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ON CONJUGACY OF SUBALGEBRAS IN GRAPH C^*-ALGEBRAS. II

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Abstract We apply a method inspired by Popa's intertwining-by-bimodules technique to investigate inner conjugacy of MASAs in graph C^*-algebras. First, we give a new proof of non-inner conjugacy of the diagonal MASA $\mathcal{D}_E$ to its non-trivial image under a quasi-free automorphism, where E is a finite transitive graph. Changing graphs representing the algebras, this result applies to some non quasi-free automorphisms as well. Then, we exhibit a large class of MASAs in the Cuntz algebra $\mathcal{O}_n$ that are not inner conjugate to the diagonal $\mathcal{D}_n$.

Keywords: Graph C^*-algebra; MASA; automorphism

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1. Introduction

This paper is devoted to investigations of conjugacy of MASAs in the C^*-algebras of finite directed graphs. The problem of conjugacy of MASAs in factor von Neumann algebras has been extensively investigated for many years, in particular with relation to Cartan subalgebras. Variety of different situations may occur. There exist factors with a unique Cartan subalgebra or with (uncountably) many ones, e.g. see [17, 21, 23].

This problem has received much less attention by researchers working with C^*-algebras. In particular, the literature on the conjugacy of subalgebras in simple purely infinite C^*-algebras is rather scarce. The present paper is the continuation of investigations of this problem initiated in [5, 14], where the question of inner conjugacy to the diagonal MASA of its images under quasi-free automorphisms was looked at in the Cuntz algebras and more generally graph C^*-algebras. The arguments from [5, 14] where based on rather ad hoc estimations, tailor made for the cases at hand. Now, we aim at developing a more general technique that may be applicable in many diverse instances. The idea is simple, see Lemma 2.3, and it is inspired by Popa's intertwining-by-bimodules technique, see

Theorem 2.2. We believe that this approach is conceptually sound and may be useful in many a different situation.

Our paper is organized as follows. Section 2 contains rather extensive preliminaries on graph C^*-algebras, traces on them, and their endomorphisms. In particular, a discussion of aspects of the classical Perron–Frobenius theory is included, in so far as it is relevant for our purpose. At the end of this section, we briefly state the key technical device we intend to use for distinguishing non-inner conjugate subalgebras. Section 3 contains a discussion of quasi-free automorphisms in relation to aspects of the Perron–Frobenius theory. In this section, we give a new, and hopefully conceptually more interesting, proof of non-inner conjugacy to the diagonal of its images under non-trivial quasi-free automorphisms, see Theorem 3.3. In Section 4, we show that by changing the graph representing the algebra in question our main result on quasi-free automorphism becomes applicable to some non quasi-free automorphisms as well. In Section 5, we exhibit a large class of MASAs of the Cuntz algebra $\mathcal{O}_n$ that are not inner conjugate to the diagonal MASA $\mathcal{D}_n$, thus generalizing the case resulting from quasi-free automorphisms. In the final Section 6, we collected proofs of a few technical lemmas needed in the preceding parts of the paper.

2. Preliminaries

2.1. Finite-directed graphs and their C^*-algebras

Let $E = (E^0, E^1, r, s)$ be a directed graph, where E^0 and E^1 are *finite* sets of vertices and edges, respectively, and $r, s : E^1 \to E^0$ are range and source maps, respectively. A *path* μ of length $|\mu| = k \geq 1$ is a sequence $\mu = (\mu_1, \ldots, \mu_k)$ of k edges μ_j such that $r(\mu_j) = s(\mu_{j+1})$ for $j = 1, \ldots, k-1$. We view the vertices as paths of length 0. The set of all paths of length k is denoted E^k, and E^* denotes the collection of all finite paths (including paths of length zero). The range and source maps naturally extend from edges E^1 to paths E^k. A *sink* is a vertex v which emits no edges, i.e. $s^{-1}(v) = \emptyset$. A *source* is a vertex w which receives no edges, i.e. $r^{-1}(w) = \emptyset$. By a *cycle* we mean a path μ of length $|\mu| \geq 1$ such that $s(\mu) = r(\mu)$. A cycle $\mu = (\mu_1, \ldots, \mu_k)$ has an exit if there is a j such that $s(\mu_j)$ emits at least two distinct edges. If α is an initial subpath of β then we write $\alpha \prec \beta$. Graph E is *transitive* if for any two vertices v, w there exists a path $\mu \in E^*$ from v to w of non-zero length. Thus, a transitive graph does not contain any sinks or sources. Given a graph E, we will denote by $A = [A(v, w)]_{v,w \in E^0}$ its *adjacency matrix*. That is, A is a matrix with rows and columns indexed by the vertices of E, such that $A(v, w)$ is the number of edges with source v and range w. If the graph E is transitive then the corresponding matrix A is *irreducible*, in the sense that for any two vertices v, w there is a positive integer k such that $A^k(v, w) > 0$. Here A^k is the k'th power of matrix A and hence $A^k(w, v)$ gives the number of paths from vertex w to vertex v.

The C^*-algebra $C^*(E)$ corresponding to a graph E is by definition, [19, 20], the universal C^*-algebra generated by mutually orthogonal projections P_v, $v \in E^0$, and partial isometries S_e, $e \in E^1$, subject to the following two relations:

(GA1) $S_e^* S_e = P_{r(e)}$,

(GA2) $P_v = \sum_{s(e)=v} S_e S_e^*$ if $v \in E^0$ emits at least one edge.

For a path $\mu = (\mu_1, \ldots, \mu_k)$, we denote by $S_\mu = S_{\mu_1} \cdots S_{\mu_k}$ the corresponding partial isometry in $C^*(E)$. We agree to write $S_v = P_v$ for a $v \in E^0$. Each S_μ is non-zero with the domain projection $P_{r(\mu)}$. Then $C^*(E)$ is the closed span of $\{S_\mu S_\nu^* : \mu, \nu \in E^*\}$. Note that $S_\mu S_\nu^*$ is non-zero if and only if $r(\mu) = r(\nu)$. In that case, $S_\mu S_\nu^*$ is a partial isometry with domain and range projections equal to $S_\nu S_\nu^*$ and $S_\mu S_\mu^*$, respectively.

The range projections $P_\mu = S_\mu S_\mu^*$ of all partial isometries S_μ mutually commute, and the abelian C^*-subalgebra of $C^*(E)$ generated by all of them is called the diagonal subalgebra and denoted $\mathcal{D}_E$. We set $\mathcal{D}_E^0 = \mathrm{span}\{P_v : v \in E^0\}$ and, more generally, $\mathcal{D}_E^k = \mathrm{span}\{P_\mu : \mu \in E^k\}$ for $k \geq 0$. C^*-algebra $\mathcal{D}_E$ coincides with the norm closure of $\bigcup_{k=0}^\infty \mathcal{D}_E^k$. If E does not contain sinks and all cycles have exits then $\mathcal{D}_E$ is a MASA (maximal abelian subalgebra) in $C^*(E)$ by [15, Theorem 5.2]. Throughout this paper, we make the following

standing assumption: all graphs we consider are finite, transitive and all cycles in these graphs admit exits.

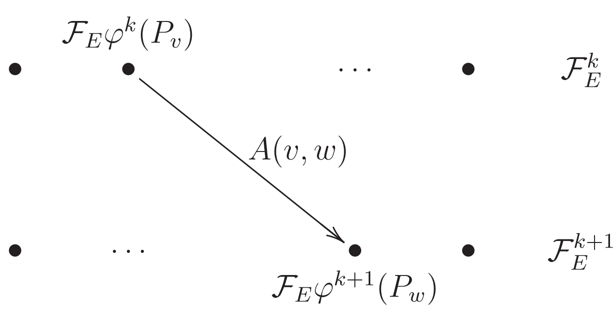
There exists a strongly continuous action γ of the circle group $U(1)$ on $C^*(E)$, called the *gauge action*, such that $\gamma_z(S_e) = zS_e$ and $\gamma_z(P_v) = P_v$ for all $e \in E^1$, $v \in E^0$ and $z \in U(1) \subseteq \mathbb{C}$. The fixed-point algebra $C^*(E)^\gamma$ for the gauge action is an AF-algebra, denoted $\mathcal{F}_E$ and called the core AF-subalgebra of $C^*(E)$. $\mathcal{F}_E$ is the closed span of $\{S_\mu S_\nu^* : \mu, \nu \in E^*, \ |\mu| = |\nu|\}$. For $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$ we denote by $\mathcal{F}_E^k$ the linear span of $\{S_\mu S_\nu^* : \mu, \nu \in E^*, \ |\mu| = |\nu| = k\}$. C^*-algebra $\mathcal{F}_E$ coincides with the norm closure of $\bigcup_{k=0}^\infty \mathcal{F}_E^k$.

We consider the usual *shift* on $C^*(E)$, [10], given by

$$\varphi(x) = \sum_{e \in E^1} S_e x S_e^*, \quad x \in C^*(E). \tag{1}$$

In general, for finite graphs without sinks and sources, the shift is a unital, completely positive map. However, it is an injective $*$-homomorphism when restricted to the relative commutant $(\mathcal{D}_E^0)' \cap C^*(E)$ of $\mathcal{D}_E^0$ in $C^*(E)$.

