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Interfacial instability of coupled-rotating inviscid fluids

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We consider the three-dimensional, cylindrical equivalent to the problem of instability between two inviscid fluids due to a velocity shear between them, known as Kelvin–Helmholtz instability. We begin by developing the solution to the linearized equations for a rotating fluid. While this solution is not in itself new, we carry the analysis further than previous treatments by including non-zero modes and considering the effect of the surface tension, particularly on the dispersion relation. We then consider a system of two fluids rotating at different rates and derive its dispersion relation, which is rather more complicated than that for a single rotating fluid. While a general analytic solution is unattainable, by investigating some special cases we show that the fundamental mode is always stable, and that Kelvin–Helmholtz instability also exists for the system. We compare our results with experiments and conclude by suggesting some hypothetical links between the theory and nature.

Key words: internal waves, waves in rotating fluids, wind–wave interactions

1. Introduction

Although in this paper we concentrate mainly on theory, the motivation behind the problem came from a series of experiments. These involved a cylindrical tank half-filled with water, above which a spinning rotor generated a wind which perturbed the water's surface (Vladusic 2001). Some of the waves suggested that Kelvin–Helmholtz instability had a role to play in their generation, and it was apparent to us that, so far as we could tell, the theory had never been adapted to this geometry; indeed, given that we now have a system with shear varying continuously with radius, and cannot consider the fluids to be in an inertial reference frame, it is arguably a new problem. The experimental results on Kelvin–Helmholtz instability are interpreted in detail in a separate paper (Bye & Ghantous 2012). Here we focus on the general linear solution for the dynamics of interfacial capillary–gravity waves between two discontinuously shearing fluids in a circular domain. While we do not pretend to be able to describe the rich variety of wave patterns that were observed in the experiment, we hope that this analytical approach, while incomplete for the obvious reason that many simplifying assumptions have been made, may help to illuminate the subject further.

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We note the body of work devoted to a similar experimental set-up, with two fluids forced at either end by differentially rotating rotors (for example Hart (1972) and Bradford, Berman & Lundgren (1981)), and a related experimental set-up, where a cylinder filled with one fluid is forced at either end by counter-rotating rotors (for example Gauthier *et al.* (2002), Nore *et al.* (2003, 2004) and Moisy *et al.* (2004)). While these studies have investigated instabilities in the motion, none has explored the theory behind inviscid interfacial instability of two distinct fluids of arbitrarily large density contrast, both co- and counter-rotating, in the general case of arbitrary rotation rates with arbitrarily large shear. Furthermore, the scope of this study differs from previous work in two important ways. First, in the wave analysis both fluids are assumed to be of infinite vertical extent, which means that shallow wave dynamics, where the velocities of each fluid are independent of depth, do not apply. Second, there are no restrictions on the ratio or direction of rotation or on the density ratio. While Bradford *et al.* (1981) treated a very tall tank, their two fluids had similar densities and the rotation rates of the two fluids were only slightly different. Their focus was on geostrophic effects, and they along with Hart (1972) experimentally modelled the general circulation on the rotating Earth by applying a small differential rotation to two fluids of small density contrast. Our experimental model, on the other hand, is an investigation of large density contrast oscillations which represent air–sea interaction under rotation on short time periods, in which the Earth's rotation is of negligible importance; crucially, they rotated the whole cylinder and rotors either at top or bottom applied the differential rotation, whereas our cylindrical tank was stationary with a rotor at the top. Nevertheless, Bradford *et al.*'s calculations did show a quasi-geostrophic interfacial instability for conditions similar to their experiments, and we have tried to compare our purely inviscid theory to their viscid one. While some other studies have investigated differentially rotating fluids in an annulus (such as Gula, Zeitlin & Plougonven (2009), who were exploring shallow-water dynamics) we treat exclusively a cylinder. The boundary conditions of these systems are different, and in particular mean that solutions involving the Bessel function of the second kind, or indeed Hankel functions, are not present in the cylinder.

We assume that our two fluids of infinite extent are bounded by a cylindrical wall of radius $r = r_0$, with an interface at $z = 0$ when the surface is unperturbed. In order to coherently develop the analysis we begin by examining only a single fluid of density ρ, which is rotating with a solid-body angular frequency Ω. This problem has already been solved by Miles (1963), but we approach it in a more straightforward manner by assuming a form for the solution by analogy with the non-rotating fluid. This is somewhat akin to Chandrasekhar's approach to exploring Kelvin–Helmholtz instability due to uniform shear in a rotating frame (Chandrasekhar 1961), but here the treatment is only for one fluid, and there is no shear or possibility of instability. While we arrive at essentially the same result as Miles did, there is a slight difference due to our inclusion of the non-axisymmetric modes from the beginning. (The difference can be removed by a simple change of variables.) We also, for the sake of completeness, include surface tension and determine the dispersion relation, as well as complete expressions for the velocities, and we retain the surface displacement due to rotation which Miles omitted. This constitutes § 2; in § 3 we introduce the second fluid. Once again this is not entirely new (Bradford *et al.* (1981) described the same velocity relations as ours) but we provide details of the solution method and describe the equations for the inviscid radial and vertical scaling parameters and, in particular, we determine the dispersion relation. Of course, a complete theory would really need to be nonlinear, as linear theory will only enable us to explore the onset of instability

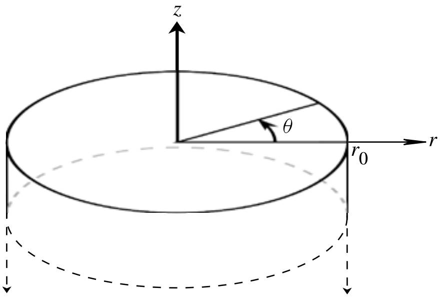

FIGURE 1. Definition sketch of the coordinate system in the cylindrical domain. The mean fluid rotation rate is $\Omega = \mathrm{d}\theta/\mathrm{d}t$.

without its development, but that problem belongs in the realm of computation rather than analysis.

Since a general solution to the dispersion relation is not possible, in § 4 we explore several special cases for which analytic solutions could be found. We also compare these with two experiments.

2. Waves on a rotating fluid

2.1. *Determining an expression for the pressure*

Consider a semi-infinite cylinder of an inviscid fluid bounded by walls at radius $r = r_0$ and with the undisturbed interface at $z = 0$ (figure 1). The equations of motion are

$$\nabla \cdot \boldsymbol{u} = 0 \tag{2.1}$$

$$-\frac{1}{\rho}\nabla p - g\hat{z} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}. \tag{2.2}$$

We separate the velocities into the wave-induced velocities (u, v, w) and solid-body rotation, Ωr, where Ω is the angular frequency of the rotating fluid and r is the radius $(r \leqslant r_0)$,

$$u_r = u \tag{2.3}$$

$$u_\theta = v + \Omega r \tag{2.4}$$

$$u_z = w. \tag{2.5}$$

From the equations of motion we have, after eliminating nonlinear terms, the following differential equations relating the velocities to the pressure,

$$\frac{\partial u}{\partial t} + \Omega\frac{\partial u}{\partial \theta} - \Omega^2 r - 2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{2.6}$$

$$\frac{\partial v}{\partial t} + 2\Omega u + \Omega\frac{\partial v}{\partial \theta} = -\frac{1}{r\rho}\frac{\partial p}{\partial \theta} \tag{2.7}$$

$$\frac{\partial w}{\partial t} + \Omega\frac{\partial w}{\partial \theta} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \tag{2.8}$$

As the solid-body term is a function of r only we can immediately incorporate it into the pressure. Assuming that the solution is separable, based on that of the non-rotating tank we propose that the pressure takes the following form:

$$p = p_a + Ae^{kz}R(r)\sin(n\theta - \omega t) - \rho g z + \rho\frac{\Omega^2 r^2}{2}, \tag{2.9}$$

where p_a is the atmospheric pressure (or in any case the pressure of a much less dense fluid above onto the fluid below) and k is a vertical scaling variable. Defining the operator

$$\mathscr{L} = \frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \theta}, \tag{2.10}$$

we can rewrite (2.8) as

$$\mathscr{L}w = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g. \tag{2.11}$$

