Adv. Appl. Probab. **54,** 404–432 (2022)
*doi:*10.1017/apr.2021.38

MOMENT-CONSTRAINED OPTIMAL DIVIDENDS: PRECOMMITMENT AND CONSISTENT PLANNING

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Abstract

A moment constraint that limits the number of dividends in an optimal dividend problem is suggested. This leads to a new type of time-inconsistent stochastic impulse control problem. First, the optimal solution in the precommitment sense is derived. Second, the problem is formulated as an intrapersonal sequential dynamic game in line with Strotz's consistent planning. In particular, the notions of pure dividend strategies and a (strong) subgame-perfect Nash equilibrium are adapted. An equilibrium is derived using a smooth fit condition. The equilibrium is shown to be strong. The uncontrolled state process is a fairly general diffusion.

Keywords: Constrained stochastic control; stochastic impulse control; Strotz's consistent planning; subgame-perfect Nash equilibrium; time-inconsistency

2020 Mathematics Subject Classification: Primary 93E20
Secondary 60J70; 91A15; 91G80

1. Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$, $x \in \mathbb{R}$, be a family of filtered probability spaces where $(\mathcal{F}_t)_{t\geq 0}$ is the augmented filtration generated by a Wiener process $W=(W_t)_{t\geq 0}$. Let $X=(X_t)_{t\geq 0}$ be a one-dimensional process given under $\mathbb{P}_x$ by

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t - dD_t, \quad X_0 = x \quad \text{almost surely (a.s.)}, \tag{1}$$

where $D=(D_t)_{t\geq 0}$ is a nondecreasing process. The associated expectations are denoted by $\mathbb{E}_x$. A classical optimal dividend problem in this setting is to suppose the following: (i) D is the accumulated dividend process paid by an insurance company to its owner, (ii) X is the surplus process of the insurance company, and (iii) the owner selects the process D that is maximal in the following stochastic control problem:

$$\begin{aligned} &U(x) := \sup_{D\in\mathcal{A}(x)} J(x;D), \quad J(x;D) := \mathbb{E}_x\left(\int_0^\tau e^{-rt}dD_t\right), \\ &\tau := \inf\{t\geq 0 : X_t \leq 0\}, \\ &\quad D\in\mathcal{A}(x) \text{ if } D \text{ is a nondecreasing adapted process that is left-continuous with} \\ &\quad \text{right limits (LCRL), with } D_0 = 0, \text{ such that } X_{\tau+}\geq 0. \end{aligned} \tag{2}$$

Received 2 July 2020; revision received 22 June 2021.
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Typically, the constant $r>0$ is viewed as a discount rate. However, r may also be viewed as a killing rate for X corresponding to a random time horizon ξ; in particular, if we let ξ be an exponentially distributed stopping time (defined on our probability space and independent of W) with expectation $\frac{1}{r}$, then we obtain

$$J(x;D)=\mathbb{E}_x\left(\int_0^\tau I_{\{t\leq\xi\}}dD_t\right)=\mathbb{E}_x\big(D_{(\tau+)\wedge\xi}\big)\,. \tag{3}$$

In this paper we take the latter viewpoint. The interpretation is that r is the instantaneous rate at which some external event with timing ξ—after which no dividends are paid to the owner—occurs; examples of such events may include liquidation or bankruptcy caused by major incidents and government expropriation. Specifically, no dividends are paid to the owner after $(\tau+)\wedge\xi$, where we interpret $\tau+$ as the time of bankruptcy due to lack of surplus. We remark that optimal dividend problems with random time horizons have previously been studied in e.g. [2, 49, 50]. We also remark that the parameter r can be reinterpret as corresponding to a situation with both discounting and the presence of an external (killing) event of the kind corresponding to ξ; in particular, if we let $r=r_1+r_2$ where $r_i>0$, $i=1,2$, then we may interpret r_1 as the discount rate and r_2 as an external killing rate. The integral in (2) should be interpreted as a Lebesgue–Stieltjes integral with respect to the LCRL integrator D, where jumps in the integrator are understood as $D_{t+}-D_t$.

The problem (2) was first studied in [46], where—under certain conditions for the functions $\mu(\cdot)$ and $\sigma(\cdot)$, notably $\mu'(x)\leq r$ for all $x\geq 0$—it was found that if an optimal policy exists, then it is to pay dividends only in order to reflect the process X at a barrier x^*, and if no optimal policy exists, then the optimal value function $U(\cdot)$ is the limit of the value function given a reflecting barrier dividend policy when sending the barrier to infinity. A criticism of this formulation of the dividend problem from an economic viewpoint is that the solution involves an unreasonably high number of dividend payments; in particular, once X reaches the barrier x^* an infinite number of dividends will be paid during any immediately following time interval, no matter how small. One way of taking this criticism into account is to introduce a fixed cost for each dividend payment, which obviously limits the optimal number of dividend payments and thus leads to a stochastic impulse control problem. In this paper we instead introduce a moment constraint that directly limits the expected number of dividend payments. Specifically, if we denote by τ_n the timing of the nth dividend payment for a discrete dividend policy D, then D is in this paper said to be admissible for a given initial surplus $x\geq 0$ in (1), which we write as $D\in\mathcal{A}(x,k)$, if $D\in\mathcal{A}(x)$ and the moment constraint $R(x;S)=\mathbb{E}_x\left(|\{n:\tau_n\leq\tau\wedge\xi\}|\right)\leq\frac{1}{k}$ is satisfied, where $k>0$. An interpretation of this constraint is that the company wants to limit the number of dividend payments not mainly because of the financial costs of paying dividends, but rather because a high number of dividend payments is undesirable for other reasons—for example, because the owners of the company expect a limited number of dividend payments, or because they involve tedious administrative work for the decision-maker. We remark that other constraints with this motivation are possible; in particular, it would be interesting to investigate the optimal dividend problem under constraints for the (expected) number of dividends paid during fixed time intervals, or the (expected) time between dividend payments. This is, however, outside the scope of the present paper.

Clearly, a dividend policy satisfying our constraint must be of impulse control type; in particular, an admissible dividend policy $D\in\mathcal{A}(x,k)$ can be represented as

$$D_t=\sum\nolimits_{n:\tau_n<t}\zeta_n,\ t\geq 0 \text{ with } D_0=0,\ \text{ where, for each } n=1,2,\ldots,$$

$$\tau_n\geq 0 \text{ is an } (\mathcal{F}_t)_{t\geq 0}\text{-stopping time such that } \tau_{n+1}>\tau_n \text{ on } \{\tau_n<\infty\}$$

https://doi.org/10.1017/apr.2021.38 Published online by Cambridge University Press

and ζ_n is an $\mathcal{F}_{\tau_n}$-measurable random variable with $0 < \zeta_n \leq X_{\tau_n}$ a.s.,

$$\text{and } \tau_n \to \infty \text{ a.s. } n \to \infty. \tag{4}$$

Note also that $\zeta_n = X_{\tau_n} - X_{\tau_n+}$. The interpretation of ζ_n is that it is the nth dividend payment. In the sequel we denote a dividend policy of impulse control type as defined in (4) by $S = (\tau_n, \zeta_n)_{n\geq 1}$—which means, using a slight abuse of notation, that $S = (\tau_n, \zeta_n)_{n\geq 1} \in \mathcal{A}(x)$ for each $x \geq 0$ by definition. This implies that $S = (\tau_n, \zeta_n)_{n\geq 1} \in \mathcal{A}(x; k)$ whenever the moment constraint is satisfied. Note that the assumption that D is adapted to the filtration generated by W implies that it is not possible (a.s.) to pay a lump sum dividend at the time of killing. In the case of a dividend policy of impulse control type $S = (\tau_n, \zeta_n)_{n\geq 1}$ we may thus write the corresponding value function, cf. (3), as

$$J(x; S) := \mathbb{E}_x\left(\sum_{n\,:\,\tau_n \leq \tau\wedge\xi} \zeta_n\right) \tag{5}$$

and the constraint as

$$R(x; S) := \mathbb{E}_x\left(|\{n : \tau_n \leq \tau \wedge \xi\}|\right) \leq \frac{1}{k}, \quad \text{where } k > 0 \text{ is a fixed parameter.} \tag{6}$$

The objective of the present paper is to study the problem of maximizing the sum of expected dividends (5) over the set of admissible dividend policies, which in particular satisfy the constraint (6). It turns out that this problem is time-inconsistent in the sense that a dividend policy which is, in the precommitment sense, optimal at time 0 will not generally be optimal at a time $t > 0$ when considering the constraint (6) using the value for the state process at t; see Remark 3.1. We remark that it could be argued that it would be more reasonable to call this problem space-inconsistent, but we have chosen to use the more established term. The main contribution of the present paper is to formulate and solve this problem both in the precommitment sense and in the game-theoretic sense of Strotz's consistent planning. In addition to the formulation and study of the particular problem of the present paper, a further contribution is that we use a Lagrangian approach that reduces our constrained control problem to a solvable unconstrained control problem. This approach is likely to be possible also for other constrained stochastic control problems.

The rest of the paper is structured as follows. Section 1.1 mentions further related literature. In Section 2 the model of the present paper is formulated in more detail and some results that will be used in the sequel are presented. In Section 3 the precommitment interpretation of the constrained dividend problem is formulated and solved, and properties of the solution are investigated. These results rely on the solution to the (unconstrained) optimal dividend problem under the assumption that a fixed cost is incurred for each dividend payment, which is therefore also recapitulated and studied in Section 3. In Section 4 we formulate and solve the constrained dividend problem as a game along the lines of Strotz's consistent planning. An equilibrium is derived using a smooth fit condition. The equilibrium is shown to be strong and its properties are investigated. A discussion of our equilibrium definition is found in Section 4.1. In Section 4.2 we define alternative notions of equilibria and show that they are equivalent to those defined earlier in Section 4. An example is studied in Section 5. Model assumptions are discussed in Appendix A. Most proofs are found in Appendix B.

1.1. Background and related literature

The study of time-inconsistent control problems goes back to a seminal paper by Strotz [47] in the 1950s, but the field has experienced considerable activity in recent years.

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Time-inconsistency in stochastic control typically arises due to the consideration of (i) non-exponential discounting, (ii) a state-dependent reward function, or (iii) nonlinearities in the expected reward, e.g. mean–variance utility; see e.g. [14, 22, 23, 24, 35] for descriptions of these kinds of problems and references. Time-inconsistency is typically studied using the precommitment approach, which means finding an optimal control policy for a given initial value of the controlled process, or the time-consistent (game-theoretic) approach along the lines of Strotz's invention. Time-inconsistency can also be studied using the notion of dynamic optimality; see [43, 44].

The game-theoretic approach is to interpret a time-inconsistent problem as an intrapersonal sequential dynamic game. The approach is formalized by defining a subgame-perfect Nash equilibrium suitable for the particular problem at hand. See Section 4.1 for an interpretation of the game in the present paper and Section 3 for our equilibrium definition. A main reference for the general theory of the game-theoretic approach to time-inconsistent stochastic control is [13]. A large literature studying particular time-inconsistent problems using the game-theoretic approach has evolved during the last few years; a short recent survey is contained in [35]. The general theory of time-inconsistent stopping is studied in e.g. [22–24, 31]. The present paper is different from most papers on time-inconsistent control in the sense that the time-inconsistency does not arise from the factors (i)–(iii) mentioned above; instead it is due to the consideration of a constraint for an otherwise time-consistent stochastic control problem.

There are many papers that study stochastic control under different kinds of constraints; here we mention only a few. Optimal dividends under ruin probability constraints are studied in [27, 30], while optimal dividends under a constraint for the ruin time is studied in [29]. A dividend problem under the constraint that the surplus process must be above a given fixed level in order for dividend payments to be admissible is studied in [40]; see also [36], where this problem is studied in a model which allows for capital injection. Optimal stopping under expectation constraints is studied in [4, 10], while stochastic control under expectation constraints is studied in [20]. Distribution-constrained optimal stopping is studied in [9, 11]. It should also be mentioned that mean–variance problems are sometimes formulated as constrained optimization problems. For example, constrained mean–variance portfolio selection (a control problem) is studied in [44], and constrained mean–variance selling strategies (a stopping problem) are studied in [43], although the main topic of these papers is the notion of dynamic optimality. In [42] a constrained portfolio selection problem is investigated using the dynamic optimality approach and a comparison is made to the precommitment approach. The game-theoretic approach to a mean–variance optimization problem under the constraint of no short selling is studied in [12]. We also mention [37], in which a conditional optimal stopping problem is studied using a game-theoretic approach.

