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Abstract

We show that the zero locus of an admissible normal function on a smooth complex algebraic variety is algebraic. In Part II of the paper, which is an appendix, we compute the Tannakian Galois group of the category of one-variable admissible real nilpotent orbits with split limit. We then use the answer to recover an unpublished theorem of Deligne, which characterizes the sl_2-splitting of a real mixed Hodge structure.

Contents

1. Introduction

Let $\bar{S}$ be a complex manifold and $\mathcal{H}$ be an integral variation of pure Hodge structure of weight $w<0$ which is defined on a Zariski open subset S of $\bar{S}$. Saito [Sai96] defines an *admissible normal function on S with respect to $\bar{S}$* to be an extension class

$$0 \to \mathcal{H} \to \mathcal{V} \to \mathbb{Z}(0) \to 0 \tag{1.1}$$

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in the category $\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}$ of variations of mixed Hodge structure on S which are admissible relative to $\bar{S}$. Via Carlson's formula [Car87], an admissible normal function corresponds to a holomorphic section $\nu : S \to J(\mathcal{H})$ of the associated family of generalized intermediate Jacobians $J(\mathcal{H}) \to S$ which satisfies a version of Griffiths horizontality and has controlled asymptotic behavior near the boundary of S in $\bar{S}$. The purpose of this paper is to prove the following.

Theorem 1.2. *With the above conventions, let $\nu : S \to J(\mathcal{H})$ be an admissible normal function with respect to $\bar{S}$. Let $\mathcal{Z} = \mathcal{Z}(\nu) = \{s \in S : \nu(s) = 0\}$ denote the zero locus of ν. Then the topological closure of $\mathcal{Z}$ in $\bar{S}$ is a closed analytic complex subspace of $\bar{S}$.*

In this paper we use the notation $\mathrm{NF}(S, \mathcal{H})^{\mathrm{ad}}_{\bar{S}}$ to denote the group of normal functions $\nu : S \to J(\mathcal{H})$ which are admissible with respect to $\bar{S}$ as above. If S has an algebraic structure, then, by Nagata and Hironaka, there exists some smooth algebraic compactification $\bar{S}$. However, for S algebraic the notion of admissibility is independent of the choice of smooth compactification $\bar{S}$ of S [Sai96, Remark 1.6(i)]. Therefore we follow [Sai96] and write $\mathrm{NF}(S, \mathcal{H})^{\mathrm{ad}}$ instead of $\mathrm{NF}(S, \mathcal{H})^{\mathrm{ad}}_{\bar{S}}$ for the group of admissible normal functions on S. The following corollary is then immediate from GAGA [Ser56].

Corollary 1.3. *If S is algebraic then the zero locus of an admissible normal function $\nu : S \to J(\mathcal{H})$ is an algebraic subvariety of S.*

For $w = -1$ the assertion of Corollary 1.3 was a conjecture of Phillip Griffiths and Mark Green. At least in the case that $w = -1$, Theorem 1.2 has also been proved by Schnell [Sch12]. In Schnell's work it is a consequence of the existence of a 'Néron model' extending the family $J(\mathcal{H}) \to S$ over $\bar{S}$. The full theorem is used in our joint work [BPS08] with Schnell where we prove the generalization of the theorem of Cattani, Deligne and Kaplan to admissible variations of mixed Hodge structure. The paper [KNU10] by Kato, Nakayama and Usui indicates a proof of the theorem using the log classifying spaces developed by those authors.

To prove Theorem 1.2, it suffices (Theorem 3.5) to consider the case that $D = \bar{S} \backslash S$ is a normal crossing divisor. Working locally on S, we are then reduced to proving the following theorem.

Theorem 1.4. *In the context of Theorem 1.2, suppose $D := \bar{S} \backslash S$ is a normal crossing divisor and that $p \in D$ is an accumulation point of $\mathcal{Z}$. Then there exists an open polydisk $P \subset \bar{S}$ containing p and an analytic subvariety A of P such that $A \cap S = \mathcal{Z} \cap P$.*

Overview

The first part of this paper is devoted to the proof of Theorem 1.4. In § 2 we review and extend the theory of variations of real mixed Hodge structure in $\mathrm{VMHS}(\Delta^{*r})^{\mathrm{ad}}_{\Delta^r}$. These have a very concrete local normal form which can be expressed in terms of matrix-valued holomorphic functions. We state several results concerning the asymptotics of these variations in § 2, in particular Theorem 2.30, which is a strong boundedness result. Some of the proofs, in particular the proof of our main boundedness result Theorem 2.30, are deferred to § 9.

In § 3, we reduce the proof of Theorem 1.4 to the analogous statement on a polydisk. In § 4 we use the results stated in § 2 to give an explicit system of equations for $\mathcal{Z}$ on a punctured polydisk Δ^{*r}.

The remainder of the first part of the paper is devoted to the proofs of the results concerning variations of real mixed Hodge structure stated in § 2.

The second part of the paper is an appendix, which proves results of Deligne on the sl_2-splitting stated in an unpublished letter to Cattani and Kaplan [Del93]. We interpret Deligne's results as the computation of the Tannakian Galois group of the category Split_1 consisting of admissible nilpotent orbits (in one variable) whose limits are split over $\mathbb{R}$. The group in question is very similar to Deligne's group $\mathcal{M}$ whose category of representations is the category of real mixed Hodge structures [Del94]. Part of this work consists in explaining what a one-variable SL_2-orbit is in terms of the representation of a real reductive group of rank 3, which we call the *Schmid group*. This is probably well known to the experts and implicit in Schmid's paper [Sch73], but to the best of our knowledge it has not yet been written down explicitly.

Notation

We use the following notation:

- Δ^r is a polydisk with holomorphic coordinates $s = (s_1, \ldots, s_r)$;
- $\Delta^{*r} \subset \Delta^r$ is the set of points where $s_1 \cdots s_r \neq 0$;
- $U^r \subset \mathbb{C}^r$ is the product of upper-half planes;
- points $z = (z_1, \ldots, z_r) \in \mathbb{C}^r$ are written $z = x + iy$ where $x = (x_1, \ldots, x_r)$, and $y = (y_1, \ldots, y_r) \in \mathbb{R}^r$;
- $\pi : U^r \to \Delta^{*r}$ is the covering map given by $s_j = e^{2\pi i z_j}$ for $j = 1, \ldots, r$;
- the underlying vector space of a filtration or mixed Hodge structure is denoted V, and if V is defined over a subring R of $\mathbb{C}$ the associated R-module is denoted V_R;
- elements of $\mathrm{GL}(V)$ act linearly on filtrations of V, e.g. $(g \cdot F)^p = g(F^p)$, and elements of $\mathrm{GL}(V)$ act on endomorphisms of V via the adjoint action, i.e. $g \cdot Y = gYg^{-1}$.

Part I. Zero Loci of Admissible Normal Functions

2. Admissible variations on the punctured polydisk

Here we collect results about the structure of variations of mixed Hodge structure on the punctured polydisk Δ^{*r} which are admissible relative to the polydisk Δ^r. In this section, all variations of mixed Hodge structure will be variations with real coefficients and the emphasis will be on asymptotics.

Deligne gradings

Given an increasing filtration W of a finite-dimensional vector space V over a field of characteristic zero, a grading of V is a semisimple endomorphism Y of V such that, for each index $k \in \mathbb{Z}$, W_k is the direct sum of W_{k-1} and the k-eigenspace $E_k(Y)$ for each index k. By a theorem of Deligne [Del71], a mixed Hodge structure (F, W) induces a unique, functorial decomposition

$$V_{\mathbb{C}} = \bigoplus_{r,s} I^{r,s}_{(F,W)} \quad \text{or simply} \quad \bigoplus_{r,s} I^{r,s} \tag{2.1}$$

of the underlying complex vector space $V_{\mathbb{C}}$ such that:

(a) $F^p = \bigoplus_{r \geqslant p} I^{r,s}$;

(b) $W_k = \bigoplus_{r+s \leqslant k} I^{r,s}$;

(c) $\bar{I}^{p,q} = I^{q,p} \mod \bigoplus_{r<q, s<p} I^{r,s}$.

Here F is the Hodge filtration and W is the weight filtration.

In particular, a mixed Hodge structure (F, W) induces a grading $Y_{(F,W)}$ of $V_{\mathbb{C}}$ by the requirement that $Y_{(F,W)}$ acts as multiplication by $p + q$ on $I^{p,q}$. We sometimes write $V^{r,s}$ instead of $I^{r,s}$ when this is the only bigrading of V in sight.

Local normal form

Let $\mathcal{V}$ be an admissible variation of graded-polarized mixed Hodge structure over a punctured polydisk Δ^{*r} with unipotent monodromy $T_j = e^{N_j}$ about $s_j = 0$, with weight filtration W. Let V be any fiber of $\mathcal{V}$ and define $\mathfrak{g}$ to be the Lie subalgebra of $\mathfrak{gl}(V)$ consisting of all elements which preserve W and act by infinitesimal isometries on $\mathrm{Gr}^W V$. Then, the limit mixed Hodge structure (F_∞, M) of $\mathcal{V}$ induces a mixed Hodge structure on $\mathfrak{g}$. We write $\mathfrak{g}^{r,s}$ for the corresponding decomposition of $\mathfrak{g}_{\mathbb{C}}$ by the complex subspaces $I^{p,q}$. There is then a distinguished vector space decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{q} \oplus \mathfrak{g}_{\mathbb{C}}^{F_\infty}, \quad \mathfrak{q} = \bigoplus_{r<0,s} \mathfrak{g}^{r,s}_{(F_\infty,M)} \tag{2.2}$$

where $\mathfrak{g}_{\mathbb{C}}^{F_\infty}$ is the stabilizer of the limit Hodge filtration. Relative to this decomposition, we can then write (cf. [Pea00, (6.11)]) the period map

$$F : U^r \to \mathcal{M}$$

of the pullback $\mathcal{V}$ to the universal cover $\pi : U^r \to \Delta^{*r}$ as

$$F(z_1, \dots, z_r) = e^{\sum_j z_j N_j} e^{\Gamma(s_1,\dots,s_r)} \cdot F_\infty \tag{2.3}$$

where $\Gamma(s)$ is a $\mathfrak{q}$-valued holomorphic function which vanishes at the origin. This is called the *local normal form of the variation* $\mathcal{V}$.

Nilpotent orbits

If $\mathcal{V} \to \Delta^{*r}$ is an admissible variation of mixed Hodge structure over the punctured polydisk Δ^{*r} with local normal form (2.3) then the associated map

$$\theta(z_1, \dots, z_r) = e^{\sum_j z_j N_j} \cdot F_\infty$$

from $\mathbb{C}^r$ into the 'compact dual' of $\mathcal{M}$ is called the nilpotent orbit of $\mathcal{V}$. (See [Pea00] for the notion of compact dual.)

In general, given a classifying space $\mathcal{M}$ with 'compact dual' $\check{\mathcal{M}}$, a nilpotent orbit with values in $\mathcal{M}$ is a holomorphic, horizontal map

$$F_{\mathrm{nilp}}(z_1, \dots, z_r) = e^{\sum_j z_j N_j} \cdot F : \mathbb{C}^r \to \check{\mathcal{M}} \tag{2.4}$$

such that:

(a) $N_1, \dots, N_r$ are nilpotent, mutually commuting elements of the Lie algebra $\mathfrak{g}_{\mathbb{R}} := \mathfrak{g}_{\mathbb{C}} \cap \mathfrak{gl}(V_{\mathbb{R}})$;

(b) there exists a constant $K > 0$ such that

$$F_{\mathrm{nilp}}(z_1, \dots, z_r) \in \mathcal{M}$$

for all $z \in \mathbb{C}^r$ with $\mathrm{Im}(z_j) > K$.

In particular, F_{nilp} defines a variation of mixed Hodge structure $\mathcal{V}_{\mathrm{nilp}}$ on the set of points in Δ^{*r} where $|s_j| < e^{-2\pi K}$. Accordingly, we say that F_{nilp} is admissible if $\mathcal{V}_{\mathrm{nilp}}$ is admissible.

Via the formula (2.4), a real nilpotent orbit is completely determined by the data $(N_1, \dots, N_r; F, W)$ consisting of nilpotent operators $N_1, \dots, N_r$ on a real vector space V together with a decreasing filtration F of $V_{\mathbb{C}}$ and a decreasing filtration W of V.

Remark 2.5. For the remainder of this paper, we will suppress the data of the graded-polarization when discussing nilpotent orbits.

Given an admissible nilpotent orbit $(N_1, \dots, N_r; F, W)$, define

$$N(z) = \sum_j z_j N_j.$$

In [Kas86] Kashiwara proved the following results concerning relative weight filtrations associated to admissible nilpotent orbits. (We refer to [Kas86] for the notion of a relative weight filtration.)

THEOREM 2.6. *Suppose $(N_1, \dots, N_r; F, W)$ is an admissible nilpotent orbit. Then the following are true.*

(a) *The relative weight filtration $M(N(v), W)$ exists for every vector $v \in \mathbb{R}^r_{\geqslant 0}$.*

(b) *For each subset $I \subset \{1, \dots, r\}$, let $C(I)$ denote the monodromy cone $\sum_{i\in I} \mathbb{R}_{>0} N_i$ in $\mathfrak{g}_{\mathbb{R}}$. Then the filtration $M(C(I), W) = M(N, W)$ is constant for $N \in C(I)$.*

(c) *Let $I = \{1, \dots, r\}$ and $M = M(C(I), W)$. Then, (F, M) is a mixed Hodge structure with respect to which each N_i is a $(-1, -1)$-morphism. More generally, if I is a subset of $\{1, \dots, r\}$ with complement I' then*

$$\theta_I = \left(\exp\left(\sum_{j\in I'} z_j N_j\right) \cdot F, M(C(I), W)\right) \tag{2.7}$$

is an admissible nilpotent orbit, and each $N_i \in I$ is a $(-1, -1)$ morphism of the mixed Hodge structure on the right-hand side of (2.7).

(d) *If I and J are subsets of $\{1, \dots, r\}$ then*

$$M(C(I), M(C(J), W)) = M(C(I \cup J), W).$$

Deligne systems

Let $(N_1, \dots, N_r; F, W)$ define an admissible nilpotent orbit and let $W^0, \dots, W^r$ be the sequence of increasing filtrations defined by the requirement that $W^0 = W$ and $W^j = M(N_j, W^{j-1})$. Then, by a theorem of Deligne [BP09, Del93, Sch01], the data $(N_1, \dots, N_r, Y_{(F,W^r)})$ defines a sequence of mutually commuting gradings (in the notation of [BP09, (3.3)])

$$Y^r = Y_{(F,W^r)}, \quad Y^{r-1} = Y(N_r, Y^r), \quad \dots \tag{2.8}$$

such that Y^k grades W^k. Furthermore, if (F, W^r) is split over $\mathbb{R}$ this construction gives the corresponding gradings of the SL_2-orbit theorem [CKS86, KNU08]. More precisely, let $(\hat{F}, W^r)$ denote the sl_2-splitting of (F, W^r), and $\{\hat{Y}^j\}$ be the corresponding system of gradings. Let $\hat{H}_j = \hat{Y}^j - \hat{Y}^{j-1}$ and $\hat{N}_j$ denote the component of N_j with eigenvalue zero with respect to $\operatorname{ad} \hat{Y}^{j-1}$ for $j = 1, \dots, r$. Then, each pair $(\hat{N}_j, \hat{H}_j)$ is an sl_2-pair which commutes with $(\hat{N}_k, \hat{H}_k)$.

Our main interest in Deligne systems will be in the grading

$$Y^0 = Y(N_1, Y(N_2, \dots, Y_{(F,W^r)})) \tag{2.9}$$

obtained by applying the construction in (2.8) recursively.

Note that the proof of Deligne's theorem is pure linear algebra. In particular, it applies to situations that do not necessarily arise from Hodge theory. In the terminology of [Sch01, Definition 2], a finite-dimensional vector space V equipped with a finite increasing filtration W^0 and r commuting nilpotent operators $N_1, \dots, N_r$ and an operator Y^r preserving W^0 is called a

Deligne system if:

(i) $W^{j+1} = M(N_{j+1}, W^j)$ exists for $j \geqslant 0$;

(ii) $W^{j+1}|_{W^i_\ell} = M(N_{j+1}, W^j|_{W^i_\ell})$, for each j and ℓ and each $i < j$;

(iii) $N_i \in W^j_{-2}\,\mathrm{End}(V)$ for $i \leqslant j$ and $N_i \in W^j_0\,\mathrm{End}(V)$ for $i \geqslant j$;

(iv) Y^r splits W^r and preserves each W^i. Moreover $[Y^r, N_i] = -2N_i$ for all i.

Deligne's theorem [Sch01, Theorem 2], shows that the data $(N_1 \ldots, N_r; W^0)$ of a Deligne system gives rise to a system of splittings Y^k of the W^k as above.

In particular, if we start with an admissible orbit $(N_1, \ldots, N_r; F, W)$ and choose vectors $v_1, \ldots, v_d \in \mathbb{R}^r$ such that the $N(v_i)$ all lie in the closure of the monodromy cone $C(\{1, \ldots, r\})$ and $\sum_{i=1}^r v_i \in \mathbb{R}^r_{>0}$, then the data

$$(N(v_1), \ldots, N(v_d); F, W) \tag{2.10}$$

again defines an admissible nilpotent orbit. Thus we obtain a system of gradings as above and, in particular, a grading $Y^0 = Y(N(v_1), Y(N(v_d), \ldots, Y_{(F,M)}))$ of W.

Remark 2.11. If (F, W) is a mixed Hodge structure then the data $(N = 0, F, W)$ determines an admissible nilpotent orbit for which the associated grading (2.9) is just $Y_{(F,W)}$.

Splittings

Let (F, W) be an $\mathbb{R}$-mixed Hodge structure with underlying vector space V, and $\mathfrak{gl}(V) = \bigoplus_{a,b} \mathfrak{gl}(V)^{a,b}$ be the bigrading (2.1) for the mixed Hodge structure induced by (F, W) on the Lie algebra $\mathfrak{gl}(V)$. Define

$$\Lambda^{-1,-1}_{(F,W)} = \bigoplus_{a,b<0} \mathfrak{gl}(V)^{a,b}. \tag{2.12}$$

Then, on account of the defining properties (a)–(c) of the bigrading (2.1) it follows that

$$I^{p,q}_{(g\cdot F,W)} = g.I^{p,q}_{(F,W)}$$

for all $g \in \exp(\Lambda^{-1,-1}_{(F,W)})$.

Theorem 2.13 (Deligne, [CKS86, Proposition 2.20]). *Given an $\mathbb{R}$-mixed Hodge structure (F, W) with underlying vector space V, there exists a unique, functorial element*

$$\delta \in \mathfrak{gl}(V_\mathbb{R}) \cap \Lambda^{-1,-1}_{(F,W)}$$

such that $(e^{-i\delta} \cdot F, W)$ is split over $\mathbb{R}$. Every morphism of (F, W) commutes with δ; thus the morphisms of (F, W) are exactly the morphisms of $(e^{-i\delta} \cdot F, W)$ which commute with this element.

The proof of Lemma 6.60 in [CKS86] contains the implicit construction of another functorial splitting operation (cf. [CKS86, (3.30)])

$$(F, W) \mapsto (e^{-\xi} \cdot F, W) \tag{2.14}$$

on the category of $\mathbb{R}$-mixed Hodge structures which is optimal for the study of nilpotent orbits. More precisely, if for any mixed Hodge structure (F, W) we define

$$\hat{Y}_{(F,W)} = Y_{(e^{-\xi}.F,W)} \tag{2.15}$$

then one of the major components of the SL_2-orbit theorem of [KNU08] can be stated as follows.

THEOREM 2.16 [KNU08]. *Let $(N_1, \dots, N_r; F, W)$ generate an admissible nilpotent orbit and let $y(m) \in \mathbb{R}^r$ be a sequence of positive real numbers such that the ratios $y_{j+1}(m)/y_j(m)$ tend to 0 for $j = 1, \dots, r$ upon formally setting $y_{r+1}(m) = 1$. Then,*

$$\lim_{m\to\infty} \hat{Y}_{(e^{iN(y(m))}\cdot F,W)} = Y(N_1, Y(N_2, \dots, \hat{Y}_{(F,W^r)})). \tag{2.17}$$

In this paper, we call the splitting operation (2.14) the sl_2-*splitting*. In [KNU08], (2.14) is called the *canonical splitting* operation and ξ is denoted as $\epsilon(F, W)$. The proof of Lemma 6.60 in [CKS86] gives a recursive formula for ξ in terms of the Hodge components of Deligne δ-splitting for (F, W). In Corollary 12.7, we prove the following result, due essentially to Deligne.

THEOREM 2.18 [Del93]. *The sl_2-splitting is the unique, functorial splitting of $\mathbb{R}$-MHS which is given by universal Lie polynomials in the Hodge components of Deligne's δ-splitting such that if $(e^{zN} \cdot F, W)$ is an admissible nilpotent orbit with limit mixed Hodge structure (F, M) which is split over $\mathbb{R}$ then the Deligne grading of the splitting of $(e^{iN} \cdot F, W)$ is a morphism of type $(0, 0)$ for (F, M).*

Remark 2.19. It follows from [CKS86, Lemma 3.12] that $(e^{iN} \cdot F, W)$ is a mixed Hodge structure whenever $(e^{zN} \cdot F, W)$ is an admissible nilpotent orbit with limit split over $\mathbb{R}$, because, in that case, the graded quotients Gr^W are SL_2-orbits.

The notion of Deligne system does not appear in [KNU08]. To extract (2.17) from [KNU08], we need the following lemma from an unpublished letter of Deligne to Cattani and Kaplan. The reader can find a proof in Remark 13.6 of Part II.

LEMMA 2.20 [BP09, Del93]. *Let $(N; F, W)$ be an admissible nilpotent orbit (in one variable) with limit mixed Hodge structure (F, M) split over $\mathbb{R}$. Then*

$$\hat{Y}_{(e^{iN}\cdot F,W)} = Y(N, Y_{(F,M)}). \tag{2.21}$$

In particular (cf. [BP09, Del93, KP03]), it follows from (2.21) and Theorem 2.18 that $Y(N, Y_{(F,M)})$ is a morphism of type $(0, 0)$ for (F, M). We will use the following extension of this result.

LEMMA 2.22. *Let $(N_1, \dots, N_r; F, W)$ define an admissible nilpotent orbit. Then,*

$$Y = Y(N_1, Y(N_2, \dots, Y(N_r, Y_{(F,W^r)})))$$

preserves the Deligne subspaces $I^{p,q}$ of (F, W^r). Furthermore, if $(\hat{F}, W^r) = (e^{-\xi} \cdot F, W^r)$ is the sl_2-splitting of (F, W^r) then

$$Y(N_1, Y(N_2, \dots, Y(N_r, Y_{(\hat{F},M)}))) = e^{-\xi} \cdot Y(N_1, Y(N_2, \dots, Y(N_r, Y_{(F,M)}))).$$

Proof. See Lemma 5.8. □

Remark 2.23. Implicit in the second part of Lemma 2.22 is the statement that ξ preserves each weight filtration W^j. This is explained in the proof of Lemma 5.8.

For future use, we record the following lemma.

LEMMA 2.24. *Let α be a morphism of type $(-1, -1)$ for the mixed Hodge structure (F, W). Then, $(\widehat{e^{i\alpha} \cdot F}, W) = (\hat{F}, W)$ for both the Deligne and the sl_2-splitting operation.*

Proof. By [CKS86, Proposition 2.20],

$$\bar{Y}_{(F',W)} = e^{-2i\delta_{(F',W)}} \cdot Y_{(F',W)}$$

for any mixed Hodge structure (F', W). Using this formula and the fact that Deligne's splitting commutes with morphisms of mixed Hodge structure it easily follows that

$$\delta_{(e^{i\alpha}\cdot F,W)} = \delta_{(F,W)} + i\alpha$$

since α is a $(-1,-1)$-morphism of both $(e^{i\alpha}\cdot F, W)$ and (F, W). To obtain the analogous assertion for the sl_2-splitting one uses the fact that ϵ is given by universal Lie polynomials in the Hodge components of δ and the fact that $[\delta^{p,q}, \alpha] = 0$ since $[\delta, \alpha] = 0$ and α is of type $(-1, -1)$. □

Limiting gradings

The (standard) vertical strip in U^r is the set of points

$$I = \{z = x + iy \mid x_j \in [0, 1], y_j \in [1, \infty) \ \ \forall j\}. \tag{2.25}$$

For a point $z = x + iy \in U^r$ we define $t_j = y_{j+1}/y_j$, where we formally set $y_{r+1} = 1$. Let S_r denote the group of permutations of $\{1, \ldots, r\}$ and let $\sigma \in S^r$ act on $\mathbb{C}^r$ by permuting coordinates. Then,

$$I = \bigcup_{\sigma\in S_r} \sigma(I') \tag{2.26}$$

where $I' = \{z \in I \mid t_j \in (0, 1] \ \ \forall j\}$ is the set of points where $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_r \geqslant y_{r+1} = 1$.

Definition 2.27. A sequence of points $z(m) = x(m) + iy(m)$ in I' is said to be *tame* if $\lim_{m\to\infty} x_j(m)$ and $\lim_{m\to\infty} t_j(m)$ exist for each index j. A tame sequence is said to be an sl_2*-sequence* if there exists:

(a) a linear transformation $T : \mathbb{R}^d \to \mathbb{R}^r$;

(b) a sequence $v(m) \in \mathbb{R}^d_{>0}$;

(c) a convergent sequence $b(m) \in \mathbb{R}^r$;

such that

$$y(m) = T(v(m)) + b(m)$$

and $\lim_{m\to\infty} v_{j+1}(m)/v_j(m) = 0$ for $j = 1, \ldots, d$ (with $v_{d+1}(m) = 1$ as usual). An sl_2-sequence is said to be strict if $d = r$, $b(m) = 0$ and T is the identity.

In particular, since (2.26) is a finite union, we have the following lemma.

Lemma 2.28. *Let $z(m) = x(m) + iy(m) \in I$ be a sequence of points. Then, there exists an element $\sigma \in S_r$ and a subsequence $z(m')$ of $z(m)$ such that $\sigma(z(m'))$ is an sl_2-sequence.*

Given an sl_2-sequence with associated linear transformation T as above, let $\{e_1, \ldots, e_d\}$ denote the standard basis of $\mathbb{R}^d$ and define $\theta^i = T(e_i)$. Then, while neither d nor the transformation $T : \mathbb{R}^d \to \mathbb{R}^r$ is uniquely determined by the sl_2-sequence $y(m)$, the associated flag defined by the increasing sequence of subspaces

$$\Theta^j = \sum_{i\leqslant j} \mathbb{R}\theta^i \tag{2.29}$$

depends only on the sequence $y(m)$. Moreover, since $y(m) \in U^r$, it can be arranged that T is injective and $\theta^i \in \mathbb{R}^r_{\geqslant 0}$, $i = 1, \ldots, d$.

