3.2 (i) $H_3(\mathcal{D}_\bullet(I))=0$ since x_1,x_2,x_3 are analytically independent.
(ii) In Lemma 3.3 we show that if x_1,x_2,x_3 is a regular sequence and if $I^2\cap J=JI$, then image$(\alpha_2)=\ker(\alpha_1)$, so $H_1(\mathcal{D}_\bullet(I))=0$.
(iii) The assumption $I^2\cap J=JI$ holds when the following hold.
  (a) I is integrally closed (see[12, p. 317] and [13, Theorem 1]).
  (b) The initial forms x_1^*,x_2^*,x_3^* in $G(I)_1$ form a regular sequence (see [27, 2.3]).
(iv) Clearly α_1 is surjective and so $H_0(\mathcal{D}_\bullet(I))=0$.

Lemma 3.3 *Assume* (3.2). *If x_1,x_2,x_3 is a regular sequence and if $I^2\cap J=JI$, then* image$(\alpha_2)=\ker(\alpha_1)$.

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Proof Let $\mathcal{K}(\mathbf{x})_\bullet$ be the Koszul complex on x_1, x_2, x_3. It is acyclic since x_1, x_2, x_3 is a regular sequence. Suppose

$$\xi = \begin{pmatrix} \overline{a} \\ \overline{b} \\ \overline{c} \end{pmatrix} \in \ker \alpha_1.$$

Then $ax_1 + bx_2 + cx_3 = a'x_1 + b'x_2 + c'x_3$ where $a', b', c' \in \mathfrak{m}I^2$. As $\mathcal{K}(\mathbf{x})_\bullet$ is acyclic, there exists $f, g, h \in A$ such that

$$\begin{pmatrix} a - a' \\ b - b' \\ c - c' \end{pmatrix} = \begin{pmatrix} -x_2 f - x_3 g \\ x_1 f - x_3 h \\ x_1 g + x_2 h \end{pmatrix}. \tag{3.3}$$

We show that f, g, h are in I. Using (3.3) we get $-x_2 f - x_3 g \in I^2 \cap J = JI$. So $-x_2 f - x_3 g = x_1 p + x_2 q + x_3 r$ where $p, q, r \in I$. Again using the fact that $\mathcal{K}(\mathbf{x})_\bullet$ is acyclic, we get that there exists $u, v, w \in A$ such that

$$\begin{pmatrix} p \\ q + f \\ r + g \end{pmatrix} = \begin{pmatrix} -x_2 u - x_3 v \\ x_1 u - x_3 w \\ x_1 v + x_2 w \end{pmatrix}.$$

So $f, g \in I$. Similarly by using the second row in (3.3), we get that $f, h \in I$. Set

$$\eta = \begin{pmatrix} \overline{f} \\ \overline{g} \\ \overline{h} \end{pmatrix} \in (I/\mathfrak{m}I)^3.$$

Notice $\alpha_2(\eta) = \xi$ since $a', b', c' \in \mathfrak{m}I^2$. Thus $\ker \alpha_1 \subseteq \operatorname{image} \alpha_2$. ∎

Let us recall the following well-known fact about complexes. Let

$$\mathcal{X}_\bullet \colon 0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to 0$$

be a complex of A-modules with $\ell(X_i)$ finite for all i. Then

$$\sum_{i=0}^{n} (-1)^i \ell(X_i) = \sum_{i=0}^{n} (-1)^i \ell(H_i(\mathcal{X}_\bullet)). \tag{3.4}$$

4 Proof of Theorem A

In this section we prove Theorem A. The setup below is used throughout. The hypothesis $\mathbf{a}(I)$ is crucial for our results. See Examples 4.2, 4.3, and 4.10 for some illustrations of Theorem A.

Setup: Let $J = (x_1, x_2)$ be a minimal reduction of I. Notice $J^{[n]} = (x_1^n, x_2^n)$ is a minimal reduction of I^n. If $\operatorname{grade}(I) = 2$, then we can take x_1, x_2 to be a regular sequence (and so x_1^n, x_2^n is also a regular sequence). For $i = 1, 2$ set $X_i = x_i^\circ$, the image of x_i in $I/\mathfrak{m}I$. So $F(J^n) = k[X_1^n, X_2^n]$ for all $n \geq 1$.

We first prove the inequality (1.1) stated in Theorem A.

Theorem 4.1 *Let $(A, \mathfrak{m})$ be a local ring and let I be an ideal with $l(I) = 2$. If $\mathbf{a}(I) < 0$, then $f_1(I) \le f_0(I) - 1$. Furthermore, if $\operatorname{grade}(I) = 2$, then equality holds.*

Proof We consider the complex (3.1) for the ideal I^n for each $n \ge 0$. Set $\mathcal{C}_\bullet[n] = \mathcal{C}_\bullet(I^n)$ for $n \ge 1$. By Observation 3.1 we have $H_i(\mathcal{C}_\bullet[n]) = 0$ for $i = 0, 2$. Using (3.4) for the complex $\mathcal{C}_\bullet[n]$ for each n, we get an equation

$$1 - 2\ell(I^n/\mathfrak{m}I^n) + \ell(I^n J^{[n]}/\mathfrak{m}I^n J^{[n]}) = -\ell(H_1(\mathcal{C}_\bullet[n])). \tag{4.1}$$

Since $\mathbf{a}(I) < 0$, by Theorem 2.2 we have $\operatorname{red}_{J^{[n]}}(I^n) = 1$ for all $n \gg 0$. So $I^n J^{[n]} = I^{2n}$ for all $n \gg 0$. Also for all $n \gg 0$ we have $f_I(n) = \ell(I^n/\mathfrak{m}I^n)$. Setting these in (4.1), we get for all $n \gg 0$,

$$1 - 2f_I(n) + f_I(2n) = -\ell(H_1(\mathcal{C}_\bullet[n])).$$

Write $f_I(n) = f_0(n+1) - f_1$. Therefore

$$1 - 2\{f_0(n+1) - f_1\} + f_0(2n+1) - f_1 = -\ell(H_1(\mathcal{C}_\bullet[n])).$$

