Proceedings of the Royal Society of Edinburgh, **137A**, 995–1035, *2007*

Generalized eigenfunction expansions for conservative scattering problems with an application to water waves

Christophe Hazard and François Loret
Propagation des Ondes: Étude Mathématique et Simulation (CNRS, ENSTA, INRIA), Unité de Mathématiques Appliquées, Ecole Nationale Supérieure de Techniques Avancées, 32 Boulevard Victor, 75739 Paris cedex 15, France
(christophe.hazard@ensta.fr; francois.loret@ensta.fr)

(MS received 18 January 2006; accepted 27 September 2006)

This paper is devoted to a *spectral* description of wave propagation phenomena in conservative unbounded media, or, more precisely, the fact that a time-dependent wave can often be represented by a continuous superposition of time-harmonic waves. We are concerned here with the question of the perturbation of such a *generalized eigenfunction expansion* in the context of scattering problems: if such a property holds for a free situation, i.e. an unperturbed propagative medium, what does it become under perturbation, i.e. in the presence of scatterers? The question has been widely studied in many particular situations. The aim of this paper is to collect some of them in an abstract framework and exhibit sufficient conditions for a perturbation result. We investigate the physical meaning of these conditions which essentially consist in, on the one hand, a *stable limiting absorption principle* for the free problem, and on the other hand, a *compactness* (or short-range) property of the perturbed problem.

This approach is illustrated by the scattering of linear water waves by a floating body. The above properties are obtained with the help of integral representations, which allow us to deduce the asymptotic behaviour of time-harmonic waves from that of the Green function of the free problem. The results are not new: the main improvement lies in the structure of the proof, which clearly distinguishes the properties related to the free problem from those which involve the perturbation.

1. Introduction

1.1. Motivation

The aim of this paper is to propose a relatively general way of establishing *generalized eigenfunction expansions* for linear scattering problems, and illustrate the method by a coupled problem arising in hydrodynamics: the scattering of water waves by a floating body. The notion of eigenfunction expansion plays an essential role in the study of wave propagation in non-dissipative continuous media, as well as in the context of quantum physics. It provides conclusively the connection between *transient* and *time-harmonic* phenomena. More precisely, it offers a representation of a transient wave as a superposition of time-harmonic waves. In many textbooks, such a representation appears as a universal truth, but is seldom rigorously justified.

https://doi.org/10.1017/S0308210506000138 Published online by Cambridge University Press

The fact is that, even for simple situations, the existing proofs are very intricate, and hardly accessible to the layman.

From a mathematical point of view, the underlying question concerns the notion of *diagonalization* of the self-adjoint operator that describes the dynamics of a propagative system: can one always find a family of eigenfunctions which compose an *orthonormal basis* of the energy space of the system? The problem is well understood in the case of a bounded propagative domain, for which the operator generally possesses a *pure point spectrum*, due to some compactness property (see, for example, [36]). Spectral theory then ensures that the eigenfunctions can be chosen so as to form a discrete orthonormal basis and hence the eigenfunction expansion of the transient vibrations of the system is expressed as a series of time-harmonic states. The present paper is rather devoted to wave propagation in unbounded media, for which the associated operator generally possesses a *continuous spectrum*. In this case, eigenfunctions have to be sought outside the finite energy space of the system, which justifies the word 'generalized'. And one may hope to find a *continuous* orthonormal basis (in a sense to be specified) of such generalized eigenfunctions. The general context in which such a family exists is now well understood [7, 8, 15] but, apart from differential operators, there is no general way to construct it. Our purpose is to focus on one particular aspect of the question, in the context of scattering by obstacles: if one knows a basis of generalized eigenfunctions for the medium called 'free' in the following, i.e. without obstacle, how can one construct a basis for the *perturbed* medium, i.e. in the presence of the obstacle?

We do not deal here with the determination of a basis in the free situation, which generally follows from the usual functional transforms. The simplest example is given by a homogeneous medium filling the whole space. In this case the Fourier transform is the appropriate tool for the diagonalization of the associated operator (see, for example, [43] in acoustics), and the eigenfunction expansion appears as a plane wave representation. Similarly, for water waves in a half-space, we shall see in § 3 that the eigenfunction expansion follows easily from the use of a horizontal Fourier transform. On the other hand, some particular inhomogeneous media can be considered as free. For instance, separation of variables allows us to deal with stratified media [45], using spectral theory of differential equations with variable coefficients [38]. The case of gratings [44] also comes within this context.

What happens, then, when an obstacle is inserted in such a free medium? From a physical point of view, a perturbation approach seems quite natural. Considering the free generalized eigenfunctions as *incident* time-harmonic waves, the generalized eigenfunctions of the scattering problem are sought as perturbations of the previous eigenfunctions, by adding a perturbation term representing a *scattered* time-harmonic wave. The first rigorous justification of this approach is due to Ikebe [24] for the Schrödinger equation. His method was then improved [2,33,37,43] and applied to many other wave propagation phenomena. The scalar wave equation has been intensively studied in numerous situations, for example, perturbed cylindrical waveguides [16, 17, 29], wedge-shaped regions [18], stratified media [11, 13, 41] and periodic waveguides [30]. Similar approaches were developed in elastodynamics [10, 12, 31, 34] and electromagnetism [4, 41] and for more abstract models [40]. In hydrodynamics, the scattering of linear water waves by a fixed body was first studied by Beale [6] (see also [21, 39]).

Collecting these different papers, one may be surprised at the apparent heterogeneousness of the methods used to reach the same goal. Indeed, the distinctive feature of a given physical situation may affect most proofs, which leads us to think that these proofs are highly 'problem dependent', that is, a slight change in the definition of the problem requires us to adapt most proofs. This is particularly true in the papers based on a *local* definition of energy (i.e. in any bounded subdomain of the propagation medium). The purpose of the present paper is to propose a unified approach based on a *weighted* definition of energy, following the notion of Hilbert riggings developed in mathematical physics [8].

Section 2 presents an abstract perturbation result for generalized eigenfunction expansions. This section, which is an improved version of the approach proposed in [20], is organized as follows. In § 2.1 we set the definition of a generalized spectral basis which furnishes a suitable functional context for generalized eigenfunction expansions and offers a general but formal point of view on integral representations. The theoretical framework is close to that proposed in [8]. In § 2.2, considering an abstract wave equation, we show the basic consequences of our definition for time–frequency analysis. The so-called limiting absorption principle guarantees the existence of time-harmonic outgoing and incoming waves. We introduce a 'stable' form of this principle that plays an essential role in § 2.3, where the question of the perturbation of a generalized spectral basis is investigated. The main result of this paper, theorem 2.11, shows that, under some compactness property, the perturbed generalized eigenfunctions are obtained by adding to the unperturbed ones, considered as *incident* waves, the outgoing or incoming wave *scattered* by the perturbation. The proof is given in § 2.4, which clarifies the connection with spectral theory of self-adjoint operators. For the sake of simplicity, the theory is presented under restrictive assumptions: only bounded operators with no point spectrum are considered. We show in § 2.5 how to relax these assumptions in order to deal with physical applications.

The method is illustrated by the two-dimensional problem of a rigid body floating on a sea of infinite depth. The section below describes the equations of the problem and presents a formal expression of the generalized eigenfunction expansion of the transient motions. Its derivation is the object of § 3. We first state the mathematical formulation of the problem in § 3.1. The free problem is studied in § 3.2, and the compactness of the perturbation is proved in § 3.3. The final results are presented in § 3.4. These results are not new: they slightly improve the known results [19]. The proofs, however, are new. In particular they clearly identify the part of the work which has to be revisited when dealing with more involved situations, for instance, the case of an elastic floating body [22].

1.2. Main results for the two-dimensional sea-keeping problem

The linearized equations that model the coupled motions of a rigid body floating at the free surface of an inviscid perfect fluid (with a potential flow) are well known (see, for example, [25]). We present here a non-dimensional expression of these equations which involves the 'acceleration potential' $\Phi = \Phi(X, t)$ (where $X = (x, y) \in \mathbb{R}^2$ denotes a point in the fluid, and t denotes the time) instead of the usual velocity potential (i.e. the former is the time derivative of the latter). The advantage of this

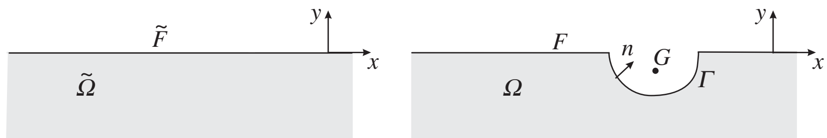

Figure 1. Free (left) and perturbed (right) fluid domains.

formulation lies in the fact that it only involves second-order time derivatives: it is well adapted to enter the abstract framework described in § 2.

For the sake of simplicity, we consider only two of the three possible rigid motions of the floating body. These motions are described by a vector $p = p(t) \in \mathbb{R}^2$, where the first component represents the *heave*, i.e. the vertical displacement of the centre of gravity $G = (x_G, y_G)$ of the body with respect to its equilibrium position, and the second component denotes the *roll*, i.e. its rotation in the plane. The *sway*, i.e. the horizontal displacement, actually plays a different role from the other displacements in the coupling process. The fact that it does not contribute to the buoyancy (that is, Archimedes') force leads to a one-way coupling: at every time t, this motion is determined entirely by knowledge of the pair (Φ, p).

We denote by $\tilde{\Omega} := \{(x, y) \in \mathbb{R}^2;\ y < 0\}$ the half-space filled by the water at rest in the absence of the body, and by $\tilde{F} := \{(x, 0) \in \mathbb{R}^2\}$ its free surface (see figure 1). The tilde will be used for all quantities related to this free situation. In the presence of the body, the fluid domain is denoted by $\Omega \subset \tilde{\Omega}$. Its boundary consists of the part F of $\tilde{F}$ located outside the body, and the immersed part Γ of its hull. At every point $X \in \Gamma$, we denote by $n = (n_x, n_y)$ the outer unit normal (exterior to Ω) and by $\nu \in \mathbb{R}^2$ the vector related to n by $\nu(X) := (n_y, (x - x_G)n_y - (y - y_G)n_x)$.

The time-dependent sea-keeping problem consists in finding a pair-(Φ, p) solution to the following coupled equations at every time $t > 0$:

$$\Delta\Phi = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$\partial_t^2\Phi + \partial_y\Phi = 0 \quad \text{on } F, \tag{1.2}$$

$$\partial_n\Phi - \mathrm{d}_t^2 p\nu = 0 \quad \text{on } \Gamma, \tag{1.3}$$

$$\mathbb{M}\,\mathrm{d}_t^2 p + \mathbb{K}p + \int_\Gamma \Phi\nu\,\mathrm{d}\gamma = 0, \tag{1.4}$$

as well as suitable initial conditions. In (1.4), $\mathbb{M}$ and $\mathbb{K}$ are 2×2 real symmetric positive definite matrices: $\mathbb{M}$ is the *mass matrix* of the body and $\mathbb{K}$ is the *hydrostatic stiffness matrix* ($\mathbb{K}p$ represents the variation of the Archimedes force due to a displacement p of the body). Without loss of generality, we assume in the following that $\mathbb{M}$ is the identity matrix: using the Cholesky factorization of $\mathbb{M} = \mathbb{L}\mathbb{L}^*$, this amounts to replacing p, ν and $\mathbb{K}$ respectively by $\mathbb{L}^*p$, $\mathbb{L}^{-1}\nu$ and $\mathbb{L}^{-1}\mathbb{K}(\mathbb{L}^*)^{-1}$.

Our aim is to express (Φ, p) by means of particular *time-harmonic* solutions to (1.1)–(1.4) which are constructed by considering this system as a perturbation of the *free* water-wave problem:

$$\left.\begin{aligned} \Delta\tilde{\Phi} &= 0 \quad \text{in } \tilde{\Omega}, \\ \partial_t^2\tilde{\Phi} + \partial_y\tilde{\Phi} &= 0 \quad \text{on } \tilde{F}. \end{aligned}\right\} \tag{1.5}$$

https://doi.org/10.1017/S0308210506000138 Published online by Cambridge University Press

A time-harmonic free wave is a solution to these equations which has the form

$$\tilde{\Phi}(X,t) = \mathrm{Re}\{\tilde{\Phi}_\omega(X)\exp(-\mathrm{i}\omega t)\},$$

for some given angular frequency $\omega > 0$. Throughout the paper, we shall use a 'spectral index' $\lambda = \omega^2$ instead of ω. Define

$$\tilde{\Phi}_{\lambda,k}(X) := \frac{1}{\sqrt{2\pi}}\exp(\lambda(\mathrm{i}kx + y)) \quad \text{for } \lambda \in \mathbb{R}^+ \text{ and } k = \pm 1 \tag{1.6}$$

(where the choice of the coefficient $1/\sqrt{2\pi}$ will be justified later). These functions obviously satisfy the time-harmonic equations corresponding to (1.5):

$$\Delta\tilde{\Phi}_{\lambda,k} = 0 \quad \text{in } \tilde{\Omega}, \tag{1.7}$$

$$\partial_y\tilde{\Phi}_{\lambda,k} - \lambda\tilde{\Phi}_{\lambda,k} = 0 \quad \text{on } \tilde{F}. \tag{1.8}$$

They represent plane surface waves of frequency $\sqrt{\lambda}$ which propagate towards $k\times\infty$.