Accordingly, we henceforth assume $\theta^1, \ldots, \theta^d \in \mathbb{R}^r_{\geqslant 0}$. With this assumption in place, it then follows from (2.10) that

$$(N(\theta^1), \ldots, N(\theta^d); F, W)$$

is also an admissible nilpotent orbit.

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We are now ready to state our main theorem concerning the asymptotic behavior of variations of mixed Hodge structure.

Theorem 2.30. *Let $\mathcal{V}$ be an admissible variation of mixed Hodge structure on a polypunctured disk Δ^{*r} with unipotent monodromy and associated local normal form (2.3). Let $z(m) = x(m) + iy(m)$ be an sl_2-sequence with corresponding flag (2.29). Then,*

$$\lim_{m\to\infty} e^{-N(x(m))} \cdot \hat{Y}_{(F(z(m)),W)} = Y(N(\theta^1), Y(N(\theta^2), \ldots, Y_{(\hat{F}_\infty, W^r)})). \tag{2.31}$$

Remark 2.32. The statement of (2.31) implicitly assumes that $y_j \to \infty$ for each j. In the case where only $y_1, \ldots, y_\ell$ diverge, the limit Hodge filtration F_∞ should be replaced by

$$\lim_{m\to\infty} e^{\sum_{j\leqslant \ell} -z_j(m)N_j} \cdot F(z(m))$$

and the weight filtration W^r should be replaced by W^ℓ. In the extreme case where $z(m)$ is bounded, $\ell = 0$ in the previous equation and Theorem 2.30 is just the continuity of the sl_2-splitting.

The proof of Theorem 2.30 will take up most of this paper. However, we would like to bring to the reader's attention the obvious fact that, as the left-hand side of the equation in the theorem depends only on the choice of flag (2.29) the right-hand side must also depend only on the choice of this flag. It is an elementary exercise using Corollary 5.12 to show that the right-hand side of (2.31) depends only on the flag Θ.

Remark 2.33. Theorem 2.30 has also been obtained independently by Kato *et al.* in a private communication in their study of classifying spaces of degenerations of mixed Hodge structure. In particular, as part of their study of log intermediate Jacobians [KNU08], they obtain an independent proof of Conjecture 1.2.

Finiteness

One immediate corollary of Theorem 2.30 is the boundedness of the function $z \mapsto \hat{Y}_{(F(z),W)}$.

Corollary 2.34. *Let $F(z)$ be the period map of an admissible variation of mixed Hodge structure over Δ^{*r}. Let I denote the standard vertical strip (2.25) for U^r. Then the function $z \mapsto \hat{Y}_{(F(z),W)}$ is bounded on I.*

Proof. Otherwise, we can find a sequence of points $z(m) \in I$ on which $\hat{Y}_{(F(z(m)),W)}$ is unbounded. Passing to an sl_2-subsequence, we then obtain a contradiction to Theorem 2.30. □

This boundedness gives rise to finiteness, which will be important for integral variations.

Corollary 2.35. *Let $\mathcal{V}$ be an admissible variation of integral mixed Hodge structure over Δ^{*r} with unipotent monodromy. Then, with the notation as in Theorem 2.34, the set $\mathcal{Y}$ of integral gradings in the image of the map $z \mapsto Y_{(F(z),W)}$ as z runs over the vertical strip I is finite.*

Proof. If $Y_{(F(z),W)}$ is an integral grading, then (clearly) it is a real grading. It follows that $\hat{Y}_{(F(z),W)} = Y_{(F(z),W)}$. (To see this, note that, in the terminology of [KNU08, § 1.2], $\delta(F, W) = 0$ and this implies that $\epsilon(F, W) = 0$.) Therefore, the set of integral gradings of the form $Y_{(F(z),W)}$ as z ranges over I is bounded and discrete. Thus it is finite. □

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3. Reductions

Our next job is to reduce Theorem 1.2 to the case that $\bar{S}$ is a polydisk and S is a punctured polydisk. We begin with a review of some features of germs.

3.1. Let X be a topological space. Write $\mathcal{S}$ for the presheaf (in fact, a sheaf) associating to every open $U \subset X$ the set of all subsets of U. (If $V \subset U$, the map $\mathcal{S}(U) \to \mathcal{S}(V)$ is given by $Z \mapsto Z \cap V$). The set of subset germs at a point $x \in X$ is the stalk $\mathcal{S}_x$. If $Z \subset X$ then the germ of Z at x is the image of Z in $\mathcal{S}_x$.

3.2. If X is a complex analytic space we write $\mathcal{A}$ for the presheaf (also a sheaf) associating to every open subset U of X the set of all complex analytic subspaces Z of U. The set of germs of complex analytic subspaces at a point $x \in X$ is the stalk $\mathcal{A}_x$. We write $\mathcal{A}^r$ for the subsheaf of reduced subspaces. There is an obvious inclusion $\mathcal{A}^r \to \mathcal{S}$. We say that a germ $Z \in \mathcal{S}_x$ is analytic if it is in the image of this inclusion.

3.3. Let X be a reduced complex analytic space, let $x \in X$ and let $Z \in \mathcal{S}_x$ be a subset germ. We say that $f \in \mathcal{O}_{X,x}$ vanishes on Z if there is an open neighborhood U of x such that f is regular on U and vanishes on a subset Z_U of U whose germ is Z. We write $\mathcal{I}_{Z,x}$ for the set of all $f \in \mathcal{O}_{X,x}$ which vanish on Z. Clearly, $\mathcal{I}_{Z,x}$ is an ideal in $\mathcal{O}_{X,x}$. By the Noetherian property of $\mathcal{O}_{X,x}$ it is, therefore, finitely generated. We define *the Zariski closure* $\mathrm{ClZar}_x\, Z$ *of* Z *at* x to be the analytic subspace germ corresponding to $\mathcal{I}_{Z,x}$. We say that Z is *Zariski dense* at $x \in X$ if $\mathrm{ClZar}_x\, Z$ is the germ associated to X. Then Z is Zariski dense at x if any $f \in \mathcal{O}_{X,x}$ regular on a neighborhood U of x and vanishing on $U \cap Z$ vanishes identically on a neighborhood of x. A subset Z of X is Zariski dense in X if it is Zariski dense at every point $x \in X$.

3.4. Let $\bar{S}$ be a complex analytic space with Zariski dense regular locus $\bar{S}_{\mathrm{reg}}$ (for example, any reduced complex analytic space). Let $S \subset \bar{S}_{\mathrm{reg}}$ be a Zariski open subset. A variation of mixed Hodge structure $\mathcal{V}$ on S is *admissible relative to* $\bar{S}$ if, for any resolution of singularities $\pi : \bar{T} \to \bar{S}$ with $T := \pi^{-1}S$ biholomorphic to S, $\mathcal{V}_T$ is admissible relative to $\bar{T}$. Note that if the above property holds for one resolution of singularities $\bar{T} \to \bar{S}$ it holds for all. This defines a category $\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}$ which is, in fact, equivalent to the category $\mathrm{VMHS}(T)^{\mathrm{ad}}_{\bar{T}}$ for any resolution $\bar{T} \to \bar{S}$. If $\mathcal{H}$ is a variation of Hodge structure of negative weight on S, we define $\mathrm{NF}(S,\mathcal{H})^{\mathrm{ad}}_{\bar{S}} = \mathrm{Ext}^1_{\mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}}(\mathbb{Z},\mathcal{H}) = \mathrm{NF}(T,\mathcal{H}_T)^{\mathrm{ad}}_{\bar{T}}$ where $\pi : \bar{T} \to T$ is any resolution with $\pi : \pi^{-1}(S) \to S$ an isomorphism.

Theorem 3.5. *Let r be a non-negative integer, then the following are equivalent.*

(a) *Let $S = \Delta^{*r}$, $\bar{S} = \Delta^r$; let $\mathcal{H}$ be a polarized variation of pure Hodge structure of negative weight, with $\mathcal{H}_{\mathbb{Z}}$-torsion free and with unipotent monodromy on S; and let $\nu \in \mathrm{NF}(S,\mathcal{H})^{\mathrm{ad}}_{\bar{S}}$. Let $\bar{\mathcal{Z}}(\nu)$ denote the closure of the zero locus $\mathcal{Z}(\nu)$ in the analytic topology of $\bar{S}$. Assume that the germ of $\mathcal{Z}(\nu)$ at 0 is Zariski dense at 0. Then, the germ of $\bar{\mathcal{Z}}(\nu)$ at 0 coincides with the germ of $\bar{S}$ at 0.*

(b) *Let $S, \bar{S}$ and $\mathcal{H}$ be as in (a), but drop the assumption that the germ of $\mathcal{Z}(\nu)$ at 0 is Zariski dense. Then the germ of $\bar{\mathcal{Z}}(\nu)$ at 0 is analytic.*

(c) *The same statement as in (a) holds without the assumption that $\mathcal{H}_{\mathbb{Z}}$ is torsion free.*

(d) *The same statement as in (b) holds without the assumption that $\mathcal{H}$ has unipotent monodromy.*

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(e) *Let a and b be non-negative integers with $a+b=r$, let $S=\Delta^{*a}\times\Delta^{b}$ and $\bar{S}=\Delta^{a+b}$. Let $\mathcal{H}$ be a variation of pure Hodge structure of negative weight on S and let $\nu\in\mathrm{NF}(S,\mathcal{H})^{\mathrm{ad}}_{\bar{S}}$. Then the germ of $\bar{\mathcal{Z}}(\nu)$ at 0 is analytic.*

(f) *Theorem 1.2 holds in the case that S has dimension r and $\bar{S}\backslash S$ is a normal crossing divisor.*

(g) *Let $\bar{S}$ be a complex analytic space of dimension r and let S be a Zariski open subset of $\bar{S}_{\mathrm{reg}}$. Let $\mathcal{H}$ be a variation of Hodge structure of negative weight on S and let $\nu\in\mathrm{NF}(S,\mathcal{H})^{\mathrm{ad}}_{\bar{S}}$. Then the topological closure $\bar{\mathcal{Z}(\nu)}$ is the underlying space of a closed complex subspace of $\bar{S}$.*

Proof. We prove the entire theorem by induction on r. The equivalence is obvious for $r=0$ (since all of the individual statements hold unconditionally).

(a) $\Rightarrow$ (b). Let Z denote the Zariski closure of $\mathcal{Z}(\nu)$ at 0. Shrinking the polydisk $\bar{S}$ if necessary, we can assume that Z is an analytic subspace of $\bar{S}$ and that Z contains $\bar{\mathcal{Z}}(\nu)$. We can also assume that $\dim Z<r$. Let ν_Z denote the restriction of ν to the regular locus of Z. By induction (g) applies to show that $\bar{\mathcal{Z}}(\nu_Z)$ is a closed complex analytic subspace of Z. Since $\mathcal{Z}(\nu_Z)=\mathcal{Z}(\nu)\cap Z_{reg}$ is Zariski dense in Z, this implies that $\bar{\mathcal{Z}}(\nu)=Z$.

(b) $\Rightarrow$ (c). Let $\mathcal{H}_{\mathrm{tors}}$ denote the torsion part of $\mathcal{H}$ and $\mathcal{H}_{\mathrm{free}}:=\mathcal{H}/\mathcal{H}_{\mathrm{tors}}$ denote the torsion-free part with $\pi:\mathcal{H}\to\mathcal{H}_{\mathrm{free}}$ the projection map. Then, for $\nu\in\mathrm{NF}(S,\mathcal{H})^{\mathrm{ad}}_{\bar{S}}$ we have $\mathcal{Z}(\nu)=\mathcal{Z}(\pi(\nu))$ (because $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z},H)=0$ for H a torsion mixed Hodge structure).

(c) $\Rightarrow$ (d). By Borel's theorem, the monodromy of $\mathcal{H}$ is quasi-unipotent. Therefore we can find a positive integer d such that the pullback of $\mathcal{H}$ to Δ^{*r} via the map $f:S\to S$ given by $(z_1,\ldots,z_r)\mapsto(z_1^d,\ldots,z_r^d)$ has unipotent monodromy. By assumption, the germ of $\bar{\mathcal{Z}}(f^*(\nu))$ at 0 is analytic. Since f is proper, the proper mapping theorem implies that the germ of $\bar{\mathcal{Z}}(\nu)$ at 0 coincides with the germ of $f(\bar{\mathcal{Z}}(\nu))$ at 0 and is analytic.

(d) $\Rightarrow$ (e). We induct on $r:=a+b$ starting with $a=b=0$ where the statement is obvious. For $i\in\{1,\ldots,r\}$ set $S_i:=\{z\in S: z_i=0\}$, and set $S_0=\Delta^{*r}$. For $i\in\{0,\ldots,r\}$ let ν_i denote the restriction of ν to S_i. Then $\mathcal{Z}(\nu)=\bigcup_{i=0}^r\mathcal{Z}(\nu_i)$, so $\bar{\mathcal{Z}}(\nu)=\bigcup_{i=0}^r\bar{\mathcal{Z}}(\nu_i)$. By hypothesis, the germ of $\bar{\mathcal{Z}}(\nu_0)$ at 0 is analytic and, by induction, for each $i>0$ the germ of $\bar{\mathcal{Z}}(\nu_i)$ at 0 is analytic. It follows that the germ of $\bar{\mathcal{Z}}(\nu)$ at 0 is analytic.

(e) $\Rightarrow$ (f). To prove that $\bar{\mathcal{Z}}(\nu)$ is analytic, it suffices to prove that its germ is analytic at each point $s\in\bar{S}$. Since $\bar{S}\backslash S$ is a normal crossing divisor this follows from (e) and from the obvious fact that the germ of $\bar{\mathcal{Z}}(\nu)$ is analytic at every point $s\in S$.

(f) $\Rightarrow$ (g). Set $C:=\bar{S}\backslash S$. By Hironaka [Hir77], we can find a proper morphism $\pi:\bar{T}\to\bar{S}$ where $\bar{T}$ is smooth, $D=\pi^{-1}(C)$ is a normal crossing divisor and $\pi:\bar{T}\backslash D\to S$ is an isomorphism. Then, setting $T:=\bar{T}\backslash D$, π induces an isomorphism $\pi^*\mathrm{NF}(S,\mathcal{H})^{\mathrm{ad}}_{\bar{S}}\cong\mathrm{NF}(T,\mathcal{H})^{\mathrm{ad}}_{\bar{T}}$ [Sai96, Remark 1.6(i)]. Let $\nu\in\mathrm{NF}(S,\mathcal{H})^{\mathrm{ad}}_{\bar{S}}$ be a normal function, and let $\mathcal{Z}_T=\{s\in\bar{T}:\pi^*(\nu)=0\}$. Suppose $\bar{\mathcal{Z}}_T\subset\bar{T}$ is complex analytic. Since π is proper, the proper mapping theorem shows that $\pi(\bar{\mathcal{Z}}_T)$ is analytic and $\bar{\mathcal{Z}}=\pi(\bar{\mathcal{Z}}_T)$.

(g) $\Rightarrow$ (a). This is obvious. $\square$

Lemma 3.6. *Let H be a pure Hodge structure of weight $w<0$ and with $H_{\mathbb{Z}}$-torsion free. Let $\nu\in\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Z},H)$ be represented by the short exact sequence*

$$0\to H\to V\to\mathbb{Z}\to 0 \tag{3.7}$$

with $V=(V_{\mathbb{Z}},F,W)$. Then $\nu=0\Leftrightarrow Y_{(F,W)}\in w\,\mathrm{End}(V_{\mathbb{Z}})$.

Proof. $\Rightarrow$. If $\nu=0$, we have $V=\mathbb{Z}\oplus H$. So, every $v\in V_{\mathbb{Z}}$ can be written as $v=r+h$ with $r\in\mathbb{Z}$ and $h\in H$. Clearly, $Y_{(F,W)}(v)=wh\in wV_{\mathbb{Z}}$.

$\Leftarrow$. Suppose $Y_{(F,W)} \in w\,\mathrm{End}(V_{\mathbb{Z}})$. Then the map $(1/w)Y_{(F,W)}$ is a morphism of mixed Hodge structure from V to H inducing a retraction of the sequence (3.7). $\square$

Corollary 3.8. *Let $S, \bar{S}, \mathcal{H}$ and ν be as in Theorem 3.5(b), and let ν be given by an extension*

$$0 \to \mathcal{H} \to \mathcal{V} \to \mathbb{Z} \to 0$$

of variations of mixed Hodge structures on $S = (\Delta^*)^n$. Let $(F(z), W)$ be the local normal form of $\mathcal{V}$ on U^n with $F(z) = e^{N(z)}e^{\Gamma(s)} \cdot F_\infty$. Then

$$\mathcal{Z}(\nu) = \{s \in S : s = e^{2\pi i z}, Y_{(F(z),W)} \in w\,\mathrm{End}\,V_{\mathbb{Z}}\}.$$

4. Analyticity of the zero locus

We now prove Theorem 1.2 assuming Theorem 2.30 and the results on the Deligne systems and the sl_2-splittings stated in § 2. In fact, we will deduce the theorem as a corollary of a more general result concerning admissible variations on punctured polydisks.

4.1. Set $S = \Delta^{*r}$, $\bar{S} = \Delta^r$ and $\pi : U^r \to S$ be as in the discussion preceding (2.3). Let $\mathcal{V} \in \mathrm{VMHS}(S)^{\mathrm{ad}}_{\bar{S}}$ with V and the local normal form of $\mathcal{V}$

$$F(z) = e^{N(z)}e^{\Gamma(s)} \cdot F_\infty$$

as in (2.3). To fix the notation, we remind the reader that $N(z) = \sum z_i N_i$ with $N_i \in \mathrm{End}\,V_{\mathbb{Q}}$ and that $V_{\mathbb{Q}}$ comes equipped with the weight filtration W. By definition, the limit mixed Hodge structure is (F_∞, M) where $M = W^r = M(N_1 + \cdots + N_r, W)$.

For $z \in U^r$, set $Y(z) = Y_{(F(z),W)}$. Let I denote the vertical strip (2.25). Then, for each integral $Y_{\mathbb{Z}} \in \mathrm{End}\,V_{\mathbb{Z}}$, set $B(Y_{\mathbb{Z}}) := \{z \in I : Y(z) = Y_{\mathbb{Z}}\}$ and $C(Y) = \pi(B(Y_{\mathbb{Z}}))$.

Theorem 4.2. *Suppose $C(Y_{\mathbb{Z}})$ is Zariski dense at the origin in $\bar{S}$ for some $Y_{\mathbb{Z}} \in \mathrm{End}\,V$. Then $C(Y_{\mathbb{Z}}) = S$. Moreover, $[N_i, Y_{\mathbb{Z}}] = 0$ for all i.*

Proof. By assumption, 0 is a limit point of $C(Y_{\mathbb{Z}})$ in the usual topology. Therefore, we can find (possibly after permuting the coordinates) an sl_2-sequence $z(m) \in I$ such that $Y(z(m)) = Y_{\mathbb{Z}}$ for all m (and $z_i(m)$ is unbounded for each i).

Write $z(m) = x(m) + iy(m)$ with x, y real and set $\mu = \lim_{m\to\infty} x(m)$. Write ξ for the sl_2-splitting of (F_∞, M). (In the notation of [KNU08], $\xi = \epsilon(F_\infty, M)$.) Then, by Theorem 2.30 and Lemma 2.22,

$$\begin{aligned} Y_{\mathbb{Z}} &= e^{N(\mu)} \cdot Y(N(\theta^1), Y(N(\theta^2), \ldots, Y_{(\hat{F}_\infty, M)})) \\ &= e^{N(\mu)}e^{-\xi} \cdot Y(N(\theta^1), Y(N(\theta^2), \ldots, Y_{(F_\infty, M)})). \end{aligned}$$

To simplify the notation, we write $Y_\infty = Y(N(\theta^1), Y(N(\theta^2), \ldots, Y_{(F_\infty,M)}))$ and $\tilde{\xi} = \xi - N(\mu)$. Then, since ξ commutes with the N_i we have

$$Y_{\mathbb{Z}} = e^{-\tilde{\xi}} \cdot Y_\infty.$$

Now, for any operator A on V, write $A^{p,q}$ for the component of A in $\mathfrak{gl}^{p,q}(V)_{(F_\infty,M)}$. By Lemma 2.22, $Y_\infty = Y_\infty^{0,0}$. In other words, Y_∞ preserves the $I^{p,q}_{(F_\infty,M)}$. Likewise, $\mathrm{ad}\,Y_\infty$ preserves the subalgebra $\mathfrak{q}$.

Lemma 4.3. *Suppose $z \in B(Y_{\mathbb{Z}})$. Then*

$$e^{\Gamma(s)} \cdot Y_\infty = e^{-N(z)} \cdot Y_{\mathbb{Z}}. \tag{4.4}$$

Proof of Lemma 4.3. Recall that by (2.2) we have $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{F_\infty} \oplus \mathfrak{q}$ where $\mathfrak{g}_{\mathbb{C}}^{F_\infty}$ stabilizes the limit Hodge filtration. Therefore, since $Y(z)$ and $e^{N(z)}e^{\Gamma(s)} \cdot Y_\infty$ both preserve $F(z)$ it follows that $Y(z) - e^{N(z)}e^{\Gamma(s)} \cdot Y_\infty$ preserves $F(z)$ and hence

$$f(z) = e^{-\Gamma(s)}e^{-N(z)} \cdot Y(z) - Y_\infty$$

takes values in $\mathfrak{g}_{\mathbb{C}}^{F_\infty}$. On the other hand, since $z \in B(Y_{\mathbb{Z}})$ and $Y_{\mathbb{Z}} = e^{-\tilde{\xi}} \cdot Y_\infty$ we also have

$$f(z) = e^{-\Gamma(s)}e^{-N(z)}e^{-\tilde{\xi}} \cdot Y_\infty - Y_\infty.$$

In particular, since $\Gamma(s)$, $N(z)$ and $\tilde{\xi}$ are elements of $\mathfrak{q}$ and $\operatorname{ad} Y_\infty$ preserves $\mathfrak{q}$ it follows that $f(z)$ takes values in

$$\mathfrak{g}_{\mathbb{C}}^{F_\infty} \cap \mathfrak{q} = 0. \qquad \square$$

LEMMA 4.5. *The two linear maps $L_\infty : \mathbb{C}^r \to \operatorname{End} V_{\mathbb{C}}$ given by $z \mapsto [N(z), Y_\infty]$ and $L_{\mathbb{Z}} : \mathbb{C}^r \to \operatorname{End} V_{\mathbb{C}}$ given by $z \mapsto [N(z), Y_{\mathbb{Z}}]$ have the same kernels.*

Proof of Lemma 4.5. This follows directly from the fact that $\tilde{\xi}$ commutes with $N(z)$. $\square$

Since $Y_{\mathbb{Z}}$ is integral, we can find a subset $\Omega \subset \{1, \ldots, r\}$ such that the $[N_j, Y_{\mathbb{Z}}]$ form a basis of $L_{\mathbb{Z}}(\mathbb{Q}^r)$ as j runs through Ω. Thus there exist rational numbers β_{ij} ($i \in \{1, \ldots, r\}, j \in \Omega$) such that

$$[N_i, Y_{\mathbb{Z}}] = \sum_{j \in \Omega} \beta_{ij}[N_j, Y_{\mathbb{Z}}].$$

Clearly $\beta_{jj} = 1$ for $j \in \Omega$. Likewise, applying $\operatorname{Ad}(e^{\tilde{\xi}})$ to the previous equation, it follows that

$$[N_i, Y_\infty] = \sum_{j \in \Omega} \beta_{ij}[N_j, Y_\infty].$$

So, since $\tilde{\xi}$ commutes with $N_1, \ldots, N_r$, setting $L_i = [N_i, Y_\infty]$, we have

$$L_\infty(z) = \sum_{j \in \Omega} \sum_{i \geqslant j} \beta_{ij} z_i L_j.$$

Multiplying (4.4) by $e^{\tilde{\xi}}$ and using the fact that $\tilde{\xi}$ commutes with $N(z)$, we see that

$$z \in B(Y) \Rightarrow e^{\tilde{\xi}}e^{\Gamma(s)} \cdot Y_\infty = e^{-N(z)} \cdot Y_\infty. \tag{4.6}$$

Since $Y_\infty = Y_\infty^{0,0}$ with respect to (F_∞, M) while each N_i is a morphism of type $(-1, -1)$, it follows that

$$(e^{N(z)} \cdot Y_\infty)^{-1,-1} = [N(z), Y_\infty].$$

Set $\gamma(s) = (e^{\tilde{\xi}}e^{\Gamma(s)} \cdot Y_\infty)^{-1,-1}$. This is a holomorphic function on $\bar{S}$ which by (4.6) must be equal to

$$(e^{-N(z)} \cdot Y_\infty)^{-1,-1} = L_\infty(-z) = -\sum_{j \in \Omega} \sum_{i=1}^{r} \beta_{ij} z_i L_j$$

for $z \in B(Y_{\mathbb{Z}})$. Since $C(Y_{\mathbb{Z}})$ is Zariski dense at the origin and $\gamma(s)$ lies in $L_\infty(\mathbb{C}^r)$ for $s \in C(Y_{\mathbb{Z}})$, $\gamma(s)$ must lie in $L_\infty(\mathbb{C}^r)$ for all $s \in \bar{S}$. The L_j, $j \in \Omega$, form a basis of $L_\infty(\mathbb{C}^r)$, and we can therefore write $\gamma(s) = \sum_{j \in \Omega} \gamma_j(s) L_j$ for $s \in \bar{S}$.

Therefore

$$z \in B(Y_{\mathbb{Z}}) \Rightarrow \gamma_j(s) = \sum_{i=1}^{r} \beta_{ij} z_i, \quad \forall j \in \Omega.$$

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We can find a positive integer b such that $b\beta_{ij} \in \mathbb{Z}$ for all i, j. So we have

$$z \in B(Y_{\mathbb{Z}}) \Rightarrow b\gamma_j(s) = \sum_{i=1}^{r} b\beta_{ij} z_i, \quad \forall j \in \Omega.$$

Exponentiating both sides we find that

$$z \in B(Y_{\mathbb{Z}}) \Rightarrow \exp(2\pi i b\gamma_j(s)) = \prod_{i=1}^{r} s_i^{b\beta_{ij}}, \quad \forall j \in \Omega. \tag{4.7}$$

By our assumption that $C(Y_{\mathbb{Z}})$ is Zariski dense at 0, (4.7) must hold identically on $\bar{S}$. The left-hand side of the equation is non-vanishing and holomorphic on a neighborhood of the origin, this forces $\beta_{ij} = 0$ for all i, j. Thus, since $\beta_{jj} = 1$ for $j \in \Omega$, we have $\Omega = \emptyset$ and $[N_i, Y_\infty] = 0$ for all i. Since $N(z)$ commutes with Y_∞ and with $\tilde{\xi}$, it commutes with $Y_{\mathbb{Z}}$. Thus we have

$$z \in B(Y_{\mathbb{Z}}) \Leftrightarrow e^{\Gamma(s)} \cdot Y_\infty = Y_{\mathbb{Z}}.$$

As the above equation is a holomorphic equation for $C(Y_{\mathbb{Z}})$, it must hold identically. Thus we have $C(Y_{\mathbb{Z}}) = S$. Moreover, since $\Gamma(0) = 0$, we have $Y_\infty = Y_{\mathbb{Z}}$.