This suggests the separable form for w

$$w = Be^{kz}\cos(n\theta - \omega t)R(r). \tag{2.12}$$

Substituting this into (2.11) gives

$$-B(\Omega n - \omega)e^{kz}R(r)\sin(n\theta - \omega t) = -\frac{kA}{\rho}e^{kz}R(r)\sin(n\theta - \omega t) \tag{2.13}$$

from which we can determine the expression for w,

$$w = \frac{Ak}{\rho(\Omega n - \omega)}e^{kz}\cos(n\theta - \omega t)R(r). \tag{2.14}$$

Similarly, the equations for u and v suggest the following forms,

$$u \sim e^{kz}U(r)\cos(n\theta - \omega t) \tag{2.15}$$

$$v \sim e^{kz}V(r)\sin(n\theta - \omega t). \tag{2.16}$$

Using these we can rewrite (2.6) and (2.7) as

$$\mathscr{L}U\cos(n\theta - \omega t) - 2\Omega V\sin(n\theta - \omega t) = -\frac{1}{\rho}\frac{\partial R}{\partial r}\sin(n\theta - \omega t) \tag{2.17}$$

$$\mathscr{L}V\sin(n\theta - \omega t) + 2\Omega U\cos(n\theta - \omega t) = -\frac{1}{\rho r}R\frac{\partial \sin(n\theta - \omega t)}{\partial \theta}, \tag{2.18}$$

and evaluating the operators gives

$$-U(\Omega n - \omega) - 2\Omega V = -\frac{A}{\rho}\frac{\partial R}{\partial r} \tag{2.19}$$

$$V(\Omega n - \omega) + 2\Omega U = -\frac{An}{\rho r}R. \tag{2.20}$$

These can be combined to give

$$U = -\frac{A}{\rho}\left(\frac{2\Omega n}{r}R + \alpha\frac{\partial R}{\partial r}\right)(4\Omega^2 - \alpha^2)^{-1} \tag{2.21}$$

$$V = \frac{A}{\rho}\left(\frac{n\alpha}{r}R + 2\Omega\frac{\partial R}{\partial r}\right)(4\Omega^2 - \alpha^2)^{-1}, \tag{2.22}$$

where $\alpha = (\Omega n - \omega)$. Applying the operator $\mathscr{L}$ to (2.1) we get

$$\frac{U}{r} + \frac{\partial U}{\partial r} + \frac{n}{r}V - \frac{Ak^2}{\rho\alpha}R = 0 \tag{2.23}$$

which, upon substitution of U and V from (2.21) and (2.22), gives

$$\frac{1}{r}\frac{\partial R}{\partial r} + \frac{\partial^2 R}{\partial r^2} - \frac{n^2}{r^2}R - \alpha^{-2}(4\Omega^2 - \alpha^2)k^2 R = 0. \tag{2.24}$$

We can rearrange this as

$$\frac{1}{r}\frac{\partial R}{\partial r} + \frac{\partial^2 R}{\partial r^2} + R\left(\gamma^2 - \frac{n^2}{r^2}\right) = 0 \tag{2.25}$$

where

$$\gamma^2 = k^2\left(1 - \frac{4\Omega^2}{\alpha^2}\right). \tag{2.26}$$

Equation (2.25) is Bessel's equation and the solution is $R(r) = \mathrm{J}_n(\gamma r)$, where we have implicitly eliminated the Bessel function of the second kind as it blows up at the origin. This is the same form as that found by Miles (1963), with the only difference being the presence of Ωn in α^2. (A change of variables can remove this, but the same change has to be made in the arguments of the sinusoids.) With the Bessel function, the pressure is

$$p = A\mathrm{e}^{kz}\mathrm{J}_n(\gamma r)\sin(n\theta - \omega t) - \rho g z + \rho\frac{\Omega^2 r^2}{2}. \tag{2.27}$$

2.2. *Finding the velocities, surface elevation and deriving the dispersion relation*

Referring back to the proportionalities (2.15) and (2.16), the velocities u and v can be recovered from (2.21) and (2.22). These are, shown together with w for the sake of completeness,

$$u = -\frac{A\mathrm{e}^{kz}\cos(n\theta - \omega t)}{\rho(4\Omega^2 - \alpha^2)}\left(\alpha\gamma \mathrm{J}_n'(\gamma r) + \frac{2\Omega n}{r}\mathrm{J}_n(\gamma r)\right) \tag{2.28}$$

$$v = \frac{A\mathrm{e}^{kz}\sin(n\theta - \omega t)}{\rho(4\Omega^2 - \alpha^2)}\left(2\Omega\gamma \mathrm{J}_n'(\gamma r) + \frac{n\alpha}{r}\mathrm{J}_n(\gamma r)\right) \tag{2.29}$$

$$w = \frac{Ak}{\rho\alpha}\mathrm{e}^{kz}\mathrm{J}_n(\gamma r)\cos(n\theta - \omega t). \tag{2.30}$$

Note that

$$\mathrm{J}_n'(\gamma r) = \frac{\mathrm{d}\mathrm{J}_n(\gamma r)}{\mathrm{d}(\gamma r)}. \tag{2.31}$$

We now seek a form for the surface equation. The surface elevation η satisfies

$$u_z|_{z=\eta} = \frac{\partial \eta}{\partial t} + (\boldsymbol{u}\cdot\nabla)\eta. \tag{2.32}$$

Discarding the nonlinear terms gives us

$$\frac{Ak}{\rho\alpha}\mathrm{J}_n(\gamma r)\cos(n\theta - \omega t) = \mathscr{L}\eta, \tag{2.33}$$

and if we assume a separable form for η as for the pressure and velocities we arrive at

$$\eta = \frac{Ak}{\rho\alpha^2}\mathrm{J}_n(\gamma r)\sin(n\theta - \omega t) + f(r). \tag{2.34}$$

The $f(r)$ term is related to the curvature of the surface due to the rotation of the fluid and will be evaluated shortly. Assuming that the curvature of the surface due to the effect of surface tension is small, the pressure difference across the surface is given by

$$\Delta p|_{z=\eta} = p_a - p|_{z=\eta} = \rho\sigma\nabla^2\eta, \tag{2.35}$$

where σ is the specific surface tension. Setting the surface tension to zero for the moment, we obtain

$$-A\mathrm{J}_n(\gamma r)\sin(n\theta - \omega t) + \rho g\eta - \rho\frac{\Omega^2 r^2}{2} = 0. \tag{2.36}$$

Substituting in the expression (2.34) for η results in the following equation:

$$\frac{Agk}{\alpha^2}\mathrm{J}_n(\gamma r)\sin(n\theta - \omega t) + \rho g f(r) = A\mathrm{J}_n(\gamma r)\sin(n\theta - \omega t) + \rho\frac{\Omega^2 r^2}{2}. \tag{2.37}$$

As f is a function of r alone it must be associated with the $\Omega^2 r^2$ term, so

$$f(r) = \frac{\Omega^2 r^2}{2g}. \tag{2.38}$$

This paraboloid is what we would expect from comparison with the unperturbed surface arising from Bernoulli's equation. Meanwhile, the remainder of (2.37) leads to the dispersion relation,

$$gk = \alpha^2 = (\Omega n - \omega)^2. \tag{2.39}$$

Now suppose that the surface tension is non-zero; from (2.35) we have

$$\begin{aligned} A\left(\frac{gk}{\alpha^2} - 1\right)\mathrm{J}_n(\gamma r)\sin(n\theta - \omega t) + \rho g f(r) - \rho\frac{\Omega^2 r^2}{2} \\ = \rho\sigma\frac{Ak}{\rho\alpha^2}\sin(n\theta - \omega t)\left(\frac{\gamma}{r}\mathrm{J}_n'(\gamma r) + \gamma^2\mathrm{J}_n''(\gamma r) - \frac{n^2}{r^2}\mathrm{J}_n(\gamma r)\right) \\ + \rho\sigma\left(\frac{f'(r)}{r} + f''(r)\right), \end{aligned} \tag{2.40}$$

and again it is clear that the terms involving f or its derivatives, being functions of r only, must be associated with the $\Omega^2 r^2$ term. Thus,

$$f''(r) + \frac{f'(r)}{r} - \frac{g}{\sigma}f(r) = -\frac{\Omega^2 r^2}{2\sigma}. \tag{2.41}$$