Thus $1 - f_0 + f_1 = -\ell(H_1(\mathcal{C}_\bullet[n]))$. Hence $1 - f_0 + f_1 \le 0$. By our assumption on J, $H_1(\mathcal{C}_\bullet[n]) = 0$ for each n if $\operatorname{grade}(I) = 2$. Hence equality holds in the above equation. This proves the result. ■

The following example shows that if $\mathbf{a}(I) < 0$, but $\operatorname{grade}(I) \ne 2$, then $f_1(I) < f_0(I) - 1$ is possible.

Example 4.2 Set $A = k[[X_1, X_2, X_3]]/(X_1^2, X_1X_2) = k[[x_1, x_2, x_3]]$. Set $I = \mathfrak{m} = (x_1, x_2, x_3)$ and $J = (x_2, x_3)$. Then J is a reduction of $\mathfrak{m}$ and $\mathfrak{m}^2 = J\mathfrak{m}$. By [26, 3.2] we get $\mathbf{a}(\mathfrak{m}) < 0$.

It can be checked that $\operatorname{grade}(\mathfrak{m}) = 1$ and x_3 is a non-zero divisor. Using COCOA [3] it can verified that the Hilbert series of $F(\mathfrak{m}) = G(\mathfrak{m})$ is

$$\frac{1 + z - z^2}{(1-z)^2}.$$

So $f_1(\mathfrak{m}) = -1$, but $f_0(\mathfrak{m}) = 1$.

The next example shows that Theorem 4.1 need not hold when $\mathbf{a}(I) > 0$.

Example 4.3 Let $(A, \mathfrak{m})$ be a two dimensional Cohen–Macaulay local ring with $\operatorname{red}(\mathfrak{m}) = 2$. Then we have $G(\mathfrak{m}) = F(\mathfrak{m})$ is Cohen–Macaulay [24, 2.1] and its Hilbert-series is

$$\frac{1 + z + cz^2}{(1-z)^2} \quad \text{where } c > 0.$$

So $f_1(\mathfrak{m}) - f_0(\mathfrak{m}) + 1 = c > 0$.

Next we analyze the case when $f_1(I) = f_0(I) - 1$. Observe that $F(I^n) = F(I)^{\langle n \rangle}$, the n-th *Veronese* subring of $F(I)$. In particular $l(I^n) = l(I)$ for each $n \ge 1$. Local cohomology commutes with the Veronese functor [11, 2.5]. In [22, 2.8] it is proved that depth $F(I^n)$ is constant for all $n \gg 0$. We prove the following.

Theorem 4.4 *Let $(A, \mathfrak{m})$ be a local ring with infinite residue field and let I be an ideal with $s = l(I) > 0$.*

(i) depth $F(I^n) > 0$ *for all* $n \gg 0$.

(ii) *There exists a minimal reduction* $J = (x_1, \dots, x_s)$ *of* I *such that* $(x_1^n)^\circ$ *is* $F(I^n)$ *regular for all* $n \gg 0$.

Proof (i) Set $E = H^0_{F(I)_+}(F(I))$. Clearly $\ell(E) < \infty$. Say $E = \bigoplus_{i=0}^{r} E_i$. Notice E is an ideal of $F(I)$ with finite length. If $E_0 \neq 0$, then $1_{F(I)} \in E$. So $E = F(I)$ will have finite length, a contradiction since $\dim F(I) = l(I) \geq 1$. Therefore $E_0 = 0$, and as a consequence we have

$$H^0_{F(I^n)_+} F(I^n) = \left(H^0_{F(I)_+}(F(I))\right)^{\langle n\rangle} = 0 \quad \text{for all } n > r.$$

Thus depth $F(I^n) > 0$ for all $n \gg 0$.

(ii) By [26, 3.8] we get that there exists a minimal reduction $J = (x_1, \dots, x_s)$ of I such that $x_1^\circ, \dots, x_s^\circ \in F(I)_1$ is an $F(I)$-filter regular sequence. Set $x = x_1$. Since x° is $F(I)$-filter regular, by Corollary A.5 we get that $(x^n)^\circ$ is $F(I^n)$-filter regular for each $n \geq 1$. By (i) depth $F(I^n) > 0$ for all $n \gg 0$. So by Remark A.2 we get that $(x^n)^\circ$ is $F(I^n)$-regular for all $n \gg 0$. ∎

Observation 4.5 As $l(I) = 2$, by computing the Hilbert polynomial of $F(I)$ and $F(I^n)$ we obtain

$$f_1(I^n) - f_0(I^n) + 1 = f_1(I) - f_0(I) + 1 \quad \text{for all } n \geq 1.$$

Observation 4.6 Let $J = (x_1, x_2)$ be a minimal reduction of I as constructed in Theorem 4.4. In particular $(x_1^n)^\circ$ is $F(I^n)$-regular for all $n \gg 0$. Set $X_j^n = (x_1^n)^\circ$ for $j = 1, 2$. We resolve $F(I^n)$ as an $F(J^{[n]}) = k[X_1^n, X_2^n]$ module and write it as:

$$0 \longrightarrow K_n \longrightarrow \bigoplus_{i=1}^{\beta_1^{[n]}} F(J^{[n]})(-1 - \alpha_i^{[n]}) \longrightarrow F(J^{[n]})^{\beta_0^{[n]}} \longrightarrow F(I^n) \longrightarrow 0.$$

Here $\alpha_i^{[n]} \geq 0$ for all i. As depth $F(I^n) \geq 1$ for all $n \gg 0$, we get $K_n = 0$ for all $n \gg 0$.