Time-inconsistent dividend problems have been studied before: the optimal dividend problem under non-exponential discounting is studied using the game-theoretic approach in [16, 17, 19, 34, 48, 51], while [18] studies this problem incorporating capital injections as well.

The precommitment approach of the present paper relies, as we have mentioned, on results for the fixed-cost dividend problem. This problem was studied first in [33], which considers a Wiener process with drift, and later in [41], which considers a more general diffusion model; see Remarks 3.4 and 3.5 and Appendix A for further references.

There is a vast literature on many different versions of the optimal dividend problem; see e.g. the literature reviews [1, 5] and the more recent surveys included in [26, 28, 36].

2. Model formulation and preliminaries

In this section we specify model assumptions and present results and notation on which the subsequent analysis relies. Unless otherwise stated we assume throughout the paper that all items in Assumption 2.1 below hold; see Appendix A for a discussion of Assumption 2.1.

Assumption 2.1.

(A.1) The functions $\mu(\cdot)$ and $\sigma(\cdot)$ are continuously differentiable and Lipschitz continuous, and $\mu'(\cdot)$ and $\sigma'(\cdot)$ are Lipschitz continuous.

(A.2) We have $\sigma^2(x) > 0$ for all $x \geq 0$.

(A.3) We have $\mu'(x) < r$ for all $x \geq 0$.

(A.4) An $\varepsilon > 0$ and an $x_a \geq 0$ such that $\mu'(x) < r - \varepsilon$ for all $x \geq x_a$ exist.

(A.5) We have $\mu(0) > 0$.

(Note that (A.3) and (A.4) are, given that (A.1) holds, equivalent to the condition that there exists an $\varepsilon > 0$ such that $\mu'(x) < r - \varepsilon$ for all $x \geq 0$.) Consider the boundary value problem

$$A_X g(x) := \mu(x)g'(x) + \frac{1}{2}\sigma^2(x)g''(x) = rg(x), \quad x > 0, \tag{7}$$

$$g'(0) > 0, \quad g(0) = 0, \quad g(\cdot) \in C^2(0, \infty). \tag{8}$$

Lemma 2.1. *Suppose (A.1)–(A.2) hold. Then a solution $g(\cdot)$ of (7)–(8) that is unique up to multiplication by a positive constant exists (in the sequel this is called a canonical solution). Moreover, the following hold:*

(i) We have $g(\cdot) \in C^3(0, \infty)$, and $g'''(\cdot)$ is Lipschitz continuous.

(ii) Adding (A.3) implies that $g'(x) > 0$ for all $x \geq 0$.

(iii) Adding (A.3)–(A.4) implies that $\lim_{x\to\infty} g'(x) = \infty$.

(iv) Adding (A.3)–(A.5) implies that a unique $x_b \in (0, \infty)$ such that $g''(x_b) = 0$, $g''(x) < 0$ for $x < x_b$, and $g''(x) > 0$ for $x > x_b$ exists.

We remark that which canonical solution g is chosen turns out to be irrelevant, since the multiplicative constant will be canceled out in all relevant expressions. We also remark that Assumption 2.1 and Lemma 2.1 both rely heavily on the analysis in [41, 46]; see Appendix A and the proof of Lemma 2.1 in Appendix B for details. Throughout this paper we use that the value function (5) and the function in the constraint (6) can respectively be written as follows, using e.g. the properties of ξ:

$$J(x; S) = \mathbb{E}_x\left(\sum_{n\,:\,\tau_n\leq\tau} e^{-r\tau_n}\zeta_n\right) \tag{9}$$

and

$$R(x; S) = \mathbb{E}_x\left(\sum_{n\,:\,\tau_n\leq\tau} e^{-r\tau_n}\right). \tag{10}$$

In the sections below it will be shown that both the precommitment and the equilibrium solutions (to be defined) are of the following kind.

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Definition 2.2. *A dividend policy of impulse control type* $S = (\tau_n, \zeta_n)_{n\geq 1}$ *(see (4)) is said to be a constant lump sum dividend barrier policy if the following hold:*

- *Each dividend is of the same size, i.e.*

$$\zeta_n = \bar{x} - \underline{x}, \quad \text{for some } \bar{x} > \underline{x} \geq 0,$$

except possibly at time 0*, when a dividend of size* $x - \underline{x}$ *is paid if* $x \geq \bar{x}$*.*

- *A dividend is paid when the process* X *reaches a fixed level, i.e.*

$$\tau_1 = \inf\{t > 0 : X_t \geq \bar{x}\}, \quad \tau_n = \inf\{t > \tau_{n-1} : X_t \geq \bar{x}\}, \quad n = 2, 3, \ldots$$

(we use the convention $\inf \emptyset = \infty$*).*

With a slight abuse of notation we denote a constant lump sum dividend barrier policy S by $(\underline{x}, \bar{x})$ and write the corresponding value function $J(x; S)$ as $J(x; \underline{x}, \bar{x})$, and similarly for e.g. the function $R(x; S)$.

We will use the following results.

Proposition 2.1. *Consider an arbitrary constant lump sum dividend barrier policy* $(\underline{x}, \bar{x})$*. The corresponding value function* $J(x; S)$ *is then continuous and can be written as*

$$J(x; \underline{x}, \bar{x}) = \begin{cases} J^0(x; \underline{x}, \bar{x}) := g(x)\frac{\bar{x}-\underline{x}}{g(\bar{x})-g(\underline{x})}, & 0 \leq x \leq \bar{x}, \\ x - \underline{x} + J^0(\underline{x}; \underline{x}, \bar{x}), & x > \bar{x}. \end{cases} \tag{11}$$

Moreover, the corresponding function $R(x; S)$ *is continuous and can be written as*

$$R(x; \underline{x}, \bar{x}) = \begin{cases} R^0(x; \underline{x}, \bar{x}) := g(x)\frac{1}{g(\bar{x})-g(\underline{x})}, & 0 \leq x \leq \bar{x}, \\ 1 + R^0(\underline{x}; \underline{x}, \bar{x}), & x > \bar{x}. \end{cases} \tag{12}$$

Lemma 2.2.

(i) *Consider an arbitrary initial surplus* $x > 0$*. Then* $R(x; \underline{x}, \bar{x})$ *is continuous and strictly decreasing in* $\bar{x}$*, with* $\lim_{\bar{x}\to\infty} R(x; \underline{x}, \bar{x}) = 0$ *for any fixed* $\underline{x} \geq 0$*. Moreover,* $R(x; \underline{x}, \bar{x})$ *is continuous and strictly increasing in* $\underline{x}$*, with* $\lim_{\underline{x}\to\bar{x}} R(x; \underline{x}, \bar{x}) = \infty$ *for any fixed* $\bar{x} > 0$*.*

(ii) *The function* $R(\bar{x}; \underline{x}, \bar{x})$ *is continuous and strictly decreasing in* $\bar{x}$*, with* $\lim_{\bar{x}\to\infty} R(\bar{x}; \underline{x}, \bar{x}) = 1$ *for any fixed* $\underline{x} > 0$*. Also,* $R(\bar{x}; 0, \bar{x}) = 1$ *for any fixed* $\bar{x} > 0$*.*

In Sections 3 and 4 it will be shown that if we let the constraint (6) vanish in the sense of sending $k \to 0$ then the precommitment and equilibrium solutions both converge to the solution of the classical dividend problem (2); see Corollary 3.1 and Theorem 4.6, respectively. For the convenience of the reader we therefore include the result below, which follows directly from [46, Theorem 4.3] and Lemma 2.1.

Proposition 2.2. *The optimal dividend policy for the unconstrained problem* (2) *reflects the state process* (1) *at the barrier*

$$x^* := x_b \tag{13}$$

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(where x_b is defined in Lemma 2.1), while being flat off $\{t \geq 0 : X_t = x^*\}$. Here it should be understood that if the initial surplus satisfies $x > x^*$, then there is an immediate dividend payment of size $x - x^*$. The optimal value function is

$$U(x) = \begin{cases} \frac{g(x)}{g'(x^*)}, & 0 \leq x \leq x^*, \\ x - x^* + U(x^*), & x > x^*. \end{cases} \tag{14}$$

Remark 2.1. Theorem 4.3 of [46] presents the solution to the problem (2) under, essentially, (A.1)–(A.2) and the following relaxed version of (A.3):

$$\mu'(x) \leq r \quad \text{for all } x \geq 0. \tag{A.3$'$}$$

In particular it essentially says the following: if $x_b = 0$ (which is equivalent to $\mu(0) \leq 0$; see [41, Lemma 2.2]), then the optimal policy is to pay all initial surplus x as a dividend immediately for all x; if $x_b \in (0, \infty)$, then the solution is as in Proposition 2.2; if $x_b = \infty$, then no optimal policy exists, but the optimal value function can be obtained by considering the value function for a reflection dividend policy at a barrier b and then sending $b \to \infty$, i.e.

$$U(x) = \frac{g(x)}{\lim_{b \to \infty} g'(b)}. \tag{15}$$

Let us explain some of the notation used in the present paper. As used above, the derivative of a one-dimensional function $f(\cdot)$ is denoted by $f'(\cdot)$. This notation is also used for the derivative of a multidimensional function with respect to the first variable, provided the latter is separated from the other variables with a semicolon; otherwise, the derivative with respect to, say, x is indicated by a subindex x. By way of example, $J'(x; \underline{x}, \bar{x})$ is the derivative of $J(x; \underline{x}, \bar{x})$ with respect to x, while $A_x(x, y)$ is the derivative of $A(x, y)$ with respect to x. We also use the general notation $f(x+) := \lim_{y \searrow x} f(y)$ and $f(x-) := \lim_{y \nearrow x} f(y)$.

3. Precommitment solution

We define the precommitment interpretation of the constrained optimal dividend problem as

$$V(x_0) = \sup_{S \in \mathcal{A}(x_0, k)} J(x_0, S), \quad \text{where } x_0 > 0 \text{ is arbitrary but fixed.} \tag{16}$$

The solution to the problem (16) is presented in Theorem 3.1. Our approach to this constrained problem relies on the Lagrangian idea. If we add the constraint as a penalty term with Lagrange parameter $\lambda > 0$, the unconstrained optimization problem reads as follows (see (9) and (10)):

$$\begin{aligned} &J(x_0, S) - \lambda \left(R(x_0, S) - \frac{1}{k} \right) \\ &= \mathbb{E}_{x_0}\left(\sum_{n : \tau_n \leq \tau} e^{-r\tau_n} \zeta_n \right) - \lambda \left(\mathbb{E}_{x_0}\left(\sum_{n : \tau_n \leq \tau} e^{-r\tau_n} \right) - \frac{1}{k} \right) \\ &= \mathbb{E}_{x_0}\left(\sum_{n : \tau_n \leq \tau} e^{-r\tau_n} (\zeta_n - \lambda) \right) + \frac{\lambda}{k}. \end{aligned}$$

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We thus see a natural connection to the well-studied dividend problem in the case where a fixed cost $c > 0$ is incurred each time a dividend is paid. Therefore, it is not surprising that our treatment relies on properties of the fixed-cost dividend problem, which in the present setting corresponds to

$$\sup_{S\in\mathcal{A}(x)} H(x; c, S), \quad \text{where } c > 0 \text{ and}$$
$$H(x; c, S) := \mathbb{E}_x\left(\sum_{n\,:\,\tau_n\le\tau} e^{-r\tau_n}(\zeta_n - c)\right) \tag{17}$$
$$(= J(x; S) - cR(x; S))\,.$$

In the case of a constant lump sum dividend barrier policy $(\underline{x}, \bar{x})$, we write the function $H(x; c, S)$ as $H(x; c, \underline{x}, \bar{x})$ and note that

$$H(x; c, \underline{x}, \bar{x}) = \begin{cases} g(x)\dfrac{\bar{x}-\underline{x}-c}{g(\bar{x})-g(\underline{x})}, & 0 \le x \le \bar{x}, \\ x - \underline{x} - c + g(\underline{x})\dfrac{\bar{x}-\underline{x}-c}{g(\bar{x})-g(\underline{x})}, & x > \bar{x}. \end{cases} \tag{18}$$

The solution to the problem (17) is presented in Proposition 3.1 below, which follows directly from [41, Theorem 2.1 and Remark 2.2(e)] together with Lemma 2.1 (another reference is [6, Theorem 2.3 and Remark 2.4]); see also Remark 3.4. Recall that (A.1)–(A.5) are assumed throughout the paper.