This completes the proof of Theorem 4.2. □

Corollary 4.8. *Suppose $\nu \in \mathrm{NF}(S, \mathcal{H})^{\mathrm{ad}}_{\bar{S}}$ and suppose that $Z(\nu)$ is Zariski dense at the origin in S with $\mathcal{H}$ and S as in Theorem 3.5. Then $\mathcal{Z}(\nu) = S$.*

Proof. We write

$$0 \to \mathcal{H} \to \mathcal{V} \to \mathbb{Z} \to 0$$

for the extension corresponding to ν. By Corollary 2.35, the density of $\mathcal{Z}(\nu)$ at the origin implies that there is a $Y_{\mathbb{Z}}$ such that $C(Y_{\mathbb{Z}})$ is also Zariski dense. Then use Theorem 4.2 to complete the proof. □

Proof of Theorem 1.2. Corollary 4.8 establishes (a) of Theorem 3.5. Therefore (g) holds as well. This directly implies Theorem 1.2. □

5. Deligne systems I

In the remainder of this paper we will work exclusively with admissible variations of $\mathbb{R}$-mixed Hodge structure.

We now reduce the proof of Lemma 2.22 to a corollary of the following sequence of lemmata.

Lemma 5.1 [Del93]. *If $(N; F, W)$ defines an admissible nilpotent orbit with limit mixed Hodge structure split over $\mathbb{R}$ then, in the notation of (2.8), the sl_2-splitting of the mixed Hodge structure $(e^{iN} \cdot F, W)$ is $(e^{i\hat{N}} \cdot F, W)$.*

Proof. We have $Y_{(e^{i\hat{N}} \cdot F, W)} = Y(N, Y_{(F,M)})$ by the second line in the proof of Theorem 2 of the appendix to [KP03]. (In [KP03], the notation N_0 is used to denote the 0-eigencomponent of N under the operator $Y(N, Y_{(F,M)})$, which is denoted by $\hat{N}$ in this paper.) It follows from Lemma 2.20 that $Y_{(e^{i\hat{N}} \cdot F, W)} = \hat{Y}_{(e^{iN} \cdot F, M)}$. Therefore the SL_2-splitting of $(e^{iN} \cdot F, W)$ is equal to $(e^{i\hat{N}} \cdot F, W)$. □

Suppose now that $(N_1, \ldots, N_r; \hat{F}_r, W)$ defines an admissible nilpotent orbit with limit mixed Hodge structure split over $\mathbb{R}$. Following the notation of (2.8), let $W^0, \ldots, W^r$ be the associated

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system of weight filtrations. Recall that, by [CKS86, Kas86],

$$(z_1, \ldots, z_{r-1}) \mapsto (e^{\sum_{j \leqslant r-1} z_j N_j} e^{iN_r} \cdot \hat{F}_r, W^0)$$

is an admissible nilpotent orbit, and hence $(e^{iN_r} \cdot \hat{F}_r, W^{r-1})$ is a mixed Hodge structure. Accordingly,

$$(z_1, \ldots, z_{r-1}) \mapsto (e^{\sum_{k \leqslant r-1} z_k N_k} \cdot \hat{F}_{r-1}, W^0)$$

is an admissible nilpotent orbit with limit mixed Hodge structure split over $\mathbb{R}$, where $(\hat{F}_{r-1}, W^{r-1}) = (e^{i\hat{N}_r} \cdot \hat{F}_r, W^{r-1})$ is the sl_2-splitting of $(e^{iN_r} \cdot \hat{F}_r, W^{r-1})$. Iterating this construction, we obtain a sequence of mixed Hodge structures

$$(\hat{F}_{j-1}, W^{j-1}) = (e^{i\hat{N}_j} \cdot \hat{F}_j, W^{j-1}) \tag{5.2}$$

and associated nilpotent orbits

$$(z_1, \ldots, z_j) \mapsto e^{\sum_{k \leqslant j} z_k N_k} \cdot \hat{F}_j, \tag{5.3}$$

where $\hat{N}_j$ is defined as in the paragraph after (2.8).

In particular, given the data $(N_1, \ldots, N_r; \hat{F}_r, W)$ of an admissible nilpotent orbit with limit mixed Hodge structure split over $\mathbb{R}$, the sequence of gradings $\hat{Y}^j$ constructed in (2.8) is given by $\hat{Y}^j = Y_{(\hat{F}_j, W^j)}$. Since $N_1, \ldots, N_j$ are $(-1,-1)$-morphisms of $(\hat{F}_j, W^j)$, it follows that

$$[N_k, \hat{H}_j] = 0 \tag{5.4}$$

for $j > k$ where, as in (2.8), $\hat{H}_j = \hat{Y}^j - \hat{Y}^{j-1}$.

Lemma 5.5. *Let $(N_1, \ldots, N_r; \hat{F}_r, W)$ define an admissible nilpotent orbit with limiting mixed Hodge structure $(\hat{F}, M)$ split over $\mathbb{R}$. Then,*

$$\hat{Y}^0 = Y(N_1, Y(N_2, \ldots, Y(N_r, Y_{(\hat{F},M)})))$$

preserves $\hat{F}$.

Proof. To begin, we recall that $(\hat{N}_1, \hat{H}_1), \ldots, (\hat{N}_r, \hat{H}_r)$ form a commuting system of sl_2-representations [Del93, Sch01]. Consequently,

$$[\hat{Y}^j, \hat{N}_k] = 0 \tag{5.6}$$

for $j < k$. Indeed, this is true by definition for $j = k-1$. Suppose that $j \leqslant k-2$. Then,

$$\begin{aligned}[\hat{Y}^j, \hat{N}_k] &= -[(\hat{Y}^{j+1} - \hat{Y}^j) + \cdots + (\hat{Y}^{k-1} - \hat{Y}^{k-2}), \hat{N}_k] \\ &= -[\hat{H}^{j+1} + \cdots + \hat{H}^{k-1}, \hat{N}_k] = 0.\end{aligned}$$

By the prior paragraphs, $\theta(z) = (e^{zN_1} \cdot \hat{F}_1, W)$ is an admissible nilpotent orbit with limit mixed Hodge structure split over $\mathbb{R}$, and hence by Lemma 2.20,

$$\hat{Y}^0(\hat{F}_1^p) \subseteq \hat{F}_1^p.$$

Using the identity $\hat{F}_r = e^{\sum_{j>1} i\hat{N}_j} \cdot \hat{F}$ and the fact that $\hat{Y}^0$ commutes with all $\hat{N}_j$, it then follows from the previous equation that $\hat{Y}^0$ preserves $\hat{F}$. □

To pass from admissible nilpotent orbits with limit mixed Hodge structure split over $\mathbb{R}$ to the general case, we now use the following sequence of lemmata.

Lemma 5.7. *Let $(N_1, \ldots, N_r; F, W)$ generate an admissible nilpotent orbit with sl_2-splitting $(\hat{F}, W^r) = (e^{-\xi} \cdot \hat{F}, W^r)$. Then, ξ preserves each associated weight filtration W^j.*

Proof. The sl_2-splitting is functorial and each W^j is a filtration of the limit mixed Hodge structure $(\hat{F}, W^r)$ by subobjects. It follows easily that ξ preserves each W^j. ☐

We now prove the result stated in Lemma 2.22.

Lemma 5.8. *Let $(N_1, \dots, N_r; F, W)$ define an admissible nilpotent orbit with sl_2-splitting $(\hat{F}, W^r) = (e^{-\xi} \cdot F, W^r)$. Then,*

$$Y(N_1, Y(N_2, \dots, Y(N_r, Y_{(\hat{F}, W^r)}))) = e^{-\xi} \cdot Y(N_1, Y(N_2, \dots, Y(N_r, Y_{(F, W^r)}))).$$

Proof. The operator ξ commutes with $N_1, \dots, N_r$ since ξ is a universal Lie polynomial in the Hodge components of Deligne's δ-splitting $(e^{-i\delta} \cdot F, W^r)$ of (F, W^r) and δ commutes with all $(-1,-1)$-morphisms of (F, W^r), and hence in particular with $N_1, \dots, N_r$. Furthermore, since

$$Y_{(e^{-\xi} \cdot F, W^r)} = e^{-\xi} \cdot Y_{(F, W^r)}$$

and ξ preserves W^{r-1} and commutes with N_r, we have (by the properties of Deligne's construction [KP03])

$$Y(N_r, Y_{(\hat{F}, W^r)}) = e^{-\xi} \cdot Y(N_r, Y_{(F, W^r)}).$$

Iterating this process, we obtain,

$$Y(N_1, Y(N_2, \dots, Y(N_r, Y_{(\hat{F}, W^r)}))) = e^{-\xi} \cdot Y(N_1, Y(N_2, \dots, Y(N_r, Y_{(F, W^r)}))).$$ ☐

Corollary 5.9. *Let $(N_1, \dots, N_r; F, W)$ define an admissible nilpotent orbit. Then,*

$$Y = Y(N_1, Y(N_2, \dots, Y(N_r, Y_{(F, W^r)})))$$

preserves the Deligne subspaces $I^{p,q}$ of (F, W^r).

Proof. Let $\hat{Y}$ denote the analog of Y obtained by replacing (F, W^r) by the sl_2-splitting $(\hat{F}, W^r)$. Then, $\hat{Y}$ is real and preserves both $\hat{F}$ and W^r. Therefore, $\hat{Y}$ preserves

$$I^{p,q}_{(\hat{F}, W^r)} = \hat{F}^p \cap \overline{\hat{F}^q} \cap W^r_{p+q}.$$

By the previous lemma, it then follows that Y preserves $I^{p,q}_{(F, W^r)}$ since $F = e^{\xi} \cdot \hat{F}$. ☐

For future reference, we now record the following three results.

Lemma 5.10. *Let $(N_1, \dots, N_r; F, W)$ be an admissible nilpotent orbit with limit mixed Hodge structure split over $\mathbb{R}$. Let $\sigma : \mathbb{R}^{r-i} \to \mathbb{R}^r$ denote the embedding $y \mapsto (0, y)$, and $\tilde{z}(m)$ be an sl_2-sequence in $\mathbb{R}^{r-i}$, with associated vectors θ^j as in (2.29). Then,*

$$(N_1, \dots, N_i, N(\sigma(\theta^1)), \dots, N(\sigma(\theta^d)); F, W) \tag{5.11}$$

is an admissible nilpotent orbit with the same limit MHS as the original nilpotent orbit. In particular, if $\hat{F}_0, \hat{F}_1, \dots,$ is the sequence (5.2) of filtrations associated with the nilpotent orbit (5.11) then

$$Y(N(\sigma(\theta^1)), \dots, Y(N(\sigma(\theta^d)), Y^r)) = Y_{(\hat{F}_i, W^i)}.$$

Proof. This is a minor variation on (2.10) and the previous remarks. ☐

Lemma 5.12. *Let $(N', N''; F, W)$ generate an admissible nilpotent orbit with associated weight filtrations $W' = M(N', W)$ and $W'' = M(N'', W')$. Let $Y'' = Y_{(F, W'')}$. Then, the pairs $(N' + N''; F, W')$ and $(N''; F, W')$ generate admissible nilpotent orbits which give the same associated gradings $Y(N' + N'', Y'')$ and $Y(N'', Y'')$ of W'.*

Proof. Since $(e^{z'N'+z''N''}\cdot F, W)$ is a nilpotent orbit it follows that N' is a $(-1,-1)$-morphism of $(e^{z''N''}\cdot F, W')$ whenever the latter is a MHS, and hence both (N'', F, W') and $(N'+N'', F, W')$ generate admissible nilpotent orbits with the same limit mixed Hodge structure. Moreover, by Lemma 2.22 we can without loss of generality assume that (F, W'') is split over $\mathbb{R}$. As such, by Lemmas 2.20 and 2.24

$$Y(N'+N'', Y'') = \hat{Y}_{(e^{iN'+iN''}\cdot F, W')} = \hat{Y}_{(e^{iN''}\cdot F, W')} = Y(N'', Y'')$$

since N' is a $(-1,-1)$-morphism of $(e^{iN''}, F, W')$. □

LEMMA 5.13. *Let W be an increasing filtration of a finite-dimensional vector space V over a field of characteristic zero. Let Y be a grading of W and N be a nilpotent endomorphism of V such that $[Y, N] = -2N$. Let $\beta \in W_{-1}(\mathfrak{gl}(V))$ and suppose that $[e^{\beta}\cdot Y, N] = -2N$. Then, $\beta \in \ker(\mathrm{ad}\, N)$.*

Proof. Observe that under the above hypothesis, $[e^{\beta}\cdot Y - Y, N] = 0$. Let $\beta = \sum_{j<0}\beta_{-j}$ with respect to the eigenvalues of $\mathrm{ad}\, Y$. If $\beta \neq 0$ then there is a smallest integer $k > 0$ such that $\beta_{-k} \neq 0$, and hence

$$e^{\beta}\cdot Y - Y = k\beta_{-k} \mod W_{-k-1}(\mathfrak{gl}(V)).$$

Applying $\mathrm{ad}\, N$ to both sides it then follows from the fact that $\mathrm{ad}\, N$ lowers the eigenspaces of $\mathrm{ad}\, Y$ by 2 that $\beta_{-k} = 0$. □

6. Deligne systems II

In [KNU08], Kato *et al.* attach to any admissible nilpotent orbit with data $(N_1, \ldots, N_r; F, W)$ an associated semisimple endomorphism $t(y)$. For later use, we now derive a formula for $t(y)$ in terms of the gradings $\hat{Y}^j$ constructed above. To this end, let us assume for the moment that $(N_1, \ldots, N_r; F, W)$ underlies a nilpotent orbit of pure Hodge structure of weight k. Let $(\hat{F}_r, W^r)$ denote the sl_2-splitting of (F, W^r), and recall that W^r in this case is the monodromy weight filtration $W(N)[-k]$ for any element N in the cone of positive linear combinations of $N_1, \ldots, N_r$. In particular, since any such N is a $(-1,-1)$-morphism of $(\hat{F}_r, W^r)$ it follows that the pair $(N, \hat{Y}_{(r)})$ where

$$\hat{Y}_{(r)} = \hat{Y}^r - k$$

defines an sl_2-pair. As above, we can iteratively define $\hat{Y}_{(j)} = \hat{Y}^j - k$ using the nilpotent orbit $(N_1, \ldots, N_j; \hat{F}_j)$. Define,

$$\tilde{t}(y) = \prod_{j=1}^{r} t_j^{\frac{1}{2}\hat{Y}_{(j)}} = \left(\prod_{j=1}^{r} t_j^{-\frac{1}{2}k}\right)\left(\prod_{j=1}^{r} t_j^{\frac{1}{2}\hat{Y}^j}\right)$$

where $t_j = y_{j+1}/y_j$, and hence $t_1 \ldots t_r = y_{r+1}/y_1 = 1/y_1$. Accordingly,

$$\tilde{t}(y) = y_1^{(\frac{1}{2}k)} \prod_{j=1}^{r} t_j^{\frac{1}{2}\hat{Y}^j}.$$

By [KNU08, Theorem 0.5], the mixed version of $t(y)$ is to be constructed as follows: if $(N_1, \ldots, N_r; F, W)$ defines an admissible nilpotent orbit then

$$\hat{Y}_{(e^{\sum iy_jN_j}\cdot F, W)} \to \hat{Y}^0$$

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provided that $t_j \to 0$ for all j. Let $t_k(y)$ denote the semisimple endomorphism $\tilde{t}(y)$ attached by the previous paragraph to the induced nilpotent orbit of pure Hodge structure of weight k on Gr^W_k. Then, $t(y)$ is constructed by multiplying each $t_k(y)$ by $y_1^{-\frac{1}{2}k}$ and then lifting the resulting semisimple element to the ambient vector space via the grading $\hat{Y}^0$. Accordingly, since the gradings $\hat{Y}^0, \ldots, \hat{Y}^r$ are mutually commuting, it follows that

$$t(y) = \prod_{j=1}^{r} t_j^{\frac{1}{2}\hat{Y}^j}. \tag{6.1}$$

Remark 6.2. For a nilpotent orbit of pure Hodge structure, the elements $\hat{Y}_{(j)}$ are infinitesimal isometries of the polarization. Consequently, although $t(y)$ is not an element of $G_{\mathbb{C}}$ since it is the twist of an automorphism of the graded-polarizations by $y_1^{-\frac{1}{2}\hat{Y}^0}$, the action of $\mathrm{Ad}(t^{-1}(y))$ preserves $G_{\mathbb{C}}$.

The following result appears in [KNU08, Proposition 10.4] with slightly different notation.

Lemma 6.3. *Let $(N_1, \ldots, N_r; F, W)$ define an admissible nilpotent orbit. Then,*

$$\mathrm{Ad}(t^{-1}(y))e^{\sum_j iy_jN_j} = e^P$$

where P is a polynomial in non-negative half-integral powers of $t_1, \ldots, t_r$ with constant term $iN_1 + i\sum_{j>1}\hat{N}_j$.

Proof. By (6.1),

$$\mathrm{Ad}(t^{-1}(y))y_kN_k = \left(\prod_{j\leqslant k-1} t_j^{-\frac{1}{2}\hat{Y}^j}\right)\left(\prod_{j\geqslant k} t_j^{-\frac{1}{2}\hat{Y}^j}\right)y_kN_k$$

where N_k is a $(-1,-1)$-morphism of $(\hat{F}_j, W^j)$ for $j = k, \ldots, r$, and hence $[N_k, \hat{Y}^j] = -2N_k$. Consequently,

$$\left(\prod_{j\geqslant k} t_j^{-\frac{1}{2}\hat{Y}^j}\right)y_kN_k = t_k \ldots t_r y_k N_k = N_k.$$

On the other hand, N_k preserves W^j for $j < k$. Therefore,

$$\left(\prod_{j<k} t_j^{-\frac{1}{2}\hat{Y}^j}\right)N_k$$

is a polynomial in non-negative, half-integral powers of t_j for $j < k$. Taking the limit as $t_1, \ldots, t_r \to 0$ it then follows that the constant term of P is $i\sum_k N_k^\sharp$ where $N_k^\sharp$ is the projection of N_k to $\bigcap_{0<j<k}\ker(\mathrm{ad}\,\hat{Y}^j)$ with respect to the mutually commuting gradings $\hat{Y}^j$. Accordingly, $N_1^\sharp = N_1$, whereas for $k > 1$, we can first project onto $\ker(\mathrm{ad}\,N_{k-1})$ to obtain $\hat{N}_k$. By (5.6), $\hat{N}_k$ commutes with $\hat{Y}^j$ for $j < k$, and hence $N_k^\sharp = \hat{N}_k$. □

Remark 6.4. For nilpotent orbits of pure Hodge structure, this statement appears in [CK89, Lemma (4.5)]; note however that in [CK89], t_j is defined to be y_j/y_{j+1} which is reciprocal to our convention.

7. Relative compactness

Let I and I' be subsets of U^r as in (2.25) and (2.26). Let $F : U \to \mathcal{M}$ be the period map of an admissible variation of mixed Hodge structure over Δ^{*r} with local normal form $F(z) = e^{N(z)}e^{\Gamma(s)} \cdot F_\infty$ as in (2.3). Let $t(y)$ be the associated family of semisimple endomorphisms (6.1). In this section, we will prove the following result, which is due to Cattani and Kaplan in the pure case [CK89, Theorem 4.7].

LEMMA 7.1. *The image of the set I' under the map*

$$\tilde{F}(z_1, \dots, z_r) = t^{-1}(y)e^{-\sum_j x_j N_j} \cdot F(z_1, \dots, z_r)$$

is a relatively compact subset of the classifying space $\mathcal{M}$.

Remark 7.2. In [CK89, Theorem (4.7)], Cattani and Kaplan define $t_j = y_j/y_{j+1}$, which is reciprocal to our convention.

For each index $j = 1, \dots, r$ let

$$\Gamma_j(s) = \Gamma(0, \dots, 0, s_{j+1}, \dots, s_r).$$

Then, for each j we have an associated partial period map

$$F_j(z_1, \dots, z_r) = e^{\sum_j z_j N_j}e^{\Gamma_j(s)} \cdot F_\infty \tag{7.3}$$

which takes values in $\mathcal{M}$ for $\mathrm{Im}(z)$ sufficiently large. Indeed, (7.3) is the nilpotent orbit obtained from $F(z)$ by degenerating $z_1, \dots, z_j$.

Remark 7.4. As in [CK89, Proposition (2.6)], it follows via [Pea00, (6.10)] that

$$\Gamma_j \in \ker(\mathrm{ad}\, N_1) \cap \cdots \cap \ker(\mathrm{ad}\, N_j). \tag{7.5}$$

To simplify future notation, we define $F_0(z) = F(z)$ and set

$$\tilde{F}_j(z_1, \dots, z_r) = t^{-1}(y)e^{-\sum_j x_j N_j} \cdot F_j(z_1, \dots, z_r) \tag{7.6}$$

for $j = 0, \dots, r$.

DEFINITION 7.7. Let $z(m) \in U^r$ be an sl_2-sequence, and suppose that $t_j(m) \to 0$. Then, we say that $z(m)$ has non-polynomial growth with respect to y_j (or z_j) if there exists a subsequence $z(m')$ of $z(m)$ such that

$$\lim_{m' \to \infty} \frac{y_{j+1}^d(m')}{y_j(m')} = 0 \tag{7.8}$$

for every $d > 0$. In particular, unless a sequence of points $z(m) \in I'$ is bounded, there exists a smallest index ι such that $z(m)$ has non-polynomial growth with respect to y_ι (since we formally define $y_{r+1}(m) = 1$). If $z(m) \in I'$ is bounded, we define $\iota = 0$.

To employ the notion of non-polynomial growth in aid of the proof of Theorem 2.30 we recall the following elementary observation about convergent sequences.

LEMMA 7.9. *Let Σ be a topological space. Then, a sequence σ_m in Σ converges to σ if and only if for every subsequence σ'_m of σ_m there exists a subsequence σ''_m of σ'_m such that $\sigma''_m \to \sigma$.*

Given an sl_2-sequence $z(m)$ and a grading Y_{lim} of W, in order to show that

$$\hat{Y}_{(F(z(m)),W)} \to Y_{\mathrm{lim}},$$

it is sufficient to show that, for every subsequence $z'(m)$ of $z(m)$, we can find a subsequence $z''(m)$ such that

$$\hat{Y}_{(F(z''(m)),W)} \to Y_{\mathrm{lim}}.$$

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In particular, since each $z'(m)$ is an sl_2-sequence, it has a corresponding smallest index ι with respect to which it has non-polynomial growth, and hence we can pass to a subsequence $z''(m)$ of $z'(m)$ for which (7.8) holds for y_ι.

As such, it is sufficient to prove Theorem 2.30 for sl_2-sequences $z(m)$ satisfying (7.8) since the right-hand side of (2.31) only depends on the original sequence $z(m)$ and the associated nilpotent orbit. Moreover, we may pass to a subsequence of $z(m)$ as necessary.

Theorem 7.10. *Let $z(m) = x(m) + iy(m) \in I'$ be an sl_2-sequence. Let ι be the smallest index with respect to which $z(m)$ has non-polynomial growth with respect to y_ι. Assume that (7.8) holds on $z(m)$. Then,*

$$\lim_{m\to\infty} \hat{Y}_{(F(z(m)),W)} - \hat{Y}_{(F_\iota(z(m)),W)} \to 0 \tag{7.11}$$

upon passage to a suitable subsequence.

Proof. Assume Lemma 7.1. If $\iota = 0$, then (7.11) is a tautology because $F_\iota = F$.

Assume therefore that $\iota > 0$ and observe that both $F(z)$ and $F_\iota(z)$ arise from period maps with the same nilpotent orbit, and hence the same associated family (6.1) of semisimple endomorphisms $t(y)$. Therefore,

$$\begin{aligned} e^{-N(x)} \cdot F(z) &= e^{iN(y)} e^{\Gamma(s)} \cdot F_\infty \\ &= e^{iN(y)} e^{\Gamma(s)} e^{-\Gamma_\iota(s)} e^{\Gamma_\iota(s)} \cdot F_\infty \\ &= \mathrm{Ad}(e^{iN(y)})(e^{\Gamma(s)} e^{-\Gamma_\iota(s)}) e^{iN(y)} e^{\Gamma_\iota(s)} \cdot F_\infty \\ &= \mathrm{Ad}(e^{iN(y)})(e^{\Gamma(s)} e^{-\Gamma_\iota(s)}) e^{-N(x)} \cdot F_\iota(z_1, \ldots, z_r). \end{aligned}$$

Consequently,

$$\begin{aligned} \tilde{F}(z) &= t^{-1}(y) e^{-N(x)} \cdot F(z) \\ &= t^{-1}(y)\, \mathrm{Ad}(e^{iN(y)})(e^{\Gamma(s)} e^{-\Gamma_\iota(s)}) t(y) t^{-1}(y) e^{-N(x)} \cdot F_\iota(z_1, \ldots, z_r) \\ &= \mathrm{Ad}(t^{-1}(y))(\mathrm{Ad}(e^{iN(y)})(e^{\Gamma(s)} e^{-\Gamma_\iota(s)})) \cdot \tilde{F}_\iota(z) \\ &= e^{B(z)} \cdot \tilde{F}_\iota(z) \end{aligned}$$

where

$$\begin{aligned} e^{B(z)} &= \mathrm{Ad}(t^{-1}(y))(\mathrm{Ad}(e^{iN(y)})(e^{\Gamma(s)} e^{-\Gamma_\iota(s)})) \\ &= \mathrm{Ad}(\mathrm{Ad}(t^{-1}(y)) e^{iN(y)})\, \mathrm{Ad}(t^{-1}(y))(e^{\Gamma(s)} e^{-\Gamma_\iota(s)}). \end{aligned}$$

By Lemma 6.3, we know that

$$\mathrm{Ad}(t^{-1}(y))(e^{\sum_j iy_j N_j}) = e^{P(t)} \tag{7.12}$$

where $P(t)$ is a polynomial in non-negative half-integral powers of $t_1, \ldots, t_r$ (with constant term). Accordingly,

$$e^{B(z)} = \mathrm{Ad}(e^{P(t)})\, \mathrm{Ad}(t^{-1}(y))(e^{\Gamma(s)} e^{-\Gamma_\iota(s)}).$$

To analyze the asymptotic behavior of $e^{B(z)}$, define $\tilde{\Gamma}(s) \in \mathfrak{q}$ to be the unique nilpotent operator satisfying

$$e^{\tilde{\Gamma}(s)} = e^{\Gamma(s)} e^{-\Gamma_\iota(s)}.$$

Then, since $\tilde{\Gamma}(0) = 0$ if $s_1, \ldots, s_\iota = 0$, it follows that there exist $\mathfrak{q}$-valued holomorphic functions $f_1, \ldots, f_\iota$ on Δ^r such that

$$\tilde{\Gamma}(s) = \sum_{j=1}^{\iota} s_j f_j.$$

Moreover, the identity $|s_j| = e^{-2\pi y_j}$ coupled with the order structure

$$y_1 \geqslant y_2 \geqslant \cdots \geqslant y_r \geqslant 1 \tag{7.13}$$

on I' implies that $|s_j| \leqslant |s_\iota|$ for $j = 1, \ldots, \iota$. Shrinking Δ^r if necessary, we can then find a constant K such that

$$|\tilde{\Gamma}(s)| < K|s_\iota|.$$

Now consider a sequence $z(m) \in I'$ such that ι is the smallest index such that $z(m)$ has non-polynomial growth with respect to $z_\iota(m)$. Then, by construction, the quantities $y_1(m), \ldots, y_\iota(m)$ must satisfy some set of mutually polynomial bounds, otherwise we contradict the definition of ι. Therefore, since $t(y)$ acts semi-simply by multiplication by monomials in half-integral powers of $t_1, \ldots, t_r$ on its eigenspaces, it follows that (after increasing K if necessary)

$$|\mathrm{Ad}(t^{-1}(y))\tilde{\Gamma}(s(m))| < K y_\iota^d(m)|s_\iota(m)| \tag{7.14}$$

for some half-integer d.