We consider two kinds of perturbations of these plane waves, denoted by

$$w^{\pm}_{\lambda,k} := (\Phi^{\pm}_{\lambda,k}, p^{\pm}_{\lambda,k}), \quad \text{where } \Phi^{\pm}_{\lambda,k} := \tilde{\Phi}_{\lambda,k} + \dot{\Phi}^{\pm}_{\lambda,k}, \tag{1.9}$$

which represents the superposition of the *incident* wave $\tilde{\Phi}_{\lambda,k}$ and the *scattered* wave $\dot{\Phi}^{\pm}_{\lambda,k}$. These pairs correspond to periodic solutions of (1.1)–(1.4) if $(\dot{\Phi}^{\pm}_{\lambda,k}, p^{\pm}_{\lambda,k})$ satisfies

$$\Delta\dot{\Phi}^{\pm}_{\lambda,k} = 0 \quad \text{in } \Omega, \tag{1.10}$$

$$\partial_y\dot{\Phi}^{\pm}_{\lambda,k} - \lambda\dot{\Phi}^{\pm}_{\lambda,k} = 0 \quad \text{on } F, \tag{1.11}$$

$$\partial_n\dot{\Phi}^{\pm}_{\lambda,k} + \lambda p^{\pm}_{\lambda,k}\cdot\nu = -\partial_n\tilde{\Phi}_{\lambda,k} \quad \text{on } \Gamma, \tag{1.12}$$

$$(\mathbb{K} - \lambda)p^{\pm}_{\lambda,k} + \int_\Gamma \dot{\Phi}^{\pm}_{\lambda,k}\nu\,\mathrm{d}\gamma = -\int_\Gamma \tilde{\Phi}_{\lambda,k}\nu\,\mathrm{d}\gamma. \tag{1.13}$$

The plus and minus signs are assigned to *outgoing* and *incoming* waves, respectively. This difference is usually specified by means of a *radiation condition* at infinity, which expresses that the energy of the scattered wave either radiates towards infinity (+) or comes from infinity (−). It is given by [25]

$$\lim_{R\to+\infty}\int_{|x|=R} |\partial_{|x|}\dot{\Phi}^{\pm}_{\lambda,k} \mp \mathrm{i}\lambda\dot{\Phi}^{\pm}_{\lambda,k}|^2\,\mathrm{d}y = 0. \tag{1.14}$$

The aim of the present paper is to understand the different properties which allow us to express the transient solution $u(t) := (\Phi(\cdot,t), p(t))$ to (1.1)–(1.4) as a continuous superposition of the time-harmonic solutions (1.9). More precisely, we show that, for a suitable inner product $\langle\cdot,\cdot\rangle$, we have

$$u(t) = \mathrm{Re}\int_{\mathbb{R}^+}\sum_{k=\pm1}\langle u^{(0)}, w^{\pm}_{\lambda,k}\rangle w^{\pm}_{\lambda,k}\mathrm{e}^{-\mathrm{i}\sqrt{\lambda}t}\,\mathrm{d}\lambda, \tag{1.15}$$

where $u^{(0)}$ only depends on the initial conditions. This formula is called the *generalized eigenfunction expansions* of $u(t)$.

https://doi.org/10.1017/S0308210506000138 Published online by Cambridge University Press

2. An abstract context for generalized eigenfunctions

We now describe the mathematical tools which will help us to prove the above expansion in § 3. Our aim is to show how the well-known discrete spectral expansion associated with a compact self-adjoint operator extends to the case of an operator that has a continuous spectrum. Such a generalization is possible in a very general context, but its description is intricate. Here we restrict ourselves to a particular situation that applies for self-adjoint operators having an *absolutely continuous* spectrum with no change of multiplicity. Our water-wave problem comes within this framework, as do many other applications.

2.1. Definition of a generalized spectral basis

2.1.1. *Functional framework*

In a Hilbert space $\mathcal{H}$ equipped with an inner product $(\cdot,\cdot)$ and the associated norm $\|\cdot\|$, consider a *bounded self-adjoint* operator $\mathcal{A}$, that is, such that

$$(\mathcal{A}u, v) = (u, \mathcal{A}v) \quad \text{for all } u, v \in \mathcal{H}.$$

The kind of pairs $(\mathcal{H}, \mathcal{A})$ we are interested in concerns models of wave propagation phenomena in *unbounded* conservative media (see § 2.2). The norm of $\mathcal{H}$ can be interpreted as the energy gauge of our system; in practice, such an energy is defined by an integral over the whole propagative medium. The elements of $\mathcal{H}$ represent the possible states v of the system: their energy $\|v\|$ is finite, which implies a sufficient decay of v at infinity. The operator $\mathcal{A}$ describes the dynamics of the system, and the above self-adjointness property expresses a reciprocity condition between action and observation. On account of the unbounded nature of the medium, the eigenfunctions of $\mathcal{A}$ with finite energy, if any, cannot describe the propagation at large distance: generalized eigenfunctions have to be searched in an 'overspace' of $\mathcal{H}$, which will be obtained by a weighted definition of energy. We introduce below an extension of the notion of 'spectral basis' composed of such generalized eigenfunctions, which requires the following ingredients.

(i) A *weighted energy* Hilbert space $\mathcal{H}_\downarrow \subset \mathcal{H}$ equipped with a norm $\|\cdot\|_\downarrow$ where the index '$\downarrow$' signifies that the states of $\mathcal{H}_\downarrow$ have a stronger decay at infinity than those of $\mathcal{H}$: these 'localized' states are obtained by introducing a suitable weight in the integral that defines $\|\cdot\|$. We assume that $\mathcal{H}_\downarrow$ is *continuously embedded* in $\mathcal{H}$ and *dense*. In this situation, its dual space $\mathcal{H}_\uparrow := (\mathcal{H}_\downarrow)'$ can be interpreted as an 'overspace' of $\mathcal{H}$ (containing non-localized states of infinite energy) when the latter is identified with its own dual, i.e.

$$\mathcal{H}_\downarrow \subset \mathcal{H} = \mathcal{H}' \subset \mathcal{H}_\uparrow. \tag{2.1}$$

Hence, the duality product[1] between $\mathcal{H}_\downarrow$ and $\mathcal{H}_\uparrow$ appears as an extension of the inner product $(\cdot,\cdot)$ since relation (2.1) means in particular that

$$\langle u, v\rangle = (u, v) \quad \text{for all } u \in \mathcal{H}_\downarrow \text{ and all } v \in \mathcal{H}. \tag{2.2}$$

[1]We actually consider a semi-duality product: $\langle \alpha u, \beta v\rangle = \alpha\bar{\beta}\langle u, v\rangle$ for $\alpha, \beta \in \mathbb{C}$. We use the notation $\langle u, v\rangle$ and $\langle v, u\rangle$ for both $u \in \mathcal{H}_\downarrow$ and $v \in \mathcal{H}_\uparrow$: they are simply conjugated to each other.

(ii) A *spectral* Hilbert space $\hat{\mathcal{H}}$, whose elements will represent the coordinates of the states of $\mathcal{H}$ in a generalized spectral basis. For the sake of simplicity, we consider here the situation where this spectral space can be chosen as a standard L^2-space, more precisely[2]

$$\hat{\mathcal{H}} := L^2(\Lambda \times K), \tag{2.3}$$

where $\Lambda :=]\lambda_-, \lambda_+[$ is a bounded open interval of $\mathbb{R}$ (whose closure will represent the spectrum of $\mathcal{A}$) equipped with the Lebesgue measure. The elements of K are intended to number the generalized eigenfunctions associated with a given $\lambda \in \Lambda$: these 'wave numbers' may compose a discrete, or continuous set K. Here we assume that K is either finite or a compact subset of $\mathbb{R}^n$ (for some $n \geqslant 1$) equipped with a finite measure $\mathrm{d}\sigma$. Both spectral variables λ and k will generally appear as indices, and we shall denote by

$$(\hat{u}, \hat{v})_{\Lambda \times K} := \int_{\Lambda \times K} \hat{u}_{\lambda,k} \overline{\hat{v}_{\lambda,k}} \, \mathrm{d}\lambda \, \mathrm{d}\sigma_k$$

the inner product in $L^2(\Lambda \times K)$. Note that if K is finite, the integral on K should be replaced by a finite sum. We keep the integral notation for simplicity.

We are now able to state the definition of a generalized spectral basis composed of non-localized states $w_{\lambda,k} \in \mathcal{H}_\uparrow$ indexed by $\lambda \in \Lambda$ and $k \in K$. For technical reasons that will be clarified later, we shall need a sufficient regularity of the vector-valued function $(\lambda, k) \mapsto w_{\lambda,k}$. We shall say that a family $\{w_{\lambda,k} \in \mathcal{H}_\uparrow;\ (\lambda, k) \in \Lambda \times K\}$ is *Λ-locally Hölder continuous* if, for every compact interval Λ' interior to Λ (i.e. $\Lambda' = [a, b]$ with $\lambda_- < a < b < \lambda_+$), on the one hand

$$\text{the map } \Lambda' \times K \ni (\lambda, k) \mapsto w_{\lambda,k} \in \mathcal{H}_\uparrow \text{ is continuous} \tag{2.4}$$

and on the other hand there exist $\alpha \in]0, 1]$ and $C(\Lambda') > 0$ such that

$$\sup_{k \in K} \|w_{\lambda,k} - w_{\lambda',k}\|_\uparrow \leqslant C(\Lambda')|\lambda - \lambda'|^\alpha \quad \text{for all } \lambda, \lambda' \in \Lambda'. \tag{2.5}$$

Note that (2.4) derives from (2.5) when K is finite.

Definition 2.1. A Λ-locally Hölder continuous family $\{w_{\lambda,k} \in \mathcal{H}_\uparrow;\ (\lambda, k) \in \Lambda \times K\}$ associated with a spectral space $L^2(\Lambda \times K)$ is said to be a generalized spectral basis[3] for the operator $\mathcal{A}$ in $\mathcal{H}$ if the transformation $\mathcal{U}$ given by

$$(\mathcal{U}v)_{\lambda,k} := \langle v, w_{\lambda,k} \rangle \quad \text{for all } v \in \mathcal{H}_\downarrow, \tag{2.6}$$

[2]Our special choice (2.3) of a spectral space is sufficient for the applications we have in mind in this paper, but not if the 'spectral multiplicity' of a point $\lambda \in \Lambda$ depends on λ. In this situation, which occurs, for instance, in waveguides, instead of (2.3), one may choose

$$\hat{\mathcal{H}} := L^2(\Lambda_1 \times K_1) \oplus L^2(\Lambda_2 \times K_2) \oplus \cdots .$$

The ideas we present here can be readily extended to this case. More generally, the spectral space of a self-adjoint operator appears as a *direct integral* of Hilbert spaces [35].

[3]This definition extends the notion of a spectral basis for a Hermitian matrix, or more generally, that of a compact self-adjoint operator $\mathcal{A}$ in $\mathcal{H}$. In the latter case, one can find a Hilbertian basis $\{w_n \in \mathcal{H};\ n \in \mathbb{N}\}$ (i.e. an infinite complete orthonormal family) composed of eigenvectors: $(\mathcal{A} - \lambda_n)w_n = 0$, where $(\lambda_n)_{n\in\mathbb{N}}$ is a real sequence which tends to 0. The transformation $\mathcal{U}$ defined

defines by density a unitary operator from $\mathcal{H}$ to $L^2(\Lambda \times K)$ (still denoted by $\mathcal{U}$) which diagonalizes $\mathcal{A}$ in the sense that[4]

$$\mathcal{A} = \mathcal{U}^* \lambda \mathcal{U}. \tag{2.7}$$

REMARK 2.2. The functions $w_{\lambda,k} \in \mathcal{H}_\uparrow$ are called *generalized eigenfunctions* since they formally satisfy

$$\mathcal{A} w_{\lambda,k} = \lambda w_{\lambda,k} \quad \text{for all } (\lambda, k) \in \Lambda \times K. \tag{2.8}$$

Indeed, suppose that $\mathcal{A}(\mathcal{H}_\downarrow) \subset \mathcal{H}_\downarrow$, which allows us to define the extension of $\mathcal{A}$ to the space $\mathcal{H}_\uparrow$ by

$$\langle v, \mathcal{A} u \rangle := \langle \mathcal{A} v, u \rangle \quad \text{for all } u \in \mathcal{H}_\uparrow \text{ and all } v \in \mathcal{H}_\downarrow.$$

Relation (2.7) can also be written as $\mathcal{U}\mathcal{A} = \lambda \mathcal{U}$, i.e.

$$\langle \mathcal{A} v, w_{\lambda,k} \rangle = \lambda \langle v, w_{\lambda,k} \rangle \quad \text{for all } v \in \mathcal{H}_\downarrow,$$

which is merely (2.8). In practice we shall use this relation only for an intuitive construction of a perturbed generalized basis (see § 2.3). Hence, the restrictive assumption $\mathcal{A}(\mathcal{H}_\downarrow) \subset \mathcal{H}_\downarrow$ is not necessary for our purposes.

The diagonal representation (2.7) provides the key for a functional calculus of $\mathcal{A}$: for every bounded function $f : \Lambda \to \mathbb{C}$, the operator $f(\mathcal{A})$ is then simply given by

$$f(\mathcal{A}) := \mathcal{U}^* f(\lambda) \mathcal{U}. \tag{2.9}$$

We show below two interpretations of this formula, which also reads

$$(f(\mathcal{A})u, v) = (f(\lambda)\mathcal{U}u, \mathcal{U}v)_{\Lambda \times K} = \int_{\Lambda \times K} f(\lambda) \langle u, w_{\lambda,k} \rangle \overline{\langle v, w_{\lambda,k} \rangle} \, \mathrm{d}\lambda \, \mathrm{d}\sigma_k \tag{2.10}$$

if u and v belong to $\mathcal{H}_\downarrow$. By permuting the integral on $\Lambda \times K$ with the second duality product (involving v), we first exhibit the *generalized eigenfunction expansion* of $f(\mathcal{A})$. The permutation of the integral with both duality products then leads to the notion of *integral representations*. However, in the present context, this latter interpretation is but formal (its justification requires more information about $\mathcal{A}$), whereas the functional scheme (2.1) yields a suitable framework for the former interpretation.