Leaving this aside for a moment, the remaining terms must also be equal, and they can be rearranged as

$$\mathrm{J}_n''(\gamma r) + \frac{1}{\gamma r}\mathrm{J}_n'(\gamma r) + \left[\frac{1}{\sigma\gamma^2}\left(\frac{\alpha^2}{k} - g\right) - \frac{n^2}{\gamma^2 r^2}\right]\mathrm{J}_n(\gamma r) = 0 \tag{2.42}$$

which is satisfied as long as

$$\frac{1}{\sigma\gamma^2}\left(\frac{\alpha^2}{k}-g\right)=1. \tag{2.43}$$

This is, in fact, our dispersion relation, written more fully as

$$(\Omega n-\omega)^2=gk+\sigma k^3\left(1-\frac{4\Omega^2}{(\Omega n-\omega)^2}\right). \tag{2.44}$$

Setting $\Omega=0$ we retrieve the dispersion relation for a non-rotating fluid, which may easily be confirmed by deriving the expressions with $\Omega=0$ (see for example Le Méhauté (1976)).

The solution to the inhomogeneous differential equation (2.41) is

$$f(r)=C_1\mathrm{I}_0\left(\sqrt{\frac{g}{\sigma}}r\right)+C_2\left\{\mathrm{i}\mathrm{I}_0\left(\sqrt{\frac{g}{\sigma}}r\right)-\frac{2}{\pi}\mathrm{K}_0\left(\sqrt{\frac{g}{\sigma}}r\right)\right\}+\frac{\Omega^2r^2}{2g}+2\frac{\Omega^2\sigma}{g^2}. \tag{2.45}$$

Here I_0 and K_0 are the modified Bessel functions of the first and second kinds, respectively, of order zero. Clearly C_2 must be zero as the solution cannot be allowed to be infinite at the origin. As for C_1, we use the condition at the cylinder wall. A contact angle, ϕ, between the fluid and the wall can be defined as (Bachelor 1967):

$$\gamma_{12}=\gamma_{31}+\gamma_{23}\cos\phi, \tag{2.46}$$

where the γ_{ij} are the surface tensions between media i and j. For the purposes of this problem we may envisage these as the cylinder wall (medium 1), air (medium 2) and the water in the tank (medium 3). (In practice, determining the contact angle is not straightforward; the problem is outlined by Ghantous (2010).)

Armed with our contact angle, the condition on the slope of the surface at the boundary becomes

$$\left.\frac{\mathrm{d}f}{\mathrm{d}r}\right|_{r=r_0}=\tan\left(\frac{\pi}{2}-\phi\right) \tag{2.47}$$

$$C_1\sqrt{\frac{g}{\sigma}}I_1\left(\sqrt{\frac{g}{\sigma}}r_0\right)+\frac{\Omega^2r_0}{g}=\tan\left(\frac{\pi}{2}-\phi\right) \tag{2.48}$$

and this uniquely defines our C_1 for any given combination of media. Our surface can now be fully described. From (2.34) and (2.45), together with the value of C_1 just determined and $C_2=0$, this is

$$\eta=\frac{Ak}{\rho\alpha^2}\mathrm{J}_n(\gamma r)\sin(n\theta-\omega t)+C_1\mathrm{I}_0\left(\sqrt{\frac{g}{\sigma}}r\right)+\frac{\Omega^2r^2}{2g}+2\frac{\Omega^2\sigma}{g^2}. \tag{2.49}$$

The oscillating part of the solution was solved under the assumption that the surface was located at $z=0$. The paraboloid part of the solution indicates that this is not in fact generally true, however the perturbations are linear waves, so can be justifiably superimposed onto the paraboloid surface. This is at least true for a surface which does not deviate much from the non-rotating surface, corresponding to a small value for the rate of rotation, Ω.

2.3. *The effect of rotation*

Returning to the oscillatory part of the solution, the wave disturbance of the surface profile, and the effect of rotation on the form of the solution, we find that the rather more complicated expression for the radial scaling γ, as compared with k in the non-rotating cylinder, actually has no bearing on the geometric form of the surface other than amplitude. Clearly for $\varOmega = 0$ the value of γ is just k, but in order to satisfy the condition at the boundary of the cylinder, where $\partial\eta/\partial r = 0$, the value of γ must be the same for all $\varOmega$ given a value for n.

One might guess that we arrive at a different solution as we approach $(\varOmega n - \omega)^2 = 4\varOmega^2$ (in this realm we also approach the limits of linear theory). This is the point at which $\gamma^2 = 0$; we may take the limit as $(\varOmega n - \omega)^2 \rightarrow 4\varOmega^2$, while keeping γ constant, and $(\varOmega n - \omega)$ must increase towards infinity, as would in fact $\varOmega$ and, importantly, k. Were the ratio

$$\frac{4\varOmega^2}{(\varOmega n - \omega)^2} \tag{2.50}$$

to become greater than one, either k would need to be imaginary, leading to inertial oscillations with depth, or γ would need to be imaginary, leading to modified Bessel functions, a solution which would no longer fit the boundary conditions, nor the requirements for our linear, first-order solution (that is, small amplitude everywhere). Beyond noting their existence we shall not treat inertial oscillations here.

Applying the dispersion relation (2.39), the expression for γ becomes

$$\gamma_R^2 = k^2\left(1 - \frac{4\varOmega^2}{kg}\right) \tag{2.51}$$

where γ_R is the value of γ set by the boundary condition at the radius r_0. Given that γ_R is a constant, this produces a quadratic in k, whose solution is

$$k = \frac{2\varOmega^2}{g} \pm \sqrt{\frac{4\varOmega^4}{g^2} + \gamma_R^2}, \tag{2.52}$$

and we may ignore the negative solution as negative k would lead to increasing pressure perturbations with depth. It is clear from this expression for k that for any non-zero $\varOmega$, and any non-zero γ_R, k is always larger than $4\varOmega^2/g$ and (2.51) is always positive. In the rapidly rotating limit $\varOmega \rightarrow \infty$, k approaches $4\varOmega^2/g$.

Turning now to the amplitude we make the following observation: as the difference $(\varOmega n - \omega)$ increases, so does k and, as we have just seen, k approaches $4\varOmega^2/g$. The amplitude of the oscillating part of the expression for η (equation (2.49)) remains unchanged, but the individual velocities u, v and w (equations (2.28)–(2.30)) increase. For w this is easy to see, since α can be replaced with $\sqrt{kg}$ by the dispersion relation, which leads to the amplitude being proportional to $\sqrt{k}$. For u, substituting in the dispersion relation (2.39), the expression for γ (2.26) and using the recurrence relation

$$\mathrm{J}_n'(\gamma r) = \mathrm{J}_{n-1}(\gamma r) - \frac{n}{\gamma r}\mathrm{J}_n(\gamma r) \tag{2.53}$$

gives, in the limit as $k \rightarrow 4\varOmega^2/g$ (implying that $\alpha \rightarrow 2\varOmega$),

$$u = \frac{Ak^{3/2}\mathrm{e}^{kz}}{\rho\sqrt{g}\gamma_R}\cos(n\theta - \omega t)J_{n-1}(\gamma_R r). \tag{2.54}$$

The horizontal velocity is now proportional to $k^{3/2}$. In this limit v behaves similarly, and since k is also large the horizontal velocities dominate the vertical velocity. Meanwhile, the pressure, remains unchanged (save for the $\varOmega^2 r^2$ term which defines the parabolic surface profile of the undisturbed surface), and the motion becomes quasi-horizontal. By substitution it is easy to show that the limiting forms of the equations for u, v and w satisfy the continuity equation (2.1).