Combining the above remarks, it then follows that

$$||e^{B(z(m))} - 1|| < K y_\iota^d(m)|s_\iota(m)| \tag{7.15}$$

for a suitable constant K.

To continue, observe that the operator norm of $\mathrm{Ad}(e^{\sum_j x_j N_j})$ is bounded on I' since $x \in [0,1]^r$ is compact. Accordingly, (for any fixed norm)

$$||\hat{Y}_{(F(z),W)} - \hat{Y}_{(F_\iota(z),W)}|| \leqslant K' ||\hat{Y}_{(e^{-N(x)}\cdot F(z),W)} - \hat{Y}_{(e^{-N(x)}\cdot F_\iota(z),W)}||$$

for some suitable constant K'. Accordingly, (7.11) is equivalent to

$$\lim_{m\to\infty} \hat{Y}_{(e^{-N(x(m))}\cdot F(z(m)),W)} - \hat{Y}_{(e^{-N(x(m))}\cdot F_\iota(z(m)),W)} \to 0.$$

In particular, since $t(y)$ is a real automorphism which preserves W,

$$\begin{aligned}
&\hat{Y}_{(e^{-N(x(m))}\cdot F(z(m)),W)} - \hat{Y}_{(e^{-N(x(m))}\cdot F_\iota(z(m)),W)} \\
&\quad = t(y(m)) \cdot (\hat{Y}_{(\tilde{F}(z(m)),W)} - \hat{Y}_{(\tilde{F}_\iota(z(m)),W)}) \\
&\quad = t(y(m)) \cdot (\hat{Y}_{(e^{B(z(m))}\cdot \tilde{F}_\iota(z(m)),W)} - \hat{Y}_{(\tilde{F}_\iota(z(m)),W)}).
\end{aligned}$$

Moreover, by Lemma 7.1, after passage to a subsequence, we can assume that $\tilde{F}_\iota(z(m))$ converges to some point in $\mathcal{M}$. By the real-analyticity of the map $(F, W) \mapsto \hat{Y}_{(F,W)}$ it then follows that

$$\hat{Y}_{(e^{B(z(m))}\cdot \tilde{F}_\iota(z(m)),W)} = e^{C(z(m))} \cdot \hat{Y}_{(\tilde{F}_\iota(z(m)),W)} \tag{7.16}$$

where $e^{C(z(m))} - 1$ satisfies a bound of the same form (7.15) as $e^{B(z(m))} - 1$. Therefore,

$$\begin{aligned}
&\hat{Y}_{(e^{-N(x(m))}\cdot F(z(m)),W)} - \hat{Y}_{(e^{-N(x(m))}\cdot F_\iota(z(m)),W)} \\
&\quad = t(y(m)) \cdot ((\mathrm{Ad}(e^{C(z(m))}) - 1)\hat{Y}_{(\tilde{F}_\iota(z(m)),W)}) \to 0
\end{aligned}$$

on account of the fact that $t(y)$ acts by half-integral powers of $t_1, \ldots, t_r$ and $y_j^\ell(m)(e^{C(z(m))} - 1) \to 0$ for $j = 1, \ldots, r$ and every half-integer ℓ. □

To continue, we now prove Theorem 2.30 in the case where $F(z)$ is a nilpotent orbit.

Lemma 7.17. *Theorem 2.30 is true for admissible nilpotent orbits.*

Proof. Let $z(m) = x(m) + iy(m) \in I'$ be an sl_2-sequence. Then,

$$y(m) = T(v(m)) + b(m)$$

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as in Definition 2.27. Accordingly,

$$e^{-N(x(m))}\cdot \hat{Y}_{(e^{N(z(m))}\cdot F_\infty,W)} = \hat{Y}_{(e^{\sum_j iv_j(m)N(\theta^j)}e^{iN(b(m))}\cdot F_\infty,W)}$$

with $\theta^1,\dots,\theta^d$ as described in (2.29).

Now, for any fixed element $b\in\mathbb{R}^r$, the data $(N(\theta^1),\dots,N(\theta^d),e^{iN(b)}\cdot F_\infty,W)$ defines an admissible nilpotent orbit with limit mixed Hodge structure $(e^{iN(b)}\cdot F_\infty,W)$. By Lemma 2.24,

$$(\widehat{e^{iN(b)}\cdot F}_\infty,W^r)=(\hat{F}_\infty,W^r)$$

and hence by (2.17) it follows that $\hat{Y}_{(e^{\sum_j iv_j(m)N(\theta^j)}e^{iN(b)}\cdot F_\infty,W)}$ converges to the grading

$$\tilde{Y}=Y(N(\theta^1),Y(N(\theta^2),\dots,Y_{(\hat{F}_\infty,W^r)}))$$

independent of b. By [KNU08, Theorem 10.8] it then follows that for a variable b confined to the interior of a compact set there is a constant c such that if $\tau_j=v_{j+1}/v_j<c$ then

$$\hat{Y}_{(e^{\sum_j iv_jN(\theta^j)}e^{iN(b)}\cdot F_\infty,W)}=\exp(u(\tau;b))\cdot\tilde{Y}$$

where $u(\tau,b)$ is a real-analytic function of $\tau=(\tau_1,\dots,\tau_r)$ and b with $u(0,b)=0$. Accordingly,

$$\hat{Y}_{(e^{\sum_j iv_j(m)N(\theta^j)}e^{iN(b(m))}\cdot F_\infty,W)}\to\tilde{Y}.$$

□

Corollary 7.18. *Let $z(m)$ be an sl_2-sequence for which (7.8) holds for either $\iota=0$ or $\iota=r$. Then, (2.31) holds along a subsequence of $z(m)$.*

Proof. For $\iota=0$ the sequence is bounded, and the statement follows from the continuity of the sl_2-splitting. For $\iota=r$ we first use Theorem 7.10 to reduce the computation of the limit (2.31) to the corresponding nilpotent orbit and then use the previous lemma. □

Corollary 7.19. *Theorem 2.30 is true for variations over Δ^*.*

Proof. This follows from the previous corollary since over Δ^* any sl_2-sequence $z(m)$ must have either $\iota=0$ or $\iota=1$. □

It remains to verify Lemma 7.1 for variations of mixed Hodge structure. For this, we will modify [KNU08, Corollary (12.8)] which asserts that if $z(m)$ is a strict sl_2-sequence then the limit

$$F_\flat=\lim_{m\to\infty}t^{-1}(y(m))\cdot F(z(m))=\lim_{m\to\infty}t^{-1}(y(m))\cdot F_r(z(m)) \tag{7.20}$$

exists, and belongs to the classifying space $\mathcal{M}$. We begin with the following result.

A sequence of points $s(m)\in\Delta^{*r}$ is a strict sl_2-sequence if $s(m)=\pi(z(m))$ for some strict sl_2-sequence $z(m)\in U^r$ where $\pi:U^r\to\Delta^{*r}$ is the covering map defined by $s_j=e^{2\pi iz_j}$ for $j=1,\dots,r$.

Lemma 7.21. *If $f:\Delta^r\to\mathbb{C}$ is a holomorphic function which vanishes along every strict sl_2-sequence $s(m)\in\Delta^{*r}$ then $f\equiv 0$.*

Proof. Strict sl_2-sequences $z(m)=x(m)+iy(m)\in U^r$ are equivalent to pairs of sequences $(x(m),t(m))$ such that $x(m)$ is a convergent sequence in $[0,1]^r$ and $t(m)$ is a sequence in $(0,1]^r$ which converges to zero. Indeed, given a strict sl_2-sequence $z(m)=x(m)+iy(m)$ we define $t(m)=(t_1(m),\dots,t_r(m))$ via the usual rule $t_j(m)=y_{j+1}(m)/y_j(m)$. Conversely,

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given $(x(m), t(m))$ as above, the sequence $z(m) = x(m) + iy(m)$ obtained by setting $y_j(m) = (t_j(m) \cdots t_r(m))^{-1}$ is a strict sl_2-sequence. In particular, since the differential of the map

$$t = (t_1, \ldots, t_r) \in (0, 1]^r \mapsto (y_1, \ldots, y_r) \in \mathbb{R}^r_{>0}$$

defined by $y_j = (t_j \ldots t_r)^{-1}$ is an isomorphism at each point $t \in (0, 1]^r$, it follows that given any point $s_o \in \Delta^{*r}$ on a strict sl_2-sequence $s(m)$, there exist strict sl_2-sequences passing through every point on a neighborhood of s_o. In particular, if f vanishes on every strict sl_2-sequence then $f \equiv 0$. □

To continue, we now prove Lemma 7.1 in the case where $F(z)$ is an admissible nilpotent orbit generated by $(N_1, \ldots, N_r; F, W)$.

Proof of Lemma 7.1 for admissible nilpotent orbits. Let $(\hat{F}, W^r) = (e^{-\xi} \cdot F, W^r)$ denote the sl_2-splitting of (F, W^r). Let $t(y)$ be the associated family of semisimple endomorphisms. Then,

$$t^{-1}(y) e^{\sum_j i y_j N_j} \cdot F = e^{P(t)} \operatorname{Ad}(t^{-1}(y))(e^{\xi}) \cdot \hat{F} \tag{7.22}$$

because $t^{-1}(y)$ fixes $\hat{F}$.

Given $A \in \operatorname{End}(V)$, let

$$A = \sum_{b \in \mathbb{Z}^r} A^b \tag{7.23}$$

where A^b is the eigencomponent of A on which $\operatorname{Ad}(t^{-1}(y)) A^b = t_1^{\frac{1}{2}b_1} \cdots t_r^{\frac{1}{2}b_r} A^b$. Then, by Lemma 5.7

$$\xi = \sum_{b \in \mathbb{Z}^r_{\geqslant 0}} \xi^b \tag{7.24}$$

i.e. $\xi^b = 0$ unless b is a vector with non-negative coordinates. Accordingly,

$$e^{\xi(t)} = \operatorname{Ad}(t^{-1}(y))\xi$$

is a polynomial in non-negative, half-integral powers of $t_1, \ldots, t_r$. Therefore, the image of any sequence $z(m)$ in I' under the map

$$z \mapsto t^{-1}(y) e^{iN(y)} \cdot F = e^{P(t)} e^{\xi(t)} \cdot \hat{F} \tag{7.25}$$

has a convergent subsequence in the compact dual $\check{\mathcal{M}}$. Now, a point in $\check{\mathcal{M}}$ belongs to $\mathcal{M}$ if and only if it induces polarized Hodge structures on Gr^W. By Lemma 7.1 for variations of pure Hodge structure, the image of I' in Gr^W via the map (7.25) is a relatively compact subset of the sum of the corresponding classifying spaces of pure Hodge structure. Therefore, the image of I' under the map (7.25) is a relatively compact subset of $\mathcal{M}$.

Let $b = (b_1, \ldots, b_r) \in \mathbb{Z}^r$. Define a partial order on the group $\mathbb{Z}^r$ by declaring that $b \geqslant 0$ if $b_j \geqslant 0$ for all j, and $b < 0$ otherwise. If $b < 0$ then $w(b) = \min\{j \mid b_j < 0\}$.

LEMMA 7.26. *Let*

$$\Gamma(s) = \sum_{b \in \mathbb{Z}^r} \Gamma^b(s) \tag{7.27}$$

be the decomposition of Γ *with respect to the action of* $\operatorname{Ad}(t^{-1}(y))$ *as above. If* $\Gamma^b \neq 0$ *and* $b < 0$ *then* $\Gamma^b_{w(b)} = 0$.

Proof. The proof of Lemma 6.3 shows that

$$\operatorname{Ad}(t^{-1}(y))(e^{N(x)}) = e^{Q(x,t)}$$

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where $Q(x,t)$ is a polynomial in half-integral powers of $t_1,\dots,t_r$ with constant term 0 and coefficients which are polynomials in $x_1,\dots,x_r$. By the above conventions (with $P(t)$ and $\xi(t)$ constructed using the nilpotent orbit attached to $F(z)$),

$$t^{-1}(y)\cdot F(z) = e^{Q(x,t)}e^{P(t)}\,\mathrm{Ad}(t^{-1}(y))(e^{\Gamma(s)})e^{\xi(t)}\cdot \hat{F}_\infty.$$

Consequently, since $Q(x,t)$, $P(t)$, $\Gamma(s)$ and $\xi(t)$ all take values in $\mathfrak{q}$ (see (2.2)) and converge along strict sl_2-sequences, it follows that (7.20) holds along strict sl_2-sequences if and only if

$$\mathrm{Ad}(t^{-1}(y))(\Gamma^b(s))\to 0$$

along strict sl_2-sequences for each index b.

Suppose now that $b<0$ and $\Gamma^b(s)\neq 0$. Let $w=w(b)$. Then,

$$t_1^{\frac{1}{2}b_1}\dots t_r^{\frac{1}{2}b_r}(\Gamma^b(s)-\Gamma^b_w(s))\to 0 \tag{7.28}$$

along any strict sl_2-sequence since

$$\Gamma^b(s)-\Gamma^b_w(s)=\sum_{k=1}^{w} s_k g_k \tag{7.29}$$

where $g_1,\dots,g_w$ are $\mathfrak{q}$-valued holomorphic functions and $b_1,\dots,b_{w-1}\geqslant 0$.

Therefore, $\mathrm{Ad}(t^{-1}(y))(\Gamma^b(s))\to 0$ along strict sl_2-sequences if and only if

$$t_1^{\frac{1}{2}b_1}\dots t_r^{\frac{1}{2}b_r}\Gamma^b_w(s)\to 0$$

along strict sl_2-sequences. In particular, via a choice of basis, $\Gamma^b_w(s)$ is represented by a matrix of holomorphic functions in the variables $s_{w+1},\dots,s_r$. Let f denote a typical matrix entry of $\Gamma^b_w(s)$. Then, the previous equation holds if and only if

$$t_1^{\frac{1}{2}b_1}\dots t_r^{\frac{1}{2}b_r}f(s)\to 0 \tag{7.30}$$

along strict sl_2-sequences. If $f\neq 0$ then by Lemma 7.21 it follows that there exists a strict sl_2-sequence $s(m)=\pi(z(m))$ on which f is non-vanishing. Furthermore, since f depends only on $s_{w+1}(m),\dots,s_r(m)$, we have the freedom to select $z_1(m),\dots,z_w(m)$ in such a way that $t_1^{\frac{1}{2}b_1}\dots t_r^{\frac{1}{2}b_r}$ becomes unbounded (since $b_w<0$). But this contradicts (7.30) since f does not vanish along $s(m)$. □

Corollary 7.31. *For any weight $b=(b_1,\dots,b_r)\in\mathbb{Z}^r$, the function $\mathrm{Ad}(t^{-1}(y))\Gamma^b(s)$ converges along any sl_2-sequence $z(m)\in I'$.*

Proof. Suppose that $z(m)$ is bounded. Then, $\mathrm{Ad}(t^{-1}(y))\Gamma^b(s)$ converges since both $\mathrm{Ad}(t^{-1}(y))$ and $\Gamma^b(s)$ converge. Likewise, if $b\geqslant 0$ then $\mathrm{Ad}(t^{-1}(y))\Gamma^b(s)$ converges along every sl_2-sequence since both $\Gamma^b(s)$ and $t_1^{\frac{1}{2}b_1}\cdots t_r^{\frac{1}{2}b_r}$ converge.

Assume therefore that $z(m)$ is unbounded and $b<0$. Let $w=w(b)$. Then, by the previous lemma, $\Gamma^b_w(s)=0$ and hence by (7.29)

$$\Gamma^b(s)=\Gamma^b(s)-\Gamma^b_w(s)=\sum_{k=1}^{w} s_k g_k \tag{7.32}$$

where $g_1,\dots,g_w$ are $\mathfrak{q}$-valued holomorphic functions. By the order structure (7.13) on I' it follows that $y_\ell\geqslant y_w$ for $\ell=1,\dots,w$ and hence

$$s_\ell t_w^{\frac{1}{2}b_w}\cdots t_r^{\frac{1}{2}b_r}\to 0 \tag{7.33}$$

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along any sl_2-sequence since $|s_\ell| = e^{-2\pi y_\ell}$. Combining (7.32) and (7.33), we obtain the convergence of $\mathrm{Ad}(t^{-1}(y))\Gamma^b(s)$ along sl_2-sequences since the remaining factor $t_1^{\frac{1}{2}b_1} \cdots t_{w-1}^{\frac{1}{2}b_{w-1}}$ converges, because $b_1, \ldots, b_{w-1} \geqslant 0$. ☐

COROLLARY 7.34. *Let $z(m)$ be an sl_2-sequence. Then, there exists $F_\sharp \in \mathcal{M}$ such that*

$$\tilde{F}(z(m)) \to F_\sharp.$$

Furthermore, the value of $F_\sharp$ depends only on the limiting values of the sequences $t_j(m)$.

Proof. By the above remarks,

$$\tilde{F}(z(m)) = e^{P(t)}(\mathrm{Ad}(t^{-1}(y))e^{\Gamma(s)})e^{\xi(t)} \cdot \hat{F}_\infty.$$

Moreover, along any sl_2-sequence, $P(t)$, $\mathrm{Ad}(t^{-1}(y))\Gamma(s)$ and $\xi(t)$ converge to limiting values in $\mathfrak{g}_{\mathbb{C}}$ which only depend on the limiting values of $t_j(m)$. This forces $\tilde{F}(z(m))$ to converge to a point $F_\sharp$ of the 'compact dual' $\check{\mathcal{M}}$ of $\mathcal{M}$. In particular, since elements of $\mathfrak{g}_{\mathbb{C}}$ preserve W, it follows from Lemma 7.1 for variations of pure Hodge structure that $F_\sharp$ is an element of $\mathcal{M}$. ☐

End of Proof of Lemma 7.1. To complete the proof of Lemma 7.1 for variations of mixed Hodge structure, observe that every sequence $z(m) \in I'$ contains an sl_2-sequence $z'(m)$. Applying the previous corollary to $F(z'(m))$ we see that the image of I' by $\tilde{F}$ is a relatively compact subset of $\mathcal{M}$. ☐

8. Polarized mixed Hodge structures

In this section we prove two technical results about deformations of admissible nilpotent orbits using the theory of polarized mixed Hodge structure outlined in [CK89].

LEMMA 8.1. *Let $(N_1, \ldots, N_k; F, W)$ generate an admissible nilpotent orbit. Then, there exists a neighborhood $\mathcal{A}$ of $0 \in \mathfrak{g}_{\mathbb{C}} \cap \ker(\mathrm{ad}\, N_1) \cap \cdots \cap \ker(\mathrm{ad}\, N_k)$ such that for all $\alpha \in \mathcal{A}$, $(N_1, \ldots, N_k; e^\alpha \cdot F, W)$ generates an admissible nilpotent orbit.*

Proof. The map

$$(z_1, \ldots, z_k) \mapsto e^{\sum_j z_j N_j} e^\alpha \cdot F$$

is horizontal since $N_j(F^p) \subseteq F^{p-1}$. Therefore, $N_j(e^\alpha \cdot F^p) \subseteq e^\alpha \cdot F^{p-1}$. Likewise, since $(N_1, \ldots, N_k; F, W)$ is admissible, all of the required relative weight filtrations exist. It remains therefore to show that there exists a constant c such that

$$(e^{\sum_j z_j N_j} e^\alpha \cdot F, W)$$

is a graded-polarized mixed Hodge structure for $\mathrm{Im}(z_1), \ldots, \mathrm{Im}(z_k) > c$.

Since $N_1, \ldots, N_k$ and α preserve W, we can assume without loss of generality that W is pure of weight ℓ. Via a Tate-twist, we can assume that $(e^{\sum_j z_j N_j} e^\alpha \cdot F, W)$ is pure and effective of weight ℓ. By [CK89, Theorem (2.3)], it is therefore sufficient to show that $e^{zN} e^\alpha \cdot F$ is a nilpotent orbit of pure Hodge structure of weight ℓ, where $N = \sum_j N_j$.

To continue, we note that since α commutes with $N_1, \ldots, N_k$ it follows that α preserves the monodromy weight filtration $W(N)$. Since we already know that $e^{zN} \cdot F$ is a nilpotent orbit of weight ℓ (polarized by some bilinear form Q), it then follows from the theory of polarized mixed Hodge structures (see [CK89]) that it is sufficient to show that $e^\alpha \cdot F$ induces a pure Hodge structure of weight $j + \ell$ on the primitive part (with respect to N) of $Gr_{j+\ell}^{W(N)[-\ell]}$ which

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is polarized by $Q_\ell(*,*) = Q(*, N^\ell *)$. But, this is an open condition on $\alpha \in \mathcal{A}$ which is true for $\alpha = 0$ since $e^{zN} \cdot F$ is a nilpotent orbit. $\square$

Given a nilpotent orbit generated by $(N_1, \ldots, N_r; F, W)$ let $t(y)$ be the associated semisimple operator (6.1). Then, the corresponding semisimple operator

$$t_\iota(y) = \prod_{j>\iota} t_j^{\frac{1}{2}\hat{Y}_j}$$

attached to the nilpotent orbit generated by $(N_{\iota+1}, \ldots, N_r; F, W^\iota)$ is obtained from $t(y)$ by setting $t_1, \ldots, t_\iota = 1$.

Lemma 8.2. *If $k \leqslant \ell$ and $\alpha \in \ker(\mathrm{ad}\, N_k)$ then each eigencomponent of α with respect to* $\mathrm{ad}\, \hat{Y}^\ell$ *belongs to* $\ker(\mathrm{ad}\, N_k)$.

Proof. By the Jacobi identity,

$$[N_k, [\hat{Y}^\ell, \alpha]] = [[N_k, \hat{Y}^\ell], \alpha] + [\hat{Y}^\ell, [N_k, \alpha]] = [2N_k, \alpha] = 0$$

since $[N_k, \hat{Y}^\ell] = 2N_k$. Consequently, each eigencomponent of α must also belong to $\ker(\mathrm{ad}\, N_k)$ since $\mathrm{ad}\, N_k$ decreases eigenvalues with respect to $\mathrm{ad}\, \hat{Y}^\ell$ by 2. $\square$

Corollary 8.3. *If α commutes with $N_1, \ldots, N_\iota$ then so does* $\mathrm{Ad}(t_\iota^{-1}(y))\alpha$.

Proof. Decompose α with respect to $\hat{Y}^{\iota+1}, \ldots, \hat{Y}^r$ and apply the previous lemma. $\square$

Lemma 8.4. *For $k \leqslant \iota$, $\mathrm{Ad}(t_\iota^{-1}(y)) y_k N_k = y_k/y_{\iota+1} N_k$ and hence*

$$\mathrm{Ad}(t_\iota^{-1}(y)) e^{i\sum_{k\leqslant\iota} y_k N_k} = e^{i\sum_{k\leqslant\iota}(y_k/y_{\iota+1})N_k}.$$

Proof. By (6.1),

$$t_\iota(y) = \prod_{j>\iota} t_j^{\frac{1}{2}\hat{Y}^j} = y_{\iota+1}^{-\frac{1}{2}\hat{Y}^{\iota+1}} \prod_{j>\iota} y_{j+1}^{-\frac{1}{2}\hat{H}_{j+1}}. \tag{8.5}$$

Accordingly, since $[\hat{Y}^{\iota+1}, N_k] = -2N_k$ for $k \leqslant \iota+1$ whereas $[N_k, \hat{H}_j] = 0$ for $j > k$ by (5.4) it follows by (8.5) that (for $k \leqslant \iota$)

$$\begin{aligned}\mathrm{Ad}(t_\iota^{-1}(y)) y_k N_k &= \mathrm{Ad}(y_{\iota+1}^{\frac{1}{2}\hat{Y}^{\iota+1}}) \prod_{j>\iota} \mathrm{Ad}(y_{j+1}^{\frac{1}{2}\hat{H}_{j+1}}) y_k N_k \\ &= \mathrm{Ad}(y_{\iota+1}^{\frac{1}{2}\hat{Y}^{\iota+1}}) y_k N_k = y_k/y_{\iota+1} N_k.\end{aligned}$$ $\square$

Given a period map $F(z_1, \ldots, z_r)$ with local normal form (2.3) let

$$F_\infty(z_{\iota+1}, \ldots, z_r) = e^{\sum_{j>\iota} z_j N_j} e^{\Gamma_\iota} \cdot F_\infty \tag{8.6}$$

be the limit mixed Hodge structure obtained by degenerating the variables $z_1, \ldots, z_\iota$ in $F(z)$. Then, $(F_\infty(z_{\iota+1}, \ldots, z_r), W^\iota)$ is an admissible variation of mixed Hodge structure. Let

$$I'_\iota = \{(z_{\iota+1}, \ldots, z_r) \in U^{r-\iota} \mid x_{\iota+1}, \ldots, x_r \in [0,1], y_{\iota+1} \geqslant \cdots \geqslant y_r \geqslant 1\}. \tag{8.7}$$

Then, by Corollary 7.34, for any sl_2-sequence $z(m) = (z_{\iota+1}(m), \ldots, z_r(m)) \in I'_\iota$, the filtration

$$\tilde{F}_\infty(z_{\iota+1}(m), \ldots, z_r(m)) = t_\iota^{-1}(y) e^{\sum_{j>\iota} i y_j N_j} e^{\Gamma_\iota(s)} \cdot F_\infty \tag{8.8}$$

converges to a filtration $F_\natural$ in the corresponding classifying space $\mathcal{M}_\iota$. Furthermore, the filtration $F_\natural$ depends only on the limiting values of $t_{\iota+1}(m), \ldots, t_r(m)$.