2.1.2. *Generalized eigenfunction expansions*

The continuity assumption (2.4) on the $w_{\lambda,k}$ (which does not concern the behaviour of $w_{\lambda,k}$ near the bounds $\lambda_\pm$ of Λ) allows us to construct some *superpositions*

by $(\mathcal{U}v)_n := (v, w_n)$ is a unitary operator from $\mathcal{H}$ to the spectral space

$$\ell^2(\mathbb{N}) := \Big\{ \hat{v} = (\hat{v}_n)_{n \in \mathbb{N}};\ \hat{v}_n \in \mathbb{C} \text{ and } \sum_{n \in \mathbb{N}} |\hat{v}_n|^2 < \infty \Big\},$$

and $\mathcal{U}$ diagonalizes $\mathcal{A}$ in the sense that $\mathcal{A}u = \sum_{n \in \mathbb{N}} \lambda_n (v, w_n) w_n$, which is merely (2.7).

[4]In formula (2.7), we abusively denote λ the operator of multiplication by the scalar function λ in the spectral space $L^2(\Lambda \times K)$.

https://doi.org/10.1017/S0308210506000138 Published online by Cambridge University Press

of these functions by means of vector-valued integrals[5]

$$v' = \int_{\Lambda\times K} \hat{v}_{\lambda,k} w_{\lambda,k}\,\mathrm{d}\lambda\,\mathrm{d}\sigma_k \in \mathcal{H}_\uparrow,$$

provided that $\hat{v} \in L^2(\Lambda\times K)$ is Λ-compactly supported (i.e. $\hat{v}_{\lambda,k} = 0$ for all $k \in K$ and λ in a vicinity of $\lambda_\pm$). It satisfies the Fubini property

$$\langle v', v\rangle = \int_{\Lambda\times K} \hat{v}_{\lambda,k}\langle v, w_{\lambda,k}\rangle\,\mathrm{d}\lambda\,\mathrm{d}\sigma_k \quad \text{for all } v \in \mathcal{H}_\downarrow.$$

By virtue of the definition (2.6) of $\mathcal{U}$, this relation reads $\langle v', v\rangle = (\hat{v}, \mathcal{U}v)_{\Lambda\times K}$, where the right-hand side is merely $(\mathcal{U}^*\hat{v}, v)$. Hence, the above vector-valued integral initially defined in $\mathcal{H}_\uparrow$ actually belongs to $\mathcal{H}$ and

$$\mathcal{U}^*\hat{v} = \int_{\Lambda\times K} \hat{v}_{\lambda,k} w_{\lambda,k}\,\mathrm{d}\lambda\,\mathrm{d}\sigma_k.$$

The expression of $\mathcal{U}^*\hat{v}$ for any $\hat{v} \in L^2(\Lambda\times K)$ follows by approximating $\hat{v}$ by its restrictions to an increasing sequence of compact subsets of Λ whose union covers Λ. Since $\mathcal{U}^*$ is continuous, the corresponding integrals admit a limit in $\mathcal{H}$, which can be seen as a *principal value* at the bounds of Λ, and will be denoted by

$$\mathcal{U}^*\hat{v} = \mathcal{H}\text{-PV}\int_{\Lambda\times K} \hat{v}_{\lambda,k} w_{\lambda,k}\,\mathrm{d}\lambda\,\mathrm{d}\sigma_k \quad \text{for all } \hat{v} \in L^2(\Lambda\times K). \tag{2.11}$$

This formula provides an explicit form of the diagonal representation (2.9):

$$f(\mathcal{A})u = \mathcal{H}\text{-PV}\int_{\Lambda\times K} f(\lambda)\langle u, w_{\lambda,k}\rangle w_{\lambda,k}\,\mathrm{d}\lambda\,\mathrm{d}\sigma_k \quad \text{for all } u \in \mathcal{H}_\downarrow, \tag{2.12}$$

which is the *generalized eigenfunction expansion* of $f(\mathcal{A})$ (for $u \in \mathcal{H}$, the same expression holds if $\langle u, w_{\lambda,k}\rangle$ is replaced by $(\mathcal{U}u)_{\lambda,k}$).

2.1.3. *Integral representations*

Considering tensor products of Hilbert spaces [5, 8] (also called direct products), formula (2.10) can be rewritten as

$$(f(\mathcal{A})u, v) = \int_\Lambda f(\lambda)\left\{\int_K \langle\!\langle \overline{w_{\lambda,k}} \otimes w_{\lambda,k}, \bar{u}\otimes v\rangle\!\rangle\,\mathrm{d}\sigma_k\right\}\mathrm{d}\lambda \quad \text{for all } u, v \in \mathcal{H}_\downarrow,$$

where the double duality product between $\mathcal{H}_\uparrow \otimes \mathcal{H}_\uparrow$ and $\mathcal{H}_\downarrow \otimes \mathcal{H}_\downarrow$ is given by

$$\langle\!\langle u_1 \otimes u_2, v_1 \otimes v_2\rangle\!\rangle := \langle u_1, v_1\rangle\langle u_2, v_2\rangle \quad \text{for all } u_1, u_2 \in \mathcal{H}_\uparrow \text{ and all } v_1, v_2 \in \mathcal{H}_\downarrow.$$

[5]The vector- or operator-valued integrals considered in this paper can be interpreted as Bochner integrals (see, for example, [46]). Here this means that the integrand $\hat{v}_{\lambda,k} w_{\lambda,k}$ can be approximated by a sequence of finitely valued functions $\{\varphi^{(n)}_{\lambda,k}\}_{n\in\mathbb{N}}$ strongly convergent in $\mathcal{H}_\uparrow$ for fixed λ and k, in such a way that

$$\lim_{n\to\infty}\int_{\Lambda\times K} \|\hat{v}_{\lambda,k} w_{\lambda,k} - \varphi^{(n)}_{\lambda,k}\|_\uparrow\,\mathrm{d}\lambda\,\mathrm{d}\sigma_k = 0.$$

In the vector case the Bochner integral offers most of the nice properties of the scalar Lebesgue integral. It obeys a norm-dominated convergence theorem, and allows the permutation of the integral with any continuous operator (the case of linear forms yields the Fubini-like relation which leads to (2.11)).

The permutation of $\int_K$ with $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ then yields the condensed expression

$$(f(\mathcal{A})u, v) = \int_\Lambda f(\lambda)\langle\!\langle \Upsilon_\lambda, \bar{u}\otimes v\rangle\!\rangle\,\mathrm{d}\lambda, \tag{2.13}$$

where

$$\Upsilon_\lambda := \int_K \overline{w_{\lambda,k}}\otimes w_{\lambda,k}\,\mathrm{d}\sigma_k \in \mathcal{H}_\uparrow\otimes\mathcal{H}_\uparrow \quad \text{for } \lambda\in\Lambda. \tag{2.14}$$

Then by permuting similarly the integral on Λ, we obtain

$$(f(\mathcal{A})u, v) = \langle\!\langle \kappa_f, \bar{u}\otimes v\rangle\!\rangle, \quad \text{where } \kappa_f := \mathrm{PV}\int_\Lambda f(\lambda)\Upsilon_\lambda\,\mathrm{d}\lambda. \tag{2.15}$$

This relation is merely the *integral representation* of $f(\mathcal{A})$ which involves its *kernel* κ_f, expressed here by means of the generalized eigenfunctions.

The permutations of integrals which lead to this representation are easily justified if f is compactly supported. Indeed, $\overline{w_{\lambda,k}}\otimes w_{\lambda,k}$ is a continuous family of $\mathcal{H}_\uparrow\otimes\mathcal{H}_\uparrow$ (thus uniformly continuous on $\Lambda'\times K$ for every compact $\Lambda'\subset\Lambda$), so κ_f belongs to $\mathcal{H}_\uparrow\otimes\mathcal{H}_\uparrow$. But, apart from this case, the justification of the last permutation requires a precise knowledge of the behaviour of Υ_λ near the bounds $\lambda_\pm$ of Λ. This behaviour determines the singularity of the kernel: in general the principal value must be understood in a weaker topology than that of $\mathcal{H}_\uparrow\otimes\mathcal{H}_\uparrow$. For instance, the kernel of the identity operator is the diagonal Dirac measure (given by $\langle\!\langle \delta_{\mathrm{diag}}, \bar{u}\otimes v\rangle\!\rangle = (u, v)$). In § 3.2 we shall deal with the case of the Green function of our free water-wave problem, i.e. the kernel of its resolvent (which belongs to $\mathcal{H}_\uparrow\otimes\mathcal{H}_\uparrow$ in this particular two-dimensional situation).

2.2. Application to time–frequency analysis

2.2.1. *An abstract wave equation*

In order to illustrate the consequences of the preceding definition of a generalized spectral basis, we consider in this section an abstract wave equation which consists in finding $u = u(t)\in\mathcal{H}$ such that

$$\mathrm{d}_t^2 u + \mathcal{A}u = 0 \quad \text{for } t > 0, \tag{2.16}$$

and which satisfies the initial conditions

$$u(0) = g \quad \text{and} \quad \mathrm{d}_t u(0) = h, \tag{2.17}$$

for some real data g and h in $\mathcal{H}$. As before, $\mathcal{A}$ is supposed bounded and self-adjoint in $\mathcal{H}$. Here we assume in addition that $\mathcal{A}$ is real, positive and invertible. In these conditions the solution to (2.16), (2.17) reads

$$u(t) = \cos(\mathcal{A}^{1/2}t)g + \mathcal{A}^{-1/2}\sin(\mathcal{A}^{1/2}t)h,$$

or equivalently,

$$u(t) = \mathrm{Re}\{\exp(-\mathrm{i}\mathcal{A}^{1/2}t)u^{(0)}\}, \quad \text{where } u^{(0)} := g + \mathrm{i}\mathcal{A}^{-1/2}h. \tag{2.18}$$

https://doi.org/10.1017/S0308210506000138 Published online by Cambridge University Press

Definition 2.1 offers an explicit form of operator $\exp(-\mathrm{i}\mathcal{A}^{1/2}t) = \mathcal{U}^* \exp(-\mathrm{i}\sqrt{\lambda}t)\mathcal{U}$ (which is unitary since the multiplication by $\exp(-\mathrm{i}\sqrt{\lambda}t)$ is unitary in $L^2(\Lambda \times K)$). Indeed, in this case formula (2.9) becomes

$$u(t) = \mathrm{Re}\left\{\mathcal{H}\text{-PV}\int_{\Lambda\times K} \langle u^{(0)}, w_{\lambda,k}\rangle w_{\lambda,k} \mathrm{e}^{-\mathrm{i}\sqrt{\lambda}t}\,\mathrm{d}\lambda\,\mathrm{d}\sigma_k\right\}, \tag{2.19}$$

which amounts to interpreting the transient wave $u(t)$ as a continuous superposition of the time-harmonic waves $w_{\lambda,k}\exp(-\mathrm{i}\sqrt{\lambda}t)$.

Consider now the case of a time-harmonic excitation starting at $t = 0$ which generates a time-dependent wave $u = u(t) \in \mathcal{H}$ solution to

$$\mathrm{d}_t^2 u + \mathcal{A}u = f\mathrm{e}^{-\mathrm{i}\sqrt{\lambda_0}t}, \tag{2.20}$$

for some given $\lambda_0 \in \Lambda$ and $f \in \mathcal{H}$, and which satisfies initial conditions such as (2.17). What can be said about the behaviour of $u(t)$ at large time? Does it reach asymptotically a time-harmonic regime? We first show an essential consequence of definition 2.1 which will yield the answer.

2.2.2. *The limiting absorption principle*

By definition the spectrum of $\mathcal{A}$ is the complement of the set composed of the $\zeta \in \mathbb{C}$ such that the resolvent of $\mathcal{A}$,

$$\mathcal{R}_\zeta := (\mathcal{A} - \zeta)^{-1}, \tag{2.21}$$

is a bounded operator in $\mathcal{H}$. Definition 2.1 contains precise information about the behaviour of $\mathcal{R}_\zeta$ in the vicinity of Λ (which will justify in particular the fact that $\bar{\Lambda}$ is the spectrum of $\mathcal{A}$). This is the object of the so-called *limiting absorption principle*, which concerns the existence of both one-sided limits

$$\mathcal{R}_\lambda^\pm := \lim_{\mathbb{C}^\pm \ni \zeta \to \lambda \in \Lambda} \mathcal{R}_\zeta \quad \text{with } \mathbb{C}^\pm := \{\zeta \in \mathbb{C};\ \pm\,\mathrm{Im}\,\zeta > 0\}, \tag{2.22}$$

for a suitable topology. Indeed, such limits cannot be bounded in $\mathcal{H}$ (otherwise λ would not be in the spectrum of $\mathcal{A}$). The functional scheme (2.1) furnishes a weaker topology: by virtue of (2.2), we can write

$$\langle \mathcal{R}_\zeta u, v\rangle = (\mathcal{R}_\zeta u, v) \quad \text{for all } u, v \in \mathcal{H}_\downarrow,$$

which amounts to considering $\mathcal{R}_\zeta$ in $\mathcal{B}(\mathcal{H}_\downarrow, \mathcal{H}_\uparrow)$, the space of bounded operators from $\mathcal{H}_\downarrow$ to $\mathcal{H}_\uparrow$.