Proposition 3.1. *For any fixed $c > 0$, a constant lump sum dividend barrier policy independent of x is optimal in* (17). *In particular, the following hold:*

(i) If the (smooth fit) equation system

$$H'\big(\bar{x}-;\, c, \underline{x}, \bar{x}\big) = 1, \quad H'\big(\underline{x};\, c, \underline{x}, \bar{x}\big) = 1, \quad \underline{x} > 0, \tag{19}$$

has a solution, denoted by $(\underline{x}_c, \bar{x}_c)$, then it is unique and it is also an optimal constant lump sum dividend barrier policy, i.e.

$$\sup_{S\in\mathcal{A}(x)} H(x; c, S) = H\big(x; c, \underline{x}_c, \bar{x}_c\big), \quad \textit{for all } x \ge 0.$$

(ii) If (19) *does not have a solution, then the (smooth fit) equation*

$$H'\big(\bar{x}-;\, c, 0, \bar{x}\big) = 1 \tag{20}$$

has a unique solution, denoted by $\bar{x}_c$, and the constant lump sum dividend barrier policy $(\underline{x}_c, \bar{x}_c) = (0, \bar{x}_c)$ is optimal, i.e.

$$\sup_{S\in\mathcal{A}(x)} H(x; c, S) = H\big(x; c, 0, \bar{x}_c\big), \quad \textit{for all } x \ge 0.$$

Proposition 3.2 below presents properties of the solution to the problem (17) which we rely on when proving the main results of the present section, Theorem 3.1 and Corollary 3.1. Most of these properties have been established before—if not exactly for the problem (17) and the setting of the present paper, then for similar problems; see Remark 3.5.

Proposition 3.2. *The optimal policy in Proposition 3.1 has the following properties:*

$$0 \le \underline{x}_c < x^* < \bar{x}_c; \tag{21}$$

$$\bar{x}_c \text{ is continuous and increasing in } c; \ \underline{x}_c \text{ is continuous and decreasing in } c; \tag{22}$$

$$\underline{x}_c, \bar{x}_c \to x^* > 0 \text{ as } c \searrow 0; \tag{23}$$

$$H(x; c, \underline{x}_c, \bar{x}_c) \to U(x) \text{ as } c \searrow 0, \text{ for any } x \ge 0; \tag{24}$$

$$\bar{x}_c > c \text{ and (hence) } \bar{x}_c \to \infty \text{ as } c \to \infty; \tag{25}$$

$$\text{there exists a } \bar{c} > 0 \text{ such that if } c \ge \bar{c}, \text{ then } \left(\underline{x}_c, \bar{x}_c\right) \text{ is a ruin policy, i.e. } \underline{x}_c = 0 \tag{26}$$

(recall that x^* and $U(x)$ correspond to the solution to the problem (2); cf. Proposition 2.2).

We also need the following result.

Lemma 3.1. *For any fixed initial surplus $x_0 > 0$ there exists a unique constant $c(x_0, k) > 0$ (depending on x_0 and k) such that $R\left(x_0; \underline{x}_c, \bar{x}_c\right) \ge \frac{1}{k}$ for $c \le c(x_0, k)$ and $R\left(x_0; \underline{x}_c, \bar{x}_c\right) \le \frac{1}{k}$ for $c \ge c(x_0, k)$, where we recall that $\left(\underline{x}_c, \bar{x}_c\right)$ is determined in Proposition 3.1. In particular,*

$$R\left(x_0; \underline{x}_{c(x_0,k)}, \bar{x}_{c(x_0,k)}\right) = \frac{1}{k}. \tag{27}$$

Let the constant lump sum dividend barrier policy determined by Proposition 3.1 using the cost $c(x_0, k)$ be denoted by $\left(\underset{\sim}{x}_k, \tilde{x}_k\right)$; i.e. let

$$\left(\underset{\sim}{x}_k, \tilde{x}_k\right) := \left(\underline{x}_{c(x_0,k)}, \bar{x}_{c(x_0,k)}\right). \tag{28}$$

Theorem 3.1. (Precommitment solution.) *For any fixed initial surplus $x_0 > 0$, the constant lump sum dividend barrier policy $\left(\underset{\sim}{x}_k, \tilde{x}_k\right)$ defined in (28) is optimal in (16), and the corresponding optimal precommitment value is*

$$V(x_0) = \begin{cases} \dfrac{\tilde{x}_k - \underset{\sim}{x}_k}{k}, & 0 \le x_0 \le \tilde{x}_k, \\ x_0 - \underset{\sim}{x}_k + \dfrac{g(\underset{\sim}{x}_k)}{g(x_0)} \frac{\tilde{x}_k - \underset{\sim}{x}_k}{k}, & x_0 > \tilde{x}_k. \end{cases} \tag{29}$$

Proof of Theorem 3.1. Using e.g. Proposition 2.1, in particular (11)–(12), and also (27)–(28), it is easy to verify that the right side of (29) is equal to $J\left(x_0; \underset{\sim}{x}_k, \tilde{x}_k\right)$, i.e. the value obtained when using the constant lump sum dividend barrier policy $\left(\underset{\sim}{x}_k, \tilde{x}_k\right)$, and the second statement therefore follows from the first statement.

Note that $\left(\underset{\sim}{x}_k, \tilde{x}_k\right) \in \mathcal{A}(x_0; k)$ by (27)–(28). Consider an arbitrary fixed dividend policy $\bar{S} = (\tau_n, \zeta_n)_{n\ge1} \in \mathcal{A}(x_0, k)$. Using $c(x_0, k) > 0$ (Lemma 3.1) and the constraint (6) (see also (9)–(10)), we obtain

$$\begin{aligned} J\left(x_0; \bar{S}\right) &\le J\left(x_0; \bar{S}\right) - c(x_0, k)\left(R\left(x_0; \bar{S}\right) - \frac{1}{k}\right) \\ &= \mathbb{E}_{x_0}\left(\sum_{n\,:\,\tau_n \le \tau} e^{-r\tau_n}\left(\zeta_n - c(x_0, k)\right)\right) + c(x_0, k)\frac{1}{k}. \end{aligned}$$

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Now we use first Proposition 3.1 and second the observation in (17) to see that

$$\sup_{S\in\mathcal{A}(x_0)} \mathbb{E}_{x_0}\left(\sum_{n:\tau_n\le\tau} e^{-r\tau_n}\,(\zeta_n - c(x_0,k))\right)$$
$$= H\big(x_0; c(x_0,k), \underline{x}_{c(x_0,k)}, \bar{x}_{c(x_0,k)}\big)$$
$$= J\big(x_0; \underline{x}_{c(x_0,k)}, \bar{x}_{c(x_0,k)}\big) - c(x_0,k)R\big(x_0; \underline{x}_{c(x_0,k)}, \bar{x}_{c(x_0,k)}\big)\,.$$

Hence, using Lemma 3.1 again we obtain

$$J(x_0;\bar{S}) \le J\big(x_0; \underline{x}_{c(x_0,k)}, \bar{x}_{c(x_0,k)}\big) - c(x_0,k)R\big(x_0; \underline{x}_{c(x_0,k)}, \bar{x}_{c(x_0,k)}\big) + c(x_0,k)\frac{1}{k}$$
$$= J\big(x_0; \underline{x}_{c(x_0,k)}, \bar{x}_{c(x_0,k)}\big)\,. \qquad \square$$

Remark 3.1. The barrier $\tilde{x}_k$ and the dividend $\tilde{x}_k - \underset{\sim}{x}_k$ in the optimal policy for the problem (16) (see Theorem 3.1) depend on the initial surplus x_0. It is therefore clear that the optimal dividend policy $(\underset{\sim}{x}_k, \tilde{x}_k)$ chosen at time 0 will not generally be optimal at a time $t > 0$ when the constraint (6) is updated with the value for the state process observed at t, and that the problem of the present paper is in this sense time-inconsistent (Figure 3 in Section 5 illustrates how the precommitment value $V(x_0)$ depends on x_0 in a specific example). A possible generalization of the present model would be to replace the right-hand side of the constraint (6) with a state-dependent function satisfying suitable conditions; note, however, that we would generally expect the problem to remain time-inconsistent if we did so, and that such a generalization is outside the scope of the present paper. We remark that the inconsistency has to do with the reward criterion considered here. If we instead consider the long-term average criterion

$$\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}_x(D_{T+})$$

with corresponding constraint

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_x\left(\sum_{n\,:\,\tau_n\le T} 1\right)\left[= \limsup_{T\to\infty}\frac{1}{T}\mathbb{E}_x|\{n:\tau_n\le T\}|\right] \le \frac{1}{k},$$

then the solution to the problem will become independent of the initial state x owing to the ergodicity of the situation. We refer to [25] for the treatment of a problem of this type in portfolio optimization.

Remark 3.2. Consider an arbitrary fixed initial surplus $x_0 > 0$. Relying on e.g. Lemma 2.2(i), Lemma 3.1, and Theorem 3.1, it is easy to see that the optimal precommitment policy $\tilde{x}_k - \underset{\sim}{x}_k$ can be determined as follows:

- Pick a cost $c_1 > 0$.
- Determine (numerically) $\big(\underline{x}_{c_1}, \bar{x}_{c_1}\big)$ according to Proposition 3.1.
- If $R\big(x_0; \underline{x}_{c_1}, \bar{x}_{c_1}\big) > \frac{1}{k}$, we know that c_1 is too low, i.e. $c_1 < c(x_0,k)$, and we set $c_1 = c_1^l$ and choose a new $c_2 > c_1^l$. If $R\big(x_0; \underline{x}_{c_1}, \bar{x}_{c_1}\big) < \frac{1}{k}$ then we know that c_1 is too high, i.e. $c_1 > c(x_0,k)$, and we set $c_1 = c_1^h$ and choose a new $c_2 < c_1^h$.

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- Iterate the steps above, always choosing (using e.g. the bisection method) c_{n+1} larger than all c_i^l, $i \leq n$, and smaller than all c_i^h, $i \leq n$, until you find a c which attains (27), i.e. with $R\big(x_0; \underset{\sim}{x}_c, \tilde{x}_c\big) = \frac{1}{k}$. Now set $(\underset{\sim}{x}_k, \tilde{x}_k) = \big(\underset{\sim}{x}_c, \tilde{x}_c\big)$; then this is the optimal precommitment policy given the initial surplus x_0.

We remark that more numerically efficient methods to find the optimal precommitment policy are of course likely to exist.

The result below follows directly from Lemma 2.2(i), Proposition 3.2, and Lemma 3.1:

Corollary 3.1. *For any fixed initial surplus $x_0 > 0$, the precommitment solution $(\underset{\sim}{x}_k, \tilde{x}_k)$ in Theorem 3.1 has the following properties:*

$0 \leq \underset{\sim}{x}_k < x^* < \tilde{x}_k$;

$\tilde{x}_k$ is continuous and increasing in k; $\underset{\sim}{x}_k$ is continuous and decreasing in k;

$\underset{\sim}{x}_k, \tilde{x}_k \to x^*$ as $k \to 0$;

$V(x_0) \to U(x_0)$ as $k \to 0$;

$\tilde{x}_k \to \infty$ as $k \to \infty$;

there exists a $\bar{k} > 0$ such that if $k \geq \bar{k}$ then $\big(\underset{\sim}{x}_k, \tilde{x}_k\big)$ is a ruin policy, i.e. $\underset{\sim}{x}_k = 0$.

Remark 3.3. The main interpretations of Corollary 3.1 are the following: (i) relaxing the constraint (6) in the sense of sending $k \to 0$ implies that the optimal precommitment solution for the constrained problem (16) converges monotonically to the solution of the unconstrained problem (2); cf. Proposition 2.2; (ii) if the constraint is sufficiently restrictive, i.e. if k is sufficiently large, then a ruin policy is optimal.