3. Two fluids with horizontal shear

In the previous section, we derived the velocities and a surface relation for a rotating fluid, which describes both a mean surface profile and surface waves, and demonstrated that the equations of motion also permit the existence of inertial waves for certain frequencies. We shall now show that when two fluids are superposed and there is a rotating shear between them, instability ensues.

3.1. *Finding the surface elevation, being the shape of the interface between the fluids*

This problem is analogous to the two-dimensional, constant shear instability problem which is described in varying levels of detail in several texts. Perhaps the clearest illustration of the solution to the basic problem is still to be found in the original paper by Lord Kelvin (Thomson 1871). (Helmholtz's paper (von Helmholtz 1868*b*), although it predates Kelvin's, is not as clear, and does not show the mathematical form that Kelvin describes. It is in German but Guthrie (von Helmholtz 1868*a*) published a translation later the same year, which is what we have consulted.) And Chandrasekhar usefully explores a variety of complications, including that of rotation (Chandrasekhar 1961), however in his case the entire system is rotating and the shear is still uniform. What we are investigating here is a rotating fluid resting on top of a second fluid rotating at a different rate. In this situation, the shear between the fluids increases with radius from the axis of rotation.

We begin by defining the properties of the upper and lower fluids, which we denote by subscripts U and L, respectively. The two fluids of densities ρ_U and ρ_L are assumed to be stably stratified ($\rho_U < \rho_L$) and rotating at the rates

$$\Omega_U = \Omega_0 - \Omega \tag{3.1}$$

$$\Omega_L = \Omega_0 + \Omega, \tag{3.2}$$

where Ω_0 is the mean rotation. The equations of motion are the same as those for a single rotating fluid (2.6)–(2.8), except the rate of rotation differs in either fluid. Above an interface placed at $z = 0$ they are

$$\frac{\partial u_U}{\partial t} + \Omega_U \frac{\partial u_U}{\partial \theta} - \Omega_U^2 r - 2\Omega_U v_U = -\frac{1}{\rho_U}\frac{\partial p_U}{\partial r} \tag{3.3}$$

$$\frac{\partial v_U}{\partial t} + 2\Omega_U u_U + \Omega_U \frac{\partial v_U}{\partial \theta} = -\frac{1}{r\rho_U}\frac{\partial p_U}{\partial \theta} \tag{3.4}$$

$$\frac{\partial w_U}{\partial t} + \Omega_U \frac{\partial w_U}{\partial \theta} = -\frac{1}{\rho_U}\frac{\partial p_U}{\partial z} - g, \tag{3.5}$$

while below the interface they're

$$\frac{\partial u_L}{\partial t} + \Omega_L \frac{\partial u_L}{\partial \theta} - \Omega_L^2 r - 2\Omega_L v_L = -\frac{1}{\rho_L}\frac{\partial p_L}{\partial r} \tag{3.6}$$

$$\frac{\partial v_L}{\partial t} + 2\Omega_L u_L + \Omega_L \frac{\partial v_L}{\partial \theta} = -\frac{1}{r\rho_L}\frac{\partial p_L}{\partial \theta} \tag{3.7}$$

$$\frac{\partial w_L}{\partial t} + \Omega_L \frac{\partial w_L}{\partial \theta} = -\frac{1}{\rho_L}\frac{\partial p_L}{\partial z} - g. \tag{3.8}$$

Once again we assume a form for the pressure, which now resembles that for a single homogeneous rotating fluid as in (2.9),

$$p_U = p_a + \rho_U \left\{ A\mathrm{e}^{-k_U z} R(r) \sin(n\theta - \omega t) - gz + \frac{\Omega_U^2 r^2}{2} \right\} \tag{3.9}$$

$$p_L = p_a + \rho_L \left\{ B\mathrm{e}^{k_L z} R(r) \sin(n\theta - \omega t) - gz + \frac{\Omega_L^2 r^2}{2} \right\}. \tag{3.10}$$

We assume that just like the lower fluid, the upper fluid is of infinite extent whose pressure, integrated over the entire column, is finite and denoted by p_a. To satisfy continuity at the surface the functions $R(r)$ and $\sin(n\theta - \omega t)$ must be identical for both fluids; continuity also requires the exponential functions to be equal at the interface, which is easily satisfied for $z = 0$, and since the perturbations are expected to decrease as we move away from the interface in either direction, the exponential function for the lower fluid must grow with positive z and that for the upper fluid decrease with positive z. Furthermore, we do not assume that k_U and k_L are, in general equal. This follows naturally from the previous section where it was found that k is dependent on the rotation rate of the fluid.

Following the same method as for the single homogeneous fluid, we find expressions for the vertical velocities

$$w_U = -\frac{A k_U}{\alpha_U} \mathrm{e}^{-k_U z} R(r) \cos(n\theta - \omega t), \tag{3.11}$$

$$w_L = \frac{B k_L}{\alpha_L} \mathrm{e}^{k_L z} R(r) \cos(n\theta - \omega t), \tag{3.12}$$

where $\alpha_U = (\Omega_U n - \omega)$ and $\alpha_L = (\Omega_L n - \omega)$. We can likewise find expressions for the radial and azimuthal velocities which, as before, are used to derive equations for $R(r)$. The equations are identical to (2.25) except we now have different (but still equal) expressions for γ above and below the surface:

$$\gamma_U^2 = k_U^2 \left(1 - \frac{4\Omega_U^2}{(\Omega_U n - \omega)^2} \right) = \gamma_L^2 = k_L^2 \left(1 - \frac{4\Omega_L^2}{(\Omega_L n - \omega)^2} \right). \tag{3.13}$$

The solutions are the Bessel functions

$$R(r) = \mathrm{J}_n(\gamma_U r) = \mathrm{J}_n(\gamma_L r), \tag{3.14}$$

where we ignore the Bessel function of the second kind as it is not finite at the centre of the cylinder of fluid, and $\gamma = \gamma_U = \gamma_L$.

The horizontal velocities are derived using the same technique as before, and are identical save for the subscripts indicating the fluid, and the sign of the exponent, so

$$u_\mathrm{L} = -\frac{B\mathrm{e}^{k_L z} \cos(n\theta - \omega t)}{4\Omega_L^2 - \alpha_L^2} \left(\alpha_L \gamma_L \mathrm{J}_n'(\gamma_L r) + \frac{2\Omega_L n}{r} \mathrm{J}_n(\gamma_L r) \right) \tag{3.15}$$

$$u_\mathrm{U} = -\frac{A\mathrm{e}^{-k_U z} \cos(n\theta - \omega t)}{4\Omega_U^2 - \alpha_U^2} \left(\alpha_U \gamma_U \mathrm{J}_n'(\gamma_U r) + \frac{2\Omega_U n}{r} \mathrm{J}_n(\gamma_U r) \right) \tag{3.16}$$

$$v_\mathrm{L} = \frac{B\mathrm{e}^{k_L z} \sin(n\theta - \omega t)}{4\Omega_L^2 - \alpha_L^2} \left(2\Omega_L \gamma_L \mathrm{J}_n'(\gamma_L r) + \frac{n\alpha_L}{r} \mathrm{J}_n(\gamma_L r) \right) \tag{3.17}$$

$$v_\mathrm{U} = \frac{A\mathrm{e}^{-k_U z} \sin(n\theta - \omega t)}{4\Omega_U^2 - \alpha_U^2} \left(2\Omega_U \gamma_U \mathrm{J}_n'(\gamma_U r) + \frac{n\alpha_U}{r} \mathrm{J}_n(\gamma_U r) \right). \tag{3.18}$$

In order to determine the surface expressions we make use once again of (2.32). We get two expressions which must be equal,

$$\begin{aligned}\eta &= -\frac{Ak_U}{\alpha_U^2}\mathrm{J}_n(\gamma r)\sin(n\theta - \omega t) + f(r) \\ &= \frac{Bk_L}{\alpha_L^2}\mathrm{J}_n(\gamma r)\sin(n\theta - \omega t) + f(r).\end{aligned} \tag{3.19}$$