Formula (2.9) applied to $f(\lambda) = (\lambda - \zeta)^{-1}$ yields the generalized eigenfunction expansion of $\mathcal{R}_\zeta$:

$$\mathcal{R}_\zeta u = \mathcal{H}\text{-PV}\int_{\Lambda\times K} \frac{\langle u, w_{\lambda,k}\rangle w_{\lambda,k}}{\lambda - \zeta}\,\mathrm{d}\lambda\,\mathrm{d}\sigma_k \quad \text{for all } u \in \mathcal{H}_\downarrow. \tag{2.23}$$

The regularity assumption (2.5) allows us to exhibit the limit of this expression when $\zeta \in \mathbb{C}^\pm$ tends to some $\lambda_0 \in \Lambda$ as follows.

Proposition 2.3 (limiting absorption principle). *In the context of definition 2.1, for every $\lambda_0 \in \Lambda$, the resolvent $\mathcal{R}_\zeta$ of $\mathcal{A}$ has one-sided limits (2.22) for the uniform*

https://doi.org/10.1017/S0308210506000138 Published online by Cambridge University Press

topology[6] *of* $\mathcal{B}(\mathcal{H}_\downarrow,\mathcal{H}_\uparrow)$. *These limits (which are locally Hölder continuous on* Λ*) are given by*

$$\mathcal{R}^{\pm}_{\lambda_0}u = \mathcal{H}_\uparrow\text{-}PV\int_{(\Lambda\setminus\{\lambda_0\})\times K}\frac{\langle u,w_{\lambda,k}\rangle w_{\lambda,k}}{\lambda-\lambda_0}\,\mathrm{d}\lambda\,\mathrm{d}\sigma_k \pm \mathrm{i}\pi\int_K\langle u,w_{\lambda_0,k}\rangle w_{\lambda_0,k}\,\mathrm{d}\sigma_k \quad \textit{for all } u\in\mathcal{H}_\downarrow, \tag{2.24}$$

where the principal value occurs at $\lambda_\pm$ *as well as* λ_0.

Let us point out that this formulation of the limiting absorption principle is not optimal in general. In many situations the limits $\mathcal{R}^{\pm}_{\lambda_0}u$ exist for a finer topology than the uniform topology of $\mathcal{B}(\mathcal{H}_\downarrow,\mathcal{H}_\uparrow)$. The latter derives naturally from our definition of a spectral basis, whereas $\mathcal{R}^{\pm}_{\lambda_0}u$ generally has a stronger decay at infinity than the functions of $\mathcal{H}_\uparrow$.

Proof. In order to split the difficulties related to the principal values at $\lambda_\pm$ and λ_0, choose a compact $\Lambda'\subset\Lambda$ containing λ_0 as an interior point, and consider the following decomposition of the resolvent:

$$\mathcal{R}_\zeta = \mathcal{R}'_\zeta + \mathcal{R}''_\zeta,$$

where $\mathcal{R}'_\zeta$ and $\mathcal{R}''_\zeta$ are spectrally truncated parts of the resolvent defined by

$$\mathcal{R}'_\zeta := \mathcal{U}^*\frac{\chi_{\Lambda'}(\lambda)}{\lambda-\zeta}\mathcal{U} \quad\text{and}\quad \mathcal{R}''_\zeta := \mathcal{U}^*\frac{\chi_{\Lambda\setminus\Lambda'}(\lambda)}{\lambda-\zeta}\mathcal{U},$$

where $\chi_{\Lambda'}$ and $\chi_{\Lambda\setminus\Lambda'}$ denote the characteristic functions of Λ' and $\Lambda\setminus\Lambda'$, respectively.

The family of functions $f_\zeta(\lambda)=(\lambda-\zeta)^{-1}\chi_{\Lambda\setminus\Lambda'}(\lambda)$ is indefinitely differentiable with respect to ζ when the latter lies in a vicinity of λ_0, uniformly with respect to $\lambda\in\Lambda$. Hence, $\mathcal{R}''_\zeta$ is also infinitely differentiable near λ_0 for the topology of $\mathcal{B}(\mathcal{H})$, thus *a fortiori* for that of $\mathcal{B}(\mathcal{H}_\downarrow,\mathcal{H}_\uparrow)$. At point λ_0, its limit value is simply given by

$$\mathcal{R}''_{\lambda_0}u = \mathcal{H}\text{-PV}\int_{(\Lambda\setminus\Lambda')\times K}\frac{\langle u,w_{\lambda,k}\rangle w_{\lambda,k}}{\lambda-\lambda_0}\,\mathrm{d}\lambda\,\mathrm{d}\sigma_k \quad\text{for all } u\in\mathcal{H}_\downarrow, \tag{2.25}$$

where the principal value is taken at the bounds $\lambda_\pm$ of Λ.

It remains do deal with $\mathcal{R}'_\zeta$, given by (2.23), where Λ is replaced by Λ', which allows us to get rid of the principal value. The existence of the one-sided limits depends on the regularity of the integrand with respect to λ. In order to point out this dependence, we introduce the family of operators $\mathcal{U}_\lambda$, for $\lambda\in\Lambda$, simply obtained by fixing the variable λ in the definition of $\mathcal{U}$:

$$(\mathcal{U}_\lambda v)_k := (\mathcal{U}v)_{\lambda,k} = \langle v,w_{\lambda,k}\rangle \quad\text{for all } v\in\mathcal{H}_\downarrow. \tag{2.26}$$

[6] More explicitly,

$$\lim_{\mathbb{C}^\pm\ni\zeta\to\lambda_0\in\Lambda}\ \sup_{v\in\mathcal{H}_\downarrow\setminus\{0\}}\frac{\|\mathcal{R}_\zeta v-\mathcal{R}^{\pm}_{\lambda_0}v\|_\uparrow}{\|v\|_\downarrow}=0.$$

The continuity assumption (2.4) implies that, for every $\lambda \in \Lambda$, the operator $\mathcal{U}_\lambda$ is continuous from $\mathcal{H}_\downarrow$ to $L^2(K)$. As for formula (2.11), it is easy to see that its adjoint is the operator of superposition of the generalized eigenfunctions associated with λ:

$$\mathcal{U}_\lambda^* f = \int_K f_k w_{\lambda,k} \, \mathrm{d}\sigma_k \in \mathcal{H}_\uparrow \quad \text{for all } f \in L^2(K).$$

Hence, $\mathcal{R}'_\zeta$ reads

$$\mathcal{R}'_\zeta = \int_{\Lambda'} \frac{\mathcal{U}_\lambda^* \mathcal{U}_\lambda}{\lambda - \zeta} \, \mathrm{d}\lambda \in \mathcal{B}(\mathcal{H}_\downarrow, \mathcal{H}_\uparrow).$$

Notice then that the local Hölder continuity (2.5) of the $w_{\lambda,k}$ implies the same regularity for $\mathcal{U}_\lambda$ in the uniform topology of $\mathcal{B}(\mathcal{H}_\downarrow, L^2(K))$. Indeed, by the Schwarz inequality, we have

$$\sup_{v \in \mathcal{H}_\downarrow \setminus \{0\}} \frac{\|\mathcal{U}_\lambda v - \mathcal{U}_{\lambda'} v\|_K}{\|v\|_\downarrow} \leqslant C(\Lambda')|\lambda - \lambda'|^\alpha \quad \text{for all } \lambda, \lambda' \in \Lambda'.$$

And of course the same property holds for $\mathcal{U}_\lambda^* \in \mathcal{B}(L^2(K), \mathcal{H}_\uparrow)$, and thus also for $\mathcal{U}_\lambda^* \mathcal{U}_\lambda \in \mathcal{B}(\mathcal{H}_\downarrow, \mathcal{H}_\uparrow)$. Hence, the Plemelj formula (also called the Sokhotskyi formula [23]) gives an explicit form of the limits of the above expression of $\mathcal{R}'_\zeta$, as well as the local Hölder continuity of these limits:

$$\mathcal{R}'^{\pm}_{\lambda_0} = \mathcal{B}(\mathcal{H}_\downarrow, \mathcal{H}_\uparrow)\text{-PV} \int_{\Lambda' \setminus \{\lambda_0\}} \frac{\mathcal{U}_\lambda^* \mathcal{U}_\lambda}{\lambda - \lambda_0} \, \mathrm{d}\lambda \pm \mathrm{i}\pi \mathcal{U}_{\lambda_0}^* \mathcal{U}_{\lambda_0}, \tag{2.27}$$

where the principal value is taken at λ_0. Combining (2.25) and (2.27) yields (2.24). □

2.2.3. *Physical significance*

We now come back to our abstract wave equation (2.20). The existence of the one-sided limits $\mathcal{R}^\pm_\lambda$ actually guarantees the asymptotic time-harmonic behaviour of its solution. More precisely $\mathcal{R}^+_\lambda$ (respectively, $\mathcal{R}^-_\lambda$) describes the *outgoing* (respectively, *incoming*) regime of propagation which is reached when $t \to +\infty$ (respectively, $-\infty$). This is the object of the *limiting amplitude principle*. We give here without proof[7] a weak formulation of this property that will not be used in the following: we want only to justify the physical interpretation of $\mathcal{R}^\pm_\lambda$.

Proposition 2.4 (limiting amplitude principle). *In the context of definition 2.1, let λ_0 be an interior point of Λ, and let $f \in \mathcal{H}_\downarrow$ chosen such that $\mathcal{U}f$ is Λ-compactly supported.*[8] *Then for each initial datum $u(0)$ and $\mathrm{d}_t u(0)$ in $\mathcal{H}$, the solution $u(t)$ to (2.20) has the following asymptotic behaviour at large time:*

$$\lim_{t \to \pm\infty} \langle u(t) - \mathrm{e}^{-\mathrm{i}\sqrt{\lambda_0} t} \mathcal{R}^\pm_{\lambda_0} f, v \rangle = 0 \quad \textit{for all } v \in \mathcal{H}_\downarrow.$$

[7] Proposition 2.4 is easily proved following the idea of Eidus [14] (see also [36]).

[8] If f is not Λ-compactly supported, the limiting amplitude principle holds [14, 36] with an additional condition on the behaviour of $\mathcal{R}_\zeta$ near the bounds of Λ.

https://doi.org/10.1017/S0308210506000138 Published online by Cambridge University Press

In other words, forgetting the factor $\exp(-\mathrm{i}\sqrt{\lambda_0}t)$, we see that $\mathcal{R}^{\pm}_{\lambda_0}f$ represents the time-harmonic outgoing (+) or incoming (−) wave generated by the excitation f. The fact that $\mathcal{R}^{\pm}_{\lambda_0} \in \mathcal{B}(\mathcal{H}_{\downarrow}, \mathcal{H}_{\uparrow})$ means that we consider localized excitations $f \in \mathcal{H}_{\downarrow}$, and the corresponding responses are non-localized but have a finite weighted energy.

2.2.4. *A 'stable' form of limiting absorption*

We see from (2.24) that

$$\operatorname{Im}\langle \mathcal{R}^{\pm}_{\lambda_0}u, u\rangle = \pm\pi \int_K |\langle u, w_{\lambda_0,k}\rangle|^2 \, \mathrm{d}\sigma_k \quad \text{for all } u \in \mathcal{H}_{\downarrow}. \tag{2.28}$$

This quantity may be interpreted as the mean energy flux[9] which is necessary to maintain the time-harmonic wave $\mathcal{R}^{\pm}_{\lambda_0}u$ (it is positive for outgoing waves, and negative for incoming waves). If this flux vanishes, can we assert that the corresponding outgoing or incoming wave has a finite energy? Such a property plays a leading role in the perturbation approach of § 2.3. In short, it guarantees that the limiting absorption holds for localized enough perturbations of the propagative medium. Thus, we give the following definition.

Definition 2.5. In the context of definition 2.1, $\mathcal{A}$ is said to satisfy a stable limiting absorption principle if the limits $\mathcal{R}^{\pm}_{\lambda}$ of proposition 2.3 satisfy in addition the 'stability condition'

$$u \in \mathcal{H}_{\downarrow} \quad \text{and} \quad \operatorname{Im}\langle \mathcal{R}^{\pm}_{\lambda}u, u\rangle = 0 \quad \Longrightarrow \quad \mathcal{R}^{\pm}_{\lambda}u \in \mathcal{H}, \tag{2.29}$$

for all $\lambda \in \Lambda$.

From a physical point of view, this condition signifies that if a time-harmonic wave does not produce or use any energy on average in time, then the total energy of the wave is necessarily *finite*. To a certain extent, this means that there is a 'no wave's land' between two kinds of time-harmonic waves maintained by a localized excitation: *stationary* waves of finite energy (which are both incoming and outgoing), and *propagative* incoming or outgoing waves of infinite energy. And the energy flux is the very tool to distinguish between them.