We observe that for each $v \in E^0$ projection $\varphi^k(P_v)$ is minimal in the centre of $\mathcal{F}_E^k$. The C^*-algebra $\mathcal{F}_E^k \varphi^k(P_v)$ is the linear span of partial isometries $S_\mu S_\nu^*$ with $|\mu| = |\nu| = k$ and $r(\mu) = r(\nu) = v$. It is isomorphic to the full matrix algebra of size $\sum_{w \in E^0} A^k(w, v)$. The multiplicity of $\mathcal{F}_E^k \varphi^k(P_v)$ in $\mathcal{F}_E^{k+1} \varphi^{k+1}(P_w)$ is $A(v, w)$, so the Bratteli diagram for $\mathcal{F}_E$ is induced from the graph E, see [3, 10, 20].



We denote

$$\mathfrak{B} := (\mathcal{D}_E^0)' \cap \mathcal{F}_E^1. \tag{2}$$

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That is, $\mathfrak{B}$ is the linear span of elements $S_e S_f^*$, $e,\ f \in E^1$, with $s(e) = s(f)$. We note that $\mathfrak{B}$ is contained in the multiplicative domain of φ. We have $\mathcal{D}_E^1 \subseteq \mathfrak{B} \subseteq \mathcal{F}_E^1$ and

$$\varphi^k(\mathfrak{B}) = (\mathcal{F}_E^k)' \cap \mathcal{F}_E^{k+1} \cong \bigoplus_{v,w\in E^0} M_{A(v,w)}(\mathbb{C}) \tag{3}$$

for all k. For $v,\ w \in E^0$, we denote

$$_vQ_w := \sum_{e\in E^1, s(e)=v, r(e)=w} P_e. \tag{4}$$

Each $_vQ_w$ is a minimal projection in the centre of $\mathfrak{B}$ and $\mathfrak{B}_vQ_w \cong M_{A(v,w)}(\mathbb{C})$. We put

$$\mathfrak{B}_E^k := \bigvee_{j=0}^{k-1} \varphi^j(\mathfrak{B}), \tag{5}$$

for $k \geq 1$, the C^*-algebra generated by $\bigcup_{j=0}^{k-1} \varphi^j(\mathfrak{B})$. In general, if A and B are both C^*-subalgebras of a C^*-algebra C, then we denote by $A \vee B$ the C^*-subalgebra of C generated by A and B. Since for all k, we have

$$\mathcal{D}_E^k = \bigvee_{j=0}^{k-1} \varphi^j(\mathcal{D}_E^1), \tag{6}$$

it is easy to see that

$$\mathcal{D}_E^k \subseteq \mathfrak{B}_E^k \subseteq \mathcal{F}_E^k. \tag{7}$$

We observe that

$$_vQ_w\varphi(_{v'}Q_{w'}) = \delta_{w,v'} \sum_{s(e)=v, r(e)=s(f)=w, r(f)=w'} P_{ef}. \tag{8}$$

This implies that

$$\begin{aligned} \mathfrak{B}_E^k &= \bigoplus_{v_1,\dots,v_{k+1}\in E^0} \mathfrak{B}_{v_1}Q_{v_2} \vee \varphi(\mathfrak{B}_{v_2}Q_{v_3}) \vee \ldots \vee \varphi^{k-1}(\mathfrak{B}_{v_k}Q_{v_{k+1}}) \\ &= \bigoplus_{v_1,\dots,v_{k+1}\in E^0} \mathfrak{B}_{v_1}Q_{v_2} \otimes \varphi(\mathfrak{B}_{v_2}Q_{v_3}) \otimes \ldots \otimes \varphi^{k-1}(\mathfrak{B}_{v_k}Q_{v_{k+1}}). \end{aligned}$$

There exist faithful conditional expectations $\Phi_{\mathcal{F}} : C^*(E) \to \mathcal{F}_E$ and $\Phi_{\mathcal{D}} : C^*(E) \to \mathcal{D}_E$ such that $\Phi_{\mathcal{F}}(S_\mu S_\nu^*) = 0$ for $|\mu| \neq |\nu|$ and $\Phi_{\mathcal{D}}(S_\mu S_\nu^*) = 0$ for $\mu \neq \nu$. We note that $\Phi_{\mathcal{D}} =$

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$\Phi_{\mathcal{D}} \circ \Phi_{\mathcal{F}}$ and

$$\Phi_{\mathcal{D}} \circ \varphi = \varphi \circ \Phi_{\mathcal{D}} \text{ on } \mathcal{D}_E,$$
$$\Phi_{\mathcal{F}} \circ \varphi = \varphi \circ \Phi_{\mathcal{F}} \text{ on } \mathcal{F}_E.$$

For an integer $m \in \mathbb{Z}$, we denote by $C^*(E)^{(m)}$ the spectral subspace of the gauge action corresponding to m. That is,

$$C^*(E)^{(m)} := \{x \in C^*(E) \mid \gamma_z(x) = z^m x, \forall z \in U(1)\}. \tag{9}$$

In particular, $C^*(E)^{(0)} = C^*(E)^\gamma$. For each $m \in \mathbb{Z}$ there is a unital, contractive and completely bounded map $\Phi^m : C^*(E) \to C^*(E)^{(m)}$ given by

$$\Phi^m(x) = \int_{z \in U(1)} z^{-m} \gamma_z(x) \mathrm{d}x. \tag{10}$$

In particular, $\Phi^0 = \Phi_{\mathcal{F}}$. We have $\Phi^m(x) = x$ for all $x \in C^*(E)^{(m)}$. If $x \in C^*(E)$ and $\Phi^m(x) = 0$ for all $m \in \mathbb{Z}$ then $x = 0$.

2.2. The trace on the core AF-subalgebra

We recall the definition of a natural trace on the core AF-subalgebra $\mathcal{F}_E$. For relevant facts from the Perron–Frobenius theory, see for example [12, 13].

Let β be the Perron–Frobenius eigenvalue of the matrix A and let $(x(v))_{v \in E^0}$ be the corresponding Perron–Frobenius eigenvector. That is, $\beta > 0$, for each $v \in E^0$ we have $x(v) > 0$, and

$$\sum_{w \in E^0} A(v, w) x(w) = \beta x(v). \tag{11}$$

We set $X := \sum_{v \in E^0} x(v)$ and define a tracial state τ on $\mathcal{F}_E$ so that

$$\tau(S_\mu S_\nu^*) = \delta_{\mu,\nu} \frac{x(r(\mu))}{X \beta^k} \tag{12}$$

for μ, $\nu \in E^k$. We have $\tau(\Phi_{\mathcal{D}}(x)) = \tau(x)$ for all $x \in \mathcal{F}_E$.

Remark 2.1. Trace τ defined above is not shift invariant, in general. That is, it may happen that $\tau(\varphi(x)) \neq \tau(x)$ for some $x \in \mathcal{F}_E$. In fact, τ is φ-invariant if and only if

$$\sum_{v \in E^0} A(v, w) = \beta$$

for each $w \in E^0$. For example, the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$$

does not satisfy this condition.

2.3. Endomorphisms determined by unitaries

Cuntz's classical approach to the study of endomorphisms of $\mathcal{O}_n$, [9], has been developed further in [7] and extended to graph C^*-algebras in [1, 4, 18].

We denote by $\mathcal{U}_E$ the collection of all those unitaries in $C^*(E)$ which commute with all vertex projections P_v, $v \in E^0$. That is

$$\mathcal{U}_E := \mathcal{U}((\mathcal{D}_E^0)' \cap C^*(E)). \tag{13}$$

If $u \in \mathcal{U}_E$ then uS_e, $e \in E^1$, are partial isometries in $C^*(E)$ which together with projections P_v, $v \in E^0$, satisfy (GA1) and (GA2). Thus, by the universality of $C^*(E)$, there exists a unital $*$-homomorphism $\lambda_u : C^*(E) \to C^*(E)$ such that*

$$\lambda_u(S_e) = uS_e \text{ and } \lambda_u(P_v) = P_v, \text{ for } e \in E^1, \quad v \in E^0. \tag{14}$$

The mapping $u \mapsto \lambda_u$ establishes a bijective correspondence between $\mathcal{U}_E$ and the semigroup of those unital endomorphisms of $C^*(E)$ which fix all P_v, $v \in E^0$. As observed in [4, Proposition 2.1], if $u \in \mathcal{U}_E \cap \mathcal{F}_E$ then λ_u is automatically injective. We say λ_u is *invertible* if λ_u is an automorphism of $C^*(E)$. If u belongs to $\mathcal{U}_E \cap \mathcal{F}_E^k$ for some k, then the corresponding endomorphism λ_u is called *localized*, [4, 6].

If $u \in \mathcal{U}(\mathfrak{B})$ then λ_u is automatically invertible with inverse λ_{u^*} and the map

$$\mathcal{U}(\mathfrak{B}) \ni u \mapsto \lambda_u \in \mathrm{Aut}(C^*(E)) \tag{15}$$

is a group homomorphism with a range inside the subgroup of *quasi-free automorphisms* of $C^*(E)$, see [24]. Note that this group is almost never trivial and it is non-commutative if graph E contains two edges e, $f \in E^1$ such that $s(e) = s(f)$ and $r(e) = r(f)$.