Once again $f(r)$ is the mean surface elevation. The two expressions for the surface give us a second equation in terms of the frequency ω,

$$\frac{-Ak_U}{(\Omega_U n - \omega)^2} = \frac{Bk_L}{(\Omega_L n - \omega)^2}. \tag{3.20}$$

From the surface condition we may derive the dispersion relation, which provides a third equation for ω. The pressure difference across the surface is

$$\Delta p|_{z=\eta} = p_U|_{z=\eta} - p_L|_{z=\eta} = T\nabla^2\eta \tag{3.21}$$

where T is the surface tension. If we substitute the expressions for pressure (3.9) and (3.10), then we have

$$\eta - \frac{T}{g(\rho_L - \rho_U)}\nabla^2\eta = \frac{\rho_L B - \rho_U A}{g(\rho_L - \rho_U)}\mathrm{J}_n(\gamma_L r)\sin(n\theta - \omega t) + \left(\frac{\rho_L\Omega_L^2 - \rho_U\Omega_U^2}{\rho_L - \rho_U}\right)\frac{r^2}{2g}. \tag{3.22}$$

Using (3.19) we get an inhomogeneous differential equation for the function $f(r)$,

$$f''(r) + \frac{f'(r)}{r} - \frac{g(\rho_L - \rho_U)}{T}f(r) = -\frac{\Omega_L^2\rho_L - \Omega_U^2\rho_U}{2T}r^2, \tag{3.23}$$

and an equation relating derivatives of the Bessel function,

$$\mathrm{J}_n''(\gamma r) + \frac{1}{\gamma r}\mathrm{J}_n'(\gamma r) + \left[\frac{\alpha_L^2(B\rho_L - A\rho_U)}{Bk_L\gamma^2 T} - \frac{g(\rho_L - \rho_U)}{\gamma^2 T} - \frac{n^2}{\gamma^2 r^2}\right]\mathrm{J}_n(\gamma r) = 0. \tag{3.24}$$

This latter equation is satisfied if

$$\gamma^2 T + g(\rho_L - \rho_U) = \frac{\alpha_L^2}{Bk_L}(B\rho_L - A\rho_U). \tag{3.25}$$

The solution of (3.23) is

$$f(r) = C_1\mathrm{I}_0\left(\sqrt{\frac{g(\rho_L - \rho_U)}{T}}r\right) + \frac{\Omega_L^2\rho_L - \Omega_U^2\rho_U}{2g(\rho_L - \rho_U)}\left[r^2 + \frac{4T}{g(\rho_L - \rho_U)}\right] \tag{3.26}$$

with the constant of integration C_1 determined by the contact angle (see § 2.2); if there is no surface tension it is zero in order to satisfy the condition that the solution be finite everywhere. $f(r)$ gives us the paraboloid mean surface, determined by the difference of inertially weighted terms for each fluid plus the effect of the surface tension.

Ignoring surface tension, if we set $\Omega_L = \Omega_U$, so that both fluids are rotating at the same rate, their velocities must be equal at the interface, and so from (3.11) and (3.12) we must have that $A = -B$, just as for two non-rotating fluids; see article 231 of Lamb (1932). The left-hand term of the surface elevation equation (3.22) then resembles the form for an internal (interfacial) wave between two fluids with no shear from rectilinear theory, with the only difference being the presence of the Bessel function. As we show in the next section, however, there are only two cases where we can recover this relationship between A and B: the first is when the mean rotation rates of each fluid are exactly equal, and the second when $n=0$ and they are equal in magnitude but counter-rotating. This contrasts with Kelvin–Helmholtz instability for non-rotating fluids, where the relationship always holds, as shown in article 232 of Lamb (1932).

3.2. *Equations for the vertical scaling and for A and B*

From (3.25) and (3.20) we arrive at

$$\gamma^2 T + g(\rho_L - \rho_U) = \frac{\rho_L \alpha_L^2}{k_L} + \frac{\rho_U \alpha_U^2}{k_U}. \tag{3.27}$$

Multiplying through, alternately, by k_L and k_U and using (3.13), we find expressions for the vertical scaling factors for each fluid,

$$k_L \left[\frac{\gamma^2 T}{\rho_L} + g(1-\chi)\right] = \alpha_L^2 \pm \alpha_L \alpha_U \chi \frac{\sqrt{\alpha_U^2 - 4\Omega_U^2}}{\sqrt{\alpha_L^2 - 4\Omega_L^2}} \tag{3.28}$$

$$k_U \left[\frac{\gamma^2 T}{\rho_L} + g(1-\chi)\right] = \alpha_U^2 \chi \pm \alpha_L \alpha_U \frac{\sqrt{\alpha_L^2 - 4\Omega_L^2}}{\sqrt{\alpha_U^2 - 4\Omega_U^2}} \tag{3.29}$$

where $\chi = \rho_U/\rho_L$ and the signs must be the same. These expressions imply a strict relationship between A and B, since from (3.20) we have the ratio

$$\xi \equiv -\frac{A}{B} = \frac{k_L \alpha_U^2}{k_U \alpha_L^2}. \tag{3.30}$$

The relations (3.28) and (3.29) enable us to express this as

$$\xi = \frac{\alpha_L \alpha_U \pm \alpha_U^2 \chi \dfrac{\sqrt{\alpha_U^2 - 4\Omega_U^2}}{\sqrt{\alpha_L^2 - 4\Omega_L^2}}}{\alpha_L \alpha_U \chi \pm \alpha_L^2 \dfrac{\sqrt{\alpha_L^2 - 4\Omega_L^2}}{\sqrt{\alpha_U^2 - 4\Omega_U^2}}}. \tag{3.31}$$

Therefore A and B are not arbitrary constants. Instead, their ratio is a function of the frequency ω, the azimuthal wavenumber n, the density ratio of the fluids χ and the rates of rotation of the two fluids Ω_L and Ω_U; note it is not a function of any of the independent space or time variables r, θ, z or t, so the differential equations remain valid. Because it is their ratio which is defined, rather than their exact value, a common, arbitrary scaling constant may be applied which disappears in the analysis. It may be seen from (3.31) that when the rates of rotation are the same, that is $\Omega_L = \Omega_U$, that $\xi = 1$. For the unique case of $n = 0$, $\Omega_L = -\Omega_U$ also returns $\xi = 1$.

3.3. *The dispersion relation*

Substituting either of the relations (3.28) or (3.29) into (3.13) gives a dispersion relation for the coupled system,

$$\left[\frac{\gamma^3 T}{\rho_L} + \gamma g(1-\chi)\right]^2 = \alpha_L^2(\alpha_L^2 - 4\Omega_L^2) \pm 2\alpha_L \alpha_U \chi \sqrt{\alpha_L^2 - 4\Omega_L^2}\sqrt{\alpha_U^2 - 4\Omega_U^2} + \alpha_U^2 \chi^2 (\alpha_U^2 - 4\Omega_U^2). \tag{3.32}$$

For given values of the radial scaling γ, gravitational acceleration g, density ratio of the fluids χ, the mode n and the rotation rates Ω_L and Ω_U, this equation determines the allowed frequencies. However, it is not straightforward to solve; one method is to move the terms other than the square roots to one side and square both sides, which gives an octic equation in ω. The solutions to this can then be introduced into the original equation to determine which ones solve it (rather than the octic itself). But since there is no algebraic method for solving octic equations in general, we avoid treating them here.