[9] In equation (2.20), the 'physical' wave is represented by the real part $\varphi(t)$ of $u(t)$, which satisfies

$$\mathrm{d}_t^2\varphi + \mathcal{A}\varphi = g(t) := \operatorname{Re}\{f\mathrm{e}^{-\mathrm{i}\sqrt{\lambda_0}t}\}.$$

Taking the inner product of this equation with $\mathrm{d}_t\varphi$ yields the energy relation

$$\mathrm{d}_t\{\tfrac{1}{2}\|\mathrm{d}_t\varphi\|^2 + \tfrac{1}{2}(\mathcal{A}\varphi, \varphi)\} = (g(t), \mathrm{d}_t\varphi),$$

where the right-hand side denotes the energy flux brought to the system. If $f \in \mathcal{H}_{\downarrow}$, we know from proposition 2.4 that the behaviour of this flux at large time is given by

$$\mathcal{F}^{\pm}(t) := \langle \operatorname{Re}\{f\mathrm{e}^{-\mathrm{i}\sqrt{\lambda_0}t}\}, \ \mathrm{d}_t \operatorname{Re}\{\mathcal{R}^{\pm}_{\lambda_0}f\mathrm{e}^{-\mathrm{i}\sqrt{\lambda_0}t}\}\rangle,$$

whose average on one period reads

$$\frac{\sqrt{\lambda_0}}{2\pi}\int_0^{2\pi/\sqrt{\lambda_0}} \mathcal{F}^{\pm}(t)\,\mathrm{d}t = \frac{\sqrt{\lambda_0}}{2}\operatorname{Im}\langle \mathcal{R}^{\pm}_{\lambda_0}f, f\rangle.$$

The mathematical interpretation of condition (2.29) concerns the link between $\mathcal{R}_\lambda^\pm$ and the unbounded inverse $(\mathcal{A}-\lambda)^{-1}$ of $\mathcal{A}-\lambda$, whose domain is the range $\mathrm{R}(\mathcal{A}-\lambda)$ of $\mathcal{A}-\lambda$:

$$(\mathcal{A}-\lambda)^{-1}(\mathcal{A}-\lambda)=\mathrm{Id}_{\mathcal{H}} \quad \text{and} \quad (\mathcal{A}-\lambda)(\mathcal{A}-\lambda)^{-1}=\mathrm{Id}_{\mathrm{R}(\mathcal{A}-\lambda)}.$$

The following lemma shows that both one-sided limits $\mathcal{R}_\lambda^\pm$ may be seen as extensions of $(\mathcal{A}-\lambda)^{-1}$.

LEMMA 2.6. *In the context of definition 2.1, if* $u\in\mathrm{R}(\mathcal{A}-\lambda)\cap\mathcal{H}_\downarrow$, *then*

$$\mathcal{R}_\lambda^\pm u=(\mathcal{A}-\lambda)^{-1}u. \tag{2.30}$$

Proof. For $u\in\mathrm{R}(\mathcal{A}-\lambda_0)\cap\mathcal{H}_\downarrow$, define $v:=(\mathcal{A}-\lambda_0)^{-1}u\in\mathcal{H}$, which satisfies $(\mathcal{A}-\lambda_0)v=u$. Applying $\mathcal{R}_{\lambda_0\pm\mathrm{i}\varepsilon}$ to this relation yields

$$v\pm\mathrm{i}\varepsilon\mathcal{R}_{\lambda_0\pm\mathrm{i}\varepsilon}v=\mathcal{R}_{\lambda_0\pm\mathrm{i}\varepsilon}u.$$

In order to obtain the limit of this equality as $\varepsilon\to0^+$, consider the quantity $v_\varepsilon^\pm:=\varepsilon\mathcal{R}_{\lambda_0\pm\mathrm{i}\varepsilon}v$, which can be written $v_\varepsilon^\pm=f_\varepsilon^\pm(\mathcal{A})v$, where

$$f_\varepsilon^\pm(\lambda):=\frac{\varepsilon}{\lambda-\lambda_0\mp\mathrm{i}\varepsilon}.$$

Hence, by (2.9),

$$\|v_\varepsilon^\pm\|^2=\|f_\varepsilon^\pm(\lambda)\mathcal{U}v\|^2_{\Lambda\times K}=\int_{\Lambda\times K}|f_\varepsilon^\pm(\lambda)|^2|(\mathcal{U}v)_{\lambda,k}|^2\,\mathrm{d}\lambda\,\mathrm{d}\sigma_k.$$

Noting that $\lim_{\varepsilon\to0}f_\varepsilon^\pm(\lambda)=0$ for all $\lambda\neq\lambda_0$ and that $|f_\varepsilon^\pm(\lambda)|\leqslant1$, we deduce by the Lebesgue theorem that $\lim_{\varepsilon\to0}\|v_\varepsilon^\pm\|=0$. The conclusion follows from the limiting absorption principle. □

The stability condition (2.29) furnishes a sufficient condition for some $u\in\mathcal{H}_\downarrow$ to belong to the range of $\mathcal{A}-\lambda$ (so that (2.30) holds).

LEMMA 2.7. *In the context of definition 2.5, for* $u\in\mathcal{H}_\downarrow$ *we have*

$$\mathrm{Im}\langle\mathcal{R}_\lambda^\pm u,u\rangle=0 \iff u\in\mathrm{R}(\mathcal{A}-\lambda). \tag{2.31}$$

Proof. First note that if $u\in\mathrm{R}(\mathcal{A}-\lambda_0)\cap\mathcal{H}_\downarrow$, then $v:=(\mathcal{A}-\lambda_0)^{-1}u$ satisfies $(\mathcal{A}-\lambda_0)v=u$, which shows that

$$\mathrm{Im}((\mathcal{A}-\lambda_0)^{-1}u,u)=\mathrm{Im}\{(v,\mathcal{A}v)-\lambda_0\|v\|^2\}=0.$$

Hence, $\mathrm{Im}\langle\mathcal{R}_{\lambda_0}^\pm u,u\rangle=0$ by lemma 2.6.

The converse is based on the stability condition (2.29). Let $u\in\mathcal{H}_\downarrow$ such that $\mathrm{Im}\langle\mathcal{R}_{\lambda_0}^\pm u,u\rangle=0$. From (2.28), this is equivalent to $\mathcal{U}_{\lambda_0}u=0$ (where the operators $\mathcal{U}_\lambda$ are defined in (2.26)). In this situation, the principal value at λ_0 in the expression (2.24) of $\mathcal{R}_{\lambda_0}^\pm u$ can be removed. As this quantity here belongs to $\mathcal{H}$, we have

$$(\mathcal{R}_{\lambda_0}^\pm u,v)=\int_\Lambda\frac{(\mathcal{U}_\lambda u,\mathcal{U}_\lambda v)_K}{\lambda-\lambda_0}\,\mathrm{d}\lambda \quad \text{for all } v\in\mathcal{H}_\downarrow,$$

https://doi.org/10.1017/S0308210506000138 Published online by Cambridge University Press

since the Hölder continuity of $\mathcal{U}_\lambda$ implies that, near λ_0,

$$\|\mathcal{U}_\lambda u\|_K \leqslant C|\lambda - \lambda_0|^\alpha. \tag{2.32}$$

As $\mathcal{H}_\downarrow$ is dense in $\mathcal{H}$, the above expression remains valid for all $v \in \mathcal{H}$ such that $\|\mathcal{U}_\lambda v\|_K$ is continuous near λ_0, in particular $\mathcal{A}v$ with $v \in \mathcal{H}_\downarrow$ since $\mathcal{U}_\lambda \mathcal{A} v = \lambda \mathcal{U}_\lambda v$. We deduce that

$$(\mathcal{R}^\pm_{\lambda_0} u, (\mathcal{A} - \lambda_0)v) = \int_\Lambda (\mathcal{U}_\lambda u, \mathcal{U}_\lambda v)_K \,\mathrm{d}\lambda = (u, v) \quad \text{for all } v \in \mathcal{H}_\downarrow,$$

which shows that $(\mathcal{A} - \lambda_0)\mathcal{R}^\pm_{\lambda_0} u = u$, and hence u belongs to $\mathrm{R}(\mathcal{A} - \lambda_0)$. □

We end this section by a simple criterion for the stability condition, which is related to the α-Hölder continuity (2.5) of the $w_{\lambda,k}$.

Proposition 2.8. *In the context of definition 2.1, if $\alpha > \frac{1}{2}$, then condition (2.29) holds.*

Proof. Let $\lambda_0 \in \Lambda$ and $u \in \mathcal{H}_\downarrow$ such that $\operatorname{Im}\langle \mathcal{R}^\pm_{\lambda_0} u, u\rangle = 0$. As in the proof of lemma 2.7, we have $\mathcal{U}_{\lambda_0} u = 0$, and thus $\mathcal{U}_\lambda u$ satisfies (2.32). As a consequence, when $\zeta = \lambda_0 \pm \mathrm{i}\varepsilon$ tends to λ_0, the one-sided limits of $\mathcal{U}\mathcal{R}_\zeta u = (\lambda - \zeta)^{-1}\mathcal{U}u$ exist in $L^2(\Lambda \times K)$ (and are equal) since

$$|\lambda - \zeta|^{-1}|\lambda - \lambda_0|^\alpha \leqslant |\lambda - \lambda_0|^{-1+\alpha},$$

which belongs to $L^2(\Lambda)$ for $\alpha > \frac{1}{2}$. As $\mathcal{U}$ is unitary, $\mathcal{R}_\zeta u$ has a limit in $\mathcal{H}$, which is merely $\mathcal{R}^+_{\lambda_0} u = \mathcal{R}^-_{\lambda_0} u$. □

2.3. Perturbation of a generalized spectral basis

We show now how to construct under suitable assumptions generalized spectral bases for an operator $\mathcal{A}$ considered as a perturbation of a 'simpler' operator $\tilde{\mathcal{A}}$, called *free*, where both $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are assumed bounded self-adjoint operators in the same space $\mathcal{H}$. We define their difference by

$$\mathcal{D} := \mathcal{A} - \tilde{\mathcal{A}}, \tag{2.33}$$

which is clearly bounded and self-adjoint in $\mathcal{H}$. We shall assume that we know a generalized spectral basis $\{\tilde{w}_{\lambda,k}\}$ for $\tilde{\mathcal{A}}$ defined in the context of definition 2.1: the transformation

$$(\tilde{\mathcal{U}}v)_{\lambda,k} := \langle v, \tilde{w}_{\lambda,k}\rangle \quad \text{for all } v \in \mathcal{H}_\downarrow \tag{2.34}$$

defines by density a unitary operator from $\mathcal{H}$ to $L^2(\Lambda \times K)$ which diagonalizes $\tilde{\mathcal{A}}$.

2.3.1. *Intuitive construction*

In order to find a generalized spectral basis for $\mathcal{A}$, we search solutions $w_{\lambda,k}$ to (2.8) in the form $w_{\lambda,k} = \tilde{w}_{\lambda,k} + \dot{w}_{\lambda,k}$. Using the above definition of $\mathcal{D}$ and the fact that $(\tilde{\mathcal{A}} - \lambda)\tilde{w}_{\lambda,k} = 0$, we see that the perturbation term $\dot{w}_{\lambda,k}$ must satisfy

$$(\mathcal{A} - \lambda)\dot{w}_{\lambda,k} = -\mathcal{D}\tilde{w}_{\lambda,k}. \tag{2.35}$$

But this equation is ill-posed if λ belongs to the spectrum of $\mathcal{A}$, which is precisely the situation we are interested in. A roundabout way to solve it consists in replacing first $\mathcal{A}-\lambda$ by $\mathcal{A}-\zeta$ for some $\zeta\in\mathbb{C}^{\pm}$, solving the corresponding equation, and then letting $\zeta\to\lambda$. This is precisely the purpose of the limiting absorption principle. If the one-sided limits (2.22) of the resolvent of $\mathcal{A}$ exist, we can consider the outgoing and incoming perturbations $\dot{w}^{\pm}_{\lambda,k}$ of the free generalized eigenfunctions $\tilde{w}_{\lambda,k}$ given by

$$\dot{w}^{\pm}_{\lambda,k} := -\mathcal{R}^{\pm}_{\lambda}\mathcal{D}\tilde{w}_{\lambda,k},$$

which formally satisfy (2.35). In these conditions, we are led to two families of generalized eigenfunctions for $\mathcal{A}$:

$$w^{\pm}_{\lambda,k} := \tilde{w}_{\lambda,k} + \dot{w}^{\pm}_{\lambda,k} = (\mathrm{Id} - \mathcal{R}^{\pm}_{\lambda}\mathcal{D})\tilde{w}_{\lambda,k}. \tag{2.36}$$

In the context of wave propagation, $\tilde{w}_{\lambda,k}$ can be interpreted as an *incident* time-harmonic wave and $\dot{w}^{\pm}_{\lambda,k}$ then represents the corresponding *scattered* outgoing or incoming wave, that is, the response of the perturbation of the propagative medium to the incident wave. Do these families define generalized spectral bases for $\mathcal{A}$? We present below sufficient conditions that guarantee such a property.

2.3.2. *Compact perturbation of the free problem*

In § 2.2 we deduce the limiting absorption principle from the *a priori* knowledge of a generalized spectral basis. Here the question is reversed: the above construction of perturbed generalized eigenfunctions is subject to the existence of the one-sided limits $\mathcal{R}^{\pm}_{\lambda}$. The idea is to derive them from the following relation between $\mathcal{R}_{\zeta}$ and the free resolvent $\tilde{\mathcal{R}}_{\zeta} := (\tilde{\mathcal{A}}-\zeta)^{-1}$:

$$\mathcal{R}_{\zeta} - \tilde{\mathcal{R}}_{\zeta} = -\mathcal{R}_{\zeta}\mathcal{D}\tilde{\mathcal{R}}_{\zeta} = -\tilde{\mathcal{R}}_{\zeta}\mathcal{D}\mathcal{R}_{\zeta} \quad \text{for all } \zeta\in\mathbb{C}^{\pm}, \tag{2.37}$$

which is easily deduced from the definition (2.33) of $\mathcal{D}$. This relation yields in particular that

$$\mathcal{R}_{\zeta} = \tilde{\mathcal{R}}_{\zeta}(\mathrm{Id} + \mathcal{D}\tilde{\mathcal{R}}_{\zeta})^{-1}, \tag{2.38}$$

which shows that the existence of $\mathcal{R}^{\pm}_{\lambda}$ depends, on the one hand, on a *limiting absorption principle* for the free problem, i.e. the existence of the limits $\tilde{\mathcal{R}}^{\pm}_{\lambda}$ and, on the other hand, on an invertibility condition for $\mathrm{Id}+\mathcal{D}\tilde{\mathcal{R}}^{\pm}_{\lambda}$ (sometimes called a 'division' property).