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LEMMA 8.9. *Let $(z_{\iota+1}(m),\dots,z_r(m))\in I'_\iota$ be an* sl_2*-sequence. Let $F_\natural\in\mathcal{M}_\iota$ be the associated limiting filtration constructed above. Then, the data $(N_1,\dots,N_\iota,F_\natural,W)$ generates an admissible nilpotent orbit.*

Proof. Since $(N_1,\dots,N_r;F_\infty,W)$ generates an admissible nilpotent orbit, all of the required relative weight filtrations exist. Accordingly, we can assume that $F(z_1,\dots,z_r)$ is an effective variation of pure Hodge structure of weight ℓ.

Let

$$P_\iota(t)=\mathrm{Ad}(t_\iota^{-1}(y))\biggl(\sum_{j>\iota} iy_jN_j\biggr). \tag{8.10}$$

Then, Lemma 6.3 applied to the nilpotent orbit generated by $(N_{\iota+1},\dots,N_r;F_\infty,W^\iota)$ asserts that $P_\iota(t)$ is a polynomial in non-negative, half-integral powers of $t_{\iota+1},\dots,t_r$. Let $(\hat F_\infty,W^r)=(e^{-\xi}\cdot F_\infty,W^r)$ be the sl_2-splitting of (F_∞,W^r). Then, by Lemma 5.7,

$$\xi_\iota(t)=\mathrm{Ad}(t_\iota^{-1}(y)\xi \tag{8.11}$$

is a polynomial in non-negative, half-integral powers of $t_{\iota+1},\dots,t_r$. Accordingly,

$$\tilde F_\infty(z_{\iota+1},\dots,z_r)=e^{P_\iota(t)}(\mathrm{Ad}(t_\iota^{-1}(y))e^{\Gamma_\iota(s)})e^{\xi_\iota(t)}\cdot\hat F_\infty \tag{8.12}$$

where $\mathrm{Ad}(t_\iota^{-1}(y))e^{\Gamma_\iota(s)}$ converges along any sl_2-sequence (apply Corollary 7.31 to the variation $(F_\infty(z_{\iota+1},\dots,z_r),W^\iota)$). Consequently,

$$F_\natural=e^{\gamma_\natural}\cdot\hat F_\infty \tag{8.13}$$

where

$$e^{\gamma_\natural}=\lim_{m\to\infty}e^{P_\iota(t)}(\mathrm{Ad}(t_\iota^{-1}(y))e^{\Gamma_\iota(s)})e^{\xi_\iota(t)}. \tag{8.14}$$

Moreover (see (7.5)), since $N_{\iota+1},\dots,N_r$, Γ_ι and ξ all belong to the subalgebra $\mathfrak{q}\cap(\bigcap_{k=1}^{\iota}\ker(\mathrm{ad}\,N_k))$, it follows from Corollary 8.3 that

$$\gamma_\natural\in\mathfrak{q}\cap\ker(\mathrm{ad}\,N_1)\cap\cdots\cap\ker(\mathrm{ad}\,N_\iota). \tag{8.15}$$

In particular, by virtue of equations (8.13) and (8.15) it follows that $N_1,\dots,N_\iota$ are horizontal with respect to $F_\natural$.

By [CK89, Theorem (2.3)], to complete the proof, it suffices to show that

$$z\mapsto e^{z(N_1+\cdots+N_\iota)}\cdot F_\natural$$

is a nilpotent orbit of pure Hodge structure. To this end, let a be a positive real number and $N_\dagger=a(N_1+\cdots+N_\iota)$. Observe that since

$$F_\iota(z)=e^{\sum_{j\leqslant\iota}z_jN_j}\cdot F_\infty(z_{\iota+1},\dots,z_r)$$

is an admissible variation of mixed Hodge structure so is

$$F_\dagger(z_\iota,\dots,z_r)=e^{z_\iota N_\dagger}e^{\sum_{j>\iota}z_jN_j}e^{\Gamma_\iota(s)}\cdot F_\infty$$

(here we can use [Kas86] to derive the existence of the required relative weight filtrations since $N_\dagger$ belongs to the interior of the cone generated by $N_1,\dots,N_\iota$). The associated nilpotent orbit of $F_\dagger$ is generated by $(N_\dagger,N_{\iota+1},\dots,N_r,F_\infty,W)$. Let $t_\dagger(y)$ be the associated semisimple operator. Then,

$$t_\dagger(y)=t_\iota^{\frac12\hat Y_\dagger}t_\iota(y).$$

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Let $P_\dagger(t) = \mathrm{Ad}(t_\dagger^{-1}(y))(iy_\iota N_\dagger + i\sum_{j>\iota} y_j N_j)$ and $\xi_\dagger(t) = \mathrm{Ad}(t_\dagger^{-1}(y))\xi$. Then,

$$\begin{aligned}\tilde{F}_\dagger(z_\iota, \ldots, z_r) &= t_\dagger^{-1}(y) e^{iy_\iota N_\dagger} e^{i\sum_{j>\iota} y_j N_j} e^{\Gamma_\iota} \cdot F_\infty \\ &= e^{P_\dagger(t)}(\mathrm{Ad}(t_\dagger^{-1}(y))e^{\Gamma_\iota})e^{\xi_\dagger(t)} \cdot \hat{F}_\infty.\end{aligned}$$

Let $z_\dagger(m) = (z_\iota(m), \ldots, z_r(m))$ be the sl_2-sequence obtained from the sequence $(z_{\iota+1}(m), \ldots, z_r(m))$ by setting $z_\iota(m) = z_{\iota+1}(m)$. Applying Corollary 7.34 to $\tilde{F}_\dagger(z_\dagger(m))$, we then obtain an associated limit filtration $F_\sharp \in \mathcal{M}$.

Regarding the filtration $F_\sharp$, we note that since $t_\iota(m) = y_{\iota+1}(m)/y_\iota(m) = 1$ along the sequence $z_\dagger(m)$ it follows that $t_\dagger(y) = t_\iota(y)$ along $z_\dagger(m)$. Therefore, the limits of $\mathrm{Ad}(t_\dagger^{-1}(y))e^{\Gamma_\iota}$ and $\xi_\dagger(t)$ along $z_\dagger(m)$ coincide with the limits of $\mathrm{Ad}(t_\iota^{-1}(y))e^{\Gamma_\iota}$ and $\xi_\iota(t)$ along the original sequence $(z_{\iota+1}(m), \ldots, z_r(m))$. Likewise, along $z_\dagger(m)$,

$$P_\dagger(t) = \mathrm{Ad}(t_\iota^{-1}(y))\left(iy_\iota N_\dagger + i\sum_{j>\iota} y_j N_j\right) = iN_\dagger + P_\iota(t)$$

since $\mathrm{Ad}(t_\iota^{-1}(y))y_\iota N_\dagger = N_\dagger$ by Lemma 8.4. Therefore,

$$F_\sharp = e^{iN_\dagger} \cdot F_\natural \in \mathcal{M}.$$

In particular, since $N_\dagger = a(N_1 + \cdots + N_\iota)$ with $a > 0$ arbitrary, it follows that $e^{z(N_1+\cdots+N_\iota)} \cdot F_\natural$ is a nilpotent orbit of pure Hodge structure. □

9. Proof of Theorem 2.30

In this section we prove Theorem 2.30 by induction on dimension r of the base Δ^{*r}. For $r = 1$, Theorem 2.30 follows from Corollary 7.19.

Accordingly, assume that $r > 1$ and let $z(m)$ be an sl_2-sequence. Let ι be the smallest index such that $y(m)$ has non-polynomial growth with respect to ι. If $\iota = 0$ or $\iota = r$, Theorem 2.30 along $z(m)$ follows from Corollary 7.18. Therefore, we can assume that $r > 1$ and $0 < \iota < r$. Therefore, by Theorem 7.10 it follows that

$$\hat{Y}_{(F(z(m), W)} - \hat{Y}_{(F_\iota(z(m)), W)} \to 0$$

and hence the proof of Theorem 2.30 is reduced to the case of period maps of the special form $F_\iota(z)$.

To complete the induction, we construct an associated flag in the spirit of (2.29) and verify that if

$$Y\left(\sum_{j\leqslant d} v_j(m) N(\theta^j), \hat{Y}_{(e^{\sum_{j>\iota} iy_j N_j} e^{\Gamma_\iota(s)} \cdot F_\infty, W^\iota)}\right) \to Y_o \tag{9.1}$$

then $e^{-N(x(m))} \cdot \hat{Y}_{(F_\iota(z(m)), W)} \to Y_o$. Having done this, we then compute the limit (9.1) using the induction hypothesis and the properties of Deligne systems.

As in (8.6) and (8.8), let us define

$$\begin{aligned}F_\infty(z_{\iota+1}, \ldots, z_r) &= e^{\sum_{j>\iota} z_j N_j} e^{\Gamma_\iota} \cdot F_\infty, \\ \tilde{F}_\infty(z_{\iota+1}, \ldots, z_r) &= t_\iota^{-1}(y) e^{\sum_{j>\iota} iy_j N_j} e^{\Gamma_\iota(s)} \cdot F_\infty.\end{aligned}$$

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Then,

$$\begin{aligned} F_\iota(z_1,\dots,z_r) &= e^{\sum_j z_j N_j} e^{\Gamma_\iota}\cdot F_\infty \\ &= e^{N(x)} e^{\sum_j iy_j N_j} e^{\Gamma_\iota}\cdot F_\infty \\ &= e^{N(x)} e^{\sum_{j\leqslant\iota} iy_j N_j} e^{\sum_{j>\iota} iy_j N_j} e^{\Gamma_\iota}\cdot F_\infty \\ &= e^{N(x)} e^{\sum_{j\leqslant\iota} iy_j N_j} t_\iota(y)\cdot \tilde{F}_\infty(z_{\iota+1},\dots,z_r) \\ &= e^{N(x)} t_\iota(y) e^{i\sum_{k\leqslant\iota}(y_k/y_{\iota+1})N_k}\cdot \tilde{F}_\infty(z_{\iota+1},\dots,z_r) \end{aligned}$$

where the last step is justified by Lemma 8.4.

By (8.12)

$$\tilde{F}_\infty(z_{\iota+1},\dots,z_r) = e^{P_\iota(t)}(\mathrm{Ad}(t_\iota^{-1}(y))e^{\Gamma_\iota(s)})e^{\xi_\iota(t)}\cdot \hat{F}_\infty.$$

Moreover, by (8.13), along the sequence $(z_{\iota+1}(m),\dots,z_r(m))$,

$$\tilde{F}_\infty(z_{\iota+1}(m),\dots,z_r(m)) \to F_\natural = e^{\gamma_\natural}\cdot \hat{F}_\infty.$$

Define

$$\mathfrak{v} = \mathfrak{q}\cap \ker(\mathrm{ad}\,N_1)\cap\cdots\cap\ker(\mathrm{ad}\,N_\iota).$$

Then, by Corollary 8.3 it follows that the function

$$e^{P_\iota(t)}(\mathrm{Ad}(t_\iota^{-1}(y))e^{\Gamma_\iota(s)})e^{\xi_\iota(t)}$$

takes values in $\mathfrak{v}$, and hence so does its limiting value $\gamma_\natural$ along $(z_{\iota+1}(m),\dots,z_r(m))$. Let

$$e^{\gamma(z_{\iota+1},\dots,z_r)} = e^{P_\iota(t)}(\mathrm{Ad}(t_\iota^{-1}(y))e^{\Gamma_\iota(s)})e^{\xi_\iota(t)}e^{-\gamma_\natural}. \tag{9.2}$$

Then, $\tilde{F}_\infty(z_{\iota+1},\dots,z_r) = e^{\gamma(z_{\iota+1},\dots,z_r)}\cdot F_\natural$, with

$$\gamma(z_{\iota+1}(m),\dots,z_r(m)) \to 0. \tag{9.3}$$

By Lemma 8.9, the data $(N_1,\dots,N_\iota;F_\natural,W)$ defines an admissible nilpotent orbit, and hence by Lemma 8.1 there exists a neighborhood $\mathfrak{v}_o$ of zero in $\mathfrak{v}$ such that for every $\nu\in\mathfrak{v}_o$ the data $(N_1,\dots,N_\iota;e^\nu\cdot F_\natural,W)$ defines an admissible nilpotent orbit. Shrinking $\mathfrak{v}_o$ as necessary, we can further assume that there exists a common constant c such that if $\mathrm{Im}(z_1),\dots,\mathrm{Im}(z_\iota)>c$ then

$$e^{\sum_{j\leqslant\iota} z_j N_j} e^\nu\cdot F_\natural \in \mathcal{M}. \tag{9.4}$$

In particular, by (9.3), there exists index m_o such that

$$\gamma(z_{\iota+1}(m),\dots,z_r(m)) \in \mathfrak{v}_o$$

whenever $m>m_o$.

Corollary 9.5. *Combining the above equations, it follows that along the given* sl_2*-sequence* $z(m)$*, we can write*

$$F_\iota(z) = e^{N(x)} t_\iota(y) e^{i\sum_{k\leqslant\iota}(y_k/y_{\iota+1})N_k} e^{\gamma(z_{\iota+1},\dots,z_r)}\cdot F_\natural \tag{9.6}$$

with $\gamma(z_{\iota+1}(m),\dots,z_r(m))$ *taking values* $\mathfrak{v}_o$ *for* $m>m_o$.

Now consider the sequence $\tilde{y}(m) = (\tilde{y}_1(m),\dots,\tilde{y}_\iota(m))$ obtained by setting

$$\tilde{y}_j(m) = y_j(m)/y_{\iota+1}(m). \tag{9.7}$$

Then, since $y(m) = (y_1(m),\dots,y_r(m))$ is an sl_2-sequence it follows that $\tilde{y}(m)$ is also an sl_2-sequence in ι-variables. Therefore (see Definition 2.27) we have

$$\tilde{y}(m) = T(\tilde{v}(m)) + \tilde{b}(m)$$

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for some linear map $T:\mathbb{R}^d\to\mathbb{R}^\iota$, some strict sl_2-sequence $\tilde{v}(m)\in\mathbb{R}^d$ and a convergent sequence $b(m)\in\mathbb{R}^\iota$. Accordingly,

$$N(\tilde{y}(m))=\sum_{j=1}^{d}N(\theta^j)\tilde{v}_j(m)+N(\tilde{b}(m))$$

where $N(\theta^1),\dots,N(\theta^d)$ and $N(\tilde{b}(m))$ belong to the real subalgebra $\mathfrak{b}_{\mathbb{R}}$ of $\mathfrak{v}$ generated by $N_1,\dots,N_\iota$.

Warning 9.8. The system of vectors θ constructed in this way may differ from the sequence appearing in Theorem 2.30. Nonetheless, the result limit gradings will be the same. This will addressed in the final paragraph of the paper.

Let $N(\tilde{b}(m))\to N(\tilde{b}_o)$ and $\mathfrak{b}_o$ be a neighborhood of $N(\tilde{b}_o)$ in $\mathfrak{b}_{\mathbb{R}}$. Then, for any $\alpha\in\mathfrak{b}_o$ and any $\nu\in\mathfrak{v}_o$, the data

$$(N(\theta^1),\dots,N(\theta^d);e^{i\alpha}e^{\nu}\cdot F_\natural,W)$$

generates an admissible nilpotent orbit (with slightly larger c to compensate for α, see (9.4)). We therefore define

$$F(v_1,\dots,v_d;\alpha,\nu)=e^{\sum_{j\leqslant d}iv_jN(\theta^j)}e^{i\alpha}e^{\nu}\cdot F_\natural.$$

As in the proof of Corollary 7.18, it then follows via Lemma 2.24 that for fixed α and ν as above,

$$\hat{Y}_{(F(v_1,\dots,v_d;\alpha,\nu),W)}\to\hat{Y}(\nu):=Y(N(\theta^1),\dots,Y(N(\theta^d),\hat{Y}_{(e^{\nu}\cdot F_\natural,W^\iota)}))$$

along any strict sl_2-sequence in the variables $v_1,\dots,v_d$.

By [KNU08, Theorem (0.5)], there exists a constant b such that if $\tau_1=v_2/v_1,\dots,\tau_d=1/v_d\in(0,b)$ then

$$\hat{Y}_{(F(v_1,\dots,v_d;\alpha,\nu),W)}=\exp(u(\tau;\alpha,\nu))\cdot\hat{Y}(\nu)\tag{9.9}$$

where $u(\tau;\alpha,\nu)$ has a convergent series expansion

$$u(\tau;\alpha,\nu)=\sum_m u_m(\alpha,\nu)\prod_{j=1}^{r}\tau_j^{m(j)}\tag{9.10}$$

with constant term 0. Furthermore, by [KNU08, Theorem (10.8)], the coefficients $u_m(\alpha,\nu)$ are analytic functions of $\alpha\in\mathfrak{b}_o$ and $v\in\mathfrak{v}_o$.

Corollary 9.11. *Let $\nu(m)=\gamma(z_{\iota+1}(m),\dots,z_r(m))$ and $\alpha(m)=N(\tilde{b}(m))$. Then,*

$$\hat{Y}_{(F_\iota(z(m)),W)}=e^{N(x(m))}t_\iota(y(m))e^{u(\tau(m);\alpha(m),\nu(m))}\cdot\hat{Y}(\nu(m))\tag{9.12}$$

where $\tau_j(m)=v_{j+1}(m)/v_j(m)$ for $v_j(m)=\tilde{v}_j(m)$.

Proof. Combine (9.6), (9.7) and (9.9). □

The remainder of the proof of Theorem 2.30 now divides into two parts depending on d.

Remark 9.13. The notation introduced in Corollary 9.11 will remain in effect for the remainder of this section.

When $d=1$. In this case, it follows from the definition of non-polynomial growth that

$$\tau_1(m)y_{\iota+1}^e(m)\to 0$$

for any half-integral power e, and hence the sequence

$$e^{\eta(m)}:=\mathrm{Ad}(t_\iota(y(m)))e^{u(\tau_1(m);\alpha(m),\nu(m))}\to 1\tag{9.14}$$

since the action of $\mathrm{Ad}(t_\iota(y))$ on $\mathfrak{g}_{\mathbb{C}}$ is bounded by a polynomial in $y_{\iota+1}^{\frac{1}{2}}$.

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On the other hand, by Lemma 8.4 and the properties of Deligne systems it follows that

$$\begin{aligned} t_\iota(y)\cdot\hat{Y}(\nu) &= Y(\mathrm{Ad}(t_\iota(y))N(\theta^1),\hat{Y}_{(t_\iota(y)e^\nu\cdot F_\natural,W^\iota)}) \\ &= Y(y_{\iota+1}N(\theta^1),\hat{Y}_{(t_\iota(y)e^\nu\cdot F_\natural,W^\iota)}). \end{aligned}$$

Now, if (N,Y_M,W) is a Deligne system then so is $(\lambda N,Y_M,W)$ for any non-zero scalar. Furthermore,

$$Y(\lambda N,Y_M)=Y(N,Y_M). \tag{9.15}$$

In particular, by the previous paragraph

$$t_\iota(y)\cdot\hat{Y}(\nu)=Y(N(\theta^1),\hat{Y}_{(t_\iota(y)e^\nu\cdot F_\natural,W^\iota)}). \tag{9.16}$$

Therefore, by (9.12), (9.14) and (9.16) it follows that

$$\begin{aligned} e^{-N(x(m))}\cdot\hat{Y}_{(F_\iota(z(m)),W)} &= e^{\eta(m)}\cdot Y(N(\theta^1),\hat{Y}_{(t_\iota(y(m))e^{\nu(m)}\cdot F_\natural,W^\iota)}) \\ &= e^{\eta(m)}\cdot Y(N(\theta^1),\hat{Y}_{(e^{\sum_{j>\iota} iy_j(m)N_j}e^{\Gamma_i(s(m))}\cdot F_\infty,W^\iota)}) \end{aligned}$$

and hence (9.1) holds for $d=1$.

By induction,

$$\hat{Y}_{(e^{\sum_{j>\iota} iy_j(m)N_j}e^{\Gamma_\iota(s(m))}\cdot F_\infty,W^\iota)}\to Y(N(\theta^2),\ldots,Y(N(\theta^{d'}),\hat{Y}_{(F_\infty,W^r)}))$$

and hence combining the above, we have

$$e^{-N(x(m))}\cdot\hat{Y}_{(F_\iota(z(m)),W)}\to Y(N(\theta^1),\ldots,Y(N(\theta^{d'}),\hat{Y}_{(F_\infty,W^r)})).$$

Remark 9.17. To complete the proof for $d=1$ we still need to resolve the discrepancy introduced by the different system of vectors θ as remarked in (9.8).

When $d>1$.

Lemma 9.18. *For fixed α and ν as in (9.9) and $\tau_1,\ldots,\tau_{d-1}\in(0,b)$*

$$\exp(u(\tau_1,\ldots,\tau_{d-1},0;\alpha,\nu)\cdot\hat{Y}(\nu)=Y\left(\sum_{j\leqslant d}\omega_jN(\theta^j),\hat{Y}_{(e^\nu\cdot F_\natural,W^\iota)}\right) \tag{9.19}$$

for any vector $(\omega_1,\ldots,\omega_d)\in\mathbb{R}^d_{>0}$ such that $\tau_j=\omega_{j+1}/\omega_j$ for $j\leqslant d-1$.

Proof. Let $v_j=y\omega_j$ for $j=1,\ldots,d$. Then, by (9.9),

$$\hat{Y}_{(F(v_1,\ldots,v_d;\alpha,\nu),W)}=\exp(u(\tau_1,\ldots,\tau_{d-1},1/(y\omega_d);\alpha,\nu))\cdot\hat{Y}(\nu).$$

On the other hand,

$$\begin{aligned} \hat{Y}_{(F(v_1,\ldots,v_d;\alpha,\nu),W)} &= \hat{Y}_{(e^{\sum_{j\leqslant d} iv_jN(\theta^j)}e^{i\alpha}e^\nu\cdot F_\natural,W)} \\ &= \hat{Y}_{(e^{iy\sum_{j\leqslant d}\omega_jN(\theta^j)}e^{i\alpha}e^\nu\cdot F_\natural,W)}. \end{aligned}$$

Comparing these two equations, it follows that

$$\exp(u(\tau_1,\ldots,\tau_{d-1},1/(y\omega_d);\alpha,\nu))\cdot\hat{Y}(\nu)=\hat{Y}_{(e^{iy\sum_{j\leqslant d}\omega_jN(\theta^j)}e^{i\alpha}e^\nu\cdot F_\natural,W)}.$$

Taking the limit as $y\to\infty$, we then obtain (9.19) using (2.17). ◻

Remark 9.20. *A priori*, $u(\tau;\alpha,\nu)$ is only defined for $\tau_1,\ldots,\tau_d\in(0,b)$. However, via the series expansion (9.10), we can extend u to a real-analytic function on a neighborhood of $\tau=0$. By continuity, the formula for $u(\tau_1,\ldots,\tau_{d-1},0)$ given above agrees with the value of $u(\tau_1,\ldots,\tau_{d-1},0)$ determined by (9.10).

For α and ν as in (9.9), let

$$u_1(\tau;\alpha,\nu) = u(\tau_1,\ldots,\tau_d;\alpha,\nu) - u(\tau_1,\ldots,\tau_{d-1},0;\alpha,\nu),$$
$$u_2(\tau;\alpha,\nu) = u(\tau_1,\ldots,\tau_{d-1},0;\alpha,\nu).$$

Then, $\exp(u(\tau;\alpha,\nu)) = \exp(u_1+u_2)$ where u_1 is divisible by τ_d in the ring of real-analytic functions of $\tau_1,\ldots,\tau_d$. Therefore,

$$\mathrm{Ad}(t_\iota(y))\exp(u(\tau;\alpha,\nu)) = \mathrm{Ad}(t_\iota(y))(e^{u_1+u_2}e^{-u_2})\,\mathrm{Ad}(t_\iota(y))e^{u_2} \tag{9.21}$$

where $e^{u_1+u_2}e^{-u_2} = e^{u_3}$ with u_3 again divisible by τ_d in the ring of real-analytic functions in $\tau_1,\ldots,\tau_d$. Consequently, as in (9.14), it follows that if $\eta(m)$ is the sequence defined by the equation

$$e^{\eta} = \mathrm{Ad}(t_\iota(y))(e^{u_3}) \tag{9.22}$$

along $z(m)$ then $\eta(m)\to 0$.

Lemma 9.23. *With the above notation, we have*

$$e^{u_2(\tau(m);\alpha(m),\nu(m))}\cdot\hat{Y}(\nu(m)) = Y\left(\sum_{j\leqslant d} v_j(m)N(\theta^j),\hat{Y}_{(e^{\nu(m)}\cdot F_\natural,W^\iota)}\right).$$

Proof. Use Lemma 9.18 with $\omega_j(m) = v_j(m)$. □

Accordingly, by the previous lemma and (9.12), (9.21), (9.22) it follows that

$$\begin{aligned} e^{-N(x(m))}\cdot\hat{Y}_{(F_\iota(z(m))),W)} &= e^{\eta(m)}t_\iota(y(m))e^{u_2(\tau(m);\alpha(m),\nu(m))}\cdot\hat{Y}(\nu(m)) \\ &= e^{\eta(m)}t_\iota(y(m))\cdot Y\left(\sum_{j\leqslant d}v_j(m)N(\theta^j),\hat{Y}_{(e^{\nu(m)}\cdot F_\natural,W^\iota)}\right). \end{aligned}$$

Therefore, using Lemma 8.4 and our freedom to rescale $N(\theta^j)\to\lambda N(\theta^j)$ it follows that

$$\begin{aligned} &t_\iota(y(m))\cdot Y\left(\sum_{j\leqslant d}v_j(m)N(\theta^j),\hat{Y}_{(e^{\nu(m)}\cdot F_\natural,W^\iota)}\right) \\ &\quad = Y\left(\sum_{j\leqslant d}v_j(m)N(\theta^j),\hat{Y}_{(e^{\sum_{j>\iota}iy_jN_j}e^{\Gamma_\iota(s)}\cdot F_\infty,W^\iota)}\right) \end{aligned} \tag{9.24}$$

and hence

$$e^{-N(x(m))}\cdot\hat{Y}_{(F_\iota(z(m))),W)} = e^{\eta(m)}\cdot Y\left(\sum_{j\leqslant d}v_j(m)N(\theta^j),\hat{Y}_{(e^{\sum_{j>\iota}iy_jN_j}e^{\Gamma_\iota(s)}\cdot F_\infty,W^\iota)}\right). \tag{9.25}$$

In particular, since $\eta(m)\to 0$, it follows that (9.1) holds for $d>1$.

Thus, to complete the proof it remains to compute the limit (9.1). To this end, observe that by induction,

$$\hat{Y}_{(e^{\sum_{j>\iota}iy_jN_j}e^{\Gamma_\iota(s)}\cdot F_\infty,W^\iota)}\to Y^\dagger = Y(N(\theta^{d+1}),\ldots,Y(N(\theta^{d'}),\hat{Y}_{(F_\infty,W^r)})). \tag{9.26}$$

Consequently, there exists a unique $W^\iota_{-1}\mathrm{gl}(V)$-valued sequence $\beta(m)$ which converges to zero such that

$$\hat{Y}_{(e^{\sum_{j>\iota}iy_jN_j}e^{\Gamma_\iota(s)}\cdot F_\infty,W^\iota)} = e^{\beta(m)}\cdot Y^\dagger$$

along $z(m)$.