We prove the following.

Theorem 4.7 *With assumptions as in Observation* 4.6, *if* $\mathbf{a}(I) < 0$, *then* for all $n \gg 0$,

$$f_1(I) - f_0(I) + 1 = -\sum_{i=1}^{\beta_1^{[n]}} \alpha_i^{[n]}$$

and $\beta_1^{[n]} = 0$ *if and only if* $\alpha_i^{[n]} = 0$ *for all* i.

For the proof of this theorem we need the following.

Lemma 4.8 *Let $R = k[X]$ and let $S = \bigoplus_{n\geq 0} S_n$ be a standard k-algebra of dimension 1 and multiplicity $p+1$. Assume $S = Ru_1 + \cdots + Ru_m$ where degree $u_i \leq 1$ and $u_1 = 1_S$. Then S has the following resolution over R*

$$0 \longrightarrow \bigoplus_{i=1}^{q} R(-1-\alpha_i) \longrightarrow R \oplus R(-1)^{p+q} \longrightarrow S \longrightarrow 0.$$

Furthermore S is free if and only if all $\alpha_i = 0$.

Proof By the hypothesis on S and as R is a Euclidean domain, we get

$$S = R \oplus R(-1)^p \oplus \Big(\bigoplus_{i=1}^{q} \frac{R}{(X^{\alpha_i})}(-1)\Big),$$

where $\alpha_i \geq 0$. The result follows. ∎

Proof of Theorem 4.7 We choose n_0 such that depth $F(I^n) \geq 1$ and $\operatorname{red}_{J^{[n]}}(I^n) = 1$ for all $n \geq n_0$. Fix $n \geq n_0$ and set $\alpha_i = \alpha_i^{[n]}$ and $F(J^{[n]}) = k[X_1^n, X_2^n]$. Since $\operatorname{red}_{J^{[n]}}(I^n) = 1$, it follows that $F(I^n)$ is generated as an $F(J^{[n]})$-module in degrees ≤ 1.

Notice that by construction, X_1^n is a non-zero divisor on $F(I^n)$ (see Observation 4.6). Set $R = F(J^{[n]})/(X_1^n) = k[X_2^n]$ and $S = F(I^n)/X_1^n F(I^n)$. Note that S is generated as an R module in degrees ≤ 1. By Lemma 4.8 the resolution of S as an R-module is

$$0 \longrightarrow \bigoplus_{i=1}^{q} R(-1-\alpha_i) \longrightarrow R \oplus R(-1)^{p+q} \longrightarrow S \longrightarrow 0.$$

Since X_1^n is a non-zero divisor on $F(I^n)$ and $F(J^{[n]})$, we get that the resolution of $F(I^n)$ as $F(J^{[n]})$-module is

$$0 \longrightarrow \bigoplus_{i=1}^{q} F(J^{[n]})(-1-\alpha_i) \longrightarrow F(J^{[n]}) \oplus F(J^{[n]})(-1)^{p+q} \longrightarrow F(I^n) \longrightarrow 0.$$

Thus $q = \beta_1^{[n]}$. Set $\phi(z) = \sum_{i=1}^{q} z^{\alpha_i+1}$. Therefore the Hilbert series of $F(I^n)$ is

$$\frac{1 + (p+q)z - \phi(z)}{(1-z)^2}.$$

So $f_0(I^n) = 1 + p$. Notice that $f_1(I^n) = p + q - \sum_{i=1}^{q}(\alpha_i + 1) = p - \sum_{i=1}^{q} \alpha_i$. Using Observation 4.5 we obtain $\sum_{i=1}^{q} \alpha_i = f_0(I^n) - f_1(I^n) - 1 = f_0(I) - f_1(I) - 1$. Also by Lemma 4.8, $q = 0$ if and only if all $\alpha_i = 0$. ∎

In view of this result we are tempted to ask the following.

Question 4.9 With notation as in Observation 4.6 Let $(A, \mathfrak{m})$ be a local ring and let I be a proper ideal with $l(I) = 2$. Is $\sum_{i=1}^{\beta_1^{[n]}} {\alpha_i}^{[n]}$ constant for all $n \gg 0$?

Finally we observe that Theorem A follows from Theorems 4.1 and 4.7.

We give an example which shows that in the case $l(I) = \operatorname{grade}(I) = 2$ and $\mathbf{a}(I) < 0$ it is possible for $F(I)$ to be not Cohen–Macaulay even though $f_1(I) = f_0(I) - 1$ and $F(I^n)$ is Cohen–Macaulay for all $n \gg 0$. The example below was constructed by Marley [16, 4.1] in his study of the associated graded ring $G(I)$.

Example 4.10 Let $A = k[X,Y]_{(X,Y)}$ and let $I = (X^7, X^6Y, XY^6, Y^7)$. Using COCOA one verifies that $e_2(I) = 0$ and that the Hilbert series of the fiber cone $F(I)$ is

$$\frac{1 + 2z + 2z^2 + 2z^3 + 2z^4 + 2z^5 - 4z^6}{(1-z)^2}.$$

From the Hilbert series it is clear that $f_0(I) = 7$, $f_1(I) = 6$, but $F(I)$ is not Cohen–Macaulay.

Remark 4.11 It is also possible to prove Theorem A directly from Theorem 4.7, Observation 4.5, and by using a result of Kishor Shah [25, Theorem 1]. We kept Theorem 4.1 in this section because it gives the inequality $f_1(I) \leq f_0(I) - 1$ very easily and more importantly gives us a natural way of trying to relate $f_i(I)$ for $i = 0, 1, 2$ which is new and important. This is done in our next section.