Before we investigate some special cases of the general equation in detail, we may make some comments on the behaviour of the system. We can determine frequency limits from (3.13), exactly as we did in § 2 but now for two fluids. For real values of ω, k_U and k_L and in order to ensure that γ is real so the boundary conditions are satisfied, we must have that

$$\alpha_U^2 = (\Omega_U n - \omega)^2 > 4\Omega_U^2 \tag{3.33}$$

and

$$\alpha_L^2 = (\Omega_L n - \omega)^2 > 4\Omega_L^2. \tag{3.34}$$

This leads to the following restrictions for ω when $\Omega_U > 0$ and $\Omega_L > 0$,

$$\omega < \Omega_U(n-2), \quad \omega > \Omega_U(n+2), \tag{3.35}$$

$$\omega < \Omega_L(n-2), \quad \omega > \Omega_L(n+2), \tag{3.36}$$

and when $\Omega_L < 0$,

$$\omega > \Omega_L(n-2), \quad \omega < \Omega_U(n+2). \tag{3.37}$$

Thus, ω must satisfy separate conditions for each fluid simultaneously. Clearly, when $\Omega_L = \pm\Omega_U$ this condition is effectively the same for the two fluids, but when the magnitudes of Ω_U and Ω_L differ then they are unrelated. In the limit as $n \to \infty$ all frequencies are allowed. Now when $\alpha_U^2 < 4\Omega_U^2$ and $\alpha_L^2 < 4\Omega_L^2$, the only way that the boundary conditions can be satisfied (meaning that γ is real) is for k_U and k_L to be imaginary, which implies the existence of inertial waves. Because there are two inequalities to satisfy, one for each fluid, it is possible for one fluid to have inertial waves while the other does not. In the present work we are interested in surface wave phenomena, so we leave inertial waves here.

3.4. *Boundary condition for the cylinder wall*

At the cylinder wall the radial velocity must be zero, which means that both (3.15) and (3.16) must both be zero. This is impossible to satisfy for any non-zero frequency ω, except for when $n = 0$, indicating that the separable solution is not generally valid. But various limiting forms of the equations exist for which the condition can be satisfied, so the separable solution in many situations should function as a good approximation of the system.

The only difference between (3.15) and (3.16) is that due to the different solid-body rotation rates, Ω_L and Ω_U. The boundary condition can be satisfied when this difference becomes negligible or when one of the Bessel terms disappears. This can happen for several different reasons:

(*a*) the special case of $n = 0$, which is satisfied exactly;

(*b*) $\Omega_L \approx \Omega_U$; the special case where both fluids rotate at the same rate is satisfied exactly;

(*c*) $r_0 \to \infty$;
(*d*) $n \to \infty$;
(*e*) $|\omega|$ is much greater than both $|\Omega_L|$ and $|\Omega_U|$;
(*f*) $|\omega|$ is much smaller than both $|\Omega_L|$ and $|\Omega_U|$.

The last two follow since in both cases the two independent boundary conditions (3.15) and (3.16) become independent of Ω_L and Ω_U (this can be seen by dividing through by Ω_L and Ω_U, respectively). At least one of these six limits is assumed to hold for each of the special cases analysed in § 4.

4. Solutions for special cases

4.1. *The axisymmetric mode*

In order to satisfy the conditions (3.35)–(3.37) for the existence of surface waves, as opposed to internal waves, the axisymmetric mode, $n = 0$, requires that ω be real. Therefore, the fundamental mode is always stable, as also reported by Bradford *et al.* (1981). When considered geometrically, this is akin to rectilinear shear where there is no initial disturbance in the direction of the flow.

Neglecting surface tension and fixing $n = 0$, the dispersion relation (3.32) becomes

$$\gamma g(1-\chi) = \omega\left(\sqrt{\omega^2 - 4\Omega_L^2} \pm \chi\sqrt{\omega^2 - 4\Omega_U^2}\right). \tag{4.1}$$

Expressing this in terms of the vertical scaling variables gives

$$\omega^2 = \frac{k_L k_U g(1-\chi)}{k_U + k_L\chi} \tag{4.2}$$

which for both uniform rotation as well as exact counter-rotation, that is $\Omega_U = \pm\Omega_L$, reduces to the familiar form for the dispersion relation,

$$\omega = \pm\sqrt{gk\left(\frac{1-\chi}{1+\chi}\right)} \tag{4.3}$$

where $k = k_U = k_L$ and the two forms of (3.13) are identical. In the high-frequency limit in which the effects of rotation are negligible, there is no restriction on the allowed frequencies.

4.2. *The limit of large density contrast*

Returning to the general solution, but again neglecting surface tension, we now examine the limit as $\chi \to 0$, that is we assume that the upper fluid is much less dense than the lower. For the lower fluid we recover relations which are the same as for a single fluid, however the relation for the vertical scaling k_U of the upper fluid (3.29) does not disappear, which suggests a fundamental difference between the limit and assuming the absence of the upper fluid to begin with. Taking the limit, equation (3.32) then becomes

$$\gamma^2 g^2 - \alpha_L^2(\alpha_L^2 - 4\Omega_L^2) = 0, \tag{4.4}$$

which is identical to what we get for a single fluid by substituting the dispersion relation (equation (2.39)) into the expression for γ (equation (2.26)). The solutions are

$$\omega = \frac{\sqrt{4\Omega_L^2 - \mathrm{i}2g\gamma} \pm \sqrt{4\Omega_L^2 + \mathrm{i}2g\gamma}}{2} + n\Omega_L \tag{4.5}$$

and

$$\omega = \frac{-\sqrt{4\Omega_L^2 - \mathrm{i}2g\gamma} \pm \sqrt{4\Omega_L^2 + \mathrm{i}2g\gamma}}{2} + n\Omega_L. \tag{4.6}$$

The two terms under the square roots are complex conjugates of one another, so two solutions will be real, and two complex, which become pure imaginary if we shift the frequency by twice the rate of rotation (being the Coriolis force). These complex solutions again do not satisfy the relations (3.28) and (3.29), for the same reason that they imply a negative real value for k_L. For this configuration there is no inertial instability for any mode, which is expected since in the limit of zero density the upper fluid has zero momentum.

The dispersion relation (4.4) of course holds for the two remaining real solutions, and can be expressed in terms of the relative wave angular frequency as

$$\frac{\omega}{n} - \Omega_L = \pm\frac{2\Omega_L}{n}\sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{1 + \left(\frac{\gamma g}{2\Omega_L^2}\right)^2}}. \tag{4.7}$$

By inspection, for large values of the non-dimensionalized radial scaling $\gamma g/\Omega_L^2$ the non-dimensional relative angular wave frequency, which we define as

$$\frac{\omega/n - \Omega_L}{\Omega_L}, \tag{4.8}$$

must be large as well. This is illustrated in figure 2 for various values of n. We may therefore say that the waves rotate more rapidly as the radial scale is compressed near the centre; this may be equivalently concluded from (3.13). Two important limits of (4.7) are

$$\omega = (n + 2)\Omega_L \quad \text{as } \frac{g\gamma}{\Omega_L^2} \to 0 \tag{4.9}$$

and

$$\omega = \sqrt{g\gamma} + \Omega_L n \quad \text{as } \frac{g\gamma}{\Omega_L^2} \to \infty. \tag{4.10}$$

The first implies that as the radial scaling variable becomes small relative to the rate of rotation, the relative frequency varies only with the rate of rotation (and the mode). The second implies that in the inverse case the relative frequency is dependent only on the radial scaling. The approach to these limits can be seen in figure 2, in which for each mode the dispersion relation is given by (4.9) for $\gamma g/\Omega_L^2 = 0$ and by (4.10) for $\gamma g/\Omega_L^2 \to \infty$. Equation (4.10) has an interesting interpretation: consider a wave rotating at a radius r. Then, by (3.13), as $\Omega_L \to 0$, $\gamma \to k$, and from (4.10) $\omega \to \sqrt{gk}$, which is the dispersion relation for rectilinear deep water waves. Hence, (4.10) provides a relationship for the short-crestedness of the gravity waves, in which since for $kr \to \infty$ the Bessel function reduces to a trigonometric form with a radial wavenumber k, the ratio of the crest length $W = \pi/(2k)$ to the wavelength $\lambda = 2\pi/k$ is $1/4$.