Proposition 2.3 provides the existence of the one-sided limits $\tilde{\mathcal{R}}^{\pm}_{\lambda}$ as operators from $\mathcal{H}_{\downarrow}$ to $\mathcal{H}_{\uparrow}$. Hence, the formal limit of relation (2.38) will make sense if the invertibility of $\mathrm{Id}+\mathcal{D}\tilde{\mathcal{R}}^{\pm}_{\lambda}$ occurs in $\mathcal{H}_{\downarrow}$. We must thus assume that $\mathcal{D}$ extends by density to an operator of $\mathcal{B}(\mathcal{H}_{\uparrow},\mathcal{H}_{\downarrow})$, which signifies that $\mathcal{A}$ differs from $\tilde{\mathcal{A}}$ by a 'localizing' operator. But this continuity is not sufficient to provide the invertibility of $\mathrm{Id}+\mathcal{D}\tilde{\mathcal{R}}^{\pm}_{\lambda}$. With the following definition of a *compact perturbation*, the question falls within the Fredholm alternative.

Definition 2.9. $\mathcal{A}$ is called a compact perturbation of $\tilde{\mathcal{A}}$ in the functional scheme (2.1) if $\mathcal{D}$ extends by density to a bounded operator from $\mathcal{H}_{\uparrow}$ to $\mathcal{H}_{\downarrow}$, and $\mathcal{D}\tilde{\mathcal{R}}^{\pm}_{\lambda}$ are compact operators in $\mathcal{H}_{\downarrow}$ for every $\lambda\in\Lambda$.

https://doi.org/10.1017/S0308210506000138 Published online by Cambridge University Press

In this context, it remains to find the values of λ for which the homogeneous equations

$$u^{\pm} + \mathcal{D}\tilde{\mathcal{R}}_{\lambda}^{\pm} u^{\pm} = 0 \quad \text{with } u^{\pm} \in \mathcal{H}_{\downarrow}, \tag{2.39}$$

admit a non-zero solution.

Proposition 2.10. *Assume that the free operator $\tilde{\mathcal{A}}$ satisfies the stable limiting absorption principle of definition 2.5. Then, for every compact perturbation $\mathcal{A}$ of $\tilde{\mathcal{A}}$, the operator $\mathrm{Id} + \mathcal{D}\tilde{\mathcal{R}}_{\lambda}^{\pm}$ is invertible in $\mathcal{H}_{\downarrow}$ if and only if λ is not an eigenvalue of $\mathcal{A}$.*

Proof. Let us first assume that λ is an eigenvalue of $\mathcal{A}$. Then there exists a non-zero $w \in \mathcal{H}$ such that $\mathcal{A}w = \lambda w$, which shows by (2.33) that

$$(\tilde{\mathcal{A}} - \lambda)w = -\mathcal{D}w.$$

As a consequence $u := \mathcal{D}w \in \mathcal{H}_{\downarrow}$ belongs to the range of $\tilde{\mathcal{A}} - \lambda$ and lemma 2.6 thus shows that $w = -\tilde{\mathcal{R}}_{\lambda}^{\pm} u$. Applying $\mathcal{D}$ to this equality gives

$$(\mathrm{Id} + \mathcal{D}\tilde{\mathcal{R}}_{\lambda}^{\pm})u = 0,$$

where $u \neq 0$ (otherwise w would also vanish). Hence, $\mathrm{Id} + \mathcal{D}\tilde{\mathcal{R}}_{\lambda}^{\pm}$ is not invertible.

The converse is based on the stability property (2.29) for the free resolvent $\tilde{\mathcal{R}}_{\lambda}^{\pm}$. Assume that $u^{\pm} \in \mathcal{H}_{\downarrow}$ is a non-zero solution to (2.39). The function $w^{\pm} := \tilde{\mathcal{R}}_{\lambda}^{\pm} u^{\pm} \in \mathcal{H}_{\uparrow}$ then satisfies

$$w^{\pm} = -\tilde{\mathcal{R}}_{\lambda}^{\pm} \mathcal{D} w^{\pm}, \tag{2.40}$$

from which we infer that

$$\langle \tilde{\mathcal{R}}_{\lambda}^{\pm} \mathcal{D} w^{\pm}, \mathcal{D} w^{\pm} \rangle = -\langle w^{\pm}, \mathcal{D} w^{\pm} \rangle \in \mathbb{R},$$

since $\mathcal{D}$ is the extension in $\mathcal{H}_{\uparrow}$ of a self-adjoint operator in $\mathcal{H}$. The stability property (2.29) for the free resolvent then shows that $\tilde{\mathcal{R}}_{\lambda}^{\pm} \mathcal{D} w^{\pm} \in \mathcal{H}$, so $w^{\pm} \in \mathcal{H}$ by (2.40). Moreover, lemma 2.7 tells us that $\mathcal{D}w^{\pm}$ belongs to the range of $\tilde{\mathcal{A}} - \lambda$, which allows us to use lemma 2.6. Applying $\tilde{\mathcal{A}} - \lambda$ to (2.40) yields

$$(\tilde{\mathcal{A}} - \lambda) w^{\pm} = -\mathcal{D} w^{\pm},$$

that is, $(\mathcal{A} - \lambda) w^{\pm} = 0$; hence, λ is an eigenvalue of $\mathcal{A}$. □

2.3.3. *Main result*

The following theorem justifies our intuitive construction of perturbed generalized eigenfunctions.

Theorem 2.11. *Assume that the free operator $\tilde{\mathcal{A}}$ is diagonalized in the sense of definition 2.1 and satisfies the additional stability property of definition 2.5. Then, for every compact perturbation $\mathcal{A}$ of $\tilde{\mathcal{A}}$ (in the sense of definition 2.9) which has no point spectrum,[10] the following statements hold.*

[10] See § 2.5 if $\mathcal{A}$ has eigenvalues.

https://doi.org/10.1017/S0308210506000138 Published online by Cambridge University Press

(i) *(Perturbed limiting absorption.) The perturbed resolvent* $\mathcal{R}_\zeta$ *admits one-sided limits*[11] $\mathcal{R}^\pm_\lambda \in \mathcal{B}(\mathcal{H}_\downarrow, \mathcal{H}_\uparrow)$ *at every point* $\lambda \in \Lambda$. *These limits are given by*

$$\mathcal{R}^\pm_\lambda = \tilde{\mathcal{R}}^\pm_\lambda(\mathrm{Id} + \mathcal{D}\tilde{\mathcal{R}}^\pm_\lambda)^{-1} \quad \textit{for all } \lambda \in \Lambda. \tag{2.41}$$

(ii) *(Perturbed spectral bases.) Both families* $\{w^\pm_{\lambda,k}; (\lambda, k) \in \Lambda \times K\}$ *given by formula (2.36) define generalized spectral bases of* $\mathcal{A}$ *in the sense of definition 2.1.*

Proof. We have already justified the existence of the right-hand side of (2.41). The fact that these are the one-sided limits in $\mathcal{B}(\mathcal{H}_\downarrow, \mathcal{H}_\uparrow)$ of the right-hand side of (2.38) (as well as the Hölder continuity of these limits) follows from standard arguments of perturbation theory, essentially based on the properties of the Neumann series [27, theorem IV-1.16]. Hence, (i) is proved.

The proof of (ii) is based on spectral theory; it is given in the next section. □

Translated into a more physical language, theorem 2.11 guarantees fundamental properties for every sufficiently localized (or 'short-range') perturbation of a free propagative medium:

(i) the existence of outgoing and incoming time-harmonic waves, which involve a similar asymptotic distribution of energy to free propagation (indeed the weighted definition of energy is the same for both free and perturbed problems);

(ii) the fact that the outgoing and incoming perturbations of a complete family of *incident* free waves yield the suitable tool for time–frequency analysis.

2.4. Connection with spectral theory

The so-called spectral theorem [9, 27] is the keystone of a *functional calculus of self-adjoint operators*. This theorem asserts that every bounded self-adjoint operator $\mathcal{A}$ admits a *spectral family* $\{\mathcal{E}_\lambda\}_{\lambda\in\mathbb{R}}$, i.e. a family of orthonormal projections which allows us to express any function $f : \mathbb{R} \to \mathbb{C}$ of $\mathcal{A}$ by the explicit formula

$$(f(\mathcal{A})u, v) = \int_{\mathbb{R}} f(\lambda)\,\mathrm{d}(\mathcal{E}_\lambda u, v) \quad \text{for all } u, v \in \mathcal{H}. \tag{2.42}$$

We denote by $\mathcal{E}_I := \chi_I(\mathcal{A})$ the spectral projection associated with any interval $I \subset \mathbb{R}$, where χ_I is the characteristic function of I. This operator is related to the resolvent of $\mathcal{A}$ by means of Stone's formula, which reads

$$(\mathcal{E}_I u, u) = \|\mathcal{E}_I u\|^2 = \frac{\pm 1}{\pi} \lim_{\delta\to 0^+} \lim_{\varepsilon\to 0^+} \int_{a+\delta}^{b-\delta} \mathrm{Im}(\mathcal{R}_{\lambda\pm\mathrm{i}\varepsilon}u, u)\,\mathrm{d}\lambda \quad \text{for all } u \in \mathcal{H}$$

for an *open* interval $I =]a, b[$. Note that

$$\mathrm{Im}(\mathcal{R}_{\lambda\pm\mathrm{i}\varepsilon}u, u) = \pm\varepsilon\|\mathcal{R}_{\lambda\pm\mathrm{i}\varepsilon}u\|^2,$$

[11] As in proposition 2.3, these limits hold for the uniform topology of $\mathcal{B}(\mathcal{H}_\downarrow, \mathcal{H}_\uparrow)$ and are locally Hölder continuous on Λ.

since $\mathcal{A}$ is self-adjoint. Hence, if I is contained in the resolvent set of $\mathcal{A}$, the quantity $\|\mathcal{R}_{\lambda\pm i\varepsilon}u\|$ remains bounded as ε tends to 0, which shows that $\|\mathcal{E}_I u\| = 0$ for every $u \in \mathcal{H}$, that is, $\mathcal{E}_I = 0$.

On the other hand, if I is such that the limiting absorption principle holds near all its points, i.e. if the one-sided limits $\mathcal{R}^{\pm}_{\lambda}$ exist in $\mathcal{B}(\mathcal{H}_{\downarrow}, \mathcal{H}_{\uparrow})$, Stone's formula becomes

$$\|\mathcal{E}_I u\|^2 = \frac{\pm 1}{\pi} \int_I \mathrm{Im}\langle \mathcal{R}^{\pm}_{\lambda} u, u\rangle \,\mathrm{d}\lambda \quad \text{for all } u \in \mathcal{H}_{\downarrow}. \tag{2.43}$$

This relation tells us that the *spectral measure* $\mathrm{d}(\mathcal{E}_\lambda u, u) = \mathrm{d}\,\|\mathcal{E}_\lambda u\|^2$ is proportional to Lebesgue measure, and its density is a continuous function of λ, which is merely the energy flux appearing in (2.28):

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\|\mathcal{E}_\lambda u\|^2 = \frac{\pm 1}{\pi}\,\mathrm{Im}\langle \mathcal{R}^{\pm}_{\lambda} u, u\rangle \quad \text{for all } u \in \mathcal{H}_{\downarrow}. \tag{2.44}$$

The spectral measure is called *absolutely continuous* in this case.

2.4.1. *Spectral measures for the free and perturbed problems*

For the free problem, limiting absorption is derived from the *a priori* knowledge of a generalized spectral basis $\{\tilde{w}_{\lambda,k}\}$: proposition 2.3 holds for $\tilde{\mathcal{R}}^{\pm}_{\lambda}$, as well as its consequence (2.28), which gives here an explicit form of the spectral density of $\tilde{\mathcal{A}}$:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\|\tilde{\mathcal{E}}_\lambda u\|^2 = \frac{\pm 1}{\pi}\,\mathrm{Im}\langle \tilde{\mathcal{R}}^{\pm}_{\lambda} u, u\rangle = \int_K |\langle u, \tilde{w}_{\lambda,k}\rangle|^2 \,\mathrm{d}\sigma_k \quad \text{for all } u \in \mathcal{H}_{\downarrow}. \tag{2.45}$$

We want to establish a similar formula for the perturbed problem. Limiting absorption was obtained in this case by means of the perturbation relation (2.41), so (2.44) holds. We first prove the following lemma.