Remark 3.4. In [41] the problem (17) is solved under essentially (A.1)–(A.2) and (A.3′). Under these less restrictive assumptions it may be that neither Case (i) nor Case (ii) in Proposition 3.1 holds; and in this case, according to [41, Theorem 2.1], it holds that no optimal policy exists, but the optimal value function can be obtained by considering the value function for a reflection dividend policy at a barrier b and then sending $b \to \infty$, as in (15).

Remark 3.5. The properties (21) and (22) are established in [41, p. 675] in a setting similar to that of the present paper. The property (23) is established in [33, Remark 2] for a model based on a Wiener process with drift.

4. Time-consistent solution

Let us start by defining the notions of a pure dividend strategy and a (pure subgame-perfect Nash) equilibrium. These definitions are motivated in Section 4.1, which also contains a discussion of the results in this section.

Definition 4.1. *An impulse control policy $S = (\tau_n, \zeta_n)_{n\geq 1}$ (see (4)) is said to be a pure Markov dividend strategy profile if for each $x \geq 0$ the following hold:*

- *Each dividend date is an exit time from a set $\mathcal{W} \subseteq [0, \infty)$ that is open in $[0, \infty)$, i.e.*

$$\tau_1 = \inf\{t \geq 0 : X_t \notin \mathcal{W}\}, \quad \tau_n = \inf\{t > \tau_{n-1} : X_t \notin \mathcal{W}\}, \quad n = 2, 3, \ldots.$$

- *Each dividend is given by $\zeta_n = \zeta\big(X_{\tau_n}\big)$, for some measurable function $\zeta(\cdot)$ satisfying $x - \zeta(x) \in \{0\} \cup \mathcal{W}$ for all $x \notin \mathcal{W}$.*

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From now on we refer to a pure Markov dividend strategy profile as a pure dividend strategy. We use the notation $\nu_h := \inf\{t \geq 0 : X_t \notin (x-h, x+h)\}$.

Definition 4.2. (Equilibrium.) *A pure dividend strategy $\hat{S}$ is said to be a (pure subgame-perfect Nash) equilibrium if for all $x > 0$ the following hold:*

$$\hat{S} \in \mathcal{A}(x, k); \tag{EqI}$$

for the process X satisfying (1) with $D_t = 0$ for all $t \geq 0$ it holds that

$$\liminf_{h \searrow 0} \frac{J(x; \hat{S}) - \mathbb{E}_x\left(e^{-r\nu_h} J(X_{\nu_h}; \hat{S})\right)}{\mathbb{E}_x(\nu_h)} \geq 0; \tag{EqII}$$

$$\text{for all } y \in [0, x], \text{ if } R(y; \hat{S}) \leq \tfrac{1}{k} - 1 \text{ then } J(x; \hat{S}) \geq J(y; \hat{S}) + x - y. \tag{EqIII}$$

Moreover, $J(\cdot\,; \hat{S})$ is said to be the equilibrium value function (corresponding to $\hat{S}$).

Remark 4.1. If $k > 1$, then the only strategy that satisfies (EqI) for each x is to never pay dividends; it is easy to see that this is the unique equilibrium in this case. With the remark above in mind, we assume in the rest of this section that

$$k \leq 1. \tag{A.6}$$

Recall also that (A.1)–(A.5) are assumed throughout the paper. We need the following result.

Lemma 4.1.

(i) The system of equations

$$R(\bar{x}; \underline{x}, \bar{x}) = \frac{1}{k}, \tag{30}$$

$$J'(\bar{x}-\,; \underline{x}, \bar{x}) = 1 \tag{31}$$

has a unique solution $(\underline{x}, \bar{x})$, for which it holds that $0 \leq \underline{x} < x^* < \bar{x}$. Moreover, if $k = 1$ then $\underline{x} = 0$, and if $k < 1$ then $\underline{x} > 0$.

(ii) Let $(\underline{x}, \bar{x})$ be the unique solution to (30)–(31). Then $\bar{x}$ is determined by the (smooth fit) equation

$$J'\left(\bar{x}-\,; \bar{x} - k\frac{g(\bar{x})}{g'(\bar{x})}, \bar{x}\right) = 1, \tag{32}$$

and

$$\underline{x} = \bar{x} - k\frac{g(\bar{x})}{g'(\bar{x})}. \tag{33}$$

(iii) Equation (32) simplifies to

$$g\left(\bar{x} - k\frac{g(\bar{x})}{g'(\bar{x})}\right) = (1-k)g(\bar{x}). \tag{34}$$

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Theorem 4.3. (Time-consistent solution.)

(i) *The constant lump sum dividend barrier strategy* $(\underset{\wedge}{x}_k, \hat{x}_k)$, *where* $\hat{x}_k$ *is determined by the (smooth fit) equation* (32) *and*

$$\underset{\wedge}{x}_k = \hat{x}_k - k\frac{g(\hat{x}_k)}{g'(\hat{x}_k)},$$

is an equilibrium.

(ii) *The equilibrium value function is given by*

$$J(x; \underset{\wedge}{x}_k, \hat{x}_k) = \begin{cases} \dfrac{g(x)}{g'(\hat{x}_k)}, & 0 \le x \le \hat{x}_k, \\[2ex] x - \underset{\wedge}{x}_k + \dfrac{g(\underset{\wedge}{x}_k)}{g'(\hat{x}_k)}, & x > \hat{x}_k. \end{cases} \tag{35}$$

Proof of Theorem 4.3. Item (ii) follows from (i), Proposition 2.1, and Lemma 4.1. Let us prove (i). The condition (EqI) (in Definition 4.2) is directly verified for each x using that $(\underset{\wedge}{x}_k, \hat{x}_k)$ satisfies (30) and $\underset{\wedge}{x}_k \ge 0$ (Lemma 4.1). By Lemma 4.1,

$$J'(\hat{x}_k - ; \underset{\wedge}{x}_k, \hat{x}_k) = (J^0)'(\hat{x}_k; \underset{\wedge}{x}_k, \hat{x}_k) = J'(\hat{x}_k + ; \underset{\wedge}{x}_k, \hat{x}_k) = 1.$$

Using e.g. (11), (13), and Lemma 2.1, it is easy to verify that

$$(A_X - r)\, J^0(\hat{x}_k; \underset{\wedge}{x}_k, \hat{x}_k) = 0 \quad \text{and} \quad (J^0)''(x; \underset{\wedge}{x}_k, \hat{x}_k) \ge 0 \text{ for } x \ge x^*.$$

Recall that $\hat{x}_k > x^*$ (Lemma 4.1) and that $r > 0$. Using the above and also that $\mu'(x) - r \le 0$ (Assumption 2.1), we find that for any $x > \hat{x}_k$,

$$\begin{aligned} &\mu(x) - r\big(x - \hat{x}_k + J(\hat{x}_k; \underset{\wedge}{x}_k, \hat{x}_k)\big) \\ &\le \mu(\hat{x}_k) - rJ(\hat{x}_k; \underset{\wedge}{x}_k, \hat{x}_k) \\ &< \mu(\hat{x}_k)(J^0)'(\hat{x}_k; \underset{\wedge}{x}_k, \hat{x}_k) + \frac{1}{2}\sigma^2(\hat{x}_k)(J^0)''(\hat{x}_k; \underset{\wedge}{x}_k, \hat{x}_k) - rJ(\hat{x}_k; \underset{\wedge}{x}_k, \hat{x}_k) \\ &= (A_X - r)\, J^0(\hat{x}_k; \underset{\wedge}{x}_k, \hat{x}_k) = 0. \end{aligned}$$

Using the observations above and (11), we obtain

$$\begin{aligned} (A_X - r)\, J(x; \underset{\wedge}{x}_k, \hat{x}_k) &= \begin{cases} 0, & 0 \le x < \hat{x}_k, \\ \mu(x) - r\big(x - \underset{\wedge}{x}_k + J^0(\underset{\wedge}{x}_k; \underset{\wedge}{x}_k, \hat{x}_k)\big), & x > \hat{x}_k, \end{cases} \\ &= \begin{cases} 0, & 0 \le x < \hat{x}_k, \\ \mu(x) - r\big(x - \hat{x}_k + J^0(\hat{x}_k; \underset{\wedge}{x}_k, \hat{x}_k)\big), & x > \hat{x}_k, \end{cases} \\ &\le 0 \end{aligned}$$

Using also a generalized Itô formula (see e.g. [45, Section 3.5]), we find that the numerator in (EqII) satisfies

$$\begin{aligned} &J(x; \underset{\wedge}{x}_k, \hat{x}_k) - \mathbb{E}_x\big(e^{-r\nu_h} J(X_{\nu_h}; \underset{\wedge}{x}_k, \hat{x}_k)\big) \\ &= -\mathbb{E}_x\left(\int_0^{\nu_h} I_{\{X_s \ne \hat{x}_k\}} e^{-rs}\, (A_X - r)\, J(X_s; \underset{\wedge}{x}_k, \hat{x}_k)\, ds\right) \\ &\ge 0, \quad \text{for any } h > 0 \text{ and any } x > 0. \end{aligned}$$

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This implies that (EqII) holds for each x. Using the fact that $R\big(\hat{x}_k; \underaccent{\hat}{x}_k, \hat{x}_k\big) = \frac{1}{k}$, $R\big(\underaccent{\hat}{x}_k; \underaccent{\hat}{x}_k, \hat{x}_k\big) + 1 = \frac{1}{k}$, and $R\big(y; \underaccent{\hat}{x}_k, \hat{x}_k\big)$ is increasing in y, we see that

$$R\big(y; \underaccent{\hat}{x}_k, \hat{x}_k\big) \le \frac{1}{k} - 1 \Rightarrow y \le \underaccent{\hat}{x}_k. \tag{36}$$

Let $K(x) := J\big(x; \underaccent{\hat}{x}_k, \hat{x}_k\big) - x$. It is easy to verify that $K(0) = 0$, $K(\underaccent{\hat}{x}_k) = K\big(\hat{x}_k\big)$, $K'\big(\hat{x}_k\big) = 0$, $K''(x) < 0$ for $x < x^*$, and $K''(x) > 0$ for $x > x^*$. Using also that $\underaccent{\hat}{x}_k < x^* < \hat{x}_k$, it is easy to see that $K(\cdot)$ is increasing on $\big[0, \underaccent{\hat}{x}_k\big)$ and that

$$\begin{aligned} &\text{if } 0 \le y \le \underaccent{\hat}{x}_k \text{ and } x \ge y, \text{ then} \\ &K(x) - K(y) = J\big(x; \underaccent{\hat}{x}_k, \hat{x}_k\big) - x - \big(J\big(y; \underaccent{\hat}{x}_k, \hat{x}_k\big) - y\big) \ge 0. \end{aligned} \tag{37}$$

From (36) and (37) it follows that (EqIII) holds. □

Remark 4.2. The equilibrium $\big(\underaccent{\hat}{x}_k, \hat{x}_k\big)$ can be found as follows. First, find the unique solution to (34) and set $\hat{x}_k$ equal to this solution. This ensures that the smooth fit equilibrium condition (32) is satisfied. Second, set

$$\underaccent{\hat}{x}_k = \hat{x}_k - k\frac{g\big(\hat{x}_k\big)}{g'\big(\hat{x}_k\big)}.$$

In line with the usual situation in game theory, there is no reason to suspect that the equilibrium $\big(\underaccent{\hat}{x}_k, \hat{x}_k\big)$ is unique; it is easy to see, however, that it is unique within the class of constant lump sum dividend barrier policies.

The following definition corresponds to an adaptation of the notion of a strong equilibrium; see Section 4.1 for a motivation. Note that a strong equilibrium is necessarily an equilibrium in the sense of Definition 4.2.