5 Results When the Analytic Spread Is Three

In this section we assume that $l(I) = 3$. If $J = (x_1, x_2, x_3)$ is a reduction of I, we also assume that x_1, x_2, x_3 is a regular sequence. The goal of this section is to prove inequality (1.1) under suitable conditions on I.

Observation 5.1 We consider the complex (3.2) for the ideal I^n for each $n \geq 0$. Set $\mathcal{D}_\bullet[n] = \mathcal{D}_\bullet(I^n)$ for $n \geq 1$. By Observation 3.1 we have $H_i(\mathcal{D}_\bullet[n]) = 0$ for $i = 0, 3$.

To analyze the case when $H_1(\mathcal{D}_\bullet[n])$ is zero, we use the following.

Definition 5.2 Let I be an ideal and let $J = (x_1, x_2, x_3)$ be a minimal reduction of I. We say the pair (I, J) satisfy V_2^∞ if $I^{2n} \cap J^{[n]} = J^{[n]} I^n$ for all $n \gg 0$. This condition has been studied by Elias [9].

Observation 5.3 By Lemma 3.3, $H_1(\mathcal{D}_\bullet[n]) = 0$ for all $n \gg 0$ *if* the pair (I, J) satisfy V_2^∞.

Observation 5.4 When $\operatorname{grade}(I) = l(I)$, then using Observation 3.2(iii) the hypothesis V_2^∞ holds when either

(i) I is asymptotically normal, *i.e.,* I^n is integrally closed for all $n \gg 0$ or
(ii) the initial forms $x_1^*, \dots, x_{l(I)}^*$ in $G(I)_1$ form a regular sequence.

We now state our second main theorem.

Theorem 5.5 *Let $(A, \mathfrak{m})$ be local and let I be an ideal in A with $l(I) = \operatorname{grade}(I) = 3$. Let $J = (x_1, x_2, x_3)$ be a minimal reduction of I and assume the pair (I, J) satisfy V_2^∞. If $\mathbf{a}(I) < 0$, then inequality (1.1) holds.*

Remark 5.6 The hypotheses of Theorem 5.5 are quite stringent. However they are necessary (see Examples 5.11, 5.13, and 5.14). Also note that if $F(I)$ is Cohen–Macaulay then inequality (1.1) holds.

We give two examples where the condition of Theorem 5.5 holds.

Example 5.7 Let $(A, \mathfrak{m})$ be a three dimensional Cohen–Macaulay local ring with I an $\mathfrak{m}$-primary ideal with reduction number two and $G(I)$ Cohen–Macaulay. Then the hypothesis of Theorem 5.5 holds by Proposition 5.4(ii). This is shown in Theorem 5.10. Using [14, Example 6.1], we can construct an interesting example of this kind as follows.

Let $T = k[[t^6, t^{11}, t^{15}, t^{31}]]$, $K = (t^6, t^{11}, t^{31})$, and $L = (t^6)$. Then it can easily be verified that $K^3 = LK^2$. Since $K^2 \cap L = LK$, $G(K)$ is Cohen–Macaulay by a result of Valabrega and Valla [27, 2.3]. It can also be seen that $t^{37} \in \mathfrak{m}K^2$, but $t^{37} \notin \mathfrak{m}LK$. Therefore $F(K)$ is not Cohen–Macaulay by a criterion due to Cortadellas and Zarzuela [7, 3.2]. One can verify that the Hilbert series of $F(K)$ is $(1 + 2z)/(1 - z)$.

Let $R = k[[X, Y, Z, W]]$ and

$$\mathfrak{q} = (y^2z - xw, x^2z^2 - yw, x^3z - y^3, x^3yw - z^4, z^5 - y^4w, xyz^3 - w^2, \\ y^5 - wx^4, x^2y^3 - z^3, x^5 - z^2, x^4y^2 - zw).$$

Set $B = R/\mathfrak{q} = k[[x, y, z, w]]$. Using COCOA, one can verify that $B \cong T$. Under this isomorphism the ideal (x, y, w) maps to K and (x) goes to L

Set $A = B[[U, V]]$. Clearly A is Cohen–Macaulay of dimension 3. Set $I = (x, y, w, U, V)$ and $J = (x, U, V)$. Clearly J is a minimal reduction of I and $I^3 = JI^2$. Furthermore, $G(I) \cong G(K)[U, V]$ and $F(I) \cong F(K)[U, V]$. So $G(I)$ is Cohen–Macaulay, while $F(I)$ has depth 2.

Before giving the next example we make the following remark.

Remark 5.8 If A satisfies the condition of the theoremthen as $\mathbf{a}(I) < 0$, we get by Theorem 2.2 that $\operatorname{red}_{J^{[n]}}(I^n) = 2$ for all $n \gg 0$. So by using the Valabrega–Valla criterion [27, 2.3] it follows that $\operatorname{grade}(G(I^n)_+, G(I^n)) \geq 3$ for all $n \gg 0$.

Example 5.9 Let $A = k[[X, Y, Z]]$ and let $\mathfrak{m} = (X, Y, Z)$. Let I be an $\mathfrak{m}$-primary ideal with $I^r = \mathfrak{m}^s$ for some $s > r$ and $G(I)$ not Cohen–Macaulay (for a specific example see [5, 4.3], see also [5, 3.8]). If J is a reduction of I, then the pair (I, J) satisfy V_2^∞, by Proposition 5.4(i). Also by Hoa's result it follows that $a(I) < 0$. Any such example is different from Example 5.7, since $G(I)$ is Cohen–Macaulay in Example 5.7.

Before proving the theorem we give a proof of Theorem B. We restate it for the convenience of the reader.