To continue, note that since $\hat{Y}_{(e^{\sum_{j>\iota} iy_j N_j} e^{\Gamma_\iota(s)} \cdot F_\infty, W^\iota)}$ arises from the sl_2-splitting of the limit mixed Hodge structure of the nilpotent orbit

$$(z_1, \ldots, z_\iota) \mapsto e^{\sum_j z_j N_j} \cdot (e^{\sum_{j>\iota} iy_j N_j} e^{\Gamma_\iota(s)} \cdot F_\infty)$$

it follows that:

(a) $[N_j, e^{\beta} \cdot Y^\dagger] = -2N_j$ for $j \leqslant \iota$;

(b) $e^{\beta} \cdot Y^\dagger$ preserves $W^0, \ldots, W^\iota$.

On the other hand, by Lemma 5.10 we know that $Y^\dagger = Y_{(\hat{F}_\iota, W^\iota)}$ arises via the limit mixed Hodge structure of a nilpotent orbit

$$(e^{\sum_{j\leqslant\iota} z_j N_j} \cdot \hat{F}_\iota, W^0)$$

(with limit MHS split over $\mathbb{R}$), so we also have:

(a′) $[N_j, Y^\dagger] = -2N_j$ for $j \leqslant \iota$;

(b′) $Y^\dagger$ preserves $W^0, \ldots, W^\iota$.

WARNING 9.27. For the remainder of this paper, $\hat{F}_\iota$ is the filtration of the previous paragraph and not the filtration obtained from (5.2) and the nilpotent orbit $(N_1, \ldots, N_r; \hat{F}_\infty, W)$.

Remark 9.28. As in the remark to Theorem 2.30, we are implicitly assuming that $y_j(m) \to \infty$ for all j. The case where some $y_j(m)$ remain bounded is handled by absorbing these factors into F_∞. The details are left to the reader.

In particular, comparing (a) and (a′) it follows from Lemma 5.13 that

$$[\beta(m), N_j] = 0, \quad j \leqslant \iota. \tag{9.29}$$

Likewise, it follows from properties (b) and (b′) that

$$\beta(m) \text{ preserves } W^j, \quad j \leqslant \iota. \tag{9.30}$$

By the functoriality of Deligne systems, it follows from (9.29) and (9.30) that

$$\begin{aligned} &Y\Bigg(\sum_{j\leqslant d} v_j(m) N(\theta^j), \hat{Y}_{(e^{\sum_{j>\iota} iy_j N_j} e^{\Gamma_\iota(s)} \cdot F_\infty, W^\iota)}\Bigg) \\ &\qquad = Y\Bigg(\sum_{j\leqslant d} v_j(m) N(\theta^j), e^{\beta(m)} \cdot Y^\dagger\Bigg) = e^{\beta(m)} \cdot Y\Bigg(\sum_{j\leqslant d} v_j(m) N(\theta^j), Y^\dagger\Bigg). \end{aligned} \tag{9.31}$$

Moreover, by the properties of Deligne systems,

$$Y\Bigg(\sum_{j\leqslant d} v_j(m) N(\theta^j), Y^\dagger\Bigg) = \hat{Y}_{(e^{i\sum_{j\leqslant d} v_j(m) N(\theta^j)} \cdot \hat{F}_\iota, W^0)}. \tag{9.32}$$

Letting $m \to \infty$ and using (2.17) to compute the right-hand side shows that

$$Y\Bigg(\sum_{j\leqslant d} v_j(m) N(\theta^j), Y^\dagger\Bigg) \to \hat{Y}(N(\theta^1), \ldots, Y(N(\theta^d), Y_{(\hat{F}_\iota, W^\iota)})).$$

Therefore, since $\beta(m) \to 0$ and

$$Y_{(\hat{F}_\iota, W^\iota)} = Y^\dagger = Y(N(\theta^{d+1}), \ldots, Y(N(\theta^{d'}), \hat{Y}_{(F_\infty, W^r)}))$$

it follows from (9.31) that

$$Y\left(\sum_{j\leqslant d} v_j(m)N(\theta^j), \hat{Y}_{(e^{\sum_{j>\iota} iy_jN_j}e^{\Gamma_\iota(s)}\cdot F_\infty, W^\iota)}\right) \to Y(N(\theta^1), \ldots, Y(N(\theta^{d'}), \hat{Y}_{(F_\infty, W^r)})).$$

Returning to (9.1) it then follows that

$$e^{-N(x(m))}\cdot \hat{Y}_{(F(z(m)),W)} \to Y(N(\theta^1), \ldots, Y(N(\theta^{d'}), \hat{Y}_{(F_\infty, W^r)})) \tag{9.33}$$

as required.

To complete the proof, we note that the elements $\theta^1, \ldots, \theta^{d'}$ constructed above depend on the sequence $z(m)$ and not the period map. Consequently, it follows from (9.33) that

$$e^{-N(x(m))}\cdot \hat{Y}_{(F(z(m)),W)} - e^{-N(x(m))}\cdot \hat{Y}_{(e^{N(z(m))}\cdot F_\infty, W)} = 0 \tag{9.34}$$

for every sl_2-sequence since both $F(z)$ and $e^{N(z)}\cdot F_\infty$ have the same limit Hodge filtration. On the other hand, by Lemma 7.17, we know that (2.31) holds for nilpotent orbits. Thus, by virtue of (9.34), (2.31) is also true for period maps.

Part II. Tannakian Categories of Nilpotent Orbits

10. Central filtrations on Tannakian categories

The main purpose of this appendix is to prove Theorem 2.18 and Lemma 2.20 from the body of the paper. These results characterize the sl_2-splitting as the unique splitting ϵ given as a universal Lie polynomial in the $\delta_{p,q}$ such that, if (N, F, W) is a one-variable nilpotent orbit with limit split over $\mathbb{R}$, and $\xi = \epsilon(e^{iN}\cdot F, W)$, then $Y(e^{\xi}e^{iN}\cdot F, W)$ is a morphism of the limit mixed Hodge structure $(F, M(W, N))$. That this is so was stated first by Deligne in an unpublished letter to Cattani and Kaplan. In this appendix, we phrase Deligne's results in the language of Tannakian categories in the hope that this will lead to clarity by making explicit the relationships between various categories of Hodge structures and nilpotent orbits.

10.1 Let $\mathbf{C}$ be a Tannakian category over a field k and let $\omega : \mathbf{C} \to \mathrm{Vect}_k$ be a fiber functor (where Vect_k denotes the category of finite-dimensional vector spaces over k). Recall from Saavedra–Rivano [SR72, p. 213], that an *exact filtration* of ω consists of an exhaustive, increasing filtration W_k of the functor ω satisfying:

(i) the associated graded $\mathrm{Gr}^W \omega$ is exact;

(ii) for every $n \in \mathbb{Z}$ and every pair of objects X, Y in $\mathbf{C}$, we have

$$W_n\omega(X\otimes Y) = \sum_{p+q=n} W_p\omega X \otimes W_q\omega Y.$$

See [SR72] for a definition where the field k is replaced with a ring A.

Saavedra–Rivano calls a filtration W *central* if it arises by applying ω to a filtration (also denoted W_k) of the identity functor on $\mathbf{C}$. (So $W_k\omega X = \omega W_k X$.) Suppose W is an exact central filtration. Then we say that an object X is pure of weight k, if $\mathrm{Gr}^W_j X = 0$ for all $j \neq k$. An object is *split* if it is a direct sum of pure subobjects. It follows from the exactness of W, that the full subcategory $S_W\mathbf{C}$ consisting of all split objects is a Tannakian subcategory of $\mathbf{C}$ [Mil07, 1.7]. It also follows that $\mathrm{Hom}_{\mathbf{C}}(X, Y) = 0$ if X and Y are two pure objects of $\mathbf{C}$ of different weights.

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10.2 Suppose $\omega : \mathbf{C} \to \mathrm{Vect}_k$ is a neutral Tannakian category with Galois group $G = \mathrm{Aut}^{\otimes}\omega$. A *grading* (or, to be precise, a $\mathbb{Z}$-*grading*) of ω is a functorial decomposition $\omega = \bigoplus_{k\in\mathbb{Z}} \omega_k$. A grading of ω amounts to the same thing as a group homomorphism $i : \mathbb{G}_m \to G$ where $\mathbb{G}_m$ denotes the multiplicative group of k. Any grading, or equivalently a group homomorphism $i : \mathbb{G}_m \to G$, gives rise to a filtration $W = W(i)$ on ω via the rule $W_n\omega X = \bigoplus_{k\leqslant n} \omega_k X$. A filtration W is said to be *splittable* if $W = W(i)$ for some grading i. In this case, i is said to be a *splitting* of W.

10.3 In [SR72, p.217], Saavedra–Rivano considers affine group schemes $P = \mathrm{Aut}^{\otimes}_W(\omega)$ and $U = \mathrm{Aut}^{\otimes !}(\omega)$ associated to a neutral Tannakian category $\omega : \mathbf{C} \to \mathrm{Vect}_k$ and an exact filtration W of ω. The group P consists of automorphisms preserving $W\omega X$, for any $X \in \mathbb{C}$, while the group U consists of automorphisms inducing the identity on $\mathrm{Gr}^W \omega$. More generally, Saavedra–Rivano considers filters P by the subgroups U_α consisting of elements $g \in P$ acting trivially on $(W_p/W_{p+\alpha})\omega X$. In this notation, $U = U_{-1}$ and we set $P = U_0$. To express the fact that the filtration U_i depends on W, we write $W_iU := U_i$.

If W is central, then $P = G = \mathrm{Aut}^{\otimes}(\omega)$.

Proposition 10.4. *Suppose $\omega : \mathbf{C} \to \mathrm{Vect}_k$ is a neutral Tannakian category equipped with a central filtration W. Let $Q = \mathrm{Aut}^{\otimes}(\omega|S_W\mathbf{C})$. Then the map $\pi : G \to Q$ induced by the inclusion of $S_W\mathbf{C}$ in $\mathbf{C}$ is faithfully flat with kernel U. Thus we have a short exact sequence*

$$1 \to U \to G \to Q \to 1$$

of k-groups. Thus, G/U is isomorphic to Q.

Proof. By [DMOS82, Proposition 2.21], $G \to Q$ is faithfully flat, because the inclusion $S_W\mathbf{C} \to \mathbf{C}$ is fully faithful and every subobject in $\mathbf{C}$ of a split object is split. It is clear that U is in the kernel of π. On the other hand, every object in $\mathbb{C}$ is a successive extension of pure objects. From this observation, it follows that U contains the kernel of π. □

10.5 Suppose the filtration W in Proposition 10.4 is splittable by a homomorphism $i : \mathbb{G}_m \to G$. Let $\mathrm{Cent}(i)$ denote the centralizer of i in G.

Proposition 10.6 (Saavedra–Rivano). *If W is splittable, then $G = U \rtimes \mathrm{Cent}(i)$. In particular, $\mathrm{Cent}(i)$ is isomorphic to Q.*

10.7 In the context of the proposition, the set of splittings $i : \mathbb{G}_m \to G$ is a pseudo-torsor under $U(k)$ (acting via conjugation). Fortunately, all the filtrations which come up in Hodge theory are, in fact, splittable. One way to see this is to use the following result, a corollary of a recent theorem of Ziegler.

Theorem 10.8. *Suppose $\omega : \mathbf{C} \to \mathrm{Vect}_k$ is a neutral Tannakian category over a field k of characteristic 0 and let W be a filtration on ω. Then W is splittable.*

Proof. This follows directly from [Zie11, Theorem 1.3] and the fact that, in characteristic 0, algebraic groups are smooth. □

11. Mixed Hodge structures

11.1 The category MHS of mixed Hodge structures over $\mathbb{R}$ equipped with the forgetful functor $\omega : \mathrm{MHS} \to \mathrm{Vect}_{\mathbb{R}}$ sending a Hodge structure to its underlying real vector space is a neutral Tannakian category. The weight filtration W induces a central filtration on MHS and the category $S_W\mathrm{MHS}$ is simply the category HS of split mixed Hodge structures.

By a classical observation of Deligne, the group $\mathbb{S} := \mathrm{Aut}^{\otimes}(\omega|\mathrm{HS})$ is the Weil restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$ of the group $\mathbb{G}_m$. This group sits in exact sequences

$$1 \longrightarrow S^1 \xrightarrow{s} \mathbb{S} \xrightarrow{t} \mathbb{G}_m \longrightarrow 1$$

$$1 \longrightarrow \mathbb{G}_m \xrightarrow{w} \mathbb{S} \longrightarrow S^1 \longrightarrow 1$$

where S^1 is the unique (up to isomorphism) non-split real form of $\mathbb{G}_m$ and the action of $\mathbb{G}_m$ via w on a split mixed Hodge structure induces the splitting of the weight filtration. The homomorphism $t : \mathbb{S} \to \mathbb{G}_m$ induces a fully faithful functor from the category of graded vector spaces to the category of split mixed Hodge structures whose essential image consists of the Tate mixed Hodge structures. The map $s \times w : S^1 \times \mathbb{G}_m \to \mathbb{S}$ presents $\mathbb{S}$ as the quotient of $S^1 \times \mathbb{G}_m$ by the diagonally embedded copy of $\mathbb{Z}/2$.

11.2 By Theorem 10.8, the central filtration W on MHS is splittable. So $\mathfrak{M} := \mathrm{Aut}^{\otimes}(\omega)$ is isomorphic to a semi-direct product $\mathfrak{U} \rtimes \mathbb{S}$ where $\mathfrak{U} = \mathrm{Aut}^{\otimes !}(\omega)$. In fact, Deligne determined the structure of $\mathfrak{U}$, which, for the convenience of the reader, we explain in Deligne's language.

Let $\mathcal{L}_{\mathbb{C}}$ denote the free Lie algebra over $\mathbb{C}$ on generators $D^{i,j}$ where i and j are negative integers. The $\mathbb{C}$-vector space underlying $\mathcal{L}_{\mathbb{C}}$ carries a unique bigrading $\mathfrak{L}_{\mathbb{C}} = \oplus \mathfrak{L}_{\mathbb{C}}(i,j)$ for which $D^{i,j}$ is in bidegree (i,j). Let

$$W_n \mathcal{L}_{\mathbb{C}} = \bigoplus_{i+j \leqslant n} \mathcal{L}_{\mathbb{C}}(i,j).$$

Then $W_n \mathcal{L}_{\mathbb{C}}$ is an ideal in $\mathcal{L}_{\mathbb{C}}$, and it is easy to see that the Lie algebra $\mathcal{L}_{\mathbb{C}}/W_n\mathcal{L}_{\mathbb{C}}$ is nilpotent and finite dimensional as a $\mathbb{C}$ vector space. There is a unique real structure on the Lie algebra $\mathcal{L}_{\mathbb{C}}$ for which $\overline{D^{i,j}} = -D^{j,i}$. Let $\mathcal{L}$ denote the corresponding real Lie algebra. Since the real structure respects the filtration W on $\mathcal{L}_{\mathbb{C}}$, W descends to a filtration on $\mathcal{L}$. Set $\mathfrak{U}(n) := \exp(\mathcal{L}/W_n\mathcal{L})$, a real algebraic unipotent group.

The bigrading on $\mathcal{L}_{\mathbb{C}}$ induces an action of $\mathbb{G}_m^2$ on $\mathcal{L}_{\mathbb{C}}$ and, thus, an action on $\mathfrak{U}(n) \otimes \mathbb{C}$. Under this action $(s,t)D^{i,j} = s^i t^j D^{i,j}$. This action descends to an action of $\mathbb{S}$ on $\mathcal{L}$ for which $tD^{i,j} = z^i(t)\bar{z}^j(t)D^{i,j}$ where $z, \bar{z}$ denote conjugate generators of the character group of $\mathbb{S}$. The action of $\mathbb{S}$ on $\mathcal{L}$ then induces an action on the real algebraic group $\mathfrak{U}(n)$. Let $\varprojlim \mathfrak{U}(n)$ denote the inverse limit of the real algebraic groups $\mathfrak{U}(n)$, a pro-unipotent real affine group scheme, and write $W_k \varprojlim \mathfrak{U}(n)$ for the kernel of the canonical homomorphism $\varprojlim \mathfrak{U}(n) \to \mathfrak{U}(k)$.

Suppose $V = (V, F, W)$ is a real mixed Hodge structure. Let $\delta = \delta_{F,W} \in \mathrm{End}(V)$ and write $\delta = \sum_{i,j<0} \delta_{i,j} \in \mathrm{End}(V)_{\mathbb{C}}$. Since δ is real, $\bar{\delta}_{i,j} = \delta_{j,i}$ This induces an action of $\varprojlim \mathfrak{U}(n)$ on V by letting $D^{i,j}$ act via $\sqrt{-1}\delta_{i,j}$. This induces a homomorphism $\varprojlim \mathfrak{U}(n) \to \mathfrak{M}$, and, since $\mathfrak{U}(n)$ is unipotent for each n, the homomorphism must factor through $\mathfrak{U}$. So we have a map $\varprojlim \mathfrak{U}(n) \to \mathfrak{U}$.

Theorem 11.3 [Del94, Deligne]. *We have* $\mathfrak{U} = \varprojlim \mathfrak{U}(n)$.

Remark 11.4. If (V, F, W) is a real mixed Hodge structure, then $(V, e^{-i\delta} \cdot F, W)$ is split over $\mathbb{R}$. The splitting $Y_{(e^{-i\delta} \cdot F, W)}$ gives a canonical $\mathbb{R}$ grading of W, and thus a homomorphism $i : \mathbb{G}_m \to \mathfrak{M}$, which induces a splitting of the sequence

$$1 \to \mathfrak{U} \to \mathfrak{M} \to \mathbb{S} \to 1.$$

If we set $\mathfrak{M}(n) := \mathfrak{M}/W_n\mathfrak{U}$, then $\mathfrak{M}(n)$ is a real algebraic group and $\mathfrak{M} = \varprojlim \mathfrak{M}(n)$.

12. Nilpotent orbits

12.1 Suppose V is a finite-dimensional real vector space. Recall, from the body of the paper or from [KNU08, p. 405], that the data of an admissible nilpotent orbit (or mixed nilpotent orbit in the terminology of [KNU08]) consists of a quadruple (V, N, F, W) where:

(i) N is a nilpotent endomorphism of V;

(ii) F is a decreasing filtration of $V_{\mathbb{C}}$;

(iii) W is an increasing filtration of V.

These data are assumed to satisfy several conditions spelled out in [KNU08]. The class of all admissible nilpotent orbits (V, N, F, W) forms a category in an obvious way (morphisms are vector space homomorphisms preserving N, F and W). Moreover, by a result of Kashiwara [Kas86, Proposition 5.2.6] this category, which we will call Nilp_1, is abelian. In fact, Nilp_1 is a neutral Tannakian category: the tensor products are defined as in [Kas86, 4.4.3] in an obvious way, the functor ω_1 from Nilp_1 to the category $\mathrm{Vect}_{\mathbb{R}}$ of finite-dimensional real vector spaces is the one that forgets everything but V. From this, it is easy to check that Nilp_1 is a neutral Tannakian category using, for example, [DMOS82, Proposition 1.20]. The filtration W equips ω_1 with a central filtration.

12.2 Unfortunately, we do not have a simple description of Nilp_1. However, Nilp_1 has a Tannakian subcategory whose fundamental group is more tractable: the full subcategory Split_1 consisting of all objects (V, N, F, W) with limit $(V, F, M(N, W))$ split over $\mathbb{R}$. Let $\mathfrak{M}_1$ denote the affine group scheme $\mathrm{Aut}^{\otimes}(\omega_1|\,\mathrm{Split}_1)$. The filtration W induces a central filtration on the neutral Tannakian category $\omega_1 : \mathrm{Split}_1 \to \mathrm{Vect}_{\mathbb{R}}$. We write $\mathbf{S}_1$ for the category $S_W\,\mathrm{Split}_1$ of all split objects in Split_1. Objects in $\mathbf{S}_1$ are called SL_2-*orbits*. Write $\mathbb{S}_1 = \mathrm{Aut}^{\otimes}(\omega_1|\mathbf{S}_1)$. Then we have a (splittable) faithfully flat homomorphism $\pi_1 : \mathfrak{M}_1 \to \mathbb{S}_1$, and, if we write $\mathfrak{U}_1$ for the kernel of π_1, we obtain a (splittable) exact sequence

$$1 \to \mathfrak{U}_1 \to \mathfrak{M}_1 \to \mathbb{S}_1 \to 1.$$

In fact, $\mathfrak{M}_1$ inherits a filtration $W_k\mathfrak{M}_1$ from W as in (10.3), and $\mathfrak{U}_1 = W_1\mathfrak{M}_1$.

For each non-negative integer n, set $\mathfrak{M}_1(n) := \mathfrak{M}_1/W_n\mathfrak{U}_1$. Then $\mathfrak{M}_1(n)$ is a real algebraic group with unipotent radical $\mathfrak{U}_1(n)$. The homomorphism $\pi_1(n) : \mathfrak{M}_1(n) \to \mathbb{S}_1$ induced by π_1 sends $\mathfrak{M}_1(n)$ onto its largest reductive quotient. We have $\mathfrak{M} = \varprojlim \mathfrak{M}_1(n)$.

Since the limit $(V, F, M(N, W))$ of an object (V, N, F, W) in Split_1 is, by definition, split over $\mathbb{R}$, we obtain a morphism $i_M : \mathbb{G}_m \to \mathfrak{M}_1$ inducing a grading of M. We write $\bar{i}_M := \pi_1 \circ i_M : \mathbb{G}_m \to \mathbb{S}_1$. Moreover, write $H := i_M(\mathbb{G}_m)$ and $\bar{H} = \bar{i}_M(\mathbb{G}_m)$.

Note that any split real mixed Hodge structure (V, F, W) gives rise to a split nilpotent orbit $(V, 0, F, W)$ in a trivial way. In other words, each split real mixed Hodge structure gives rise to a constant split nilpotent orbit. This association gives rise to a tensor functor $\mathrm{Const} : \mathrm{HS} \to \mathbb{S}_1$ and, thus, to a homomorphism $\mathrm{const} : \mathbb{S}_1 \to \mathbb{S}$.

The following proposition is proved in [CKS86, Lemma 3.12].

Proposition 12.3. *Suppose (V, N, F, W) is an object in* Split_1 *and z is a complex number in the upper-half plane. Then $(V, e^{zN} \cdot F, W)$ is a real mixed Hodge structure.*

For every z in the upper-half plane, the proposition gives a functor $\mathrm{sp}_z : \mathrm{Split}_1 \to \mathrm{MHS}$ compatible with $\otimes$, the filtration W, and the forgetful functors to $\mathrm{Vect}_{\mathbb{R}}$. (We call it sp_z for 'specialization' because it corresponds to specializing the nilpotent orbit to the point z in the upper-half plane.) The functor sp_z induces, in turn, a group homomorphism $\mathrm{sp}_z : \mathfrak{M} \to \mathfrak{M}_1$

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compatible with the W filtration of both groups. For most purposes, it suffices to consider the case $z = i$.

The following is the main theorem of the appendix.

THEOREM 12.4. *Concerning $\mathfrak{M}_1$ and its relationship to $\mathfrak{M}$, we have the following results.*

(i) *The group $\mathbb{S}_1$ is isomorphic to $(\mathbb{G}_m \times S^1 \times \mathrm{SL}_2)/\mu_2$ where μ_2 is embedded diagonally in the product and S^1 denotes the unique non-split real form of $\mathbb{G}_m$.*

(ii) *The homomorphism $\mathrm{sp}_i : \mathfrak{M} \to \mathfrak{M}_1$ is injective.*

(iii) *The restriction of sp_i to $\mathfrak{U}$ induces an isomorphism of $\mathfrak{U}$ with $\mathfrak{U}_1$, thus we get the following commutative diagram.*

$$\begin{tikzcd} 1 \arrow[r] & \mathfrak{U} \arrow[r] \arrow[d, "\cong"] & \mathfrak{M} \arrow[r, "\pi"] \arrow[d, "\mathrm{sp}_i"] & \mathbb{S} \arrow[r] \arrow[d, "\overline{\mathrm{sp}}_i"] & 1 \\ 1 \arrow[r] & \mathfrak{U}_1 \arrow[r] & \mathfrak{M}_1 \arrow[r, "\pi_1"] & \mathbb{S}_1 \arrow[r] & 1 \end{tikzcd}$$

(iv) *We have $\mathrm{Cent}(H) \cap \mathfrak{U}_1 = \{1\}$. On the other hand, $\overline{\mathrm{sp}}_i(w(\mathbb{G}_m))$ is central in $\mathbb{S}_1$.*

COROLLARY 12.5. *There is a unique splitting $\sigma : \mathbb{S} \to \mathfrak{M}$ such that $\mathrm{sp}_i \circ \sigma(w(\mathbb{G}_m))$ commutes with H.*

Proof. Since $\mathrm{Cent}(H) \cap \mathfrak{U}_1 = \{1\}$, the restriction of π_1 to $\mathrm{Cent}(H)$ gives an injection $\mathrm{Cent}(H) \to \mathbb{S}_1$. Let $\bar{H}$ denote the image of H in $\mathbb{S}_1$. We then obtain a commutative diagram of group homomorphisms

$$\begin{tikzcd} \mathrm{Cent}(H) \cap \mathfrak{M} \arrow[r, "\pi"] \arrow[d] & \mathrm{Cent}\ \bar{H} \cap \mathbb{S} \arrow[d] \\ \mathrm{Cent}(H) \arrow[r, "\pi_1"] & \mathrm{Cent}\ \bar{H} \end{tikzcd}$$

where all arrows are injective and we regard $\mathfrak{M}$ as being contained in $\mathfrak{M}_1$ via sp_i.

It follows from [SGA3, Exp. XV, Lemma 7.2] that the horizontal arrows are surjective (and, therefore, isomorphisms). Thus, since $w(\mathbb{G}_m)$ is contained in $\mathrm{Cent}(\bar{H}) \cap \mathbb{S}$, there exists a unique section of π over $w(\mathbb{G}_m)$ commuting with H. This gives a grading $i : \mathbb{G}_m \to \mathfrak{M}$ of W. The centralizer of i in $\mathfrak{M}$ gives a splitting $\sigma : \mathbb{S} \to \mathfrak{M}$ of π. It is clearly the unique splitting σ such that $\sigma(w(\mathbb{G}_m))$ commutes with H. □

LEMMA 12.6. *Suppose (V, N, F, W) is a one-variable admissible nilpotent orbit with limit split over $\mathbb{R}$. Set $F_{(0)} = e^{iN} \cdot F$. Then $Y(\hat{F}_{(0)}, W)$ is a morphism of the limit mixed Hodge structure $(V, F, M(N, W))$.*

Proof. This follows from [KNU08, 10.1.3]. □

The following is a restatement of Theorem 2.18.