The shift φ globally preserves $\mathcal{U}_E$, $\mathcal{F}_E$ and $\mathcal{D}_E$. For $k \geq 1$, we denote

$$u_k := u\varphi(u)\cdots\varphi^{k-1}(u). \tag{16}$$

For each $u \in \mathcal{U}_E$ and all $e \in E^1$, we have $S_e u = \varphi(u)S_e$, and thus

$$\lambda_u(S_\mu S_\nu^*) = u_{|\mu|} S_\mu S_\nu^* u_{|\nu|}^* \tag{17}$$

for any two paths μ, $\nu \in E^*$.

2.4. The Popa criterion

In the analysis of uniqueness of Cartan subalgebras of tracial von Neumann algebras, Popa's intertwining-by-bimodules technique has been extremely successful. This method goes back to [22], but has been polished over the years and recently even extended to type III case [16]. The following result contains its essential ingredient.

Theorem 2.2 (S. Popa). *Let M be a von Neumann algebra equipped with a faithful normal trace τ. Let A, B be von Neumann subalgebras of M, and let $\Phi_B : M \to B$ be a τ-preserving conditional expectation. Then the following two conditions are equivalent.*

* The reader should be aware that in some papers (e.g. in [9]), a different convention is used, namely $\lambda_u(S_e) = u^* S_e$.

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(1) *There exist non-zero projections* $p \in A$, $q \in B$, *a non-zero partial isometry* $v \in pMq$ *and a* $*$*-homomorphism* $\Phi : pAp \to qBq$ *such that* $xv = v\Phi(x)$ *for all* $x \in pAp$.

(2) *There is no sequence of unitaries* $w_n \in \mathcal{U}(A)$ *such that*

$$||\Phi_B(xw_ny)||_2 \underset{n\to\infty}{\longrightarrow} 0, \quad \forall x, y \in M. \tag{18}$$

This beautiful theorem is inapplicable to graph C^*-algebras, of course. However, the following simple fact remains valid in the C^*-algebraic setting.

Lemma 2.3. *Let* M *be a unital* C^**-algebra, and let* A, B *be its* C^**-subalgebras containing the unit of* M. *Let* $\Phi_B : M \to B$ *be a conditional expectation, and let* τ *be a trace on* B. *If there is a sequence of unitaries* $w_n \in \mathcal{U}(A)$ *such that* (18) *holds then there is no unitary* $v \in \mathcal{U}(M)$ *such that* $vAv^* \subseteq B$.

Proof. Indeed, let $w_n \in \mathcal{U}(A)$ be as in the lemma and suppose $v \in \mathcal{U}(M)$ is such that $vAv^* \subseteq B$. Then

$$1 = ||vw_nv^*||_2 = ||\Phi_B(vw_nv^*)||_2 \underset{n\to\infty}{\longrightarrow} 0,$$

which gives a contradiction. □

3. Quasi-free automorphisms

In this section, we apply Lemma 2.3 with $M = C^*(E)$, τ the canonical trace on $\mathcal{F}_E$, $B = \mathcal{D}_E$, and $\Phi_B = \Phi_{\mathcal{D}}$. We keep the standing assumptions on the graph E. Note that for unitaries $u \in \mathfrak{B}$ and $d \in \mathcal{D}_E^1$, we have

$$\lambda_u(d\varphi(d)\cdots\varphi^{k-1}(d)) = \mathrm{udu}^*\varphi(\mathrm{udu}^*)\cdots\varphi^{k-1}(\mathrm{udu}^*).$$

Lemma 3.1. *Let* $u \in \mathfrak{B}$ *be a unitary such that* $u\mathcal{D}_E^1u^* \neq \mathcal{D}_E^1$, *and let* $d \in \mathcal{D}_E^1$ *be a unitary such that* $\mathrm{udu}^* \notin \mathcal{D}_E^1$. *Then we have*

$$\lim_{k\to\infty} ||\Phi_{\mathcal{D}}(\mathrm{udu}^*\varphi(\mathrm{udu}^*)\cdots\varphi^{k-1}(\mathrm{udu}^*))||_2 = 0.$$

Proof. We set $d_{v,w} := d \cdot_v Q_w$. Since $\mathfrak{B} \cdot_v Q_w$ is a full matrix algebra, it has a unique tracial state $\tau_{v,w}$. We denote by $||\cdot||_{2,v,w}$ the 2-norm induced by this trace. In view of Corollary 6.2, we have

$$\begin{aligned}
&\Phi_{\mathcal{D}}(\mathrm{udu}^*\varphi(\mathrm{udu}^*)\cdots\varphi^{k-1}(\mathrm{udu}^*))\\
&\quad= \sum_{v_1,v_2,\ldots,v_{k+1}\in E^0} \Phi_{\mathcal{D}}(\mathrm{udu}^*_{v_1}Q_{v_2}\varphi(\mathrm{udu}^*_{v_2}Q_{v_3})\cdots\varphi^{k-1}(\mathrm{udu}^*_{v_k}Q_{v_{k+1}}))\\
&\quad= \sum_{v_1,v_2,\ldots,v_{k+1}\in E^0} \Phi_{\mathcal{D}}(ud_{v_1,v_2}u^*)\varphi(\Phi_{\mathcal{D}}(ud_{v_2,v_3}u^*))\cdots\varphi^{k-1}(\Phi_{\mathcal{D}}(ud_{v_k,v_{k+1}}u^*))
\end{aligned}$$

We define non-negative numbers $\{\lambda_{v_1,v_2,\ldots,v_{k+1}}\}_{v_1,v_2,\ldots,v_{k+1}\in E^0}$ by

$$\begin{aligned}\lambda_{v_1,v_2,\ldots,v_{k+1}} &= \tau({}_{v_1}Q_{v_2}\varphi({}_{v_2}Q_{v_3})\cdots\varphi^{k-1}({}_{v_k}Q_{v_{k+1}}))\\ &= A(v_1,v_2)A(v_2,v_3)\cdots A(v_k,v_{k+1})\frac{x(v_{k+1})}{X\beta^k}.\end{aligned}$$

We remark that $A(v_1,\, v_2)A(v_2,\, v_3)\cdots A(v_k,\, v_{k+1})$ is the total number of paths of length k which pass through $v_1,\, v_2,\, \ldots,\, v_{k+1}$ in this order. Since ${}_{v_1}Q_{v_2}\varphi({}_{v_2}Q_{v_3})\cdots\varphi^{k-1}({}_{v_k}Q_{v_{k+1}})$ is a central minimal projection of $\mathfrak{B}_E^k$, for any $x\in\mathfrak{B}_E^k$, we have

$$\tau(x)=\sum_{v_1,v_2,\ldots,v_{k+1}\in E^0}\lambda_{v_1,v_2,\ldots,v_{k+1}}\tau_{v_1,v_2,\ldots,v_{k+1}}(x\{{}_{v_1}Q_{v_2}\varphi({}_{v_2}Q_{v_3})\cdots\varphi^{k-1}({}_{v_k}Q_{v_{k+1}})\})$$

where $\tau_{v_1,v_2,\ldots,v_{k+1}}$ is a unique tracial state on a full matrix algebra

$$\mathfrak{B}_E^k\{{}_{v_1}Q_{v_2}\varphi({}_{v_2}Q_{v_3})\cdots\varphi^{k-1}({}_{v_k}Q_{v_{k+1}})\}.$$

Then since

$$\begin{aligned}&\tau_{v_1,v_2,\ldots,v_{k+1}}(a_1\varphi(a_2)\cdots\varphi^{k-1}(a_k)\{{}_{v_1}Q_{v_2}\varphi({}_{v_2}Q_{v_3})\cdots\varphi^{k-1}({}_{v_k}Q_{v_{k+1}})\})\\ &\quad=\tau_{v_1,v_2}(a_1\cdot_{v_1}Q_{v_2})\tau_{v_2,v_3}(a_2\cdot_{v_2}Q_{v_3})\cdots\tau_{v_k,v_{k+1}}(a_k\cdot_{v_k}Q_{v_{k+1}})\end{aligned}$$

for all $a_1,\, a_2,\, \ldots,\, a_k\in\mathfrak{B}$, we have

$$\begin{aligned}&||a_1\varphi(a_2)\cdots\varphi^{k-1}(a_k)||_2^2\\ &\quad=\sum_{v_1,v_2,\ldots,v_{k+1}\in E^0}\lambda_{v_1,v_2,\ldots,v_{k+1}}\quad||a_1\cdot_{v_1}Q_{v_2}||_{2,v_1,v_2}^2||a_2\cdot_{v_2}Q_{v_3}||_{2,v_2,v_3}^2\cdots||a_k\\ &\qquad\cdot_{v_k}Q_{v_{k+1}}||_{2,v_k,v_{k+1}}^2.\end{aligned}$$