Theorem 5.10 *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d = 3$. Let I be an $\mathfrak{m}$-primary ideal with $G(I)$ Cohen–Macaulay and $\operatorname{red}(I) \leq 2$. Then inequality (1.1) holds.*

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Proof As $G(I)$ is Cohen–Macaulay we have that $\mathbf{a}(I) = \operatorname{red}(I) - 3 \leq -1$. Let J be a minimal reduction of I such that $\operatorname{red}_J(I) = \operatorname{red}(I)$. As $G(I)$ is Cohen–Macaulay, the pair (I, J) satisfy V_2^∞. So the result follows from Theorem 5.5. ∎

We now prove Theorem 5.5. The notation is as in Observation 5.1.

Proof We use the complex $\mathcal{D}_\bullet[n]$. By Observation 5.3 and (3.4) we get

$$(5.1) \quad -1 + 3\ell\Big(\frac{I^n}{\mathfrak{m}I^n}\Big) - 3\ell\Big(\frac{I^{2n}}{\mathfrak{m}I^{2n}}\Big) + \ell\Big(\frac{I^{2n}J^{[n]}}{\mathfrak{m}I^{2n}J^{[n]}}\Big) = \ell(H_2(\mathcal{D}_\bullet[n])), \ \forall n \gg 0.$$

Since $\mathbf{a}(I) < 0$ we have by (2.2) $\operatorname{red}_{J^{[n]}}(I^n) = 2$ for all $n \gg 0$. So $I^{2n}J^{[n]} = I^{3n}$ for all $n \gg 0$. Also for all $n \gg 0$ we have $f_I(n) = \ell(I^n/\mathfrak{m}I^n)$. Setting these in (5.1) we get

$$-1 + 3f_I(n) - 3f_I(2n) + f_I(3n) \geq 0.$$

Since

$$f_I(n) = f_0\binom{n+2}{2} - f_1(n+1) + f_2,$$

an easy computation yields $-1 + 3f_I(n) - 3f_I(2n) + f_I(3n) = -1 + f_0 - f_1 + f_2$, and the result follows. ∎

Our second result, Theorem 5.5, has three hypothesis, namely

(5.2a) $\quad \operatorname{grade}(I) = l(I)$,

(5.2b) $\quad$ the pair (I, J) satisfies V_2^∞,

(5.2c) $\quad \mathbf{a}(I) < 0$.

We show that if any of the hypotheses in (5.2) are not satisfied, then inequality (1.1) need not hold. In the first example only hypothesis (5.2a) is not satisfied, in fact we have $l(I) - \operatorname{grade}(I) = 1$.

Example 5.11 Let $A = k[[X, Y, U, V]]/(XY, Y^3) = k[[x, y, u, v]]$ and $I = \mathfrak{m} = (x, y, u, v)$. One can readily see that $\operatorname{grade}(\mathfrak{m}) = 2$ while $l(\mathfrak{m}) = \dim A = 3$. Set $J = (x, u, v)$. Then $I^3 = JI^2$. So by [26, 3.2] we get $\mathbf{a}(I) < 0$. The pair (I, J) satisfies V_2^∞ by Proposition 5.12. In fact in Proposition 5.12 we show $\mathfrak{m}^{2n} \cap J^{[n]} = J^{[n]}\mathfrak{m}^n$ for all $n \geq 1$. However the Hilbert series of $F(\mathfrak{m}) = G(\mathfrak{m})$ (by COCOA) is

$$\frac{1 + z - z^3}{(1 - z)^3}$$

So $f_0(I) = 1$, $f_1(I) = -2$ and $f_2(I) = -3$. Thus $f_2(I) \not\geq f_1(I) - f_0(I) + 1$.

Proposition 5.12 *We have A, $\mathfrak{m}$, J as in Example* 5.11.

$$\mathfrak{m}^{2n} \cap J^{[n]} = J^{[n]}\mathfrak{m}^n \quad \textit{for all } n \geq 1.$$

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Proof Fix $n \geq 1$. We claim that

$$\text{if } c \in k[[x,u,v]] \text{ and } x^n c \in \mathfrak{m}^{2n}, \text{ then } c \in \mathfrak{m}^n. \tag{5.3}$$

Let us assume (5.3) and prove our result. Fix $n \geq 1$. Let $\xi \in \mathfrak{m}^{2n} \cap J^{[n]}$. Write $\xi = \alpha x^n + \beta u^n + \gamma v^n$. Set $(T, \mathfrak{q}) = (A/(x^n), \mathfrak{m}/(x^n))$. Note

$$T = \frac{k[[x,y,u,v]]}{(xy, y^3, x^n)} = \Big(\frac{k[[x,y]]}{(xy,y^3,x^n)}\Big)[[u,v]].$$

Thus u^*, v^* are $G(\mathfrak{q})$-regular. So $\overline{\beta}\overline{u}^n + \overline{\gamma}\,\overline{v}^n \in \overline{\mathfrak{m}}^{2n} \cap (\overline{u}^n, \overline{v}^n) = \overline{\mathfrak{m}}^n(\overline{u}^n, \overline{v}^n)$. It follows that $\overline{\beta}, \overline{\gamma} \in \overline{\mathfrak{m}}^n$. Thus $\xi = \alpha x^n + \beta u^n + \gamma v^n + \theta x^n$ where $\beta, \gamma \in \mathfrak{m}^n$, $\theta \in A$. Since $xy = 0$ we may assume $\alpha + \theta \in k[[x,u,v]]$. Notice $(\alpha+\theta)x^n \in \mathfrak{m}^{2n}$. So by (†) we get $(\alpha + \theta) \in \mathfrak{m}^n$. It follows that $\xi \in \mathfrak{m}^n J^{[n]}$.