Definition 4.4. (Strong equilibrium.) *An equilibrium $\hat{S}$ is strong if the condition* (EqII) *in Definition 4 can be replaced by the following:*

$$\begin{aligned} &\textit{for the process } X \textit{ satisfying } (1) \textit{ with } D_t = 0 \textit{ for all } t \ge 0, \textit{ there exists} \\ &\textit{an } \bar{h} > 0 \textit{ such that } J\big(x; \hat{S}\big) \ge \mathbb{E}_x\Big(e^{-r\nu_h} J\big(X_{\nu_h}; \hat{S}\big)\Big), \quad \textit{for all } h \in [0, \bar{h}]. \end{aligned} \tag{EqII$'$}$$

It is easy to see that the arguments that imply that the condition (EqII) holds in the proof of Theorem 4.3 also imply that $J\big(x; \underaccent{\hat}{x}_k, \hat{x}_k\big) - \mathbb{E}_x\big(e^{-r\tau} J\big(X_\tau; \underaccent{\hat}{x}_k, \hat{x}_k\big)\big) \ge 0$ holds for an arbitrary stopping time τ. In particular we therefore obtain the following result.

Theorem 4.5. *The equilibrium $\big(\underaccent{\hat}{x}_k, \hat{x}_k\big)$ in Theorem 4.3 is strong.*

We also find the following.

Theorem 4.6. *The equilibrium $\big(\underaccent{\hat}{x}_k, \hat{x}_k\big)$ in Theorem 4.3 has the following properties:*

$$0 \le \underaccent{\hat}{x}_k < x^* < \hat{x}_k; \tag{38}$$

$$\hat{x}_k \textit{ is continuous and increasing in } k; \ \underaccent{\hat}{x}_k \textit{ is continuous and decreasing in } k; \tag{39}$$

$$\underaccent{\hat}{x}_k, \hat{x}_k \to x^* \textit{ as } k \to 0; \tag{40}$$

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$$J\left(x; \underset{\wedge}{x}_k, \hat{x}_k\right) \to U(x) \text{ as } k \to 0, \text{ for any } x \geq 0; \tag{41}$$

$$\text{the equilibrium is a ruin strategy, i.e. } \underset{\wedge}{x}_k = 0, \text{ if and only if } k = 1. \tag{42}$$

4.1. Motivation of equilibrium definition and discussion

The maximization of the value function $J(x; S)$ in (5) under the constraint (6) is time-inconsistent in the sense that the optimal policy depends on x; see Remark 3.1. The game-theoretic approach is to suppose that the decision-maker, or controller, in a time-inconsistent problem is a person with time-inconsistent preferences and to reinterpret the problem as an intrapersonal sequential dynamic game. For the problem of the present paper this means identifying each $x \in (0, \infty)$ with an agent, called the x-agent, who decides whether a dividend should be paid, and of what size, if the current value of the process X is x; and letting all x-agents play a sequential dynamic game against each other regarding how to pay dividends from the surplus process X. (We remark that similar interpretations for regular time-inconsistent stochastic control can be found in e.g. [13, 14, 35], and in [22–24] for time-inconsistent stopping.) The interpretation of Definition 4 is therefore that each x-agent must make his decision at x without randomization, and based only on the current value x. From this perspective it is clear that Definition 4 corresponds to a pure Markov strategy profile. (Recall that, in general, a pure Markov strategy depends only on past events that are payoff-relevant and determines the action of an agent without randomization, and that a strategy profile is a complete specification of the strategies of all agents in a game.)

The items in Definitions 4.2 and 4.4 have the following interpretations:

- The condition (EqI) ensures that an equilibrium $\hat{S}$ is admissible from the viewpoint of every x-agent. In particular, the constraint (6) is, from the viewpoint of every x-agent, satisfied.

- The condition (EqII) is an adaptation of the usual first-order equilibrium condition in time-inconsistent stochastic control, studied for a general model in continuous time in [13]. The interpretation is that if (EqII) holds, then any x-agent prefers using $\hat{S}$ over doing nothing until τ_h. More precisely, the interpretation of (EqII) is that an x-agent's criterion for not deviating from $\hat{S}$ by not paying a dividend at x when $\hat{S}$ prescribes paying a dividend at x is that the instantaneous expected rate of change relative to $\mathbb{E}_x(\nu_h)$ obtained by deviating is nonpositive. In line with [13, Remark 3.5] and [23, Section 2.1], we remark that this kind of first-order equilibrium may correspond to a stationary point that is not a maximum, in the sense that the numerator in (EqII) can be negative for each fixed $h > 0$ and still be in line with (EqII) by vanishing with order $o\big(\mathbb{E}_x(\nu_h)\big)$. Clearly, it is not entirely satisfactory in every situation that an equilibrium may correspond to a stationary point that is not a maximum in this sense. Motivated by an observation of this kind, the notion of a strong equilibrium was defined in a time-inconsistent stochastic control framework in [32]. The condition (EqII) is an adaptation of this notion to the problem of the present paper, and the interpretation is obvious based on the discussion above. In particular, the more restrictive condition (EqII′) implies not only that (EqII) holds but also that the sequence in (EqII) cannot approach 0 from below, and hence the issue of non-maximal stationary points mentioned above is not present under this condition. Similarly, since (EqII) does allow for non-maximal stationary points, we may, in the spirit of [32], call it a weak equilibrium condition. The relationship between (EqII) and

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(EqII′) and alternative notions of weak and strong equilibria based on the concatenation of control strategies is further studied in Section 4.2.

- Condition (EqIII) means that if, for an x-agent, it is admissible to pay a dividend, then the action prescribed for the x-agent by $\hat{S}$ is more desirable, from the viewpoint of that x-agent, than paying any (alternative) admissible dividend.

A conclusion of Theorem 4.6 is that the equilibrium solution for the constrained problem converges monotonically to the optimal solution for the unconstrained problem when the constraint (6) vanishes in the sense of sending $k \to 0$. It is clearly desirable that an equilibrium solution is optimal in the absence of time-inconsistency, and the fact just mentioned thus further supports our definition of equilibrium.

Equation (32) can be said to be a smooth fit equilibrium condition. We remark that a smooth fit equilibrium condition for a time-inconsistent stopping problem was found in [23].

4.2. Alternative notions of equilibria

Here we define alternative notions of equilibria based on the concatenation of control strategies. We also show that these notions of equilibria are equivalent to those defined above. The notion of equilibrium in Definition 4.7 (below) is in the spirit of the kind of equilibrium that is in [32] termed a weak equilibrium. To contrast with the notion of a weak equilibrium, the notion of a strong equilibrium was defined in [32] in the context of a time-inconsistent stochastic control problem. Definition 4.8 (below) is in the spirit of such a strong equilibrium. See the discussion above and [32] for the background and motivation of these terms.

We start by defining the concatenation $S \otimes_{\nu_h} \hat{S}$ of two pure dividend strategy profiles $S = (\rho_n, \varsigma_n)_{\{n\geq 1\}}$ and $\hat{S} = (\tau_n, \zeta_n)_{\{n\geq 1\}}$ at ν_h—which is the exit time from $(x-h, x+h)$ of the state process when dividends are paid according to S—as the impulse control policy corresponding to following S until (and including) ν_h and then following $\hat{S}$ from ν_h. Note that $S \otimes_{\nu_h} \hat{S}$ is not exactly an impulse control of the kind defined in (4), since it allows the possibility of two jumps at ν_h; in particular, if the starting value x is such that S dictates that the state process is immediately sent to a state $y < x - h$, then $\nu_h = 0$, and if y is such that $\hat{S}$ dictates an immediate jump, then two jumps occur between $t = 0$ and $t = 0+$. We define $J\left(x; S \otimes_{\nu_h} \hat{S}\right)$ and $R\left(x; S \otimes_{\nu_h} \hat{S}\right)$ in analogy with $J(x; S)$ and $R(x; S)$; cf. (5) and (6).

Let us make some initial observations. Let U denote the union of non-dividend-payment sets for S (cf. Definition 4.1). If $x \in U$, i.e. S does not pay an immediate dividend, then the openness of U implies that for any sufficiently small (fixed) h (i.e. h is smaller than the shortest distance from x to the boundary of U), it holds that $0 < \nu_h < \rho_1$ a.s., so that no dividends are paid according to S and no immediate dividend is paid, and

$$J(x; \hat{S}) - J\left(x; S \otimes_{\nu_h} \hat{S}\right) = J(x; \hat{S}) - \mathbb{E}_x \left(e^{-r\nu_h} J(X_{\nu_h}; \hat{S})\right) \tag{43}$$

by the strong Markov property. We remark that X_{ν_h} is the value of the state process at ν_h when no dividends have been paid, and that (43) therefore coincides with the numerator in (EqII). Similarly, if $x \in U^C$, then for any sufficiently small $h > 0$, it holds that $\rho_2 > \rho_1 = \nu_h = 0$ a.s., so that

$$J(x; \hat{S}) - J\left(x; S \otimes_{\nu_h} \hat{S}\right) = J(x; \hat{S}) - \varsigma(x) - J\left(x - \varsigma(x); \hat{S}\right) \tag{44}$$

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and

$$R\left(x; S \otimes_{\nu_h} \hat{S}\right) = R\left(x - \varsigma(x); \hat{S}\right) + 1. \tag{45}$$

Definition 4.7. *A pure dividend strategy $\hat{S} \in \mathcal{A}(x, k)$ is said to be a (concatenation) equilibrium if for all $x > 0$,*

$$\liminf_{h \searrow 0} \frac{J(x; \hat{S}) - J\left(x; S \otimes_{\nu_h} \hat{S}\right)}{\mathbb{E}_x(\nu_h)} \geq 0$$

for all pure dividend strategies $S \in \mathcal{A}(x, k)$ for which there exists an $\bar{h} > 0$ such that

$$R\left(x; S \otimes_{\nu_h} \hat{S}\right) \leq \frac{1}{k}$$

for all $h \in [0, \bar{h}]$.

Remark 4.3. The requirement that the deviation strategy S in the concatenation has to be pure is in line with the literature; cf. e.g. [13, Definition 3.4].

Proposition 4.1. *The notions of equilibrium in Definition 4.2 and Definition 4.7 are equivalent.*

Proof. Note that the condition (EqI) is part of both definitions. Recall that dividends corresponding to $\hat{S}$ are paid according to a function, which we denote by $\zeta(\cdot)$, satisfying certain conditions; cf. Definition 4.2. Denote by $\varsigma(\cdot)$ the corresponding function for S.

1. Consider an arbitrary x. Let S pay no immediate dividend, i.e. $x \in U$, so that the numerator in the limit in Definition 4.7 and in (EqII) are the same for any sufficiently small (fixed) h; cf. (43). Hence, if $\hat{S}$ is an equilibrium according to Definition 4.7, then (EqII) holds (i.e. Definition 4.7 $\Rightarrow$ (EqII)).

2. Consider an arbitrary x. Let S pay an immediate dividend $\varsigma(x)$, i.e. $x \in U^C$. Define $y = x - \varsigma(x)$. Using $R\left(x; S \otimes_{\nu_h} \hat{S}\right) \leq \frac{1}{k}$ (Definition 4.7) and (45) we find

$$R\left(y; \hat{S}\right) \leq \frac{1}{k} - 1.$$

   Note that we may define S and $\varsigma(x)$ so that we attain any $y = x - \varsigma(x) \in [0, x]$ satisfying the above condition. Now, for a sufficiently small h, the numerator in the limit in Definition 4.7 satisfies

$$\begin{aligned} J(x; \hat{S}) - J\left(x; S \otimes_{\nu_h} \hat{S}\right) &= J(x; \hat{S}) - \varsigma(x) - J\left(x - \varsigma(x); \hat{S}\right) \\ &= J(x; \hat{S}) - (x - y) - J\left(y; \hat{S}\right), \end{aligned}$$

   by (44). Moreover, if $\hat{S}$ is an equilibrium according to Definition 10, then the expressions above are nonnegative. Hence, Definition 4.7 $\Rightarrow$ (EqIII).

3. Consider an arbitrary S satisfying the conditions of Definition 4.7 and an arbitrary x. Suppose $x \in U$. Then the numerator in Definition 4.7 satisfies (43), for any sufficiently small h. This implies that (EqII) $\Rightarrow$ Definition 4.7 for any S with $x \in U$.