Observations of the rapidly rotating $n = 2$ mode were recently made in a series of experiments described by Vladusic (2001), Bye, Hughes & Vladusic (2005) and Bye & Ghantous (2012). In a tank of radius $r_0 = 0.19$ m, a rotating disc drove a column of air of 0.07 m depth which sat over water of depth 0.06 m. The rotation rates were $\omega/n = 9.8$ rad s^{-1} and $\Omega_L = 1.5$ rad s^{-1}. This corresponds approximately

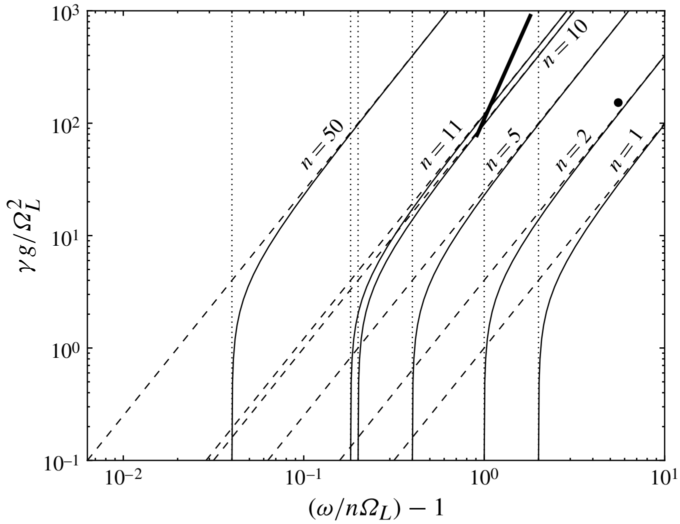

FIGURE 2. Dispersion relation (4.7) (solid lines) for the non-dimensional radial scaling $\gamma g/\Omega_L^2$ as a function of the non-dimensional relative angular frequency $(\omega/n\Omega_L)-1$. The dotted lines are (4.9), applicable for $\gamma g/\Omega_L^2 \to 0$, and the dashed lines are (4.10), applicable for $\gamma g/\Omega_L^2 \to \infty$. The black circle is the observation for the $n=2$ mode in the cylindrical tank, and the thick black line the observations for the $n=11$ mode. This line crosses several modes on the graph, a reflection of both the closeness of the modes in this part of the solution space and the fact that, as suggested by observations, the mode was not a purely $n=11$ mode.

to a three-quarter-wavelength resonance for which by (3.15) the boundary condition, $u_L(r_0)=0$, yields $\gamma r_0 = 6.653$ (when evaluated exactly, or 6.706 when applying the large omega approximation valid here) and $\omega/n \approx 10.82$ rad s^{-1}. The measured value of ω/n is about 10 % lower, which we attribute to the effect of viscosity slowing the motion. The high speed of the wave indicated that it was rotating close to the speed of the air above, suggesting that this wave was being maintained by the direct pressure resonance described by Phillips (1957).

At a range of other measured rotation rates in the lower fluid, between $\Omega_L = 1$ and 3.4 rad s^{-1}, a less impressive $n=11$ mode was observed, which was also a three-quarter-wavelength radial response, with $\gamma r_0 \approx 17.5$ (17.6 when applying the large ω approximation), rotating at an angular speed which increased with Ω_L from $\omega/n = 2.8$ to 6.5 rad s^{-1}. This precludes pressure resonance, and the mode was probably driven by nonlinear transfer from within the wave spectrum, which is also the main growth mechanism in wind–wave generation in the ocean. A photograph of this mode is shown in figure 3; note that it is not possible to definitively ascertain the mode.

In the experiments, the reflection of the wave at the rim of the cylinder (as a Bessel function of the second kind or a Hankel function) gave rise to leading spiral waveforms which for the $n=2$ mode were strikingly beautiful to behold (Bye *et al.* 2005). The vertical scales were $k_L = 28$ m^{-1} for the $n=2$ mode and $k_L = 45$ m^{-1} for the $n=11$ mode, indicating that the deep water approximation was well satisfied in the experimental rig.

4.3. *The limit of large n*

Another response was observed in the cylindrical rig with a smaller air gap of 0.035 m, in which the rather larger number of 50 wavelengths could be counted around the rim.

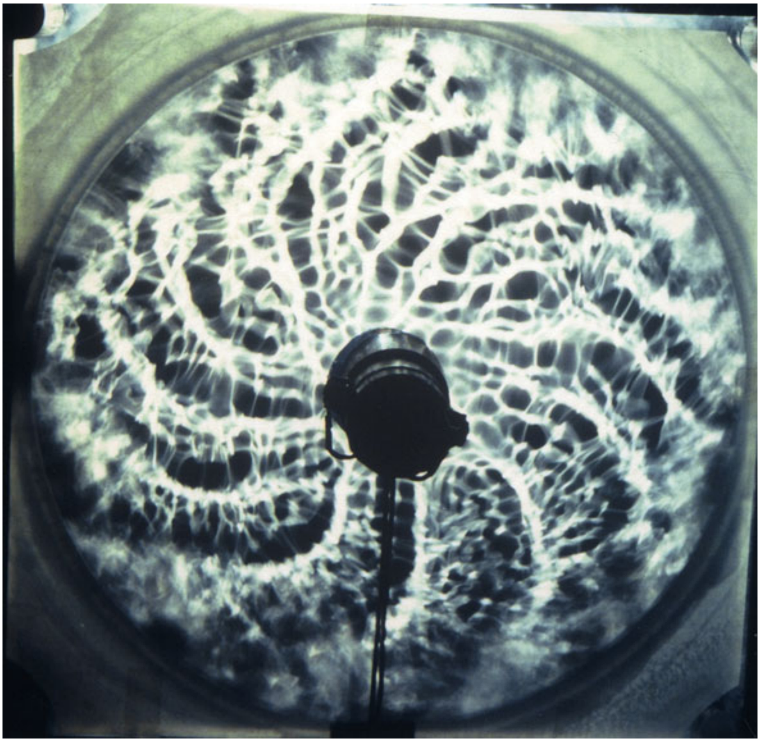

FIGURE 3. (Colour online) An image of the $n = 11$ mode in the cylindrical tank. The image was obtained by an overhead camera, with a light source situated centrally under the tank directed upwards onto opaque paper on the tank's base. The central boss is the electric motor which drove the rotating disc (Vladusic 2001). The increase in angle $\Delta\theta$ in the leading spiral over the radius is about a right angle, which is similar to the theoretical estimate of $\Delta\theta = \gamma r_0/n$ for $\gamma r \gg 1$ which yields 88° (Abramowitz & Stegun 1964).

Such a system may be approximated by taking the limit as $n \to \infty$ in the dispersion relation (3.32), which leads to

$$\gamma g(1-\chi) + \sigma\gamma^3 = +(\alpha_L^2 \pm \chi\alpha_U^2) \tag{4.11}$$

where σ is the specific surface tension with respect to the lower fluid. By comparing this with the limiting forms of (3.28) and (3.29) we see that in the limit of large n, $\gamma = k_L = k_U$ (indeed, this can be seen directly by taking the limit in (3.13)). The solutions to (4.11) are, taking the sign to be positive,

$$\omega = \frac{n(\Omega_L + \chi\Omega_U)}{1+\chi} \pm \sqrt{\frac{g\gamma(1-\chi)}{1+\chi} + \frac{\sigma\gamma^3}{1+\chi} - \frac{n^2\chi(\Omega_L - \Omega_U)^2}{(1+\chi)^2}}. \tag{4.12}$$

This yields complex solutions when the difference in rotation rates of the fluids is sufficiently high, according to the relation

$$(\Omega_L - \Omega_U)^2 > \frac{g\gamma(1-\chi^2) + \sigma\gamma^3(1+\chi)}{\chi n^2}, \tag{4.13}$$

which implies that there exist unstable waves which grow with time. These equations may be related to the regular, rectilinear Kelvin–Helmholtz relations with the following substitutions: $k=\gamma$, $n=kr_0$, where r_0 is the radius of the domain (the boundary of the cylindrical tank), $v_L(r_0)=r_0\Omega_L$, and $v_U(r_0)=r_0\Omega_U$ (see, for example, Thomson (1871) or Proudman (1953)). Bye & Ghantous (2012) showed that the dispersion relation (4.11) was satisfied by the experimental results, which appear to be the first detailed demonstration of Kelvin–Helmholtz instability for a large density contrast system.