Thus, we see that

$$\begin{aligned}&||\Phi_{\mathcal{D}}(\mathrm{u}d\mathrm{u}^*\varphi(\mathrm{u}d\mathrm{u}^*)\cdots\varphi^{k-1}(\mathrm{u}d\mathrm{u}^*))||_2^2=||\Phi_{\mathcal{D}}(\mathrm{u}d\mathrm{u}^*)\varphi(\Phi_{\mathcal{D}}(\mathrm{u}d\mathrm{u}^*))\cdots\varphi^{k-1}(\Phi_{\mathcal{D}}(\mathrm{u}d\mathrm{u}^*))||_2^2\\ &\quad=\sum_{v_1,v_2,\ldots,v_{k+1}\in E^0}\lambda_{v_1,v_2,\ldots,v_{k+1}}||\Phi_{\mathcal{D}}(ud_{v_1,v_2}u^*)||_{2,v_1,v_2}^2||\varphi(\Phi_{\mathcal{D}}(ud_{v_2,v_3}u^*))||_{2,v_2,v_3}^2\cdots\\ &\qquad\cdots||\varphi^{k-1}(\Phi_{\mathcal{D}}(ud_{v_k,v_{k+1}}u^*))||_{2,v_k,v_{k+1}}^2.\end{aligned}$$

By the hypothesis of the lemma, there exist two vertices $w_1,\, w_2$ such that

$$\begin{aligned}0&<||\mathrm{u}d\mathrm{u}^*\cdot_{w_1}Q_{w_2}-\Phi_{\mathcal{D}}(\mathrm{u}d\mathrm{u}^*\cdot_{w_1}Q_{w_2})||_{2,w_1,w_2}^2\\ &=||\mathrm{u}d\mathrm{u}^*\cdot_{w_1}Q_{w_2}||_{2,w_1,w_2}^2+||\Phi_{\mathcal{D}}(\mathrm{u}d\mathrm{u}^*\cdot_{w_1}Q_{w_2})||_{2,w_1,w_2}^2\\ &\quad-2\mathrm{Re}\tau_{w_1,w_2}(\{\mathrm{u}d\mathrm{u}^*\cdot_{w_1}Q_{w_2}\}^*\Phi_{\mathcal{D}}(\mathrm{u}d\mathrm{u}^*\cdot_{w_1}Q_{w_2}))\\ &=1+||\Phi_{\mathcal{D}}(\mathrm{u}d\mathrm{u}^*\cdot_{w_1}Q_{w_2})||_{2,w_1,w_2}^2-2\mathrm{Re}\tau_{w_1,w_2}(\Phi_{\mathcal{D}}(\mathrm{u}d\mathrm{u}^*\cdot_{w_1}Q_{w_2})^*\Phi_{\mathcal{D}}(\mathrm{u}d\mathrm{u}^*\cdot_{w_1}Q_{w_2}))\\ &=1-||\Phi_{\mathcal{D}}(\mathrm{u}d\mathrm{u}^*\cdot_{w_1}Q_{w_2})||_{2,w_1,w_2}^2\end{aligned}$$

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and hence

$$c := ||\Phi_{\mathcal{D}}(udu^* \cdot_{w_1} Q_{w_2})||^2_{2,w_1,w_2} < 1. \tag{19}$$

For $i = 0, 1, \ldots, k$, we denote by $M^i_{k,v}$ the set of all paths μ such that

(i) $|\mu| = k$,

(ii) $r(\mu) = v$,

(iii) in path μ, edges from w_1 to w_2 occur exactly i times.

We remark that $M^i_{k,v} \cap M^j_{k,v} = \emptyset$ if $i \neq j$. Thus, we have $\sum_{i=0}^k |M^i_{k,v}| = \sum_{w\in E^0} A^k(w, v)$, where $|M^i_{k,v}|$ denotes the cardinality of $M^i_{k,v}$. We claim that for all v and i

$$\lim_{k\to\infty} \frac{|M^i_{k,v}|}{\beta^k} = 0. \tag{20}$$

At first, we note that because of (19) the full matrix algebra $\mathfrak{B} \cdot_{w_1} Q_{w_2}$ is not isomorphic to $\mathbb{C}$, and hence $A(w_1, w_2) \geq 2$. Let A_1 be the matrix defined in (26) in § 6 for $(i_1, j_1) = (w_1, w_2)$, and let E_1 be the corresponding graph. E_1 may be viewed as a subgraph of E obtained by removing all but one edge in E^1 that begin at w_1 and end at w_2. Set $N^i_{k,v} := M^i_{k,v} \cap E^*_1$. It is easy to see that

$$|M^i_{k,v}| = |N^i_{k,v}| \cdot A(w_1, w_2)^i.$$

But now, by virtue of Theorem 6.6, we have

$$\frac{|M^i_{k,v}|}{\beta^k} = A(w_1, w_2)^i \cdot \frac{|N^i_{k,v}|}{\beta^k} \leq A(w_1, w_2)^i \cdot \frac{\sum_w A^k_1(v, w)}{\beta^k} \underset{k\to\infty}{\longrightarrow} 0,$$

and the claim holds.

Now, since $||\varphi^{j-1}(\Phi_{\mathcal{D}}(ud_{v_j,v_{j+1}}u^*))||^2_{2,v_j,v_{j+1}} \leq 1$ and $c = ||\Phi_{\mathcal{D}}(udu^* \cdot_{w_1} Q_{w_2})||^2_{2,w_1,w_2}$, for each i_0, we have

$$||\Phi_{\mathcal{D}}(udu^*\varphi(udu^*)\cdots\varphi^{k-1}(udu^*))||^2_2 \leq \sum_{v\in E^0} \frac{x(v)}{X\beta^k} \sum_{i=0}^k |M^i_{k,v}| c^i$$

$$= \sum_{v\in E^0} \frac{x(v)}{X\beta^k} \sum_{i=0}^{i_0} |M^i_{k,v}| c^i + \sum_{v\in E^0} \frac{x(v)}{X\beta^k} \sum_{i=i_0+1}^{k} |M^i_{k,v}| c^i,$$

and hence

$$\limsup_{k\to\infty} ||\Phi_{\mathcal{D}}(udu^*\varphi(udu^*)\cdots\varphi^{k-1}(udu^*))||^2_2 = \limsup_{k\to\infty} \sum_{v\in E^0} \frac{x(v)}{X\beta^k} \sum_{i=i_0+1}^{k} |M^i_{k,v}| c^i.$$

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Since

$$\sum_{v\in E^0}\frac{x(v)}{X\beta^k}\sum_{i=i_0+1}^{k}|M^i_{k,v}|c^i = c^{i_0}\sum_{v\in E^0}\frac{x(v)}{X\beta^k}\sum_{i=i_0+1}^{k}|M^i_{k,v}|c^{i-i_0}$$
$$\le c^{i_0}\sum_{v\in E^0}\frac{x(v)}{X\beta^k}\sum_{i=i_0+1}^{k}|M^i_{k,v}| \le c^{i_0}\sum_{v\in E^0}\frac{x(v)}{X\beta^k}\sum_{w\in E^0}A^k(w,v)$$
$$= c^{i_0}\frac{1}{X\beta^k}\sum_{w\in E^0}\sum_{v\in E^0}A^k(w,v)x(v) = c^{i_0},$$

we may conclude that

$$\limsup_{k\to\infty}||\Phi_{\mathcal{D}}(\mathrm{udu}^*\varphi(\mathrm{udu}^*)\cdots\varphi^{k-1}(\mathrm{udu}^*))||_2^2 \le c^{i_0}.$$

Since i_0 was arbitrary, the lemma is proved. □

Keeping the hypothesis of Lemma 3.1, we have the following.

Lemma 3.2. *For all x, $y\in\mathcal{F}_E$ we have*

$$\lim_{k\to\infty}||\Phi_{\mathcal{D}}(x\cdot \mathrm{udu}^*\varphi(\mathrm{udu}^*)\cdots\varphi^{k-1}(\mathrm{udu}^*)\cdot y)||_2 = 0.$$

Proof. To prove the lemma, it suffices to consider elements x, $y\in\mathcal{F}_E^p$ for an arbitrary positive integer p. We have

$$x\cdot \mathrm{udu}^*\varphi(\mathrm{udu}^*)\cdots\varphi^{k-1}(\mathrm{udu}^*)\cdot y$$
$$= (x\cdot \mathrm{udu}^*\varphi(\mathrm{udu}^*)\cdots\varphi^{p-1}(\mathrm{udu}^*)\cdot y)\cdot\varphi^p(\mathrm{udu}^*\varphi^1(\mathrm{udu}^*)\cdots\varphi^{k-1}(\mathrm{udu}^*)).$$

Therefore, it is enough to show that

$$\lim_{k\to\infty}||\Phi_{\mathcal{D}}(x\cdot\varphi^p(\mathrm{udu}^*\varphi(\mathrm{udu}^*)\cdots\varphi^{k-1}(\mathrm{udu}^*)))||_2 = 0$$

for all $x\in\mathcal{F}_E^p$. However, we have

$$\Phi_{\mathcal{D}}(x\cdot\varphi^p(\mathrm{udu}^*\varphi(\mathrm{udu}^*)$$
$$\cdots\varphi^{k-1}(\mathrm{udu}^*))) = \Phi_{\mathcal{D}}(x)\cdot\varphi^p(\Phi_{\mathcal{D}}(\mathrm{udu}^*\varphi(\mathrm{udu}^*)\cdots\varphi^{k-1}(\mathrm{udu}^*)))$$

by Lemma 6.1 and

$$\lim_{k\to\infty}||\varphi^p(\Phi_{\mathcal{D}}(\mathrm{udu}^*\varphi(\mathrm{udu}^*)\cdots\varphi^{k-1}(\mathrm{udu}^*)))||_2 = 0$$

by Lemma 3.1 and Lemma 6.4. Thus, the claim follows. □

Now we are ready to prove the main result of this section. We keep the standard assumptions on the graph E.