LEMMA 2.12. *For $\lambda \in \Lambda$ and $u \in \mathcal{H}_{\downarrow}$, we denote $\tilde{u}^{\pm} \in \mathcal{H}_{\downarrow}$ defined by*

$$\tilde{u}^{\pm} = (\mathrm{Id} + \mathcal{D}\tilde{\mathcal{R}}^{\pm}_{\lambda})^{-1} u = (\mathrm{Id} - \mathcal{D}\mathcal{R}^{\pm}_{\lambda}) u, \tag{2.46}$$

so that (2.41) equivalently reads $\mathcal{R}^{\pm}_{\lambda} u = \tilde{\mathcal{R}}^{\pm}_{\lambda}\tilde{u}^{\pm}$. Then

$$\mathrm{Im}\langle \mathcal{R}^{\pm}_{\lambda} u, u\rangle = \mathrm{Im}\langle \tilde{\mathcal{R}}^{\pm}_{\lambda}\tilde{u}^{\pm}, \tilde{u}^{\pm}\rangle.$$

Proof. For $\zeta \in \mathbb{C}\setminus\mathbb{R}$, define

$$u_\zeta := (\mathrm{Id} + \mathcal{D}\tilde{\mathcal{R}}_\zeta)^{-1} u = (\mathrm{Id} - \mathcal{D}\mathcal{R}_\zeta) u \in \mathcal{H},$$

where the second equality is readily deduced from (2.37) which also shows that $\mathcal{R}_\zeta u = \tilde{\mathcal{R}}_\zeta u_\zeta$. Hence,

$$(\mathcal{R}_\zeta u, u) = (\tilde{\mathcal{R}}_\zeta u_\zeta, (\mathrm{Id} + \mathcal{D}\tilde{\mathcal{R}}_\zeta) u_\zeta) = (\tilde{\mathcal{R}}_\zeta u_\zeta, u_\zeta) + (\tilde{\mathcal{R}}_\zeta u_\zeta, \mathcal{D}\tilde{\mathcal{R}}_\zeta u_\zeta),$$

where the last inner product is real for $\mathcal{D}$ is self-adjoint. As a consequence,

$$\mathrm{Im}(\mathcal{R}_\zeta u, u) = \mathrm{Im}(\tilde{\mathcal{R}}_\zeta u_\zeta, u_\zeta).$$

The conclusion then follows from both free and perturbed limiting absorption principles. □

By virtue of (2.44), lemma 2.12 tells us that the spectral measure $\mathrm{d}\|\mathcal{E}_\lambda u\|^2$ of $\mathcal{A}$ is related to that of $\tilde{\mathcal{A}}$ by the relation $\mathrm{d}\|\mathcal{E}_\lambda u\|^2 = \mathrm{d}\|\tilde{\mathcal{E}}_\lambda \tilde{u}^\pm\|^2$. It remains to show the link with perturbed generalized eigenfunctions.

Proposition 2.13. *With the assumptions of theorem 2.11, the spectral density of $\mathcal{A}$ is given by*

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\|\mathcal{E}_\lambda u\|^2 = \int_K |\langle u, w^\pm_{\lambda,k}\rangle|^2 \,\mathrm{d}\sigma_k \quad \textit{for all } u \in \mathcal{H}_\downarrow. \tag{2.47}$$

Proof. By the definition (2.36) of $w^\pm_{\lambda,k}$, we have

$$\langle u, w^\pm_{\lambda,k}\rangle = \langle u, (\mathrm{Id} - \mathcal{R}^\pm_\lambda \mathcal{D})\tilde{w}_{\lambda,k}\rangle,$$

which shows by transposition that

$$\langle u, w^\pm_{\lambda,k}\rangle = \langle (\mathrm{Id} - \mathcal{D}\mathcal{R}^\mp_\lambda)u, \tilde{w}_{\lambda,k}\rangle = \langle \tilde{u}^\mp, \tilde{w}_{\lambda,k}\rangle,$$

where $\tilde{u}^\pm$ is defined in (2.46). Hence, we deduce from (2.45) that

$$\int_K |\langle u, w^\pm_{\lambda,k}\rangle|^2 \,\mathrm{d}\sigma_k = \int_K |\langle \tilde{u}^\mp, \tilde{w}_{\lambda,k}\rangle|^2 \,\mathrm{d}\sigma_k = \frac{\pm 1}{\pi}\,\mathrm{Im}\langle \tilde{\mathcal{R}}^\pm_\lambda \tilde{u}^\pm, \tilde{u}^\pm\rangle.$$

Lemma 2.12 then yields the result. □

2.4.2. *Proof of theorem 2.11*

Spectral theory, together with proposition 2.13, provides the isometric character of the operation of decomposition on the $w^\pm_{\lambda,k}$ (corollary 2.14). The fact that it is surjective follows from the same property for the free problem (proposition 2.15).

Corollary 2.14. *With the assumptions of theorem 2.11, both transformations*

$$(\mathcal{U}^\pm v)_{\lambda,k} := \langle v, w^\pm_{\lambda,k}\rangle \quad \textit{for all } v \in \mathcal{H}_\downarrow, \tag{2.48}$$

extend by density as isometric operators from $\mathcal{H}$ to $L^2(\varLambda \times K)$ which satisfy

$$\mathcal{U}^\pm f(\mathcal{A}) = f(\lambda)\mathcal{U}^\pm. \tag{2.49}$$

Proof. Formula (2.47) means that, for every interval $I \subset \varLambda$,

$$\|\mathcal{E}_I u\|^2 = \int_{\varLambda\times K} \chi_I(\lambda)|(\mathcal{U}^\pm u)_{\lambda,k}|^2 \,\mathrm{d}\lambda\,\mathrm{d}\sigma_k \quad \text{for all } u \in \mathcal{H}_\downarrow.$$

In particular we have $\mathcal{E}_\mathbb{R} = \mathrm{Id}$. Thus,

$$\|u\|^2 = \|\mathcal{U}^\pm u\|^2_{\varLambda\times K} \quad \text{for all } u \in \mathcal{H}_\downarrow.$$

So $\mathcal{U}^\pm$ is isometric, and extends to the whole space $\mathcal{H}$ since $\mathcal{H}_\downarrow$ is dense. The fact that this transformation diagonalizes $\mathcal{A}$ is a direct consequence of formula (2.42), where $\mathrm{d}(\mathcal{E}_\lambda u, v)$ is deduced from (2.47) by the polarization principle:

$$(f(\mathcal{A})u, v) = \int_{\varLambda\times K} f(\lambda)(\mathcal{U}^\pm u)_{\lambda,k}\overline{(\mathcal{U}^\pm v)_{\lambda,k}}\,\mathrm{d}\lambda\,\mathrm{d}\sigma_k, \tag{2.50}$$

which reads $(f(\mathcal{A})u, v) = (f(\lambda)\mathcal{U}^{\pm}u, \mathcal{U}^{\pm}v)_{\Lambda\times K}$; hence,

$$f(\mathcal{A}) = (\mathcal{U}^{\pm})^* f(\lambda)\mathcal{U}.$$

Relation (2.49) is a slightly more precise version of this diagonalization formula, which can be written equivalently as

$$\mathcal{U}^{\pm} f(\mathcal{A}) = \mathcal{P}^{\pm} f(\lambda)\mathcal{U}^{\pm},$$

where $\mathcal{P}^{\pm} := \mathcal{U}^{\pm}(\mathcal{U}^{\pm})^*$ is the orthogonal projection on the range $\mathrm{R}(\mathcal{U}^{\pm})$ of $\mathcal{U}^{\pm}$. To remove $\mathcal{P}^{\pm}$ from the above relation, note that

$$\|f(\mathcal{A})v\| = \|\mathcal{U}^{\pm} f(\mathcal{A})v\|_{\Lambda\times K} = \|f(\lambda)\mathcal{U}^{\pm}v\|_{\Lambda\times K} \quad \text{for all } v \in \mathcal{H},$$

which is readily deduced from (2.50) by taking $u = v$ and $f(\mathcal{A})\bar{f}(\mathcal{A}) = f(\mathcal{A})(f(\mathcal{A}))^*$ instead of $f(\mathcal{A})$. So (2.49) is proved. □

Proposition 2.15. *With the assumptions of theorem 2.11, both transformations $\mathcal{U}^{\pm}$ are unitary from $\mathcal{H}$ to $L^2(\Lambda \times K)$.*

Proof. We know that $\mathcal{U}^{\pm}$ is isometric: it remains to prove that it is surjective, or equivalently that $(\mathcal{U}^{\pm})^*$ is injective. Suppose that $(\mathcal{U}^{\pm})^*\hat{u} = 0$ for some $\hat{u} \in L^2(\Lambda \times K)$. Then

$$(\hat{u}, \mathcal{U}^{\pm}v)_{\Lambda\times K} = 0 \quad \text{for all } v \in \mathcal{H}.$$

For every interval $I \subset \Lambda$, one can replace v by $\mathcal{E}_I v$ in this formula. Noting that $\mathcal{U}^{\pm}\mathcal{E}_I = \chi_I(\lambda)\mathcal{U}^{\pm}$ by (2.49), we deduce that $(\chi_I\hat{u}, \mathcal{U}^{\pm}v)_{\Lambda\times K} = 0$ for all $v \in \mathcal{H}$, which means that $(\mathcal{U}^{\pm})^*\chi_I\hat{u} = 0$. Following (2.11), this relation reads

$$\int_{\Lambda\times K} \chi_I(\lambda)\hat{v}_{\lambda,k} w^{\pm}_{\lambda,k}\,\mathrm{d}\lambda\,\mathrm{d}\sigma_k = 0$$

for every I interior to Λ. Hence,

$$\int_K \hat{v}_{\lambda,k} w^{\pm}_{\lambda,k}\,\mathrm{d}\sigma_k = 0 \quad \text{for a.e. } \lambda \in \Lambda.$$

Using the definition (2.36) of $w^{\pm}_{\lambda,k}$, this property becomes

$$(\mathrm{Id}_{\mathcal{H}_\uparrow} - \mathcal{R}^{\pm}_\lambda\mathcal{D})\tilde{u}_\lambda = 0, \quad \text{where } \tilde{u}_\lambda := \int_K \hat{v}_{\lambda,k}\tilde{w}_{\lambda,k}\,\mathrm{d}\sigma_k \in \mathcal{H}_\uparrow, \tag{2.51}$$

for almost every $\lambda \in \Lambda$. Applying $\mathcal{D}$, we deduce that

$$(\mathrm{Id}_{\mathcal{H}_\downarrow} - \mathcal{D}\mathcal{R}^{\pm}_\lambda)\mathcal{D}\tilde{u}_\lambda = 0.$$

Notice then that, from (2.41) and (2.46), we have

$$\mathcal{R}^{\pm}_\lambda\mathcal{D} = \tilde{\mathcal{R}}^{\pm}_\lambda(\mathrm{Id}_{\mathcal{H}_\downarrow} - \mathcal{D}\mathcal{R}^{\pm}_\lambda)\mathcal{D}.$$

Hence, $\mathcal{R}^{\pm}_\lambda\mathcal{D}\tilde{u}_\lambda = 0$, so $\tilde{u}_\lambda = 0$ by (2.51). In other words, we have proved that

$$\int_{\Lambda\times K} \chi_I(\lambda)\hat{v}_{\lambda,k}\tilde{w}_{\lambda,k}\,\mathrm{d}\lambda\,\mathrm{d}\sigma_k = 0$$

for every $I \subset \Lambda$, that is $\tilde{\mathcal{U}}^*(\chi_I\hat{u}) = 0$. But $\tilde{\mathcal{U}}$ is unitary, so $\tilde{\mathcal{U}}^*$ is injective, and thus $\hat{u} = 0$. □

2.5. From theory to applications

How does the preceding abstract framework apply in practical situations? For most models of wave propagation phenomena which enter the framework of our abstract wave equation (2.16) (in particular our water-wave problem), $\mathcal{A}$ is *unbounded* and may have a *point spectrum* besides the continuous spectrum investigated in the present paper. So far this situation was excluded. We show here that the content of §§ 2.1 and 2.2 easily extends to such operators, but not the perturbation approach of § 2.3. We give a trick to overcome this difficulty.

2.5.1. *Unbounded operators*

In the context of spectral theory, dealing with unbounded operators amounts to allowing unbounded functions $f : \varLambda \to \mathbb{C}$ in the functional calculus offered by formula (2.9). To do so, it suffices to restrict the latter to the elements of the domain of $f(\mathcal{A})$ which is naturally characterized by means of the inverse spectral transformation $\mathcal{U}^*$:

$$\left.\begin{aligned} f(\mathcal{A})v &= \mathcal{U}^* f(\lambda)\mathcal{U}v \quad \text{for all } v \in \mathrm{D}(f(\mathcal{A})),\\ \mathrm{D}(f(\mathcal{A})) &= \mathcal{U}^*\{\hat{v} \in L^2(\varLambda \times K);\ f(\lambda)\hat{v} \in L^2(\varLambda \times K)\}.\end{aligned}\right\} \tag{2.52}$$

This formula actually provides the extension of definition 2.1 to an unbounded self-adjoint operator $A := f(\mathcal{A})$. Indeed, suppose that f is a smooth real-valued diffeomorphism so that it defines a change of variable $\mu := f(\lambda)$. Noting that

$$\int_{\varLambda\times K} |\hat{v}_{\lambda,k}|^2 \,\mathrm{d}\lambda\,\mathrm{d}\sigma_k = \int_{f(\varLambda)\times K} |\hat{v}_{f^{-1}(\mu),k}|^2 c_\mu \,\mathrm{d}\mu\,\mathrm{d}\sigma_k \quad \text{with } c_\mu := \left|\frac{\mathrm{d}f^{-1}}{\mathrm{d}\mu}(\mu)\right|,$$

we infer that the transformation

$$\hat{v}_{\lambda,k} \xrightarrow{C} (C\hat{v})_{\mu,k} = c_\mu^{1/2}\hat{v}_{f^{-1}(\mu),k}$$

is unitary from $L^2(\varLambda \times K)$ to $L^2(f(\varLambda) \times K)$. Hence, (2.52) can be rewritten

$$A = U^*\mu U \in \mathrm{D}(A),$$

where $U := C\mathcal{U}$ is unitary from $\mathcal{H}$ to $L^2(f(\varLambda) \times K)$. This transformation takes the form

$$(Uv)_{\mu,k} := \langle v, w'_{\mu,k}\rangle \quad \text{for all } v \in \mathcal{H}_\downarrow \quad \text{with } w'_{\mu,k} := c_\mu^{1/2} w_{f^{-1}(\mu),k}, \tag{2.53}$$

which shows that the family

$$\{w'_{\mu,k} \in \mathcal{H}_\uparrow; (\mu,k) \in f(\varLambda) \times K\}$$

is a generalized spectral basis for the unbounded operator A. The normalization coefficient $c_\mu^{1/2}$ is smooth, so it does not affect the regularity assumptions of definition 2.1, which then holds for unbounded operators: $\varLambda$ is simply allowed to be unbounded. It is readily seen that the matter of §§ 2.1 and 2.2, that is, generalized eigenfunction expansions and limiting absorption, extends to unbounded operators with natural precautions concerning the domain. More precisely, all the properties mentioned in theses lines hold for $\mathcal{A}$ if and only if they hold for $A = f(\mathcal{A})$.