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4. Consider an arbitrary S satisfying the conditions of Definition 4.7 and an arbitrary x. Suppose $x \in U^C$. Then, for any sufficiently small h, the numerator in Definition 10 is

$$J\big(x; \hat{S}\big) - \varsigma(x) - J\Big(x - \varsigma(x); \hat{S}\Big)\,;$$

cf. (44). Moreover, by (45),

$$R\Big(x; S \otimes_{\nu_h} \hat{S}\Big) = R\Big(x - \varsigma(x); \hat{S}\Big) + 1 \le \frac{1}{k},$$

which (assuming (EqIII) holds) implies that $J\big(x;\ \hat{S}\big) \ge J\Big(x - \varsigma(x); \hat{S}\Big) + \varsigma(x)$. This implies that the numerator in Definition 4.7 is nonnegative. Hence, (EqIII) $\Rightarrow$ Definition 4.7 for any S with $x \in U^C$. □

Definition 4.8. *A pure dividend strategy $\hat{S} \in \mathcal{A}(x, k)$ is said to be a strong (concatenation) equilibrium if for all $x > 0$,*

$$J\big(x; \hat{S}\big) - J\Big(x; S \otimes_{\nu_h} \hat{S}\Big) \ge 0$$

for all pure dividend strategies $S \in \mathcal{A}(x, k)$ such that

$$R\Big(x; S \otimes_{\nu_h} \hat{S}\Big) \le \frac{1}{k}$$

for all $h \in [0, \bar{h}]$, for some $\bar{h} > 0$.

Proposition 4.2. *The notions of strong equilibrium in Definition 4.4 and Definition 4.8 are equivalent.*

Proof. This statement can be proved using exactly the same arguments as in the proof above, which we note never relied on sending $h \to 0$. □

5. An example

In this section we suppose that the uncontrolled surplus process is a Wiener process with positive drift, and let $\mu > 0$ and $\sigma > 0$ denote its drift and volatility, respectively. Using elementary calculations (see e.g. [46, Section 5]), we find the canonical solution of (7)–(8) to be

$$g(x) = e^{\alpha_1 x} - e^{\alpha_2 x}$$
$$\text{where } \alpha_1 := -\frac{\mu}{\sigma^2} + \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}} \text{ and } \alpha_2 := -\frac{\mu}{\sigma^2} - \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma^2}}.$$

It is now easy (cf. (13)) to verify that

$$x^* = \frac{\log\left(\alpha_2^2/\alpha_1^2\right)}{\alpha_1 - \alpha_2} \in (0, \infty).$$

In all illustrations in this section we use the parameter values $\mu = 0.06$, $\sigma^2 = 0.03$, and $r = 0.02$. This implies that $x^* = 1.1405$. The strict concavity–convexity of $g(\cdot)$, in the sense of Lemma 2.1(iv), is illustrated in Figure 1.

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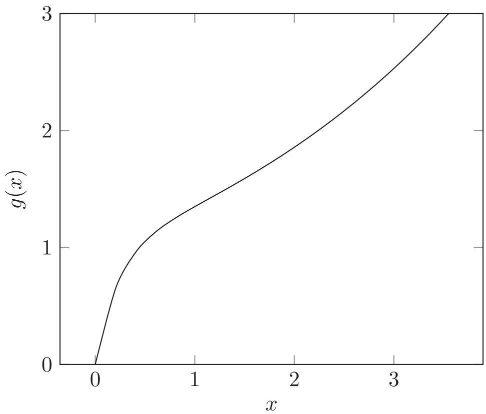

FIGURE 1. The canonical solution of (7)–(8) corresponding to a Wiener process with positive drift.

Let us first see how the optimal precommitment policy $(\underline{x}_k, \bar{x}_k)$ for an arbitrary fixed initial surplus $x_0 > 0$ is found. First, we use (18) to find

$$H\big(x; c, \underline{x}, \bar{x}\big) = \begin{cases} (e^{\alpha_1 x} - e^{\alpha_2 x})\dfrac{\bar{x} - \underline{x} - c}{e^{\alpha_1 \bar{x}} - e^{\alpha_2 \bar{x}} - e^{\alpha_1 \underline{x}} + e^{\alpha_2 \underline{x}}}, & 0 \le x \le \bar{x}, \\ x - \underline{x} - c + \big(e^{\alpha_1 \underline{x}} - e^{\alpha_2 \underline{x}}\big)\dfrac{\bar{x} - \underline{x} - c}{e^{\alpha_1 \bar{x}} - e^{\alpha_2 \bar{x}} - e^{\alpha_1 \underline{x}} + e^{\alpha_2 \underline{x}}}, & x > \bar{x}. \end{cases}$$

This implies that (19) is satisfied if

$$\big(\alpha_1 e^{\alpha_1 \bar{x}} - \alpha_2 e^{\alpha_2 \bar{x}}\big)\frac{\bar{x} - \underline{x} - c}{e^{\alpha_1 \bar{x}} - e^{\alpha_2 \bar{x}} - e^{\alpha_1 \underline{x}} + e^{\alpha_2 \underline{x}}} = 1, \tag{46}$$

$$\big(\alpha_1 e^{\alpha_1 \underline{x}} - \alpha_2 e^{\alpha_2 \underline{x}}\big)\frac{\bar{x} - \underline{x} - c}{e^{\alpha_1 \bar{x}} - e^{\alpha_2 \bar{x}} - e^{\alpha_1 \underline{x}} + e^{\alpha_2 \underline{x}}} = 1, \quad \text{for } \underline{x} > 0, \tag{47}$$

and that (20) is satisfied if

$$\big(\alpha_1 e^{\alpha_1 \bar{x}} - \alpha_2 e^{\alpha_2 \bar{x}}\big)\frac{\bar{x} - c}{e^{\alpha_1 \bar{x}} - e^{\alpha_2 \bar{x}}} = 1. \tag{48}$$

Second, we similarly obtain

$$R(x_0; \underline{x}, \bar{x}) = \begin{cases} \dfrac{e^{\alpha_1 x_0} - e^{\alpha_2 x_0}}{e^{\alpha_1 \bar{x}} - e^{\alpha_2 \bar{x}} - e^{\alpha_1 \underline{x}} + e^{\alpha_2 \underline{x}}}, & 0 \le x_0 \le \bar{x}, \\ 1 + \dfrac{e^{\alpha_1 \underline{x}} - e^{\alpha_2 \underline{x}}}{e^{\alpha_1 \bar{x}} - e^{\alpha_2 \bar{x}} - e^{\alpha_1 \underline{x}} + e^{\alpha_2 \underline{x}}}, & x_0 > \bar{x}. \end{cases} \tag{49}$$

In order to find the optimal precommitment policy $\big(\underline{x}_k, \bar{x}_k\big)$, we now consider some constant $c_1 > 0$ and determine $\big(\underline{x}_{c_1}, \bar{x}_{c_1}\big)$ as the solution to the dividend problem with fixed cost c_1 according to Proposition 3.1. This means that we let $\big(\underline{x}_{c_1}, \bar{x}_{c_1}\big)$ be the solution to (46)–(47) if it exists, and if it does not then we set $\big(\underline{x}_{c_1}, \bar{x}_{c_1}\big) = \big(0, \bar{x}_{c_1}\big)$ where $\bar{x}_{c_1}$ is the solution to (48). An illustration is presented in Figure 2.

We now evaluate the function in (49) with $\big(\underline{x}, \bar{x}\big) = \big(\underline{x}_{c_1}, \bar{x}_{c_1}\big)$ and then iterate this procedure with higher or lower costs in accordance with Remark 3.2 until we find a cost c such that (49)

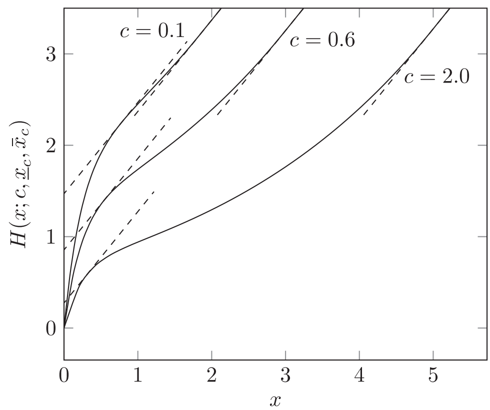

FIGURE 2. The optimal value function of the fixed-cost dividend problem (see (17)) for $c = 0.1$, $c = 0.6$, and $c = 2.0$ for which the optimal dividend policies are (0.7670, 1.8528), (0.5453, 2.9769), and (0.3183, 4.9580), respectively. The dashed lines indicate smooth fit.

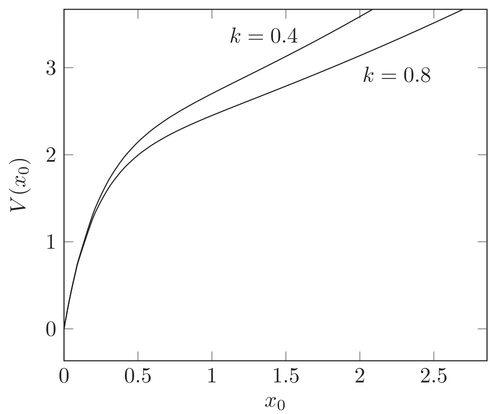

FIGURE 3. The precommitment value (see (29)) as a function of x_0 for $k = 0.4$ and $k = 0.8$.

evaluated at $(\underline{x}_c, \bar{x}_c)$ is equal to $\frac{1}{k}$, which then means that $(\underset{\check{}}{x}_k, \tilde{x}_k) = (\underline{x}_c, \bar{x}_c)$ for the particular initial surplus x_0 at hand. An illustration of the precommitment value as a function of x_0 is presented in Figure 3.

Let us now find the equilibrium dividend strategy $(\underset{\hat{}}{x}_k, \hat{x}_k)$. In the rest of the section we suppose that $k \leq 1$; cf. Remark 4.1. First, note that (34) becomes

$$e^{\alpha_1\left(\bar{x} - k\frac{e^{\alpha_1\bar{x}} - e^{\alpha_2\bar{x}}}{\alpha_1 e^{\alpha_1\bar{x}} - \alpha_2 e^{\alpha_2\bar{x}}}\right)} - e^{\alpha_2\left(\bar{x} - k\frac{e^{\alpha_1\bar{x}} - e^{\alpha_2\bar{x}}}{\alpha_1 e^{\alpha_1\bar{x}} - \alpha_2 e^{\alpha_2\bar{x}}}\right)} = (1-k)\left(e^{\alpha_1\bar{x}} - e^{\alpha_2\bar{x}}\right). \tag{50}$$

Following Remark 4.2 we now (i) ensure that the equilibrium smooth fit condition (32) holds by solving (50) and setting $\hat{x}_k$ equal to this solution, and (ii) set

$$\underset{\hat{}}{x}_k = \hat{x}_k - k\frac{g(\hat{x}_k)}{g'(\hat{x}_k)} = \hat{x}_k - k\frac{e^{\alpha_1\hat{x}_k} - e^{\alpha_2\hat{x}_k}}{\alpha_1 e^{\alpha_1\hat{x}_k} - \alpha_2 e^{\alpha_2\hat{x}_k}}.$$

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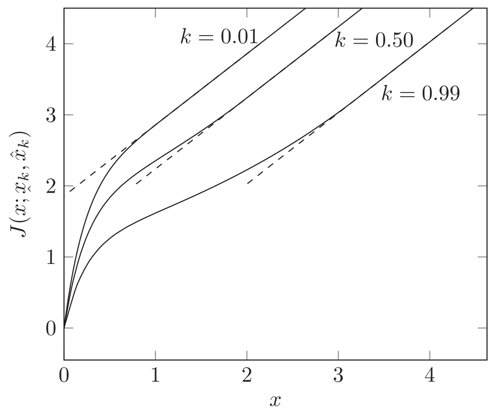

FIGURE 4. The equilibrium value functions (see (35)) for $k = 0.01$, $k = 0.5$, and $k = 0.99$, for which the equilibrium dividend strategies are (1.1206, 1.1507), (0.3757, 1.9893), and (0.0059, 3.2056), respectively. The dashed lines indicate smooth fit.