COROLLARY 12.7. *The sl_2-splitting is the unique functorial splitting on the category of real mixed Hodge structures given by a universal Lie polynomial ϵ in the $\delta_{p,q}$ satisfying the following property. If (N, F, W) is a one-variable nilpotent orbit with limit split over $\mathbb{R}$ and we set $\xi = \epsilon(e^{iN} \cdot F, W)$, then $Y(e^{-\xi} e^{iN} \cdot F, W)$ is a morphism of $(F, M(N, W))$.*

Proof. Universal Lie polynomials in the $\delta_{p,q}$ are in one–one correspondence with elements of the completed Lie algebra $\hat{\mathcal{L}} := \varprojlim \mathcal{L}/W_n\mathcal{L}$. By the Baker–Campbell–Hausdorff theorem, the map $\zeta \mapsto e^{\zeta}$ gives an isomorphism from $\hat{\mathcal{L}}$ to $\mathfrak{U}_1(\mathbb{R})$.

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Gradings $i : \mathbb{G}_m \to \mathfrak{M}$ of the weight filtration form a torsor under $\mathfrak{U}(\mathbb{R})$. If we let i_δ denote the grading $Y(e^{-i\delta} \cdot F, W)$, then any other grading $i_\zeta : \mathbb{G}_m \to \mathfrak{M}$ of the weight filtration is given by $Y(e^{\zeta} e^{-i\delta} \cdot F, W)$ where $\zeta \in \hat{\mathfrak{L}}$ is a universal Lie polynomial in the $\delta_{p,q}$. The grading is defined over $\mathbb{R}$ if and only if the universal Lie polynomial ζ is.

Note that there is a one–one correspondence between splittings $\sigma : \mathbb{S} \to \mathfrak{M}$ and gradings $i : \mathbb{G}_m \to \mathfrak{M}$. This correspondence sends the splitting σ to its restriction to $w(\mathbb{G}_m) \subset \mathbb{S}$. The inverse of the correspondence sends the grading i to its centralizer (in $\mathfrak{M}$), which is isomorphic to $\mathbb{S}$. Under this correspondence, the $\mathbb{S}$ representation given by the split mixed Hodge structure $(e^{\zeta} e^{-i\delta} \cdot F, W)$ is sent to the splitting $Y(e^{\zeta} e^{-i\delta} \cdot F, W)$.

Corollary 12.5 shows that there is a unique splitting $\sigma : \mathbb{G}_m \to \mathfrak{M}$ such that $\operatorname{sp}_i(\sigma)$ commutes with H. Thus, there can be only one real universal Lie polynomial ζ in the $\delta_{p,q}$ such that, when $\delta = \delta(e^{iN} \cdot F, W)$, $Y(e^{\zeta} e^{-i\delta} e^{iN} \cdot F, W)$ commutes with the grading $Y(F, M)$ of the limit mixed Hodge structure. Now, we can see that any ζ such that $Y(e^{\zeta} e^{-i\delta} e^{iN} \cdot F, W)$ is a morphism of the limit mixed Hodge structure commutes with the limit grading $Y(F, M)$. The result now follows immediately from Lemma 12.6. □

13. Deligne's splittings

13.1 Suppose V is a vector space over a field F of characteristic 0, N is a nilpotent operator on V and W is an exhaustive, separated increasing filtration of V (indexed by integers) such that $NW_i \subset W_{i-1}$. The triple (V, N, W) is called *admissible* if the relative weight filtration $M = M(N, W)$ exists. (By [Del80], M is unique if it does exist.)

A grading Y_M of M is said to be *compatible* with N and W if:

(i) $[Y_M, N] = -2N$;

(ii) for all i, $Y_M(W_i) \subset W_i$.

If Y_M is such a grading, let $G(W, Y_M)$ denote the set of gradings of W which commute with Y_M. It is not hard to see that $G(W, Y_M)$ is non-empty. In fact, if we let P denote the subgroup of $\mathrm{GL}(V)$ consisting of g such that $gY_M g^{-1} = Y_M$ and $(g-1)W_i \subset W_{i-1}$, then $G(W, Y_M)$ is a principle homogeneous space for P.

Suppose $Y_W \in G(W, Y_M)$. Then set $H = H(Y_W) = Y_M - Y_W$. Let N_i denote the ith eigencomponent for N under the action of $\operatorname{ad} Y_W$. Since N preserves W, $N = \sum_{i \in \mathbb{Z}_{\leqslant 0}} N_i$. It follows from the definition of the relative weight filtration, that the pair (N_0, H) is an sl_2-pair. Let $N_+ = N_+(Y_W)$ denote the unique element of $\operatorname{End} V$ such that (N_0, H, N^+) is an sl_2-triple.

Theorem 13.2 (Deligne). *Suppose that (V, N, W) is an admissible triple and Y_M is a grading of M compatible with N and W. Then there exists a unique grading $Y_W = Y(N, Y_M) \in G(W, Y_M)$ with respect to which, for each $i < 0$, $[N^+, N_i] = 0$.*

Proof. The result was proved originally in a letter from Deligne to Cattani and Kaplan. For a published proof, see [Pea06]. □

13.3 Suppose (V, N, F, W) is an object in Nilp_1. Set $Y_M = Y_{(F,M)}$. Then Y_M is a morphism of (V, F, M) and the weight filtration W is a filtration of (V, F, M) by subobjects. Therefore, Y_M preserves W, and, as N is a $(-1, -1)$ morphism of the limit mixed Hodge structure (V, F, M), $[Y_M, N] = -2N$. So, by Theorem 13.2, (V, N, F, M) gives rise to a grading $Y_W = Y(N, Y_{(F,M)})$ of W and an sl_2-triple (N_0, H, N_+).

Because of the uniqueness of Y_W, the construction of the sl_2-triple is functorial. In other words, Theorem 13.2 gives rise to a functor $\mathrm{Nilp}_1 \rightsquigarrow \mathrm{Rep}\,\mathrm{sl}_2$ associating to every admissible nilpotent orbit (V, N, F, W) the representation $\rho_V : \mathrm{sl}_2 \to \mathrm{End}\,V$ determined by the sl_2-triple (N_0, H, N_+). It follows that there is a homomorphism $\mathrm{SL}_2 \to \mathrm{Aut}^{\otimes}(\omega_1)$ where $\omega_1 : \mathrm{Nilp}_1 \to \mathrm{Vect}_{\mathbb{R}}$ is the forgetful functor. By restriction, we obtain a homomorphism $h_{\mathrm{SL}_2} : \mathrm{SL}_2 \to \mathfrak{M}_1 = \mathrm{Aut}^{\otimes}(\omega_1 |\, \mathrm{Split}_1)$.

On the other hand, if (V, N, F, W) is in Split_1, then, by definition, the limit mixed Hodge structure $(V, F, M(N, W))$ is split. Thus there is an action of Deligne's group $\mathbb{S}$ on V. From this, it follows that there is a group homomorphism $h_{\mathrm{lim}} : \mathbb{S} \to \mathfrak{M}_1$.

13.4 Let T denote the split mixed Hodge structure $\mathbb{R} \oplus \mathbb{R}(1)$. So $T_{\mathbb{R}} = \mathbb{C}$ with real basis $e := 1$ in $T_{\mathbb{C}}^{(0,0)}$ and $f := 2\pi i \in T_{\mathbb{C}}^{(-1,-1)}$. The action of $\mathbb{S}$ on T induces an embedding $i_T : \mathbb{S} \to \mathrm{GL}(T)$. This in turn induces an action of $\mathbb{S}$ on $\mathrm{SL}(T)$ given by

$$a(s)(\gamma) := i_T(s)\gamma i_T(s)^{-1}.$$

With respect to the ordered basis (e, f), the Lie algebra $\mathrm{sl}(T)$ is, of course, identified with sl_2. It has real basis

$$n_0 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad n_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Moreover, from the Hodge structure on T, $\mathrm{sl}(T)$ inherits a split mixed Hodge structure, and, thus, an action of $\mathbb{S}$. It is easy to see that this action on $\mathrm{sl}_2(T)$ is simply the Lie derivative on the above action on a on $\mathrm{SL}_2(T)$.

Lemma 13.5. *Let (V, N, F, W) be an object in* Split_1. *Then:*

(i) $Y_W = Y(N, Y_{(F,M)})$ *is a morphism of the limit mixed Hodge structure* $V_{(F,M)}$*;*

(ii) $N_i \in \mathrm{End}\, V_{(F,M)}^{(-1,-1)}$ *for all i and each N_i is real;*

(iii) N^+ *is a real element of* $\mathrm{End}\, V_{(F,M)}^{(1,1)}$.

Proof. (i) Consider the action $\rho_{\mathrm{lim}} : \mathbb{S} \to \mathrm{Aut}\,V$ of $\mathbb{S}$ on V induced by the split mixed Hodge structure (F, M). The map Y_M is a morphism of the Hodge structure (V, F, M). Therefore $\mathbb{S}$ fixes Y_M. On the other hand, N induces a morphism $V \to V(1)$ relative to the Hodge structure (V, F, M). Therefore, N is real of type $(-1, -1)$ relative to the mixed Hodge structure (V, F, M). Consequently, N is fixed by $s(S^1) \subset \mathbb{S}$. Now, it follows from the uniqueness of $M = M(N, W)$ and of $Y(N, Y_{(F,M)})$ that, if g is any endomorphism of $V_{\mathbb{C}}$ fixing W, $Y_{(F,M)}$ and N, we have $gY_W g^{-1} = Y(gNg^{-1}, gY_{(F,M)}g^{-1}) = Y_W$. The filtration W on V is by sub-mixed-Hodge structures of (V, F, M). Therefore, W is fixed by $\mathbb{S}$. So Y_W is fixed by all $g \in s(S^1)$. Consequently, $Y_W \in \oplus\, \mathrm{End}\, V_{(F,M)}^{(p,p)}$. Moreover, since N and $Y_{(F,M)}$ are real, the uniqueness of $Y(N, Y_{(F,M)})$ shows that Y_W is also real. Since Y_W commutes with Y_M, we have $Y_W \in \mathrm{End}\, V_{(F,M)}^{(0,0)}$ as desired. It therefore follows that Y_W is a morphism of (V, F, M).

(ii) Let $\rho_W : \mathbb{G}_m \to \mathrm{Aut}\,V$ denote the action of $\mathbb{G}_m$ induced by the grading Y_W. Then, by (i), we see that ρ_W commutes with the action of $\mathbb{S}$ induced by the Hodge structure (F, M). Therefore, the group $\mathbb{G}_m \times \mathbb{S}$ acts on V via $\rho_W \times \rho_{\mathrm{lim}}$. We have already noted that $N \in \mathrm{End}\, V_{(F,M)}^{(-1,-1)}$ in (i). It follows from the fact that ρ_W commutes with ρ_{lim} that $N_i \in \mathrm{End}\, V_{(F,M)}^{(-1,-1)}$ as well. The uniqueness of the N_i shows that they are real.

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(iii) Since Y_M and Y_W are both morphisms of the split mixed Hodge structure (V, F, M), $H = Y_M - Y_W$ is as well. Since N_+ is uniquely determined by the pair (N_0, H), and N_0 and H are both fixed by the action $\rho_{\lim} \circ s : S^1 \to \operatorname{Aut} V$, N_+ is also fixed by this S^1 action. Moreover, since N_0 and H are real N_+ is also real. Therefore N_+ is a real element of $\oplus \operatorname{End} V^{(p,p)}_{(F,M)}$. On the other hand, since $[H, Y_W] = [N_0, Y_W] = 0$, it follows that $[N_+, Y_W] = 0$. Therefore $[Y_M, N_+] = [H + Y_W, N_+] = [H, N_+] + [Y_W, N_+] = [H, N_+] = 2N_+$. This shows that $N_+ \in \operatorname{End} V^{(1,1)}_{(F,M)}$. □

Remark 13.6. Part (i) of Lemma 13.5 together with Corollary 12.7 implies that $Y(N, Y_{(F,M)})$ and $\hat{Y}(e^{iN} \cdot F, W)$ coincide (because both splittings are morphisms of the limit mixed Hodge structure). This proves Lemma 2.20 (once Theorem 12.4 is established).

COROLLARY 13.7. *In $\mathfrak{M}_1$, we have*

$$h_{\lim}(s) h_{\mathrm{SL}_2}(\gamma) h_{\lim}(s)^{-1} = h_{\lim}(a(s)(\gamma)). \tag{13.7.1}$$

Proof. If (V, N, F, W) is in Split_1, we obtain a group homomorphism $\rho_V : \mathfrak{M}_1 \to \mathrm{Aut}(V)$. To prove the proposition, it suffices to show that, for any such V, the equation (13.7.1) holds when ρ_V is applied to both sides. This follows from Lemma 13.5 because the action a of $\mathbb{S}$ on SL_2 is exactly the one which gives n_0, h and n_+ the Hodge types of the lemma. □

COROLLARY 13.8. *Let $\mathbb{S}$ act on SL_2 via a. Then there is a unique morphism*

$$\rho_1 : \mathrm{SL}_2 \rtimes \mathbb{S} \to \mathfrak{M}_1$$

such that $h_{\mathrm{SL}_2}(\gamma) = \rho_1(\gamma, 1)$ for $\gamma \in \mathrm{SL}_2$ and $h_{\lim}(\alpha) = \rho_1(1, \alpha)$ for $\alpha \in \mathbb{S}$.

Proof. This follows from Corollary 13.7 and the definition of the semi-direct product. □

PROPOSITION 13.9. *Consider the homomorphism $i_W : \mathbb{G}_m \to \mathrm{SL}_2 \rtimes \mathbb{S}$ given by*

$$i_W(\alpha) = \left(\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}, w(\alpha) \right).$$

Set $P_W := i_W(\mathbb{G}_m)$. Then P_W is central in $\mathrm{SL}_2 \rtimes \mathbb{S}$.

Proof. We first show that P_W centralizes SL_2. This follows from the fact that the action of $w(\alpha)$ on SL_2 coincides with the adjoint action of the matrix

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}.$$

On the other hand, $\mathbb{S}$ centralizes the subgroup D of diagonal matrices in SL_2. From this it follows that P_W centralizes $\mathbb{S}$. Therefore P_W centralizes $\mathrm{SL}_2 \rtimes \mathbb{S}$. □

COROLLARY 13.10. *Let S denote the subgroup of $\mathrm{SL}_2 \rtimes \mathbb{S}$ consisting of elements of the form $(1, s(\beta))$ for $\beta \in S^1$. Then S is isomorphic to S^1 and central in $\mathrm{SL}_2 \rtimes \mathbb{S}$. Moreover the center of $\mathrm{SL}_2 \rtimes \mathbb{S}$ is the product $P_W S = P_W \times S$.*

Proof. Clearly S is isomorphic to S^1, and S is central in $\mathrm{SL}_2 \rtimes \mathbb{S}$ because S acts trivially on SL_2. It is also clear that $S \cap P_W = \{1\}$. Therefore $SP_W = S \times P_W$ is a subgroup of $\mathrm{SL}_2 \rtimes \mathbb{S}$. To see that $S \times P_W$ is the connected component of the center, note that $\mathrm{SL}_2 \rtimes \mathbb{S}$ is reductive (as it is an extension of reductive groups). It is of complex rank 3 since ranks are additive in extensions, and it has semi-simple rank 1 since SL_2 is its derived subgroup. Therefore the connected component of the center must be a rank 2 torus. So it must be $P_W S$. □

Corollary 13.11. *Consider the morphism* $\pi : \mathrm{SL}_2 \times P_W \times S \to \mathrm{SL}_2 \rtimes \mathbb{S}$ *given by* $(\gamma, \alpha, \tau) \mapsto \gamma\alpha\tau$. *It is a surjective homomorphism of algebraic groups with kernel a diagonally embedded copy of* μ_2. *In other words, the homomorphism* π *sets up an isomorphism*

$$\mathrm{SL}_2 \rtimes \mathbb{S} \cong (\mathrm{SL}_2 \times \mathbb{G}_m \times S^1)/\mu_2.$$

Proof. Suppose G is a (connected) reductive group with derived subgroup G^{der} and with $Z(G)^0$ the connected component of the center. Then it is well known from the theory of reductive groups that the morphism $G^{\mathrm{der}} \times Z(G)^0 \to G$ given by $(g, z) \mapsto gz$ is a surjective and faithfully flat homomorphism with kernel isomorphic to $Z(G)^0 \cap Z(G^{\mathrm{der}})$. (The kernel consists of pairs (g, g^{-1}) where $g \in Z(G)^0 \cap Z(G^{\mathrm{der}})$.)

So, set $G = \mathrm{SL}_2 \rtimes \mathbb{S}$. Then $G^{\mathrm{der}} = \mathrm{SL}_2$, $Z(G)^0 = P_W S$ and the intersection $Z(G)^0 \cap Z(G^{\mathrm{der}})$ is simply $Z(G^{\mathrm{der}}) \cong \mu_2$. It is embedded diagonally in the product $\mathrm{SL}_2 \times P_W \times S$. □

13.12 Set $G = \mathrm{SL}_2 \rtimes \mathbb{S}$, and let $\tilde{G} = \mathrm{SL}_2 \times \mathbb{G}_m \times S^1$ so that $G = \tilde{G}/\mu_2$ where μ_2 is acting diagonally. If B_{SL_2} denotes the upper triangular matrices in SL_2, then $\tilde{B} := B_{\mathrm{SL}_2} \times \mathbb{G}_m \times S^1$ is a Borel subgroup of $\tilde{G}$. Let $\tilde{T}$ denote the maximal torus in $\tilde{G}$ generated by the diagonal matrices in the SL_2 factor and the connected component of the center $Z(\tilde{G})^0 = \mathbb{G}_m \times S^1$. Write $\mathrm{diag} : \mathbb{G}_m \to \mathrm{SL}_2$ for the map $z \mapsto \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1}\end{smallmatrix}\right)$. If we fix an isomorphism $\mathrm{split}_{S^1} : \mathbb{G}_m \otimes \mathbb{C} \to S^1_{\mathbb{C}}$, then we get an isomorphism $\mathrm{split}_{\tilde{T}} = \mathrm{diag} \times \mathrm{id} \times \mathrm{split}_{S^1} : \mathbb{G}^3_{m\mathbb{C}} \to \tilde{T}_{\mathbb{C}}$. This gives an identification of $\mathbb{Z}^3$ with $X^*(\tilde{T}_{\mathbb{C}})$ sending $(a, b, c) \in \mathbb{Z}$ to the character $\chi = \chi_{(a,b,c)}$ with $\chi(\mathrm{split}_{\tilde{T}}(z_1, z_2, z_3)) = z_1^a z_2^b z_3^c$.

Let T denote the image of $\tilde{T}$ in G, then the canonical map $X^*(T) \to X^*(\tilde{T})$ identifies $X^*(T)$ with the triples $(a, b, c) : 2 | a + b + c$. Let B denote the image of $\tilde{B}$ in G. The action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $X^*(\tilde{T})$ sends (a, b, c) to $(a, b, -c)$. It follows easily from the theory of representation of reductive groups that irreducible representations of $G_{\mathbb{C}}$ are classified by weights which are positive with respect to B; that is, complex representations of G are classified by triples (a, b, c) such that $a \geqslant 0$ and $2 | a + b + c$. Write $V(a, b, c)$ for the representation associated to the triples (a, b, c). If $c = 0$, then $V(a, b, c)$ is defined over $\mathbb{R}$. Otherwise, it follows from the theory of representations of real reductive groups (see [Tit71]) that there is a real representation $E(a, b, c)$ of G such that $E(a, b, c) \otimes \mathbb{C} = V(a, b, c) \oplus V(a, b, -c)$. Thus, if we set $E(a, b, 0) = V(a, b, 0)$, we find that the representations of G are classified by triples (a, b, c) such that $2 | a + b + c$ and $a, c \geqslant 0$. To sum up, we obtain the following theorem.

Theorem 13.13. *For each triple* $(a, b, c) \in \mathbb{Z}^3$ *with* $2 | a + b + c$ *and* $a, c \geqslant 0$, *there is a unique irreducible representation* $E(a, b, c)$ *of* G *such that* $E(a, b, c) \otimes \mathbb{C}$ *has a component of highest weight* (a, b, c) *under the above identification of* $\mathbb{Z}^3$ *with* $X^*(\tilde{T})$.

13.14 Composing $\rho_1 : G \to \mathfrak{M}_1$ with $\pi_1 : \mathfrak{M}_1 \to \mathbb{S}_1$, we obtain a morphism $\bar{\rho_1} : G \to \mathbb{S}_1$. Thus we obtain a functor $\bar{\rho_1}^* : \mathbf{S}_1 \to \mathrm{Rep}\, G$. We want to describe the image of certain SL_2 orbits under this functor. To do this, first consider the standard SL_2-orbit $\mathbf{Std} := (V, N, F, W)$ where

$$\begin{aligned} V &= \mathbb{C}^2 \quad \text{with basis } e = (1, 0),\ f = (0, 1), \\ N &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ W_i V &= \begin{cases} 0, & i < 1, \\ V, & i \geqslant 1, \end{cases} \end{aligned}$$

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$$F^pV_{\mathbb{C}} = \begin{cases} V_{\mathbb{C}}, & i < 1, \\ \mathbb{C}e, & i = 1, \\ 0, & i > 1. \end{cases}$$

Then the limit mixed Hodge structure associated to **Std** is $\mathbb{R} \oplus \mathbb{R}(-1)$ and the sl_2 action is the standard representation. It follows that the action of $S \subset G$ on V is trivial, and the relative weight filtration M is given by

$$M_iV = \begin{cases} 0, & i < 0, \\ \mathbb{R}f, & i \in [0, 1], \\ V, & i \geqslant 2. \end{cases}$$

Therefore, $Y_M(e) = 2, Y_M(f) = 0$. So, $i_W(\alpha) \in G$ acts on V by

$$\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} \alpha^2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}.$$

It follows that $\bar{\rho}_1^* \mathbf{Std}$ is isomorphic to $E(1, 1, 0)$.

For each non-negative integer a, let $S(a)$ denote the symmetric product $\mathrm{Sym}^a \mathbf{Std}$. Then $\bar{\rho}_1^* S(a) = E(a, a, 0)$.

The representations of G associated to the constant variations are very easy to describe. The homomorphism $G \to \mathbb{S}$ obtained by the composition of $\bar{\rho}_1$ with $\mathrm{const} : \mathbb{S}_1 \to \mathbb{S}$ is simply the canonical homomorphism $G = \mathrm{SL}_1 \rtimes \mathbb{S} \to \mathbb{S}$. It follows that the Tate Hodge structure $\mathbb{R}(k)$ corresponds to the constant SL_2-orbit $E(0, 2k, 0)$.

Suppose p, q are integers with $p > q$. Let $E(p, q)$ denote an irreducible real Hodge structure of weight $p + q$ with underlying vector space H and with $\dim H^{p,q} = 1$. Then $S \subset G$ acts on $H_{\mathbb{C}}$ with weights $\pm p - q$ and $P_W \subset G$ acts on V via the character $z \mapsto z^{p+q}$. Thus $\bar{\rho}_1^* E(p, q) \cong E(0, p + q, p - q)$.

Corollary 13.15. *The tensor functor $\mathbf{S}_1 \to \mathrm{Rep}\, G$ is essentially surjective.*

Proof. Since G is a real reductive group, all representations of G are decomposable into irreducibles. Thus, it suffices to show that every irreducible representation of G is in the essential image of $\mathbf{S}_1 \to \mathrm{Rep}\, G$.

To do this, it suffices to note that, when $2 | a + b + c$ and $a, c \geqslant 0$ then,

$$E(a, b, c) = \begin{cases} \bar{\rho}_1^* \left(S(a) \otimes \mathbb{R}\left(\dfrac{b - a}{2} \right) \right), & c = 0, \\ \bar{\rho}_1^* \left(S(a) \otimes E\left(\dfrac{b + c - a}{2}, \dfrac{b - c - a}{2} \right) \right), & c > 0. \end{cases} \tag{13.15.1}$$

□

Theorem 13.16. *The homomorphism $G \to \mathbb{S}_1$ is an isomorphism.*

Proof. It is equivalent to show that the tensor functor $\bar{\rho}_1^* : \mathbf{S}_1 \to \mathrm{Rep}\, G$ is an equivalence of categories. To do this, we need to show that $\bar{\rho}_1^*$ is fully faithful and essentially surjective. That $\bar{\rho}_1^*$ is faithful is clear (because the functor $\mathbf{S}_1 \to \mathrm{Vect}_{\mathbb{R}}$ is faithful). That $\bar{\rho}_1^*$ is essentially surjective is the content of Corollary 13.15.

To show that $\bar{\rho}_1^*$ is full, it suffices to show that $\mathrm{Hom}_{\mathbb{S}_1}(V_1, V_2) \to \mathrm{Hom}_G(V_1, V_2)$ is surjective when $\mathbf{V}_i = (V_i, N_i, W_i, F_i)$ are two pure objects in $\mathbf{S}_1$ of the same weight. But this is clear because the $G = \mathrm{SL}_2 \rtimes \mathbb{S}$ action on V_i determines both N_i and F_i: the action of SL_2 determines N_i and the action of $\mathbb{S}$ determines F. □

13.17 From now on we use Theorem 13.16 to identify $\mathbb{S}_1$ with $G = \mathrm{SL}_2 \rtimes \mathbb{S}$. We want to explain explicitly how to get an SL_2-orbit from a representation $\rho : \mathbb{S}_1 \to \mathrm{Aut}(V)$. As we said above, N is determined by the action of SL_2. It is the image of $n_0 \in \mathrm{sl}_2$ in $\mathrm{End}\, V$. Furthermore, the Hodge filtration F is determined by the $\mathbb{S}$ action as is the relative weight filtration M. The weight filtration W is the filtration of V corresponding to the central co-character i_W of Proposition 13.9. To check this, it suffices to check that it is true on the constant variations and on the standard SL_2-orbit **Std**. If V is an SL_2-orbit, both M and W are canonically split by Y_M and $Y_W = Y(N, Y_M)$. Moreover Y_W is a morphism of SL_2-orbits. We say that an element $v \in V$ is pure of weight m for M and w for W if $Y_M(v) = mv$ and $Y_W(v) = wv$.