We now prove (5.3). Let $S = k[[x,u,v]]$ be considered as a subring of A. Any element $a \in A$ can be written as

$$a = \phi_0^{(a)}(x,u,v) + \phi_1^{(a)}(x,u,v)y + \phi_2^{(a)}(x,u,v)y^2, \tag{5.4}$$

where $\phi_i^{(a)}(x,u,v) \in S$ for $i = 0,1,2$.

So A, as an S-module, is generated by $1, y, y^2$. Thus $\dim S = \dim A = 3$. It follows that $S \cong k[[X,U,V]]$.

Notice $A/(y) = S$ and the natural map $\pi\colon A \to S$ is a splitting (as S-modules) of the inclusion $\imath\colon S \to A$. Set $L = Sy + Sy^2$. Then $A = S \oplus L$ as a S-module. It follows that $\phi_0^{(a)}(x,u,v)$ in (5.4) is uniquely determined by a.

Let $\mathfrak{n}$ be the unique maximal ideal of S. Then

$$\mathfrak{m}^i = \mathfrak{n}^i \oplus (\mathfrak{m}^i \cap L) \quad \text{for all } i \geq 1. \tag{5.5}$$

Set $c = \phi_0^{(c)}(x,u,v) + \phi_1^{(c)}(x,u,v)y + \phi_2^{(c)}(x,u,v)y^2$. *Notice that by hypothesis on c we get* $\phi_1^{(c)}(x,u,v) = \phi_2^{(c)}(x,u,v) = 0$. By uniqueness of ϕ_0^* we get

$$\phi_0^{(x^n c)}(x,u,v) = x^n \phi_0^{(c)}(x,u,v).$$

Since $x^n c \in \mathfrak{m}^{2n}$, we get by (5.5) that $x^n\phi_0^{(c)}(x,u,v) \in \mathfrak{n}^{2n}$. Clearly x^* is $G_{\mathfrak{n}}(S)$-regular. So $\phi_0^{(c)}(x,u,v) \in \mathfrak{n}^n$. By (5.5) again we get that $c \in \mathfrak{m}^n$.

This proves 5.3. As stated earlier this finishes the proof of the proposition. ∎

In the second example, only hypothesis (5.2b) is not satisfied. We adapt an example from [8, 6.2]. If K is an ideal in A, let $\widetilde{K} = \bigcup_{n\geq 1}(K^{n+1} \colon K^n)$ be the Ratliff–Rush closure of K.

Example 5.13 Let $A = \mathbb{Q}[[X,Y,Z]]$. Let $I = (X^4, X^3Y, XY^3, Y^4, Z)$. The ideal $J = (X^4, Y^4, Z)$ is a minimal reduction of I, in fact $I^3 = JI^2$. So by [26, 3.2] we get $\mathbf{a}(I) < 0$. Set $B = \mathbb{Q}[[X,Y]]$ and $q = (X^4, X^3Y, XY^3, Y^4)$. One can show $\widetilde{q} \neq q$. However notice $G(I) = G(q)[Z^*]$. So Z^* is $G(I)$-regular. In particular $\widetilde{I} = I$. By

[22, 7.9] we get depth $G(I^n) = 1$ for all $n \gg 0$. So by Remark 4.8, I does not satisfy V_2^{∞}. The Hilbert series of $F(I)$ is

$$\frac{1 + 2z + 2z^2 - z^3}{(1-z)^3}.$$

So $f_0(I) = 4$, $f_1(I) = 3$ and $f_2(I) = -1$. Thus $f_2(I) \not\geq f_1(I) - f_0(I) + 1$.

In the third example hypothesis (5.2c) is not satisfied. Instead of (5.2b), the hypothesis

$$I^{2n} \cap (x_1^n, x_2^n) = I^n(x_1^n, x_2^n) \quad \text{for all } n \gg 1. \tag{5.6}$$

is satisfied. Recall $J = (x_1, x_2, x_3)$ is a minimal reduction of I. The hypothesis (5.6) is equivalent to depth $G(I^n) \geq 2$ for all $n \gg 0$ [9, 2.4].

Example 5.14 Let $A = k[X,Y,Z]_{(X,Y,Z)}$ and $I = (X^3, XY^4Z, XY^5, Z^5, Y^7)$. Set $u = Z^5, V = 5X^3 + 3Y^7$ and $w = X^3 - 3XY^4Z + 2Z^5$. Set $J = (u, v, w)$. Using COCOA we can check that $I^6 = JI^5$. So J is a minimal reduction of I. The Hilbert series of $G(I)$, $G(I/(u))$, $G(I/(u,v))$ is

$$\begin{aligned} H(G(I), z) &= (1-z)H(G(I/(u)), z) = (1-z)^2 H(G(I/(u,v)), z), \\ &= \frac{77 + 15z + 8z^2 + 2z^3 + 2z^4 + z^5}{(1-z)^3}, \\ H(G(I/(u,v,w)), z) &= 77 + 28z \end{aligned}$$

So u^*, v^* is a $G(I)$-regular sequence. Note that depth $G(I) = 2$. It follows that depth $G(I^n) \geq 2$ for all $n \geq 1$. So hypothesis (5.6) is satisfied.

We prove that $\mathbf{a}(I) \geq 0$. Set $G = G(I)$.

$$a_i(I) = \max\{n \mid H^i(G)_n \neq 0\} \quad i = 0, 1, 2, 3$$

Proposition 5.15 *Note that* $a_0(G) = a_1(G) = -\infty$. *As* $\operatorname{red}_J(I) = 5$ *we get by* [26, 3.2] *that* $a_3(G) \leq 2$. *By* [16, 2.1(a)] *we have* $a_2(G) < a_3(G) \leq 2$.