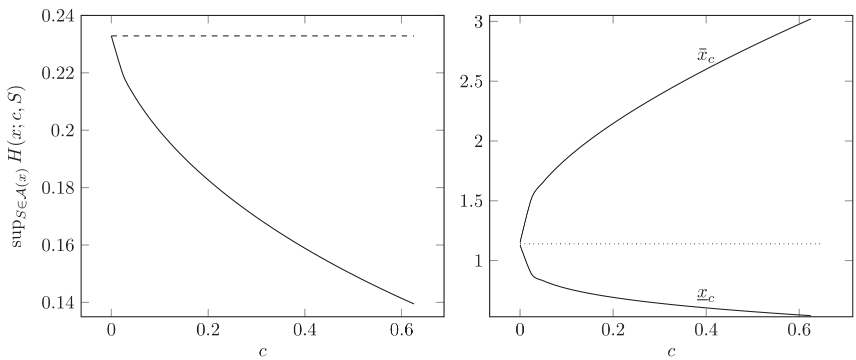

FIGURE 5. The first graph illustrates the value for the fixed-cost dividend problem as a function of c. The second graph illustrates the corresponding dividend policy as a function of c.

An illustration of the equilibrium value function and the equilibrium smooth fit principle is presented in Figure 4.

5.1. Sensitivity with respect to k and c

A natural question when studying impulse control problems is what happens for vanishing fixed costs, in particular what happens to the derivative of the optimal value function. To the best of our knowledge, such an analysis has not been carried out for the exact optimal dividend problem (17). However, slightly different problems without absorption are studied in [38] and the references therein, and it is investigated what happens when the fixed cost c is sent to zero; see also [3, 21, 39]. One main finding is that while the value of the problem converges to the one without fixed costs as $c \searrow 0$ (as in Proposition 3.2), the derivative with respect to c converges to $-\infty$. The interpretation is that small fixed costs have large effects on the value. Figure 5 illustrates these properties for the optimal dividend problem (17). (In this section we consider a fixed initial surplus $x = x_0 = 0.025$. The dashed line in each graph below indicates

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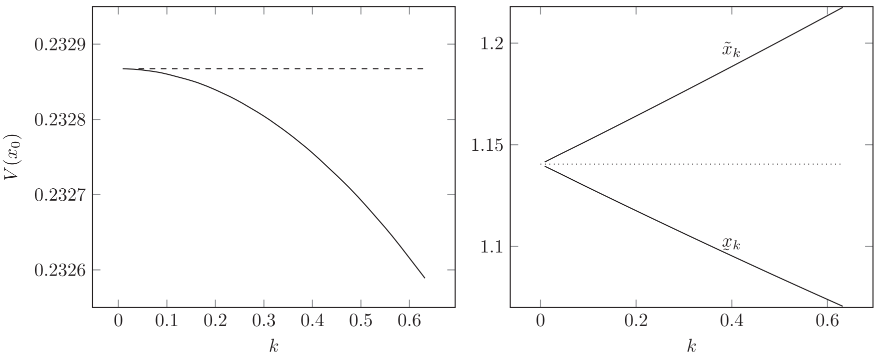

FIGURE 6. The first graph illustrates the optimal precommitment value as a function of k. The second graph illustrates the corresponding dividend policy as a function of k.

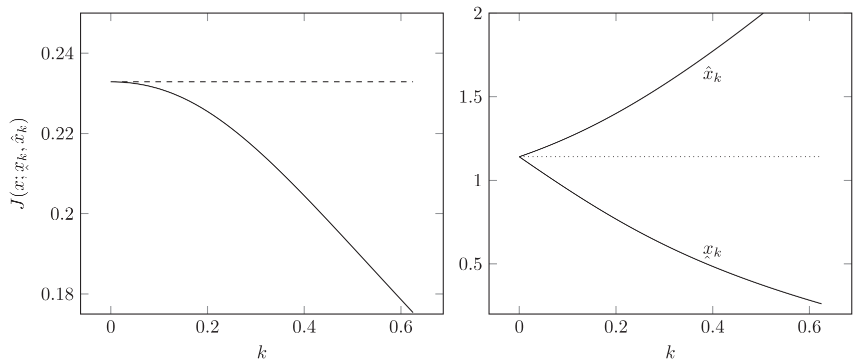

FIGURE 7. The first graph illustrates the equilibrium value as a function of k. The second graph illustrates the corresponding dividend strategy as a function of k.

the optimal value without costs—see (14)—while the dotted line is the optimal dividend barrier without costs—see (13).)

In contrast to the findings for small fixed costs c, Figures 6 and 7 suggest that small values of k in our constraint (6) have only small effects on the value for both the precommitment and the equilibrium formulation. So as not to overburden the present paper, we leave the theoretical investigation of these findings for future research.

Appendix A. Model discussion

The conditions (A.1)–(A.2) are standard assumptions that guarantee the existence of a smooth canonical solution to (7)–(8) and a strong (unique) solution to (1). Adding (A.3)–(A.5) guarantees that the canonical solution $g(\cdot)$ has the properties of Lemma 2.1(ii)–(iv), which are all important ingredients in several of the proofs underlying the main results of the present paper; in particular, the strict concavity–convexity property of $g(\cdot)$, in the sense of Lemma 2.1(iv), is crucial.

In [41, Theorem 2.1], the fixed-cost dividend problem (17) was solved essentially under (A.1)–(A.3′), i.e. under weaker assumptions than those of the present paper; see Remark 3.4 for further details. A motivation for (A.3′) is provided in [41, Remark 2.1]. In [7, 8], the fixed-cost

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dividend problem is studied under further relaxations of (A.3′). We leave for future research the question of how to relate such relaxations to the findings of the present paper.

If (A.5) does not hold, i.e,. if $\mu(0) \leq 0$, then the optimal policy in the (unconstrained) problem (2) is to pay all initial surplus x as a dividend immediately; see [46, Theorem 4.3]. Hence, in this case the optimal solution does not violate the constraint (6) (assuming that $k \leq 1$; cf. Remark 4.1), and the constrained dividend problem is in this case not time-inconsistent. There is thus no need to investigate this case along the lines of the present paper.

Appendix B. Proofs

Proof of Lemma 2.1. First note that (A.1) implies that $|\mu(x)| + |\sigma(x)| \leq K(1+x)$ for all $x \geq 0$ and some $K > 0$. A reference for the existence of a unique canonical solution satisfying 1 can be found in [41, p. 671]. Now use $g(0) = 0$, $g'(0) > 0$, and [46, Lemma 4.2(a)] to see that (A.3) holds; however, let us remark that we can replace (A.3) with the relaxed assumption (A.3′) (see Remark 2.1), and (ii) still holds.

Item (iii) follows from [6, Proposition 2.5]; here as well, (A.3) can be replaced with (A.3′).

From [41, Lemma 2.2] and (A.1)–(A.3) (where again (A.3) can be replaced with (A.3′)) it follows that there exists a point $x_b \in [0, \infty]$ such that $g(\cdot)$ is concave on $[0, x_b)$ and convex on $[x_b, \infty)$. The assumption (A.5) implies that $x_b > 0$; see [41, Lemma 2.2]. Clearly (iii) implies that $x_b < \infty$. It directly follows that $g''(x_b) = 0$. Now, by (7), it is easy to find that

$$g'''(x) = 2\frac{r - \mu'(x)}{\sigma^2(x)} g'(x) - 2\frac{\mu(x) + \frac{1}{2}(\sigma^2(x))'}{\sigma^2(x)} g''(x).$$

Hence, relying on (A.1)–(A.3), we apply [46, Lemma 4.1] to $g'(\cdot)$ and obtain (iv); we remark that a similar argument is made in the proof of [46, Lemma 4.2]. □

Proof of Proposition 2.1. Recall (9) and (10). The result can be shown using arguments analogous to those leading up to Equation (4.4) in [3]. Note that τ_1 is the first hitting time of $\bar{x}$. Using the strong Markov property, we find that the value of a lump sum dividend barrier strategy $(\underline{x}, \bar{x})$, which, for brevity, we denote in this proof by $J(x)$, satisfies, for $x \leq \bar{x}$,

$$\begin{aligned} J(x) &= \mathbb{E}_x\left(e^{-r\tau_1} J(X_{\tau_1}) I_{\{\tau_1 < \infty\}}\right) \\ &= \mathbb{E}_x\left(e^{-r\tau_1} \left(\bar{x} - \underline{x} + J(\underline{x})\right) I_{\{\tau_1 < \infty\}}\right) \\ &= \left(\bar{x} - \underline{x} + J(\underline{x})\right) \frac{g(x)}{g(\bar{x})}, \end{aligned}$$

where in the last equality we relied on properties of hitting times for diffusions absorbed at 0 found e.g. in [15, p. 18]. Thus, in particular,

$$J(\underline{x}) = (\bar{x} - \underline{x} + J(\underline{x})) \frac{g(\underline{x})}{g(\bar{x})}.$$

Solving for $J(\underline{x})$ in the latter equation and substituting it in the former equation gives us (11) in the case $x \leq \bar{x}$; the remaining case is trivial in view of the definition of a lump sum dividend barrier strategy (Definition 2.2). Using analogous arguments one can show also that (12) holds. □

Proof of Lemma 2.2. The claims can be verified using (12), Lemma 2.1 (which e.g. implies that $\lim_{x\to\infty} g(x) = \infty$), and the fact that $\bar{x} > \underline{x} \geq 0$. □

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Proof of Proposition 3.2. We use (18) to see that $H(\bar{x}_c;\, c, \underline{x}_c, \bar{x}_c) > 0$ and hence $\bar{x}_c - \underline{x}_c - c > 0$. Moreover, $H(\bar{x}_c;\, c, \underline{x}_c, \bar{x}_c) - H(\underline{x}_c;\, c, \underline{x}_c, \bar{x}_c) = \bar{x}_c - \underline{x}_c - c > 0$. Using also that $\underline{x}_c \geq 0$, we find that (25) holds.

Suppose a function $f : \mathbb{R} \to \mathbb{R}$ satisfies

- $f'(\underline{y}) = f'(\bar{y}) = 1$ for some $0 \leq \underline{y} < \bar{y}$, and
- $f''(y) < 0$ for $y \in (0, y^*)$ and $f''(y) > 0$ for $y \in (y^*, \infty)$ for some $y^* > 0$.

Then it directly follows that $0 \leq \underline{y} < y^* < \bar{y}$. Note that Lemma 2.1 implies that $g(\cdot)$ satisfies the second item above with $y^* = x^*$ (recall that $x^* := x_b$). It follows (cf. (18)) that $H(\,\cdot\,;\, c, \underline{x}_c, \bar{x}_c)$ satisfies the second item above; moreover, $H(\,\cdot\,;\, c, \underline{x}_c, \bar{x}_c)$ satisfies the first item above (with $\underline{x}_c = \underline{y}$ and $\bar{x}_c = \bar{y}$) within Case (i) (of Proposition 3.1). Hence, (21) holds in Case (i). It can be similarly shown that (21) holds in Case (ii).

Consider Case (ii). From (21) and (25) we know that $\bar{x}_c > x^*, c$. Hence, using also (18) and (20), we see that $\bar{x}_c$ is the unique solution to the equation

$$A(\bar{x}, c) := g'(\bar{x})(\bar{x} - c) - g(\bar{x}) = 0, \qquad \bar{x} > x^*, c. \tag{51}$$

Note that $A_{\bar{x}}(\bar{x}, c) = g''(\bar{x})(\bar{x} - c) > 0$ and $A_c(\bar{x}, c) = -g'(\bar{x}) < 0$ for all $\bar{x} > x^*, c$ (Lemma 2.1). Hence, by the implicit function theorem, $\bar{x}_c = \bar{x}_c(c)$ for a function $\bar{x}_c(\cdot)$ satisfying

$$\bar{x}_c{}'(c) = -\frac{A_c(\bar{x}_c, c)}{A_{\bar{x}}(\bar{x}_c, c)} = \frac{g'(\bar{x}_c)}{g''(\bar{x}_c)(\bar{x}_c - c)} > 0.$$

It follows that $\bar{x}_c$ is (strictly) increasing and continuous in c within Case (ii). In Case (i) it similarly holds that $(\underline{x}_c, \bar{x}_c)$ is the unique solution to the system of equations

$$\begin{aligned} B(\underline{x}, \bar{x}, c) &:= g'(\underline{x})(\bar{x} - \underline{x} - c) - g(\bar{x}) + g(\underline{x}) = 0, \\ C(\underline{x}, \bar{x}, c) &:= g'(\bar{x})(\bar{x} - \underline{x} - c) - g(\bar{x}) + g(\underline{x}) = 0 \\ &\text{for } \bar{x} > x^*, c \text{ and } x^* > \underline{x} > 0. \end{aligned} \tag{52}$$

Note that, for all $\bar{x} > x^*, c$ and $x^* > \underline{x} > 0$, the following hold:

- $B_{\underline{x}}(\underline{x}, \bar{x}, c) = g''(\underline{x})(\bar{x} - \underline{x} - c) < 0$,
- $B_{\bar{x}}(\underline{x}, \bar{x}, c) = C_{\underline{x}}(\underline{x}, \bar{x}, c) = g'(\underline{x}) - g'(\bar{x})$,
- $B_c(\underline{x}, \bar{x}, c) = -g'(\underline{x}) < 0$,
- $C_{\bar{x}}(\underline{x}, \bar{x}, c) = g''(\bar{x})(\bar{x} - \underline{x} - c) > 0$,
- $C_c(\underline{x}, \bar{x}, c) = -g'(\bar{x}) < 0$.