14. The main theorem

14.1 For each z in the upper-half plane, $\mathfrak{h}$, Proposition 12.3 gives us a homomorphism $\mathrm{sp}_z : \mathfrak{M} \to \mathfrak{M}_1$. On the other hand, if (V, N, F, W) is an object in $\mathbf{S}_1$ then $(V, e^{zN}F, W)$ is split. Thus we have a homomorphism $\bar{\mathrm{sp}}_z : \mathbb{S} \to \mathbb{S}_1$ making the diagram

$$\begin{tikzcd} \mathfrak{M} \arrow[r, "\pi"] \arrow[d, "\mathrm{sp}_z"] & \mathbb{S} \arrow[d, "\bar{\mathrm{sp}}_z"] \\ \mathfrak{M}_1 \arrow[r, "\pi_1"] & \mathbb{S}_1 \end{tikzcd}$$

commute. To describe $\bar{\mathrm{sp}}_z$ explicitly, let $s_i : S^1 \to \mathrm{SL}_2$ denote the homomorphism of real algebraic groups given by

$$(x, y) \mapsto \begin{pmatrix} x & -y \\ -x & y \end{pmatrix}$$

where we regard S^1 as $\mathrm{Spec}\, \mathbb{R}[x, y]/(x^2 + y^2 - 1)$. For $z = u + iv \in \mathfrak{h}$, set

$$A_z = \begin{pmatrix} 1 & 0 \\ u & v \end{pmatrix}, \quad s_z(\sigma) = A_z \sigma A_z^{-1}.$$

We then have a homomorphism $S_z : S^1 \to \mathbb{S}_1 = \mathrm{SL}_2 \rtimes \mathbb{S}$ given by $S_z(\sigma) = (s_z(\sigma), s(\sigma))$. If we represent $\mathbb{S}$ via the quotient map $w \times s : \mathbb{G}_m \times S^1 \to \mathbb{S}$ with kernel μ_2, we have $\bar{\mathrm{sp}}_z(\lambda, \sigma) = i_W(\lambda) S_z(\sigma)$. Note that $\bar{\mathrm{sp}}_z(-1, -1) = 1$, so the map factors through to $\mathbb{S}$.

14.2 For each integer $k \leqslant -2$, let E_k denote the $\mathbb{S}_1$ representation $E(-k-2, k, 0)$. So E_{-2} corresponds to the constant nilpotent orbit $\mathbb{R}(1)$ and E_{-3} is a 2-dimensional non-constant nilpotent orbit of weight -3. In general, $E_k = (\mathrm{Sym}^{-k-2} E(1, -1, 0)) \otimes \mathbb{R}(1)$. Set $E = \oplus E_k$. Since E_k corresponds to a pure nilpotent orbit of weight k, E is a graded vector space. In fact, the action of $\mathbb{G}_m$ on E via $i_W : \mathbb{G}_m \to \mathbb{S}_1$ induces the grading.

14.3 Here is one way to view E as a representation of $\mathbb{S}_1$. Recall from (13.4) the mixed Hodge structure $T \cong \mathbb{R} \oplus \mathbb{R}(1)$ with basis e in the $\mathbb{R}$ factor and $f = (2\pi i)$ in the $\mathbb{R}(1)$ factor. Then $\mathrm{Sym}^* T$ can be viewed as the polynomial ring $\mathbb{C}[e, f]$. It inherits an sl_2 action described in (13.4), which, together with the $\mathbb{S}$ action, induces a representation of $\mathbb{S}_1 = \mathrm{SL}_2 \rtimes \mathbb{S}$. It is isomorphic to the SL_2-orbit $E(1, -1, 0)$.

Then, if we let $\mathbb{R}(1)$ denote the constant SL_2-orbit, $E = (\mathrm{Sym}^* T) \otimes \mathbb{R}(1)$. For each integer, $k \leqslant -2$, write n_k for $e^{-k+2} \otimes (2\pi i) \in E$. Since $n_+ e = 0$, $n_+ n_k = 0$ for all $k \leqslant -2$. The element $e \in T$ is in $T^{(0,0)}_{(F,M)}$, so $n_k \in E^{(-1,-1)}_{(F,M)}$. Similarly $f \in T^{(-1,-1)}_{(F,M)}$, so $E_k = M_{-1} E_k$ for all k. Thus $E = M_{-1} E$. Similarly, since e has weight -1 for W, n_k has weight $-k + 2 - 2 = -k$ for W.

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14.4 Let $\mathfrak{L}_1$ denote the free Lie algebra on the graded vector space E. From E, $\mathfrak{L}_1$ inherits a grading: $\mathfrak{L}_1 = \bigoplus \mathfrak{L}_1(i)$. In fact, the group $\mathbb{S}_1$ acts on $\mathfrak{L}_1$. For each integer m, we let $W_m\mathfrak{L}_1 := \bigoplus_{i \leqslant m} \mathfrak{L}_1(i)$. Then it is rather easy to see that $W_n\mathfrak{L}_1$ is an ideal in $\mathfrak{L}_1$ stable under the action of $\mathbb{S}_1$. Moreover, the Lie algebras $\mathfrak{L}_1/W_m\mathfrak{L}_1$ are finite dimensional, nilpotent Lie algebras equipped with compatible actions of $\mathbb{S}_1$. Let $K_1(n)$ denote the unipotent Lie group associated with $\mathfrak{L}_1/W_n\mathfrak{L}_1$. Then $\mathbb{S}$ acts on $K_1(n)$ and, thus, on the affine group scheme $K_1 := \varprojlim K_1(n)$. Set $G_1 = K_1 \rtimes \mathbb{S}_1$.

Lemma 14.5. *Let $\mathbf{U}_1$ denote the category of finite-dimensional real vector spaces V equipped with a linear map $\rho_E : E \to \operatorname{End} V$ such that:*

(i) *$\rho_E(u)$ is nilpotent for all $u \in E$;*

(ii) *$\rho_E(W_k E) = 0$ for $k \ll 0$.*

Then the functor $\rho \mapsto \rho_{|E}$ is an equivalence from the category $\operatorname{Rep} K_1$ to the category $\mathbf{U}_1$.

Proof. We leave this exercise in unraveling the definition of K_1 to the reader. □

14.6 Suppose $\rho : G_1 \to \operatorname{Aut} V$ is a representation of G_1 on a finite-dimensional vector space V. The restriction of ρ to K_1 induces a representation of the Lie algebra $\mathfrak{L}_1$ which is trivial on $W_n\mathfrak{L}_1$ for some n. For $k \leqslant -2$, let $N_k \in \operatorname{End} V$ denote the image of $n_k \in E \subset \mathfrak{L}_1$. The restriction of ρ to the semi-direct factor $\mathbb{S}_1$ of G_1 induces a representation of the Lie algebra sl_2. Let N_0, H and N_+ denote the images of the elements n_0, h and n_+ respectively. The $\mathbb{S}_1$-action on V gives V the structure of an SL_2-orbit $V = (V, N_0, F, W)$. Setting $M = M(N_0, W)$ and using the definition of G_1 as a semi-direct product, we find that $N_k \in \operatorname{End} V_{(F,M)}^{(-1,-1)}$ and $Y_W(N_k) = -kN_k$. Since $N_+(n_k) = 0$, $[N_+, N_k] = 0$ as well.

14.7 Consider the category $\mathbf{C}_1$ consisting of SL_2-orbits V equipped with a family of nilpotent operators N_k $(k \leqslant -2)$ such that $N_k \in \operatorname{End} V_{(F,M)}^{(-1,-1)}$, $Y_W(N_k) = -2N_k$ and $[N_+, N_k] = 0$. The morphisms in $\mathbf{C}_1$ are simply morphisms of SL_2-orbits respecting the N_k. The category $\mathbf{C}_1$ has the structure of a neutral Tannakian category in an obvious way. By (14.6), we have a functor $\operatorname{Res} : \operatorname{Rep} G_1 \to \mathbf{C}_1$ of Tannakian categories.

Proposition 14.8. *The functor $\operatorname{Res} : \operatorname{Rep} G_1 \to \mathbf{C}_1$ is an equivalence.*

Proof. Since $G_1 = K_1 \rtimes \mathbb{S}_1$, to give a representation R of G_1 is the same thing as to give representations R_K and R_S of K_1 and $\mathbb{S}_1$ respectively such that, for $s \in \mathbb{S}_1$ and $k \in K_1$, $R_S(s)R_K(k)R_S(s)^{-1} = R_K(s.k)$ (where $s.k$ denotes the effect of s acting on k). Since (by Lemma 14.5) representations of K are determined by their restrictions to E, to give a representation of G_1 on a finite-dimensional vector space V is actually the same thing as giving representations $R_E : E \to \operatorname{End} V$ satisfying the conditions of the lemma along with the condition that

$$R_S(s)R_E(e)R_S(s)^{-1} = R_E(s.e) \quad \text{for all } e \in E, s \in \mathbb{S}_1. \tag{14.8.1}$$

Suppose then that V in $\mathbf{C}_1$ is given. Then by definition we have a representation $R_S : \mathbb{S}_1 \to \operatorname{Aut} V$ which determines operators N_0 and N_+ on V along with nilpotent operators N_k $(k \leqslant -2)$ satisfying $[N_+, N_k] = 0$ for all k. Using the fact that $n_+.n_k$ and $[N_+, N_k]$ are both 0 for $k \leqslant -2$, it is not hard to see that $R_E(n_0^i.n_k) := (\operatorname{ad} N_0)^i N_k$ for $k \leqslant -2$ unambiguously defines a map $R_E : E \to \operatorname{End} V$ satisfying (14.8.1). Thus, from an object V in $\mathbf{C}_1$, we have a representation R of G_1. It is now simple to check that $\operatorname{Res}(R) = V$. Thus, we have an equivalence of categories. □

14.9 By Deligne's theorem (Theorem 13.2) along with Lemma 13.5, we have a Tannakian functor $h_1^* : \mathrm{Split}_1 \to \mathbf{C}_1$ sending a nilpotent orbit (V, N, F, W) with limit split over $\mathbb{R}$ to the associated SL_2-orbit (V, N_0, F, W) along with the data of the N_k for $k \leqslant -2$. By Proposition 14.8, this produces a homomorphism $h_1 : G_1 \to \mathfrak{M}_1$ of affine group schemes.

Theorem 14.10. *The homomorphism $h_1 : G_1 \to \mathfrak{M}_1$ is an isomorphism.*

Proof. We need to show that the functor $\mathrm{Split}_1 \to \mathbf{C}_1$ is fully faithful and essentially surjective. It is obvious that the functor is faithful, and full is also easy. So suppose V is an object in $\mathbf{C}_1$. Explicitly, V consists of the data (V, N_0, F, W) of an SL_2-orbit together with operators N_k for $k \leqslant -2$ satisfying the conditions in (14.7). Set $N = N_0 + \sum_{k\leqslant -2} N_k$. Then we claim that (V, N, F, W) is an object in Split_1.

It follows directly from the definition of the relative weight filtration, that the relative weight filtration $M(N, W)$ exists and is equal to $M(N_0, W)$. Since $\mathrm{Gr}^W N = \mathrm{Gr}^W N_0$, $(\mathrm{Gr}^W, e^{izN} \cdot F) = (\mathrm{Gr}^W, e^{izN_0} \cdot F)$ is a split mixed Hodge structure for all z in the upper-half plane. Thus, $(V, e^{izN} \cdot F, W)$ is a mixed Hodge structure for all such z.

To see that (V, N, F, W) is an admissible nilpotent orbit, it remains to check that $N(F^P) \subset F^{p-1}$ for all p. This follows from the fact that all the N_i (including N_0) are in $\mathrm{End}\, V_{(F,M)}^{(-1,-1)}$.

Since $M(N, W) = M(N_0, W)$, the object (V, N, F, W) has limit split over $\mathbb{R}$ and is, thus, in Split_1. It is easy to see that $h_1^*(V, N, F, W)$ is the original object V in $\mathbf{C}_1$ that we started out with. So, $h_1^* : \mathrm{Split}_1 \to G_1$ is an equivalence. □

14.11 Suppose z is a complex number in the upper-half plane. Then we have $\mathrm{sp}_z : \mathfrak{M} \to \mathfrak{M}_1$ and, by (14.1), $\mathrm{sp}_z(\mathfrak{U}) \subset \mathfrak{U}_1$. We thus have the following commutative diagram.

$$\begin{tikzcd}
1 \arrow[r] & \mathfrak{U} \arrow[r] \arrow[d, "\mathrm{sp}_z"] & \mathfrak{M} \arrow[r, "\pi"] \arrow[d, "\mathrm{sp}_z"] & \mathbb{S} \arrow[r] \arrow[d, "\overline{\mathrm{sp}}_z"] & 1 \\
1 \arrow[r] & \mathfrak{U}_1 \arrow[r] & \mathfrak{M}_1 \arrow[r, "\pi_1"] & \mathbb{S}_1 \arrow[r] & 1
\end{tikzcd}$$

We want to prove that the restriction of sp_z to $\mathfrak{U}$ is an isomorphism onto $\mathfrak{U}_1$. To begin, note the following.

Lemma 14.12. *The affine group schemes $\mathfrak{U}$ and $\mathfrak{U}_1$ are abstractly isomorphic by an isomorphism preserving the filtration W.*

Proof. It suffices to see that the Lie algebras $\mathfrak{L}$ and $\mathfrak{L}_1$ are isomorphic by an isomorphism preserving the filtration. Now, $\mathfrak{L}_1$ is the free Lie algebra on the graded vector space E while $\mathfrak{L}_{\mathbb{C}}$ is the free Lie algebra on the symbols $D^{i,j}$ with $i, j < 0$. It is easily seen that the real form $\mathfrak{L}$ of $\mathfrak{L}_{\mathbb{C}}$ is nothing but the free Lie algebra on the graded vector space $V = \bigoplus V_k$ where $\dim V_k = \{(i, j) : i + j = k, i < 0, j < 0\}$. Moreover, the grading on $\mathfrak{L}$ is the one induced from the grading on V. But V is isomorphic to E as a graded vector space, since $\dim E_k = \dim \mathrm{Sym}^{-k-2} \mathbb{R}^2 = -k - 1 = \dim V_k$. □

Since $\mathfrak{U}$ and $\mathfrak{U}_1$ are pro-unipotent, it is plausible to show that sp_z induces an isomorphism by showing that the map on the abelianizations induced by sp_z is an isomorphism.

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Lemma 14.13. *Suppose $\mathfrak{g}$ is a nilpotent Lie algebra over a field k and $\mathfrak{h}$ is a subalgebra. Suppose $\mathfrak{h}$ surjects onto $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$. Then $\mathfrak{h}=\mathfrak{g}$.*

Proof. This is essentially [Jac79, Exercise 11, p. 29]. □

Now, since $\mathfrak{U}/W_k\mathfrak{U}$ and $\mathfrak{U}_1/W_k\mathfrak{U}_1$ are both nilpotent algebraic groups over $\mathbb{R}$, to show that sp_z induces an isomorphism, it suffices to show that sp_z is surjective. By Lemma 14.13, it suffices to show that sp_z is surjective on the abelianizations. This is, in fact, what we are going to do.

Lemma 14.14. *Suppose $\mathbf{H}$ is an SL_2-orbit, and let $\mathbb{R}$ denote the constant SL_2 orbit of weight 0. Then $\mathrm{Ext}^1_{\mathrm{Split}_1}(\mathbb{R},\mathbf{H})=\mathrm{Hom}(E,\mathbf{H})^{\mathbb{S}_1}$. If $\mathbf{H}$ is irreducible and pure of weight $-k-2$, then we have*

$$\mathrm{Ext}^1_{\mathrm{Split}_1}(\mathbb{R},\mathbf{H})=\begin{cases}\mathbb{R} & \mathbf{H}=E_k,\\ 0 & \textit{else.}\end{cases}$$

Proof. First note that we can assume that $\mathbf{H}$ is irreducible of weight n for some integer n. If V is an extension of $\mathbb{R}$ by $\mathbf{H}$, then $W_{n-1}\mathfrak{M}_1$ acts trivially on V. So, $\mathrm{Ext}^1_{\mathrm{Split}_1}(\mathbb{R},\mathbf{H})=\mathrm{Ext}^1_{\mathfrak{M}_1(n-1)}(\mathbb{R},\mathbf{H})$. We can compute the latter extension group by means of the inflation-restriction sequence for the short exact sequence of algebraic groups

$$1\to\mathfrak{U}_1(n-1)\to\mathfrak{M}_1(n-1)\to\mathbb{S}_1\to 1.$$

We see that the extension group is $H^1(\mathfrak{M}_1(n-1),\mathbf{H})=H^1(\mathfrak{U}_1(n-1),\mathbf{H})^{\mathbb{S}_1}=\mathrm{Hom}(\mathfrak{U}_1(n-1)^{\mathrm{ab}},\mathbf{H})^{\mathbb{S}_1}=\mathrm{Hom}(E,\mathbf{H})^{\mathbb{S}_1}$. Here we use the fact that the abelianization of $\mathfrak{U}_1(n)$ is simply $\bigoplus_{k\geqslant n}E_k$.

The E_k are all irreducible as $\mathbb{S}_1$ representations even when base-changed to $\mathbb{C}$. So $\mathrm{Hom}(E_k,E_k)^{\mathbb{S}_1}=\mathbb{R}$. Since E_k has weight $-k-2$, the formula for the extension group in the case that $\mathbf{H}$ is irreducible of weight $-k-2$ follows. □

Proposition 14.15. *Let $\mathbb{R}$ denote the constant SL_2-orbit of weight 0, and suppose z is a number in the upper-half plane. Let V be a non-zero element in the extension group $\mathrm{Ext}^1_{\mathrm{Split}_1}(\mathbb{R},E_k)$. Then, on V, each $\delta_{p,q}$ with $p+q=-k$, $p,q<0$ is non-zero.*

Proof. We have $N=N_0+N_k$ and $F(z)=e^{zN}\cdot F$. Since V is split over $\mathbb{R}$, there is a unique element u of $V^{(0,0)}_{(F,M)}$ projecting onto 1 in $\mathbb{R}$. By multiplying the class of V in the extension group $\mathrm{Ext}^1_{\mathrm{Split}_1}(\mathbb{R},E_k)\cong\mathbb{R}$, we can assume that $N_k(u)=e^{-k-2}\otimes(2\pi i)\in E_k$.

We have $e\in T^{(0,0)}_{(F,M)}$ and $f\in T^{-1,-1}_{(F,M)}$. It follows that

$$\mathbb{C}(e+zf)^a(e+\bar zf)^b\otimes(2\pi i)=(E_k)^{(-b-1,-a-1)}_{F(z)}.$$

Now, we have

$$e=\frac{z(e+\bar zf)-\bar z(e+zf)}{z-\bar z}.$$

So

$$e^{-k-2}\otimes(2\pi i)=(z-\bar z)^{-k-2}\sum_{a+b=-k-2}\binom{-k-2}{a}z^a(-\bar z)^b(e+\bar zf)^a(e+zf)^b.$$

We could use this to compute the $\delta_{p,q}$ directly. However, we only need to show that each $\delta_{p,q}$ is non-zero. For this, pick p,q with $p,q<0$ and $p+q=-k$. Let $\mathbf{H}$ denote the irreducible Hodge substructure of $(E_k)_{F(z)}$ with a $\mathbf{H}^{(p,q)}_{F(z)}\neq 0$. Thus $\mathbf{H}$ is generated as a $\mathbb{C}$ vector space by $(e+\bar zf)^{-q-1}(e+zf)^{-p-1}\otimes(2\pi i)$ and $(e+\bar zf)^{-p-1}(e+zf)^{-q-1}\otimes(2\pi i)$. Note that

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$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{R},\mathbf{H}) = \mathbf{H}_{\mathbb{C}}/(F^0(z)\mathbf{H}+\mathbf{H}_{\mathbb{R}})$. But $F^0(z)\mathbf{H}=0$, so $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{R},\mathbf{H}) = \mathbf{H}_{\mathbb{C}}/\mathbf{H}_{\mathbb{R}}$. The image of the extension class of V in $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{R},\mathbf{H})$ is then the projection of $zN_k(u) = ze^{-k-2}\otimes(2\pi i)$ onto $\mathbf{H}_{\mathbb{C}}/\mathbf{H}_{\mathbb{R}}$. But, since each component in the above expression for $e^{-k-2}\otimes(2\pi i)$ is non-zero and $e^{-k-2}\otimes(2\pi i)$ is real, the image of V in $\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{R},\mathbf{H})$ is easily seen to be non-zero. Therefore $\delta_{p,q}\neq 0$. □

Corollary 14.16. *The map* $\mathrm{sp}_z:\mathfrak{U}\to\mathfrak{U}_1$ *is an isomorphism.*

Proof. It suffices to show that, for each integer $n\leqslant 0$, the map of unipotent real algebraic groups $\mathrm{sp}_z:\mathfrak{U}(n)\to\mathfrak{U}_1(n)$ is an isomorphism. For this it suffices to show that the maps on the Lie algebras are isomorphisms. Since the Lie algebras are nilpotent and of the same dimension over $\mathbb{R}$, it suffices to show that the map induced on the abelianizations is surjective. So, set $\mathfrak{L}(n):=\mathfrak{L}/W_n\mathfrak{L}$ and $\mathfrak{L}_1(n):=\mathfrak{L}_1/W_n\mathfrak{L}_1$. It suffices to show that the map $s_z:\mathfrak{L}^{\mathrm{ab}}(n)\to\mathfrak{L}_1^{\mathrm{ab}}(n)$ induced by sp_z is an isomorphism from each n. This map is $\mathbb{S}$-equivariant, where $\mathbb{S}$ is acting on $\mathfrak{L}_1(n)$ via conjugation. For $p\leqslant q<0$ write $H(p,q)$ for the unique irreducible real pure Hodge structure with $H(p,q)^{(p,q)}\neq 0$. Then, with the given $\mathbb{S}$-action, $\mathfrak{L}^{\mathrm{ab}}(n)$ is a direct sum of the $H(p,q)$ such that $p+q>-n$ with each irreducible factor appearing exactly once. Suppose the map s_z has a kernel. Then there is some (p,q) with $H(p,q)$ in the kernel. Set $p+q=k>n$. Then, if V denotes a non-zero extension in $\mathrm{Ext}^1_{\mathrm{Split}_1}(\mathbb{R},E_k)$, we will have $\delta_{p,q}=0$ for the Hodge structure $V_{(F(z),W)}$. This contradicts Proposition 14.15. □

Proof of Theorem 12.4. The only thing left to prove is part (iv) of the theorem. Recall that $i_M:\mathbb{G}_m\to\mathfrak{M}_1$ denotes the homomorphism inducing the splitting of the relative weight filtration M and H denotes the image of i_M. Now all elements of E_k are in M_{-1} (see § 14.3). So the grading induced by M on the free Lie algebra $\mathfrak{L}_1$ on E puts all elements of $\mathfrak{L}_1$ in negative weight. Therefore, for each n, the intersection of H with $\mathfrak{U}_1$ is trivial.

The fact that $\overline{\mathrm{sp}}_i(w(\mathbb{G}_m))$ is central in $\mathbb{S}_1$ was proved in Proposition 13.9. □

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References

BP09 P. Brosnan and G. J. Pearlstein, *The zero locus of an admissible normal function*, Ann. of Math. (2) **170** (2009), 883–897.

BPS08 P. Brosnan, G. Pearlstein and C. Schnell, *The locus of Hodge classes in an admissible variation of mixed Hodge structure*, C. R. Math. Acad. Sci. Paris **348** (2010), 657–660.

Car87 J. A. Carlson, *The geometry of the extension class of a mixed Hodge structure*, in *Algebraic geometry, Bowdoin 1985, Brunswick, Maine, 1985*, Proceedings of Symposia in Pure Mathematics, vol. 46 (American Mathematical Society, Providence, RI, 1987), 199–222.

CK89 E. Cattani and A. Kaplan, *Degenerating variations of Hodge structure*, Astérisque **9** (1989), 67–96; Actes du Colloque de Théorie de Hodge (Luminy, 1987).

CKS86 E. Cattani, A. Kaplan and W. Schmid, *Degeneration of Hodge structures*, Ann. of Math. (2) **123** (1986), 457–535.

Del71 P. Deligne, *Théorie de Hodge. II*, Publ. Math. Inst. Hautes Études Sci. (1971), 5–57.

Del80 P. Deligne, *La conjecture de Weil. II*, Publ. Math. Inst. Hautes Études Sci. (1980), 137–252.

Del93 P. Deligne, Letter to E. Cattani and A. Kaplan, 1993.

Del94 P. Deligne, *Structures de Hodge mixtes réelles*, in *Motives, Seattle, WA, 1991*, Proceedings of Symposia in Pure Mathematics, vol. 55 (American Mathematical Society, Providence, RI, 1994), 509–514.

DMOS82 P. Deligne, J. S. Milne, A. Ogus and K.-y. Shih, *Hodge cycles, motives, and Shimura varieties*, Lecture Notes in Mathematics, vol. 900 (Springer, Berlin, 1982).

Hir77 H. Hironaka, *Bimeromorphic smoothing of a complex-analytic space*, Acta Math. Vietnam. **2** (1977), 103–168.

Jac79 N. Jacobson, *Lie algebras* (Dover Publications Inc., New York, 1979), (republication of the 1962 original).

KP03 A. Kaplan and G. Pearlstein, *Singularities of variations of mixed Hodge structure*, Asian J. Math. **7** (2003), 307–336.

Kas86 M. Kashiwara, *A study of variation of mixed Hodge structure*, Publ. Res. Inst. Math. Sci. **22** (1986), 991–1024.

KNU08 K. Kato, C. Nakayama and S. Usui, SL(2)*-orbit theorem for degeneration of mixed Hodge structure*, J. Algebraic Geom. **17** (2008), 401–479.

KNU10 K. Kato, C. Nakayama and S. Usui, *Moduli of log mixed Hodge structures*, Proc. Japan Acad. Ser. A Math. Sci. **86** (2010), 107–112.

Mil07 J. S. Milne, *Quotients of Tannakian categories*, Theory Appl. Categ. **18** (2007), 654–664.

Pea00 G. J. Pearlstein, *Variations of mixed Hodge structure, Higgs fields, and quantum cohomology*, Manuscripta Math. **102** (2000), 269–310.

Pea06 G. Pearlstein, SL_2*-orbits and degenerations of mixed Hodge structure*, J. Differential Geom. **74** (2006), 1–67.

Sai96 M. Saito, *Admissible normal functions*, J. Algebraic Geom. **5** (1996), 235–276.

Sch73 W. Schmid, *Variation of Hodge structure: the singularities of the period mapping*, Invent. Math. **22** (1973), 211–319.

Sch01 C. Schwarz, *Relative monodromy weight filtrations*, Math. Z. **236** (2001), 11–21.

Sch12 C. Schnell, *Complex analytic Néron models for arbitrary families of intermediate Jacobians*, Invent. Math. **188** (2012), 1–81.

Ser56 J.-P. Serre, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier (Grenoble) **6** (1955–1956), 1–42.

SGA3 M. Demazure and A. Grothendieck (eds), *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3)*, Lecture Notes in Mathematics, vol. 152 (Springer, Berlin, 1970).

SR72 N. Saavedra Rivano, *Catégories Tannakiennes*, Lecture Notes in Mathematics, vol. 265 (Springer, Berlin, 1972).

Tit71 J. Tits, *Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque*, J. Reine Angew. Math. **247** (1971), 196–220.

Zie11 P. Ziegler, *Graded and filtered fiber functors on Tannakian categories*, Preprint (2011), arXiv:1111.1981.

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