By [10, 2.4] we have

$$a_i(I^n) \leq \left[\frac{a_i(I)}{n}\right] \quad \text{for } i = 0, 1, 2. \quad \textit{and} \quad a_3(I^n) = \left[\frac{a_3(I)}{n}\right].$$

Notice by Proposition 5.15 we get

$$\begin{aligned} &a_i(I^3) \leq 0 \quad \text{for } i = 0, 1, 2. \\ &a_3(I^3) \leq -1 \quad \text{if } a_3(I) < 0 \\ &a_3(I^3) = 0 \quad \text{if } a_3(I) \geq 0. \end{aligned}$$

By [26, 3.2] it follows that

$$\operatorname{red}_{J^{[3]}}(I^3) = 2 \quad \text{if } a_3(I) < 0 \quad \text{and} \quad \operatorname{red}_{J^{[3]}}(I^3) = 3 \quad \text{if } a_3(I) \geq 0.$$

However using COCOA we have verified that $I^9 \neq I^6 J^{[3]}$. Thus $\mathbf{a}(I) = a_3(I) \geq 0$. The fiber coefficients are $f_0(I) = 17$, $f_1(I) = 34$, $f_2(I) = 17$. So

$$f_2(I) - f_1(I) + f_0(I) - 1 = -1.$$

6 Results When the Analytic Spread Is High

In this section we extend Theorem A and Theorem 5.5 to the cases when $l(I) \geq 3$ and $l(I) \geq 4$, respectively. The main tool is the use of Rees-superficial sequences. The utility of a Rees-superficial element in the study of fiber cones was first demonstrated in [14].

Definition 6.1 An element $x \in I$ is said to be *Rees-superficial* if there exists $r_0 \geq 1$ such that $(x) \cap I^r \mathfrak{m}^s = xI^{r-1}\mathfrak{m}^s$ for all $r \geq r_0$ and $s \geq 0$.

The following was proved in [14, 2.8] for $\mathfrak{m}$-primary I in a local ring A. The same proof works in general.

Proposition 6.2 *Let $(A, \mathfrak{m})$ be a local ring and I an ideal. Let $x \in I$ be a nonzero divisor in R which is also Rees-superficial for I. Set $(B, \mathfrak{n}) = (A/(x), \mathfrak{m}/(x))$ and $K = I/(x)$. Then*

$$\ell\Big(\frac{I^n}{\mathfrak{m}I^n}\Big) - \ell\Big(\frac{I^{n-1}}{\mathfrak{m}I^{n-1}}\Big) = \ell\Big(\frac{K^n}{\mathfrak{n}K^n}\Big) \quad \textit{for all } n \gg 0.$$

In particular $f_i(K) = f_i(I)$ for $i = 0, \ldots, l(I) - 2$.

The existence of a Rees-superficial element which is also regular follows from the the following special case of a lemma due to Rees [23, 1.2].

Lemma 6.3 *Let $(A, \mathfrak{m})$ be local and let I be an ideal in A. Let $\mathcal{P}$ be a finite set of primes not containing $I\mathfrak{m}$. Then there exists $x \in I$ and $r_0 \geq 1$ such that*

(i) $x \notin \mathfrak{P}$ *for all* $\mathfrak{P} \in \mathcal{P}$;
(ii) $(x) \cap I^r \mathfrak{m}^s = xI^{r-1}\mathfrak{m}^s$ *for all* $r \geq r_0$ *and* $s \geq 0$.

Remark 6.4 If $x \in I$ is Rees-superficial and a non-zero divisor, then it is easy to check that x is I-superficial.

We say $x_1, \ldots, x_r \in I$ is a *Rees-superficial* sequence if x_i is Rees superficial for the $A/(x_1, \ldots, x_{i-1})$-ideal $I/(x_1, \ldots, x_{i-1})$ for $i = 1, \ldots, r$.

Remark 6.5 (i) If $\operatorname{grade}(I) \geq r$, then using Lemma 6.3 we can show that there exists a Rees-superficial sequence $x_1, \ldots, x_r$ in I which is also a regular sequence.
(ii) In this case we can further prove (by using Proposition 6.2 repeatedly) that if $K = I/(x_1, \ldots, x_r)$, then $f_i(K) = f_i(I)$ for $i = 0, \ldots, l(I) - r - 1$.

We state our main results.

Theorem 6.6 *Let $(A, \mathfrak{m})$ be local and let I be an ideal in A with $l = l(I) \geq 2$ and* $\operatorname{grade}(I) \geq l(I) - 2$. *Assume either* $\operatorname{reg}^2(G(I)) \leq 1$ *or* $\operatorname{red}(I) \leq 1$. *Then* $f_1(I) \leq f_0(I) - 1$. *Furthermore equality holds if* $\operatorname{grade}(I) = l(I)$.

Proof If $l(I) = 2$, then we are done by Theorem A. If $l \geq 3$ and as $\operatorname{grade}(I) \geq l(I) - 2$, by Remark 6.5(i) we can choose a Rees-superficial sequence $x_1, \dots, x_{l-2}$ which is also a regular sequence (and so an I-superficial sequence). Set $K = I/(x_1, \dots, x_{l-2})$. Then $l(K) = 2$.

If $\operatorname{reg}^2(G(I)) \leq 1$, then by Observation 2.1 we get $\mathbf{a}(K) + 2 = \operatorname{reg}^2(G(K)) \leq 1$. So $\mathbf{a}(K) < 0$. If $\operatorname{red}(I) \leq 1$, then $\operatorname{red}(K) \leq 1$. So by [26, 3.2] we get $\mathbf{a}(K) < 0$. Thus at any rate $\mathbf{a}(K) < 0$.