The Jacobian matrix of $(\underline{x}, \bar{x}) \mapsto (B(\underline{x}, \bar{x}, c), C(\underline{x}, \bar{x}, c))^T$, where ^{T} denotes the transpose (cf. (52)), is

$$\begin{pmatrix} g''(\underline{x})(\bar{x} - \underline{x} - c) & g'(\underline{x}) - g'(\bar{x}) \\ g'(\underline{x}) - g'(\bar{x}) & g''(\bar{x})(\bar{x} - \underline{x} - c) \end{pmatrix},$$

and hence its determinant is

$$g''(\bar{x})g''(\underline{x})(\bar{x} - \underline{x} - c)^2 - (g'(\underline{x}) - g'(\bar{x}))^2 < 0,$$

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where we used that $g''(\underline{x}) < 0$ and $g''(\bar{x}) > 0$ (to see this, use $\bar{x} > x^* > \underline{x}$ and Lemma 2.1). The Jacobian matrix is therefore invertible. Note that $g'(\underline{x}_c) = g'(\bar{x}_c)$ (cf. (52)), and use the implicit function theorem to see that $\left(\underline{x}_c, \bar{x}_c\right) = \left(\underline{x}_c, \bar{x}_c\right)(c)$ for a function $\left(\underline{x}_c, \bar{x}_c\right)(\cdot)$ satisfying

$$\left(\frac{d}{dc}\left(\underline{x}_c, \bar{x}_c\right)(c)\right)^T = -\begin{pmatrix} g''\left(\underline{x}_c\right)\left(\bar{x}_c - \underline{x}_c - c\right) & 0 \\ 0 & g''\left(\bar{x}_c\right)\left(\bar{x}_c - \underline{x}_c - c\right) \end{pmatrix}^{-1} \begin{pmatrix} -g'\left(\underline{x}_c\right) \\ -g'\left(\bar{x}_c\right) \end{pmatrix}$$
$$= \frac{1}{g''\left(\bar{x}_c\right)g''\left(\underline{x}_c\right)\left(\bar{x}_c - \underline{x}_c - c\right)^2} \begin{pmatrix} g''\left(\bar{x}_c\right)\left(\bar{x}_c - \underline{x}_c - c\right)g'\left(\underline{x}_c\right) \\ g''\left(\underline{x}_c\right)\left(\bar{x}_c - \underline{x}_c - c\right)g'\left(\bar{x}_c\right) \end{pmatrix}$$
$$= \begin{pmatrix} \dfrac{g'\left(\underline{x}_c\right)}{g''\left(\underline{x}_c\right)\left(\underline{x}_c - \bar{x}_c - c\right)} \\ \dfrac{g'\left(\bar{x}_c\right)}{g''\left(\bar{x}_c\right)\left(\underline{x}_c - \bar{x}_c - c\right)} \end{pmatrix}.$$

Hence, $\underline{x}_c$ is (strictly) decreasing and continuous in c while $\bar{x}_c$ is (strictly) increasing and continuous in c within Case (i). Using the observations above it is easy to see that if the statements

$$\begin{aligned}&(1) \text{ there exists a } \bar{c} \text{ such that if } c < \bar{c} \text{ then we are in Case (i), and if} \\ &c \geq \bar{c} \text{ then we are in Case (ii), and (2) } (\underline{x}_c, \bar{x}_c)(c) \to (0, \left(\bar{x}_c\right)\left(\bar{c}\right)) \text{ as } c \nearrow \bar{c},\end{aligned} \tag{53}$$

hold, then (22) holds, and so does (26). Recall that $\left(\underline{x}_c, \bar{x}_c\right)$ is the unique solution to (52) in Case (i); but this is equivalent to

$$g'\left(\underline{x}_c\right) = g'\left(\bar{x}_c\right) = \frac{g\left(\bar{x}_c\right) - g\left(\underline{x}_c\right)}{\bar{x}_c - \underline{x}_c - c} \left(> \frac{g(\bar{x}_c) - g(\underline{x}_c)}{\bar{x}_c - \underline{x}_c}\right). \tag{54}$$

Similarly, in Case (ii) note that (51) is equivalent to

$$g'(\underline{x}_c) = \frac{g(\bar{x}_c)}{\bar{x}_c - c} \left(= \frac{g(\bar{x}_c) - g(0)}{\bar{x}_c - 0 - c} > \frac{g(\bar{x}_c) - g(0)}{\bar{x}_c - 0}\right). \tag{55}$$

Note that (54) means that, for any c, the derivatives $g'\left(\underline{x}_c\right) = g'(\bar{x}_c)$ strictly dominate the slope of a line between the points $\left(\underline{x}_c, g(\underline{x}_c)\right)$ and $(\bar{x}_c, g(\bar{x}_c))$, and that the difference between these derivatives and the slope is strictly decreasing in c. A similar observation can be made for (55). Using the observations above and recalling that $g(\cdot)$ is strictly concave on $(0, x^*)$ and strictly convex on (x^*, ∞) it is easy to see that (53) holds, which thus implies that (22) and (26) hold, and moreover that

$$g'(\underline{x}_c) = g'\left(\bar{x}_c\right) = \frac{g\left(\bar{x}_c\right) - g\left(\underline{x}_c\right)}{\bar{x}_c - \underline{x}_c - c} \to g'\left(x^*\right) \quad \text{as } c \searrow 0,$$

and

$$\underline{x}_c \nearrow x^* \quad \text{and} \quad \bar{x}_c \searrow x^* \quad \text{as } c \searrow 0.$$

It thus also follows that (23) and (24) hold (to see this use also e.g. (14) and (18)). □

Proof of Lemma 3.1. The claims can be verified using Lemma 2.2(i) and properties of the function $c \mapsto \left(\underline{x}_c, \bar{x}_c\right)$ corresponding to Proposition 3.2. □

Proof of Lemma 4.1. Proposition 2.1 and the fact that the functions in (30)–(31) are defined only for $\bar{x} > \underline{x} \geq 0$ are used throughout the proof.

Let us prove (i): Suppose $k = 1$. It is easy to see that (30) holds if and only if $\underline{x} = 0$. Using this and (31) it is easy to see that the claim holds if and only if the equation

$$A(\bar{x}) := g'(\bar{x})\bar{x} - g(\bar{x}) = 0$$

has exactly one solution in $(0, \infty)$ and this solution is strictly larger than x^*. To see that this is the case it suffices to note the following:

- $\lim_{\bar{x}\searrow 0} A(\bar{x}) = A(0) = 0$ and $\lim_{\bar{x}\to\infty} A(\bar{x}) = \infty$ (use differentiation and Lemma 2.1 to find that the second statement holds);
- $A'(\bar{x}) = g''(\bar{x})\bar{x}$, which, by Lemma 2.1, means that $A'(\bar{x}) < 0$ for $\bar{x} \in (0, x^*)$ and $A'(\bar{x}) > 0$ for $\bar{x} > x^*$.

Now suppose $k < 1$ (recall that $k > 0$; see (6)). It follows from (30) that $\underline{x} > 0$. Recalling that $\bar{x} > \underline{x} \geq 0$ by definition, it is easy to see that (31) holds if and only if

$$g'(\bar{x}) = \frac{g(\bar{x}) - g(\underline{x})}{\bar{x} - \underline{x}}. \tag{56}$$

Now recall from Lemma 2.1 that $g''(x) < 0$ if $x \in (0, x^*)$ and $g''(x) > 0$ if $x \in (x^*, \infty)$, and $\lim_{x\to\infty} g'(x) = \infty$, from which it is easy to see the following:

- If $\underline{x} \geq x^*$ then no $\bar{x} > \underline{x}$ such that (56) holds exists.
- If $\underline{x} < x^*$ then a unique $\bar{x} > \underline{x}$ such that (56) holds exists, and $\bar{x} > x^*$.
- There exists a continuous strictly decreasing function
$$\bar{x}(\cdot) \tag{57}$$
such that (56) holds if and only if $\bar{x} = \bar{x}(\underline{x})$ for a fixed $\underline{x} \in (0, x^*)$, where $\bar{x}(\underline{x}) > x^*$. Moreover, sending $\underline{x} \nearrow x^*$ implies that $\bar{x}(\underline{x}) \searrow x^*$.

From the observations above and properties of the function $\bar{x} \mapsto R\big(\bar{x}; \underline{x}, \bar{x}\big)$ (Lemma 2.2), it follows that the schedule below gives a unique solution to (30)–(31) in the case $k < 1$:

- Pick an $\underline{x}_1 \in (0, x^*)$.
- Determine $\bar{x}_1 = \bar{x}(\underline{x}_1)$ by verifying (56) (which implies that (31) holds).
- If $R\big(\bar{x}_1; \underline{x}_1, \bar{x}_1\big) > \frac{1}{k}$, then we know that $\underline{x}_1$ is too high and $\bar{x}_1$ is too low (cf. that $\bar{x}(\cdot)$ is decreasing) to be a solution to (30), and we set $\underline{x}_1 = \underline{x}_1^h$ and choose a new $\underline{x}_2 < \underline{x}_1^h$. Analogously, if $R\big(\bar{x}_1; \underline{x}_1, \bar{x}_1\big) < \frac{1}{k}$, then we know that $\underline{x}_1$ is too low and we set $\underline{x}_1 = \underline{x}_1^l$ and choose a new $\underline{x}_2 > \underline{x}_1^l$.
- Iterate the steps above, always choosing (using e.g. the bisection method) $\underline{x}_{n+1}$ smaller than all $\underline{x}_i^h$, $i \leq n$, and larger than all $\underline{x}_i^l$, $i \leq n$, until you find an $\underline{x}$ such that, with $\bar{x} = \bar{x}(\underline{x})$, it holds that $R\big(\bar{x}; \underline{x}, \bar{x}\big) = \frac{1}{k}$ (which by Lemma 2.2 is possible).

Note that this schedule can easily be modified in order to find the solution to (30)–(31) in the case $k = 1$, using that in this case (30) holds if and only if $\underline{x} = 0$.

Let us now prove (ii). It is easy to verify that (30) holds if and only if $(\underline{x}, \bar{x})$ is a solution to the equation

$$g(\underline{x}) = (1 - k)g(\bar{x}). \tag{58}$$

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Inserting (58) into (56) yields (33), and inserting (33) into (58) yields

$$g\left(\bar{x} - k\frac{g(\bar{x})}{g'(\bar{x})}\right) - (1-k)g(\bar{x}) = 0.$$

The claim can now be verified using (11). Item (iii) follows from (11). □

Proof of Theorem 4.6. Recall that $(\underset{\wedge}{x}_k, \hat{x}_k)$ solves (30)–(31). Hence, using Lemma 4.1 we obtain (38). To see that (40) and (41) hold, recall the properties of the function (57) and adapt the arguments after (57) in the obvious way, recalling also (56) and properties of the function $\bar{x} \mapsto R(\bar{x}; \underline{x}, \bar{x})$ (Lemma 2.2). Item (39) is proved similarly. Item (42) follows from Theorem 4.3 and Lemma 4.1. □

Acknowledgements

We thank two anonymous reviewers for helpful suggestions and remarks. In particular, Section 4.2 originates from a suggestion by one of the reviewers.

Funding Information

There are no funding bodies to thank in relation to the creation of this article.

Competing Interests

There were no competing interests to declare which arose during the preparation or publication process for this article.

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