Theorem 3.3. *Let $u \in \mathfrak{B}$ be a unitary such that $u\mathcal{D}_E^1 u^* \neq \mathcal{D}_E^1$, and let $d \in \mathcal{D}_E^1$ be a unitary such that $udu^* \notin \mathcal{D}_E^1$. Then for all $x, y \in C^*(E)$, we have*

$$\lim_{k\to\infty} ||\Phi_{\mathcal{D}}(x \cdot udu^*\varphi(udu^*)\cdots\varphi^{k-1}(udu^*)\cdot y)||_2 = 0.$$

Thus, in view of Lemma 2.3, $\mathcal{D}_E$ and $\lambda_u(\mathcal{D}_E)$ are not inner conjugate in $C^*(E)$.

Proof. By the polarization identity, it suffices to compute the above limit in the case $y = x^*$. Furthermore, we may assume that x belongs to the dense $*$-subalgebra of $C^*(E)$ generated by partial isometries corresponding to finite paths. That is, in the case, x is a finite sum of the form

$$x = \sum_{\mu\in E^*} a_\mu S_\mu^* + x_0 + \sum_{\nu\in E^*} S_\nu b_\nu,$$

with x_0, a_μ, $b_\nu \in \mathcal{F}_E$. Applying conditional expectation $\Phi_{\mathcal{F}}$ on the core AF-subalgebra first, we get

$$\begin{aligned}
&\Phi_{\mathcal{F}}(x\cdot udu^*\varphi(udu^*)\cdots\varphi^{k-1}(udu^*)\cdot x^*)\\
&\quad= \sum_{|\mu|=|\mu'|} a_\mu S_\mu^*\cdot udu^*\varphi(udu^*)\cdots\varphi^{k-1}(udu^*)\cdot S_{\mu'}a_{\mu'}^*\\
&\qquad + x_0\cdot udu^*\varphi(udu^*)\cdots\varphi^{k-1}(udu^*)\cdot x_0^*\\
&\qquad + \sum_{|\nu|=|\nu'|} S_\nu b_\nu\cdot udu^*\varphi(udu^*)\cdots\varphi^{k-1}(udu^*)\cdot b_{\nu'}^* S_{\nu'}^*.
\end{aligned}$$

Thus, we must show the following three cases:

$$\text{(1)}\quad \lim_{k\to\infty} ||\Phi_{\mathcal{D}}\left(\sum_{|\mu|=|\mu'|} a_\mu S_\mu^*\cdot udu^*\varphi(udu^*)\cdots\varphi^{k-1}(udu^*)\cdot S_{\mu'}a_{\mu'}^*\right)||_2 = 0,$$

$$\text{(2)}\quad \lim_{k\to\infty} ||\Phi_{\mathcal{D}}(x_0\cdot udu^*\varphi(udu^*)\cdots\varphi^{k-1}(udu^*)\cdot x_0^*)||_2 = 0,$$

$$\text{(3)}\quad \lim_{k\to\infty} ||\Phi_{\mathcal{D}}\left(\sum_{|\nu|=|\nu'|} S_\nu b_\nu\cdot udu^*\varphi(udu^*)\cdots\varphi^{k-1}(udu^*)\cdot b_{\nu'}^* S_{\nu'}^*\right)||_2 = 0.$$

Ad (1). Consider two paths μ and μ' with $|\mu| = |\mu'|$. For any $x \in \mathfrak{B}$ and for any $l-1 \geq |\mu| = |\mu'|$, we see that

$$\varphi^{l-1}(x)S_{\mu'} = \sum_{|\nu|=l-1} S_\nu x S_\nu^* S_{\mu'} = \sum_{|\nu'|=l-1-|\mu'|} S_{\mu'}S_{\nu'}xS_{\nu'}^*S_{\mu'}^*S_{\mu'} = S_{\mu'}\varphi^{l-1-|\mu|}(x).$$

On the other hand, for any $l-1 < |\mu| = |\mu'|$, since both $S_\mu S_\mu^*$ and $S_{\mu'}S_{\mu'}^*$ are minimal projections of $\mathfrak{B}_E^{|\mu|}$ and $x\varphi(x)\cdots\varphi^{l-1}(x) \in \mathfrak{B}_E^{|\mu|}$, we have

$$\begin{aligned}
S_\mu^* x\varphi(x)\cdots\varphi^{l-1}(x)S_{\mu'} &= S_\mu^*(S_\mu S_\mu^*)x\varphi(x)\cdots\varphi^{l-1}(x)(S_{\mu'}S_{\mu'}^*)S_{\mu'}\\
&= S_\mu^*(tS_\mu S_{\mu'}^*)S_{\mu'} = t\delta_{r(\mu),r(\mu')}P_{r(\mu)}
\end{aligned}$$

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for some scalar $t \in \mathbb{C}$ with $|t| \leq ||x||^{l-1}$. Therefore, for any $k > |\mu| = |\mu'|$, we see that

$$S_\mu^* \cdot \mathrm{udu}^* \varphi(\mathrm{udu}^*) \cdots \varphi^{k-1}(\mathrm{udu}^*) \cdot S_{\mu'} = t\delta_{r(\mu),r(\mu')} P_{r(\mu)} \cdot \mathrm{udu}^* \varphi(\mathrm{udu}^*) \cdots \varphi^{k-1-|\mu|}(\mathrm{udu}^*)$$

for some scalar $t \in \mathbb{C}$ with $|t| \leq ||\mathrm{udu}^*||^{k-1} = 1$. Since $P_{r(\mu)} \in \mathcal{D}$, the claim follows from Lemma 3.1.

Ad (2). This is shown in Lemma 3.2.

Ad (3). If $\nu \neq \nu'$ then

$$\Phi_{\mathcal{D}}(S_\nu b_\nu \cdot \mathrm{udu}^* \varphi(\mathrm{udu}^*) \cdots \varphi^{k-1}(\mathrm{udu}^*) \cdot b_{\nu'}^* S_{\nu'}^*) = 0.$$

Thus

$$\begin{aligned} &||\Phi_{\mathcal{D}} \left(\sum_{|\nu|=|\nu'|} S_\nu b_\nu \cdot \mathrm{udu}^* \varphi(\mathrm{udu}^*) \cdots \varphi^{k-1}(\mathrm{udu}^*) \cdot b_{\nu'}^* S_{\nu'}^* \right) ||_2 \\ &\quad \leq \sum_{|\nu|=|\nu'|} ||\Phi_{\mathcal{D}}(\varphi^{|\nu|}(b_\nu \cdot \mathrm{udu}^* \varphi(\mathrm{udu}^*) \cdots \varphi^{k-1}(\mathrm{udu}^*) \cdot b_{\nu'}^*))||_2, \end{aligned}$$

and this tends to 0 as k increases to infinity by the same argument as in the proof of Lemma 3.2. □

4. An application — changing graphs

The same graph C^*-algebra may often be presented by many different graphs, and the property of being quasi-free is usually not preserved when passing from one graph to another. This makes Theorem 3.3 applicable to a much wider class of automorphisms than quasi-free ones. We illustrate this phenomenon with the following two examples.

4.1. Out-splitting

Given a graph E satisfying our standing assumption, we consider its out-split graph $E_s(\mathcal{P})$, as defined by Bates and Pask in [2]. Namely, for each vertex $v \in E^0$, we partition the set of edges emitted by v, that is $s^{-1}(v)$, into $m(v)$ non-empty disjoint subsets $E_v^1, \ldots, E_v^{m(v)}$. Denote by $\mathcal{P}$ the resulting partition of E^1. The out-split graph $E_s(\mathcal{P})$ has the following vertices, edges, source and range functions:

$$\begin{aligned} E_s(\mathcal{P})^0 &= \{v^i \mid v \in E^0,\, 1 \leq i \leq m(v)\}, \\ E_s(\mathcal{P})^1 &= \{e^j \mid e \in E^1,\, 1 \leq j \leq m(r(e))\}, \\ s(e^j) &= s(e)^i, \text{ and } r(e^j) = r(e)^j. \end{aligned}$$

As shown in [2, Theorem 3.2], the C^*–algebras $C^*(E)$ and $C^*(E_s(\mathcal{P}))$ are isomorphic by an isomorphism which maps the diagonal MASA $\mathcal{D}_E$ of $C^*(E)$ onto the diagonal MASA $\mathcal{D}_{E_s(\mathcal{P})}$ of $C^*(E_s(\mathcal{P}))$. However, the groups of quasi-free automorphims of $C^*(E)$ and