By Theorem A we get $f_1(K) \leq f_0(K) - 1$. The result follows since by Remark 6.5(ii) we have $f_i(I) = f_i(K)$ for $i = 0, 1$. ■

Corollary 6.7 *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d \geq 2$. If I is $\mathfrak{m}$-primary and $e_2(I) = 0$, then $f_1(I) = f_0(I) - 1$.*

Proof First assume that $d = 2$. By Narita's result [17] we get that if J is any reduction of I, then $\operatorname{red}_{J^{[n]}}(I^n) = 1$ for all $n \gg 0$. Also as $\operatorname{grade}(I) = 2$, by Theorem A we get $f_1(I) = f_0(I) - 1$.

When $d \geq 3$ we choose a Rees-superficial sequence $x_1, \dots, x_{d-2}$ which is also an A-regular sequence (and so an I-superficial sequence). Set $K = I/(x_1, \dots, x_{d-2})$. Note that $e_2(K) = e_2(I) = 0$. Also $f_i(I) = f_i(K)$ for $i = 0, 1$. ■

Next we give an application of Theorem 5.10.

Theorem 6.8 *Let $(A, \mathfrak{m})$ be a Cohen–Macaulay local ring of dimension $d \geq 3$ and let I be an $\mathfrak{m}$-primary ideal. If $G(I)$ is Cohen–Macaulay and* $\operatorname{red}(I) \leq 2$, *then* $f_2(I) \geq f_1(I) - f_0(I) + 1$.

Proof When $d = 3$, then the result follows from Theorem 5.10. If $d \geq 4$, we choose a Rees-superficial sequence $x_1, \dots, x_{d-3}$ which is also an A-regular sequence (and so an I-superficial sequence). Set $K = I/(x_1, \dots, x_{d-3})$. Note that $G(K)$ is Cohen–Macaulay and the reduction number of K is ≤ 2. By Theorem 5.10 we have $f_2(K) \geq f_1(K) - f_0(K) + 1$. Also as $f_i(I) = f_i(K)$ for $i = 0, 1, 2$ (Remark 6.5(ii)) we get the result. ■

A Appendix: Minimal Reductions and Filter-Regular Elements

In this section we prove that if $x^\circ \in F(I)_1$ is $F(I)$ filter-regular, then $(x^n)^\circ \in F(I^n)_1$ is $F(I^n)$ filter-regular. This is used in proof of Theorem 4.4.

Proposition A.1 *Recall that the following assertions are equivalent:*

- *$x^\circ \in F(I)_1$ is $F(I)$-filter regular;*
- *$(0 :_{F(I)} x^\circ)_n = 0$ for all $n \gg 0$;*
- *x° is $F(I)/H^0_{F(I)_+}(F(I))$ regular.*

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For proof of the above equivalence see [26, 2.1]. It is perhaps better to see [1, Exercise 18.3.8].

Remark A.2 By Proposition A.1 we get that if depth $F(I) > 0$ and x° is $F(I)$ filter-regular, then x° is $F(I)$ regular.

For the definition of a filter-regular sequence see [1, 18.3.7]. We, however, are only interested in a filter-regular element.

The relation between minimal reductions and filter-regular sequences first appeared in the work of Trung [26]. We state one of his results [26, 3.8] in the form we need.

Lemma A.3 *Let I be an ideal with $s = l(I) > 0$. Then there exists a minimal reduction $J = (x_1, \dots, x_s)$ of I such that $x_1^\circ, \dots, x_s^\circ \in F(I)_1$ is an $F(I)$-filter regular sequence.*

We give an ideal-theoretic criterion for an element $x^\circ \in F(I)_1$ to be $F(I)$-filter regular.

Proposition A.4 *Let I be an ideal with $s = l(I) > 0$ and let $x \in I \setminus \mathfrak{m}I$. The following conditions are equivalent:*

(i) *x° is $F(I)$ filter-regular.*
(ii) *$(\mathfrak{m}I^{j+1} \colon x) \cap I^j = \mathfrak{m}I^j$ for all $j \gg 0$.*

Proof (i) $\Rightarrow$ (ii) We assume $(0 \colon x^\circ)_n = 0$ for all $n \geq c$. Clearly $\mathfrak{m}I^j \subseteq (\mathfrak{m}I^{j+1} \colon x) \cap I^j$ for all j. If $a \in I^j \setminus \mathfrak{m}I^j$ and $xa \in \mathfrak{m}I^{j+1}$, then we have $x^\circ \bullet a^\circ = 0$. It follows that $j < c$.

(ii) $\Rightarrow$ (i) Conversely, assume $(\mathfrak{m}I^{j+1} \colon x) \cap I^j = \mathfrak{m}I^j$ for all $j \geq c$. Say $a^\circ \in F(I)_j$ is non-zero and $x^\circ \bullet a^\circ = 0$. Then $a \in (\mathfrak{m}I^{j+1} \colon x) \cap I^j$. It follows that $j < c$. So x° is $F(I)$ filter-regular. ■

Corollary A.5 *(Assume the hypothesis of Proposition* A.4*) If $x^\circ \in F(I)_1$ is $F(I)$ filter-regular, then $(x^n)^\circ \in F(I^n)_1$ is $F(I^n)$ filter-regular.*

Proof Since x° is $F(I)$ filter-regular, by Proposition A.4, there exists $c > 0$ such that

$$(\mathfrak{m}I^{j+1} : x) \cap I^j = \mathfrak{m}I^j \quad \text{for all } j \geq c.$$

So for $j \geq c$ we have $(\mathfrak{m}I^{j+n} : x^n) \cap I^j = \mathfrak{m}I^j$. Therefore for $j \geq c$ we obtain

$$(\mathfrak{m}I^{n(j+1)} : x^n) \cap I^{nj} = (\mathfrak{m}I^{(nj+n)} : x^n) \cap I^{nj} = \mathfrak{m}I^{nj}.$$

Thus by Proposition A.4 we get that $(x^n)^\circ$ is $F(I^n)$ filter-regular. ■

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