4.4. *The limits of small contrast in density and rotation rate*

Equation (3.32) can be applied to a system where the difference in solid-body rotation rates between the fluids and the ratio of densities are small. We make the following substitutions:

$$\Omega_L=\Omega, \quad \Omega_U=\Omega-\epsilon, \quad \rho_L=\rho, \quad \rho_U=\rho-\mu, \quad \epsilon,\mu>0 \tag{4.14}$$

where, on assuming quasi-geostrophic, baroclinic conditions, ϵ/Ω and μ/ρ are both small. With these substitutions we replicate the experimental conditions of Bradford *et al.* (1981), where the upper fluid is only slightly less dense and the lower fluid is driven to rotate only slightly faster than the upper one. There are important differences, however, between their approach and ours here, due to viscosity. In their theoretical treatment they explore the limit of zero viscosity, but this is quite different to our inviscid-from-the-start approach.

Taking the positive sign in (3.32) and making the substitutions we get, after some rearranging,

$$\begin{aligned}&\gamma^3\sigma+\gamma g(1-\chi)\\&\quad=\alpha\sqrt{\alpha^2-4\Omega^2}+\alpha_U\chi\sqrt{(\alpha^2-4\Omega^2)\left(1-\frac{2\epsilon(\Omega n^2-\omega n-4\Omega)}{\alpha^2-4\Omega^2}\right)}\end{aligned} \tag{4.15}$$

where $\alpha=\alpha_L=\Omega n-\omega$. Assuming small ϵ we may take a binomial expansion and the right-hand side becomes

$$\sqrt{\alpha^2-4\Omega^2}\left[\alpha+\alpha_U\chi\left(1-\frac{\epsilon(\Omega n^2-\omega n-4\Omega)}{\alpha^2-4\Omega^2}\right)\right]. \tag{4.16}$$

Evaluating α_U and eliminating terms of order ϵ^2 gives

$$\gamma^3\sigma+\gamma g(1-\chi)=\alpha\sqrt{\alpha^2-4\Omega^2}\left[1+\chi-\chi\epsilon\left(\frac{\Omega n^2-\omega n-4\Omega}{\alpha^2-4\Omega^2}+\frac{n}{\alpha}\right)\right]. \tag{4.17}$$

We define the non-dimensional relative azimuthal wave frequency,

$$W=\frac{\dfrac{\omega}{n}-\Omega}{\Omega} \tag{4.18}$$

and for convenience we make the transformation

$$W\rightarrow-\frac{2}{n}-\phi, \quad \phi\ll 1. \tag{4.19}$$

Substituting into (4.17) and eliminating the term of order $\phi\epsilon$,

$$\phi-\left[\gamma^3\sigma+\gamma g\frac{\mu}{\rho}\right]\frac{\sqrt{\phi}}{4\Omega^2(2-\mu/\rho)}-\frac{(1+\mu/\rho)\,\epsilon(n-2)}{2\Omega\,(2-\mu/\rho)}=0 \tag{4.20}$$

where we have also substituted for $\chi = 1 - \mu/\rho$. If we now take the limit where $\mu/\rho \to 0$, ignoring the terms of order $\mu\epsilon$ and substituting for $\gamma = x/r_0$ where r_0 is the cylinder radius leads us to

$$\phi - a\sqrt{\phi} - b = 0 \tag{4.21}$$

where

$$a = \frac{x(1 + x^2 I)}{4F\,(2 - \mu/\rho)} \frac{\mu}{\rho} \tag{4.22}$$

and

$$b = \frac{A(n-2)}{2\,(2 - \mu/\rho)} \frac{\mu}{\rho}. \tag{4.23}$$

Here

$$A = \frac{\epsilon\rho}{\Omega\mu} \tag{4.24}$$

is a modified Rossby number,

$$F = \frac{r_0 \Omega^2}{g} \tag{4.25}$$

is a Froude number and

$$I = \frac{\sigma\rho}{g r_0^2 \mu} \tag{4.26}$$

is the interfacial tension number. Solving (4.21) we find that

$$\sqrt{\phi} = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b} \tag{4.27}$$

which indicates that for $n \geqslant 2$ (i.e. b always positive) $\sqrt{\phi}$ is real and hence the mode is stable, whereas for $n < 2$, $\sqrt{\phi}$ may be complex, indicating an imaginary component to the frequency and an instability. This does not correspond to Bradford *et al.*'s results, either numerical or experimental, which show that the $n = 2$ mode may also be unstable. Neutral stability occurs for $a^2 = -4b$, at which

$$AF^2 = \frac{x^2 (1 + x^2 I)^2 (\mu/\rho)}{32\,(2 - \mu/\rho)\,(2 - n)}. \tag{4.28}$$

When $n = 2$, AF^2 is undefined, and we may conclude that the analysis is invalid for such a mode; for $n > 2$ we would require that $AF^2 < 0$, which contradicts our assumptions. For $\phi \ll 0$, on using (4.19) (leading to $\omega/\Omega \sim -1$) and the boundary condition (3.15) (applicable since $\Omega_L \approx \Omega_U$), we obtain

$$x = -\frac{J_1(x)}{J_1'(x)}, \tag{4.29}$$

the solution of which is $x \approx 2.40$ for the first zero, corresponding to the first radial mode observed by Bradford *et al.* (1981). Equation (4.28) may then be evaluated, and we find that $AF^2 \approx 0.0056$, much less than the order ~ 1 solutions found by Bradford *et al.* A plot of the curve (4.28) is shown in figure 4.

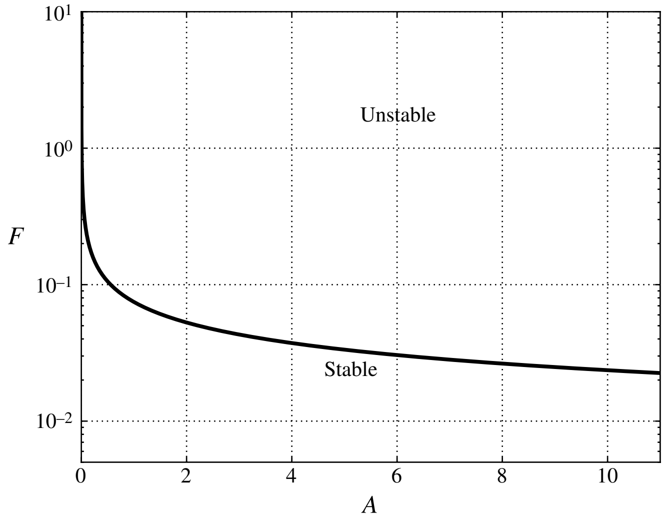

FIGURE 4. The neutral stability relation obtained from (4.28) in which $AF^2 \approx 0.0056$. Here A is the modified Rossby number and F is the Froude number.

5. Conclusion

We have presented what is essentially a normal mode analysis for the interfacial instability of the inviscid coupled differential and counter-rotation of two fluids, for which the dispersion relation was found to be (3.32). We have explored the solution space analytically where possible, and attempted to compare our results with those of another study by Bradford *et al.* (1981).

The properties of the dispersion relation are diverse, as illustrated in § 4, and may have application outside of the cylindrical rig in which the experimental observations which inspired the study were made. In particular, the rotating waves have crests along which the amplitude varies radially, in distinction to rectilinear capillary–gravity waves in which there is no cross-wave variation in crest amplitude. An open question is whether waves of this kind exist in the open ocean, where they may be generated by the vorticity of the wind field. Equation (4.10) is an important limit in this regard as it occurs when the angular velocity of the current (which we associate here with the rotation rate Ω_L) is small relative to the radial wavenumber (radial scaling) γ, which may be a relatively common feature in the wind–sea. This would give rise (see § 4.2) to a rotating wave in which the crest length is of similar magnitude to the wavelength in a rectilinear wave of the same period. While it remains unclear how the boundary conditions would be satisfied, we note that in the open ocean, and indeed in nature in general, our familiar mathematical idealizations are never strictly true. While it is improbable that a cylindrical system would be replicated in the ocean, it is not inconceivable that a situation that is at least in part qualitatively similar might eventuate.

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