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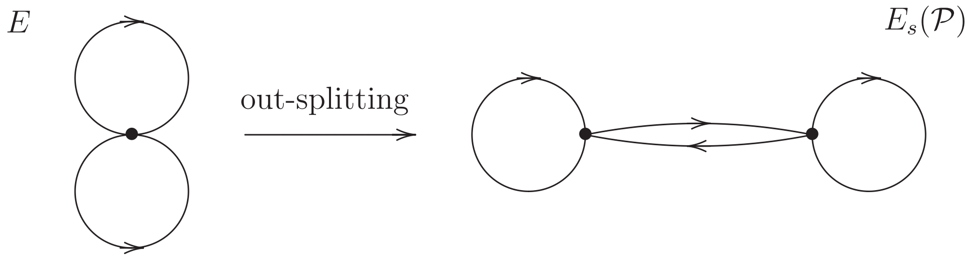
$C^*(E_s(\mathcal{P}))$ may be different. For example, in the following case

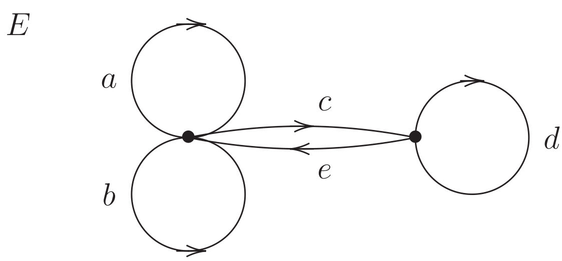


the groups $\mathcal{U}(\mathfrak{B})$ in $C^*(E_s(\mathcal{P}))$ and $C^*(E)$ are isomorphic to $U(1) \times U(1) \times U(1) \times U(1)$ and $U(2)$, respectively.

Thus, the isomorphism $C^*(E) \cong C^*(E_s(\mathcal{P}))$ may identify a quasi-free automorphism of one of the two algebras with a non quasi-free automorphism of the other. In this way, our Theorem 3.3 leads to non-trivial examples of non quasi-free automorphisms of graph algebras that map the diagonal MASA onto another MASA that is not inner conjugate to it.

4.2. Two graphs for $\mathcal{O}_2$

Consider the following graph:



Then the graph algebra $C^*(E)$ is isomorphic to the Cuntz algebra $\mathcal{O}_2 = C^*(T_1, T_2)$, [8]. Here $C^*(T_1, T_2)$ is the universal C^*-algebra for the relations $1 = T_1T_1^* + T_2T_2^* = T_1^*T_1 = T_2^*T_2$. That is, it is a graph algebra for the graph consisting of one vertex and two edges attached to it. The isomorphism between $C^*(E) = C^*(S_a, S_b, S_c, S_d, S_e)$ and $\mathcal{O}_2 = C^*(T_1, T_2)$ is obtained by the identification

$$S_a = T_{11}T_1^*, \quad S_b = T_{121}T_1^*, \; S_c = T_{122}T_2^*, \; S_d = T_{22}T_2^*, \; S_e = T_{21}T_1^*.$$

The inverse map is given by

$$T_1 = S_a + (S_b + S_c)(S_d + S_e)^*, \quad T_2 = S_d + S_e.$$

Note that this isomorphism carries the diagonal MASA $\mathcal{D}_E$ of $C^*(E)$ onto the standard diagonal MASA $\mathcal{D}_2$ of $\mathcal{O}_2$. Indeed, it follows from the above definition that every product x of the generators of $C^*(E)$ is mapped onto an element of the form $T_\alpha T_\beta^*$ in $\mathcal{O}_2$, with α, β some words on the alphabet $\{1, 2\}$. Thus, the range projection xx^* of that product is mapped onto a projection in the diagonal MASA $\mathcal{D}_2$ of $\mathcal{O}_2$. Hence, the image of $\mathcal{D}_E$ is contained in $\mathcal{D}_2$. But under an isomorphism, the image of a MASA in $C^*(E)$ is a MASA in $\mathcal{O}_2$. Thus, the image of MASA $\mathcal{D}_E$ of $C^*(E)$ is the entire MASA $\mathcal{D}_2$ of $\mathcal{O}_2$, as claimed.

Now, using the isomorphism above, we may find a quasi-free automorphism of $C^*(E)$ such that the corresponding automorphism of $\mathcal{O}_2$ is not quasi-free and yet carries the diagonal MASA $\mathcal{D}_2$ of $\mathcal{O}_2$ onto a MASA which is not inner conjugate to $\mathcal{D}_2$. Indeed, let

$$\begin{bmatrix} \xi_{aa} & \xi_{ab} \\ \xi_{ba} & \xi_{bb} \end{bmatrix}$$

be a unitary matrix whose all entries are non-zero complex numbers. Then

$$u = \xi_{aa} S_a S_a^* + \xi_{ab} S_a S_b^* + \xi_{ba} S_b S_a^* + \xi_{bb} S_b S_b^* + S_c S_c^* + S_d S_d^* + S_e S_e^*$$

is a unitary in $\mathcal{F}_E^1$ satisfying the hypothesis of Theorem 3.3. The above isomorphism transports the quasi-free automorphism λ_u of $C^*(E)$ onto an automorphism λ_U of $\mathcal{O}_2$ corresponding to the unitary

$$U = \xi_{aa} T_{11} T_{11}^* + \xi_{ab} T_{11} T_{121}^* + \xi_{ba} T_{121} T_{11}^* + \xi_{bb} T_{121} T_{121}^* + T_{122} T_{122}^* + T_2 T_2^*.$$

Clearly, λ_U is not a quasi-free automorphism of $\mathcal{O}_2$, since U does not belong to the linear span of $T_i T_j^*$, i, $j = 1, 2$. Theorem 3.3 implies that there is no unitary $w \in \mathcal{O}_2$ satisfying $w\mathcal{D}_2 w^* = \lambda_U(\mathcal{D}_2)$.

5. Certain MASAs in $\mathcal{O}_n$ not inner conjugate to the diagonal $\mathcal{D}_n$

In this section, we consider the Cuntz algebra $\mathcal{O}_n$, with $2 \leq n < \infty$. As usual, we view it as the graph C^*-algebra of the graph E_n with one vertex and n edges. Let $\lambda_u \in \mathrm{End}(\mathcal{O}_n)$. Suppose w_k is a sequence of unitaries in a commutative C^*-subalgebra A of $\mathcal{O}_n$. We ask under what circumstances the sequence w_k satisfies the condition of Lemma 2.3 for $M = \mathcal{O}_n$, A, $B = \mathcal{D}_n$, and τ the canonical trace on the UHF-subalgebra $\mathcal{F}_n$. Clearly, this is the case if and only if

$$||\Phi_{\mathcal{D}_n}(S_\alpha S_\beta^* w_k S_\mu S_\nu^*)||_2 \underset{k\to\infty}{\longrightarrow} 0, \tag{21}$$

for all paths α, β, μ, ν. Let

$$w_k = \sum_{m\in\mathbb{Z}} w_k^{(m)} \tag{22}$$

be the standard Fourier series of w_k (with respect to the decomposition of $\mathcal{O}_n$ into spectral subspaces $\mathcal{O}_n^{(m)}$ for the gauge action). Then (21) is equivalent to the requirement that

$$||\Phi_{\mathcal{D}_n}(S_\alpha S_\beta^* w_k^{(m)} S_\mu S_\nu^*)||_2 \underset{k\to\infty}{\longrightarrow} 0, \tag{23}$$

for all paths α, β, μ, ν, and all $m \in \mathbb{Z}$. Of course, it suffices to consider the case $m = |\beta| + |\nu| - |\alpha| - |\mu|$. Clearly, for all $x \in \mathcal{O}_n$ and all paths α we have

$$||\Phi_{\mathcal{D}_n}(S_\alpha x S_\alpha^*)||_2 = n^{-|\alpha|/2} ||\Phi_{\mathcal{D}_n}(x)||_2. \tag{24}$$

Thus, it suffices to consider condition (23) in the following three cases:

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(ZL1) $\nu = \emptyset$, $\beta \neq \emptyset$ and $m = |\beta| - |\alpha| - |\mu|$,
(ZL2) $\alpha = \emptyset$, $\mu \neq \emptyset$ and $m = |\beta| + |\nu| - |\mu|$,
(ZL3) $\alpha = \nu = \emptyset$ and $m = |\beta| - |\mu|$.

Lemma 5.1. *If (23) holds for all α, β, μ, ν as in (ZL3), it holds for all α, β, μ, ν as in (ZL1) and (ZL2).*

Proof. Consider condition (ZL1) first. By (ZL3), we have

$$||\Phi_{\mathcal{D}_n}(S_\beta^* w_k^{(m)} S_\mu S_\alpha)||_2 \underset{k\to\infty}{\longrightarrow} 0.$$

Thus, by identity (24), we also have

$$\begin{aligned} ||\Phi_{\mathcal{D}_n}(S_\alpha S_\beta^* w_k^{(m)} S_\mu)||_2 &= ||\Phi_{\mathcal{D}_n}(S_\alpha S_\alpha^* S_\alpha S_\beta^* w_k^{(m)} S_\mu)||_2 = ||\Phi_{\mathcal{D}_n}(S_\alpha S_\beta^* w_k^{(m)} S_\mu S_\alpha S_\alpha^*)||_2 \\ &= n^{-|\alpha|/2}||\Phi_{\mathcal{D}_n}(S_\beta^* w_k^{(m)} S_\mu S_\alpha)||_2 \underset{k\to\infty}{\longrightarrow} 0. \end{aligned}$$

Now, consider condition (ZL2). By (ZL3), we have

$$||\Phi_{\mathcal{D}_n}(S_\nu^* S_\beta^* w_k^{(m)} S_\mu)||_2 \underset{k\to\infty}{\longrightarrow} 0.$$

Thus, by identity (24), we also have

$$\begin{aligned} ||\Phi_{\mathcal{D}_n}(S_\beta^* w_k^{(m)} S_\mu S_\nu^*)||_2 &= ||\Phi_{\mathcal{D}_n}(S_\beta^* w_k^{(m)} S_\mu S_\nu^* S_\nu S_\nu^*)||_2 = ||\Phi_{\mathcal{D}_n}(S_\nu S_\nu^* S_\beta^* w_k^{(m)} S_\mu S_\nu^*)||_2 \\ &= n^{-|\nu|/2}||\Phi_{\mathcal{D}_n}(S_\nu^* S_\beta^* w_k^{(m)} S_\mu)||_2 \underset{k\to\infty}{\longrightarrow} 0. \end{aligned}$$ □

Now, we describe a construction of a large family of MASAs of the Cuntz algebra $\mathcal{O}_n$ which are contained in the core UHF-subalgebra $\mathcal{F}_n$ and are not inner conjugate to the diagonal MASA $\mathcal{D}_n$. MASAs obtained by applying to $\mathcal{D}_n$ quasi-free automorphisms not preserving $\mathcal{D}_n$ provide very special examples of this more general construction.

We start with a sequence $\{r_k\}_{k=1}^\infty$ of positive integers and denote $R_1 := 0$ and $R_k := \sum_{j=1}^{k-1} r_j$ for $k \geq 2$. For each k pick a $0 < c_k < 1$ so that

$$\prod_{k=1}^\infty c_k = 0.$$

For each k let d_k be a unitary in $\varphi^{R_k}(\mathcal{D}_n^{r_k})$ and U_k a unitary in $\varphi^{R_k}(\mathcal{F}_n^{r_k})$ such that

$$||\Phi_{\mathcal{D}_n}(U_k d_k U_k^*)||_2 \leq c_k. \tag{25}$$

Such unitaries may be found through easy matrix considerations. Given these data, we define $\mathcal{A}$ to be the C^*-subalgebra of $\mathcal{O}_n$ generated by the union of all algebras $U_k \varphi^{R_k}(\mathcal{D}_n^{r_k}) U_k^*$.

Theorem 5.2. *Every C^*-algebra $\mathcal{A}$, defined as above, is a MASA in $\mathcal{O}_n$ that is not inner conjugate to $\mathcal{D}_n$.*

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Proof. Let $\mathcal{A}$ be as above. Then clearly $\mathcal{A}$ is a MASA in the core UHF-subalgebra $\mathcal{F}_n$ of $\mathcal{O}_n$, for example see [11]. We will show that $\mathcal{A}$ is a MASA in the entire $\mathcal{O}_n$ as well.

Indeed, let x be in $\mathcal{A}' \cap \mathcal{O}_n$, and let $x = \sum_{m\in\mathbb{Z}} x^{(m)}$ be its standard Fourier series. Then for each m we have $x^{(m)} \in \mathcal{A}' \cap \mathcal{O}_n$. Consider a fixed $m > 0$. Both $x^{(m)*}x^{(m)}$ and $x^{(m)}x^{(m)*}$ are in $\mathcal{A}' \cap \mathcal{F}_n = \mathcal{A}$. Since both these elements are positive and for every projection $q \in \mathcal{A}$ we have $||qx^{(m)}x^{(m)*}|| = ||qx^{(m)}x^{(m)*}q|| = ||x^{(m)*}qx^{(m)}|| = ||qx^{(m)*}x^{(m)}||$, it easily follows that $x^{(m)*}x^{(m)} = x^{(m)}x^{(m)*}$. That is, the element $x^{(m)}$ of $\mathcal{O}_n$ is normal. Now, denote $v = S_1^m$ and $a = x^{(m)}(S_1^m)^*$. Then $a \in \mathcal{F}_n$ and we have $x^{(m)} = av$. Since av is normal, we have

$$\tau(a^*a) = \tau(vv^*a^*a) = \tau(avv^*a^*) = \tau(v^*a^*av) = n^m\tau(a^*a).$$

Thus, $\tau(a^*a) = 0$ and hence $a = 0$. Consequently, $x^{(m)} = 0$ for all $m > 0$. A similar argument shows that $x^{(m)} = 0$ for all $m < 0$. Thus, $x = x^{(0)}$ belongs to $\mathcal{A}' \cap \mathcal{F}_n = \mathcal{A}$, and $\mathcal{A}$ is a MASA in $\mathcal{O}_n$ as claimed.

To show that $\mathcal{A}$ is not inner conjugate in $\mathcal{O}_n$ to $\mathcal{D}_n$, we verify that condition (ZL3) holds for

$$w_k := \prod_{j=1}^{k} U_j d_j U_j^*.$$

Since each w_k is in $\mathcal{F}_n$, it suffices to check it with $m = 0$. So fix β, μ with $|\beta| = |\mu|$. Take t so large that $t \geq |\beta|$ and consider $k > t$. Since $\prod_{j=t+1}^{k} U_j d_j U_j^*$ is in the range of injective endomorphism $\varphi^{|\mu|}$, we have

$$\begin{aligned} ||\Phi_{\mathcal{D}_n}(S_\beta^* w_k S_\mu)||_2 &= ||\Phi_{\mathcal{D}_n}\left(S_\beta^*\Big(\prod_{j=1}^{t} U_j d_j U_j^*\Big)\varphi^{|\mu|}\left(\varphi^{-|\mu|}\Big(\prod_{j=t+1}^{k} U_j d_j U_j^*\Big)\right)S_\mu\right)||_2 \\ &= ||\Phi_{\mathcal{D}_n}\left(S_\beta^*\Big(\prod_{j=1}^{t} U_j d_j U_j^*\Big)S_\mu\varphi^{-|\mu|}\Big(\prod_{j=t+1}^{k} U_j d_j U_j^*\Big)\right)||_2 \end{aligned}$$

Thus, we have by Lemma 6.1, that

$$\begin{aligned} &||\Phi_{\mathcal{D}_n}(S_\beta^* w_k S_\mu)||_2 \\ &= ||\Phi_{\mathcal{D}_n}(S_\beta^*\left(\prod_{j=1}^{t} U_j d_j U_j^*)S_\mu\right)\Phi_{\mathcal{D}_n}\left(\varphi^{-|\mu|}\left(\prod_{j=t+1}^{k} U_j d_j U_j^*\right)\right)||_2 \\ &\leq ||\Phi_{\mathcal{D}_n}\left(S_\beta^*\left(\prod_{j=1}^{t} U_j d_j U_j^*\right)S_\mu\right)|| \cdot ||\Phi_{\mathcal{D}_n}\left(\varphi^{-|\mu|}\left(\prod_{j=t+1}^{k} U_j d_j U_j^*\right)\right)||_2 \\ &= ||\Phi_{\mathcal{D}_n}\left(S_\beta^*\left(\prod_{j=1}^{t} U_j d_j U_j^*\right)S_\mu\right)|| \cdot ||\prod_{j=t+1}^{k}\Phi_{\mathcal{D}_n}(U_j d_j U_j^*)||_2 \end{aligned}$$

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$$= ||\Phi_{\mathcal{D}_n}\left(S_\beta^*\left(\prod_{j=1}^{t} U_j d_j U_j^*\right) S_\mu\right)|| \cdot \prod_{j=t+1}^{k} ||\Phi_{\mathcal{D}_n}(U_j d_j U_j^*)||_2$$

$$\leq ||\Phi_{\mathcal{D}_n}\left(S_\beta^*\left(\prod_{j=1}^{t} U_j d_j U_j^*\right) S_\mu\right)|| \cdot \prod_{j=t+1}^{k} c_k \underset{k\to\infty}{\longrightarrow} 0. \qquad \square$$

We remark that it is not immediately clear which of the MASAs considered in Proposition 5.2 are outer conjugate in $\mathcal{O}_n$ to $\mathcal{D}_n$.

6. Technical lemmas

In this section, we collect a few technical facts used in the proofs above.

6.1. The conditional expectations

Lemma 6.1. *Let A and B be unital C^*-subalgebras of a finite-dimensional C^*-algebra, such that $ab = ba$ for all $a \in A$, $b \in B$. Let D_A and D_B be MASAs of A and B, respectively, so that $D := D_A \vee D_B$ is a MASA of $A \vee B$. Let τ be a faithful tracial state on $A \vee B$, and let E_D, E_{D_A} and E_{D_B} be τ-preserving conditional expectations from $A \vee B$ onto D, D_A and D_B, respectively. Then we have*

$$E_D(ab) = E_{D_A}(a)E_{D_B}(b)$$

for all $a \in A$, $b \in B$.

Proof. If A is a full matrix algebra (i.e., the centre of A is trivial) then $A \vee B \cong A \otimes B$ and $\tau(ab) = \tau(a)\tau(b)$ for all $a \in A$, $b \in B$. Thus, in this case, the claim obviously holds.

In the general case, let $\{p_1, \ldots, p_n\}$ be the minimal central projections in A. Then

$$A \vee B = \bigoplus_{i=1}^{n} (A \vee B)p_i \cong \bigoplus_{i=1}^{n} Ap_i \otimes Bp_i.$$

The τ-preserving conditional expectation E_i from $(A \vee B)p_i$ onto $(D_A \vee D_B)p_i$ satisfies

$$E_i(ab) = E_{D_A p_i}(a)E_{D_B p_i}(b)$$

for all $a \in Ap_i$ and $b \in Bp_i$, by the preceding argument. Since

$$E_D(x) = \sum_{i=1}^{n} E_i(xp_i),$$

the claim follows. $\square$

Corollary 6.2. *For all $x_1, x_2, \ldots, x_k \in \mathfrak{B}$, we have*

$$\Phi_{\mathcal{D}}(x_1\varphi(x_2)\cdots\varphi^{k-1}(x_k)) = \Phi_{\mathcal{D}}(x_1)\varphi(\Phi_{\mathcal{D}}(x_2))\cdots\varphi^{k-1}(\Phi_{\mathcal{D}}(x_k)).$$

Proof. Since $\mathfrak{B}, \varphi(\mathfrak{B}), \dots, \varphi^{k-1}(\mathfrak{B})$ are mutually commuting unital finite-dimensional C^*-algebras, by Lemma 6.1, we have

$$\Phi_{\mathcal{D}}(x_1\varphi(x_2)\cdots\varphi^{k-1}(x_k)) = \Phi_{\mathcal{D}}(x_1)\Phi_{\mathcal{D}}(\varphi(x_2))\cdots\Phi_{\mathcal{D}}(\varphi^{k-1}(x_k)).$$

The claims follow since the conditional expectation $\Phi_{\mathcal{D}}$ commutes with the shift φ. □

6.2. The Perron-frobenius theory

Let A be an $n \times n$ matrix with non-negative integer entries. We assume that A is irreducible in the sense that for each pair of indices (i, j) there exists a positive integer k such that $A^k(i, j) > 0$. Let β be the Perron–Frobenius eigenvalue and let $(x(1), x(2), \dots, x(n))$ be the corresponding Perron–Frobenius eigenvector. That is, $\beta > 0$, $x(i) > 0$ for all indices $i = 1, \dots, n$, and

$$\sum_j A(i,j)x(j) = \beta x(i).$$

In this subsection, for a (not necessary square) matrix B we write $B \geq 0$ if $B(i, j) \geq 0$ for all (i, j). Likewise, we write $B > 0$ if $B(i, j) > 0$ for all (i, j). For a column vector $y \geq 0$, we set

$$\lambda(y, A) = \max\{\lambda \geq 0 \mid Ay \geq \lambda y\}.$$

The following lemma is part of the classical Perron–Frobenius theory, hence its proof is omitted.

Lemma 6.3. *For an irreducible matrix A, as above, we have*

$$\beta = \max\{\lambda(y, A) \mid y \geq 0, ||y|| = 1\}.$$

Lemma 6.4. *Let $\beta' > 0$ be the Perron–Frobenius eigenvalue of the transpose matrix ${}^tA$. Let $\{y(v)\}_v$ be the Perron–Frobenius eigenvector of ${}^tA$. That is,*

$$\sum_v A(v,w)y(v) = \beta' y(w).$$

Set $m = \min_v y(v)$ and $M = \max_v y(v)$. For any $f \in \mathcal{D}_E$, we have

$$\tau(\varphi(f)) \leq \frac{\beta' M}{\beta m}\tau(f).$$

Hence we have

$$||\varphi^p(f)||_2^2 \leq \Big(\frac{\beta' M}{\beta m}\Big)^p ||f||_2^2.$$

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Proof. We may assume that $f = S_\mu S_\mu^*$. We see that

$$\begin{aligned}\tau(\varphi(S_\mu S_\mu^*)) &= \sum_{e, r(e)=s(\mu)} \tau(S_{e\mu} S_{e\mu}^*) = \sum_v A(v, s(\mu)) \frac{x(r(\mu))}{X\beta^{|\mu|+1}} \\ &\leq \sum_v A(v, s(\mu)) \frac{y(v)}{m} \frac{x(r(\mu))}{X\beta^{|\mu|+1}} = \beta' \frac{y(s(\mu))}{m} \frac{x(r(\mu))}{X\beta^{|\mu|+1}} \\ &\leq \beta' \frac{M}{m} \frac{x(r(\mu))}{X\beta^{|\mu|+1}} = \frac{\beta' M}{\beta m} \tau(S_\mu S_\mu^*). \end{aligned}$$ □

Lemma 6.5. *For an irreducible matrix A, as above, we set $X = \sum_i x(i)$, $\alpha = \min_i x(i)$, and $\alpha' = \max_i x(i)$. Then, for every positive integer k, we have*

$$0 < \frac{X}{\alpha'} \leq \frac{\sum_{i,j} A^k(i,j)}{\beta^k} \leq \frac{X}{\alpha}.$$

Proof. Since $x(j)/\alpha' \leq 1 \leq x(j)/\alpha$ for all j and $\sum_j A^k(i, j)x(j) = \beta^k x(i)$ for all i, we have

$$\frac{X}{\alpha'} = \frac{\sum_{i,j} A^k(i,j)x(j)}{\beta^k \alpha'} \leq \frac{\sum_{i,j} A^k(i,j)}{\beta^k} \leq \frac{\sum_{i,j} A^k(i,j)x(j)}{\beta^k \alpha} = \frac{X}{\alpha}.$$ □

For an irreducible matrix A, as above, and a fixed pair of indices (i_1, j_1) we set

$$A_1(i,j) := \begin{cases} A(i,j) & \text{if } (i,j) \neq (i_1, j_1) \\ 1 & \text{if } (i,j) = (i_1, j_1) \end{cases} \tag{26}$$

Theorem 6.6. *Let A be an irreducible matrix, as above. Assume that $A(i_1, j_1) \geq 2$. Then A_1 is an irreducible matrix such that $A_1 \leq A$ and we have*

$$\lim_{k\to\infty} \frac{\sum_{i,j} A_1^k(i,j)}{\beta^k} = 0,$$

with β the Perron–Frobenius eigenvalue of A.

Proof. It is clear that A_1 is irreducible and $A_1 \leq A$. Let β_1 be the Perron–Frobenius eigenvalue of A_1, with the corresponding Perron–Frobenius eigenvector $(x_1(1), \ldots, x_1(n))$. We have

$$\frac{\sum_{i,j} A_1^k(i,j)}{\beta^k} = \frac{\sum_{i,j} A_1^k(i,j)}{\beta_1^k} \cdot \frac{\beta_1^k}{\beta^k}.$$

Thus, in view of Lemma 6.5, it suffices to show that $\beta_1 < \beta$.

Now, for each pair of indices (i, j) we can find an $l_{i,j}$ such that $A_1^{l_{i,j}}(i, j) < A^{l_{i,j}}(i, j)$. Indeed, denote by E_1 a graph with the adjacency matrix A_1. We may view E_1 as a subgraph of E. Given (i, j) we can find a path $\mu \in E^* \setminus E_1^*$ with source in vertex i and range in vertex j. To this end, take a path μ_1 from i to i_1, a path μ_2 from j_1 to j, and an edge $e \in E^1 \setminus E_1^1$ from i_1 to j_1. Then put $\mu := \mu_1 e \mu_2$. Setting $l_{i,j} := |\mu|$ we have

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$A_1^{l_{i,j}}(i, j) < A^{l_{i,j}}(i, j)$, as desired. Let k be an integer such that $k > l_{i,j}$ for all i, j. Then we have

$$\sum_{j=1}^{k} A_1^j < \sum_{j=1}^{k} A^j.$$

Now, we set $\overline{A} = \sum_{j=1}^k A^j$, $\overline{A}_1 = \sum_{j=1}^k A_1^j$, $\overline{\beta} = \sum_{j=1}^k \beta^j$, and $\overline{\beta}_1 = \sum_{j=1}^k \beta_1^j$. We have $\overline{A}x = \overline{\beta}x$ and $\overline{A}_1 x_1 = \overline{\beta}_1 x_1$. To prove the theorem, it suffices to show that $\overline{\beta}_1 < \overline{\beta}$. Thus, without loss of generality, we may simply assume that $A_1 < A$.

Let I be the $n \times n$ matrix with $I(i, j) = 1$ for all i, j. Since $A > A_1$, we have

$$A \geq A_1 + I.$$

With $X_1 := \sum_j x_1(j) > 0$, we see that

$$Ax_1 \geq (A_1 + I)x_1 = \beta_1 x_1 + \begin{pmatrix} X_1 \\ \vdots \\ X_1 \end{pmatrix}.$$

We can take a small $\epsilon > 0$ such that

$$\beta_1 x_1 + \begin{pmatrix} X_1 \\ \vdots \\ X_1 \end{pmatrix} \geq (\beta_1 + \epsilon)x_1$$

This means that $\lambda(x_1, A) \geq \beta_1 + \epsilon > \beta_1$. Since $\beta \geq \lambda(x_1, A)$, we may finally conclude that $\beta > \beta_1$. □

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