2.5.2. *About a possible point spectrum*

The present spectral analysis cannot provide any information about the possible eigenvalues of the system, which reveal the existence of trapped waves (or bound states). It concerns only the continuous spectrum (more precisely the absolutely continuous spectrum) which corresponds to *propagative waves*, whereas the point spectrum is related to localized *vibrations* of the system. Fortunately, both regimes exist independently.

Indeed, suppose that $\mathcal{A}$ has a point spectrum Λ_{p}. Let $\mathcal{H}_{\mathrm{p}}$ denote the subspace of $\mathcal{H}$ spanned by the associated eigenvectors, and let $\mathcal{H}_{\mathrm{c}}$ denote its orthogonal complement:

$$\mathcal{H} = \mathcal{H}_{\mathrm{c}} \overset{\perp}{\oplus} \mathcal{H}_{\mathrm{p}}.$$

Both these subspaces are invariant under $\mathcal{A}$, which shows that $\mathcal{A}$ can be split as a direct sum,

$$\mathcal{A} = \mathcal{A}_{\mathrm{c}} \oplus \mathcal{A}_{\mathrm{p}},$$

where the restriction $\mathcal{A}_{\mathrm{c}} := \mathcal{A}|_{\mathcal{H}_{\mathrm{c}}}$ (respectively, $\mathcal{A}_{\mathrm{p}} := \mathcal{A}|_{\mathcal{H}_{\mathrm{p}}}$) has a pure continuous spectrum (respectively, pure point spectrum). Definition 2.1 then extends as follows, under the natural assumption $\mathcal{H}_{\mathrm{p}} \subset \mathcal{H}_{\downarrow}$ (trapped waves are always localized states).

Definition 2.16. Let $\Lambda_{\mathrm{c}} := \Lambda \setminus \Lambda_{\mathrm{p}}$. A Λ_{c}-locally Hölder continuous family

$$\{w_{\lambda,k} \in \mathcal{H}_{\uparrow};\ (\lambda, k) \in \Lambda_{\mathrm{c}} \times K\}$$

associated with a spectral space $L^2(\Lambda \times K)$ is said to be a generalized spectral basis for the continuous part of $\mathcal{A}$ if the transformation

$$(\mathcal{U}v)_{\lambda,k} := \langle v, w_{\lambda,k} \rangle \quad \text{for all } v \in \mathcal{H}_{\downarrow},$$

defines by density a unitary operator from $\mathcal{H}_{\mathrm{c}} = (\mathrm{N}(\mathcal{U}))^{\perp}$ to $L^2(\Lambda \times K)$ which satisfies

$$\mathcal{A}_{\mathrm{c}} \oplus 0 = \mathcal{U}^* \lambda \mathcal{U}.$$

To obtain a spectral representation of $\mathcal{A}$, not only of its continuous part, we simply have to exhibit an orthonormal family of eigenvectors which spans $\mathcal{H}_{\mathrm{p}}$. The complete spectral transformation then appears as the sum of two orthogonal contributions: the continuous part described by the $w_{\lambda,k}$ and a discrete part similar to the finite-dimensional case (see footnote 3).

Definition 2.16 offers a 'restricted' functional calculus which involves only the continuous part of $\mathcal{A}$; formula (2.12) is now the generalized eigenfunction expansion of $(f(\mathcal{A}_{\mathrm{c}}) \oplus 0)u$. Hence, the content of § 2.2 holds if we are only interested in the spectrally continuous part of the solution to our abstract wave equation (which amounts to considering data in $\mathcal{H}_{\mathrm{c}}$), bearing in mind the fact that $\mathcal{R}_{\zeta}$ now denotes $(\mathcal{A}_{\mathrm{c}} - \zeta)^{-1} \oplus 0$. Note that outside Λ_{p} a complete limiting absorption principle for $\mathcal{A}$ is readily deduced from the restricted version, since the discrete part of the resolvent, i.e. $(\mathcal{A}_{\mathrm{p}} - \zeta)^{-1}$, is analytic outside the eigenvalues. All the results concerning the energy flux then also hold for the complete resolvent of $\mathcal{A}$, provided that λ is not an eigenvalue, because the discrete part of the resolvent does not contribute to the energy flux.

Finally, the perturbation approach of § 2.3 holds with minor changes if $\mathcal{A}$ has a point spectrum. The perturbed limiting absorption principle (theorem 2.11(i)) is valid outside eigenvalues: this is exactly what proposition 2.10 tells us. And the perturbed generalized spectral bases (theorem 2.11(ii)) have to be understood in the sense of definition 2.16. Indeed, the expression of the spectral density given in proposition 2.13 is valid outside eigenvalues, and corollary 2.14 now involves $f(\mathcal{A}_c) \oplus 0$ instead of $f(\mathcal{A})$.

2.5.3. *How to use theorem 2.11*

In § 2.3 the assumption that $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are bounded operators defined in the same Hilbert space cannot be removed without introducing technical complications that reduce the generality of this approach. But, for most scattering problems, the operators which describe the free and perturbed dynamics of the system, say $\tilde{A}$ and A, are not only unbounded but also defined in different Hilbert spaces, say $\tilde{\mathcal{H}}$ and $\mathcal{H}$. Indeed, the presence of the scatterer generally implies a different description of the possible states of the system, which is particularly clear in coupled problems. The following idea, which applies in many practical situations (and is illustrated in § 3) simply consists in *comparing bounded functions of A and $\tilde{A}$ in a 'cumulative' functional framework* into which both free and perturbed problems can be carried.

If both A and $\tilde{A}$ are positive, a convenient choice of bounded function is to consider their respective resolvents R_ζ and $\tilde{R}_\zeta$ at a given point $\zeta = -\alpha$ with $\alpha > 0$:

$$\mathcal{A} := R_{-\alpha} = (A + \alpha)^{-1} \quad \text{and} \quad \tilde{\mathcal{A}} := \tilde{R}_{-\alpha} = (\tilde{A} + \alpha)^{-1}.$$

These operators are clearly bounded positive and self-adjoint. As mentioned above, the limiting absorption principle holds for A if and only if it holds for $\mathcal{A}$, and likewise for $\tilde{A}$ and $\tilde{\mathcal{A}}$. Here the link is explicit since the resolvent $\mathcal{R}_\xi$ of $\mathcal{A}$ is related to R_ζ by

$$\mathcal{R}_\xi = -\xi^{-1}(\mathrm{Id} + \xi^{-1} R_\zeta) \quad \text{with } \xi := (\zeta + \alpha)^{-1}, \tag{2.54}$$

which follows from the relation

$$r_\xi \circ r_{-\alpha} = -\xi^{-1}(1 + \xi^{-1} r_\zeta), \quad \text{where } r_\zeta(\lambda) := (\lambda - \zeta)^{-1}.$$

As a consequence, if they exist, the one-sided limits of these resolvents satisfy

$$\mathcal{R}_\mu^{\pm} = -\mu^{-1}(\mathrm{Id} + \mu^{-1} R_\lambda^{\mp}) \quad \text{with } \mu := (\lambda + \alpha)^{-1}, \tag{2.55}$$

where the change of sign is due to the fact that $\xi \in \mathbb{C}^{\pm}$ if and only if $\zeta = \xi^{-1} - \alpha \in \mathbb{C}^{\mp}$. Note that this relation shows that the stability condition (2.29) holds for $\mathcal{R}_\mu^{\pm}$ if and only if it holds for $R_\lambda^{\pm}$.

We shall be able to say that $\mathcal{A}$ is a perturbation of $\tilde{\mathcal{A}}$ if both spaces $\tilde{\mathcal{H}}$ and $\mathcal{H}$ can be identified with two subspaces of the same 'cumulative' Hilbert space $\boldsymbol{\mathcal{H}}$, that is,

$$\boldsymbol{\mathcal{H}} = \tilde{\mathcal{H}} \mathbin{\tilde{\oplus}} \tilde{\mathcal{H}}_0 = \mathcal{H} \oplus \mathcal{H}_0. \tag{2.56}$$

In such a case, the operators $\tilde{\mathcal{A}}$ and $\mathcal{A}$ can be carried into $\boldsymbol{\mathcal{H}}$ by defining

$$\tilde{\boldsymbol{\mathcal{A}}} := \tilde{\mathcal{A}} \mathbin{\tilde{\oplus}} 0 \quad \text{and} \quad \boldsymbol{\mathcal{A}} := \mathcal{A} \oplus 0,$$

https://doi.org/10.1017/S0308210506000138 Published online by Cambridge University Press

which are bounded self-adjoint operators in $\boldsymbol{\mathcal{H}}$. Their difference

$$\boldsymbol{\mathcal{D}} := \boldsymbol{\mathcal{A}} - \tilde{\boldsymbol{\mathcal{A}}} = \{(A+\alpha)^{-1} \oplus 0\} - \{(\tilde{A}+\alpha)^{-1} \tilde{\oplus} 0\} \tag{2.57}$$

is thus also bounded and self-adjoint. In order to apply theorem 2.11, the cumulative space must fit into a weighted functional scheme such as (2.1), i.e.

$$\boldsymbol{\mathcal{H}}_{\downarrow} \subset \boldsymbol{\mathcal{H}} \subset \boldsymbol{\mathcal{H}}_{\uparrow},$$

where both spaces $\boldsymbol{\mathcal{H}}_{\downarrow}$ and $\boldsymbol{\mathcal{H}}_{\uparrow}$ are related to their free and perturbed analogues by

$$\boldsymbol{\mathcal{H}}_{\downarrow/\uparrow} = \tilde{\mathcal{H}}_{\downarrow/\uparrow} \tilde{\oplus} \tilde{\mathcal{H}}_0 = \mathcal{H}_{\downarrow/\uparrow} \oplus \mathcal{H}_0. \tag{2.58}$$

In these conditions, it is clear that a family $w_{\mu,k} \in \mathcal{H}$ is a generalized spectral basis for $\mathcal{A}$ in the sense of definition 2.1 if and only if $\boldsymbol{w}_{\mu,k} := w_{\mu,k} \oplus 0 \in \boldsymbol{\mathcal{H}}$ is a generalized spectral basis for $\boldsymbol{\mathcal{A}}$ in the sense of definition 2.16 (and the same holds for the free problem). Hence, the perturbation approach of § 2.3 will apply if one is able to prove the compactness property of definition 2.9 for $\boldsymbol{\mathcal{D}}$.

To sum up, the justification of generalized eigenfunction expansions for a scattering problem consists in the following steps:

(i) find a generalized spectral basis $\tilde{w}_{\lambda,k}$ for the unbounded operator $\tilde{A}$ (using well-known functional transformations), and verify the stability condition (2.29);

(ii) find a cumulative functional scheme as described above and prove that

$$\boldsymbol{\mathcal{D}}\widetilde{\boldsymbol{\mathcal{R}}}^{\pm}_{\mu} = \boldsymbol{\mathcal{D}}(\tilde{\mathcal{R}}^{\pm}_{\mu} \tilde{\oplus} (-\mu^{-1}\mathrm{Id}))$$

is compact in $\boldsymbol{\mathcal{H}}_{\downarrow}$ for every $\mu \in r_{-\alpha}(\Lambda)$.

Then both families

$$\boldsymbol{w}^{\pm}_{\lambda,k} := (\mathrm{Id} - \boldsymbol{\mathcal{R}}^{\mp}_{\mu}\boldsymbol{\mathcal{D}})\tilde{\boldsymbol{w}}_{\lambda,k} \in \boldsymbol{\mathcal{H}}_{\uparrow}, \quad \text{where } \tilde{\boldsymbol{w}}_{\lambda,k} := \tilde{w}_{\lambda,k} \tilde{\oplus} 0,$$

are generalized spectral bases for $A \oplus 0$. Note that the change of sign in this formula is due to (2.55). Denoting by $\mathcal{P} : \boldsymbol{\mathcal{H}} \to \mathcal{H}$ and $\mathcal{P}_0 : \boldsymbol{\mathcal{H}} \to \mathcal{H}_0$ the restricted projections defined by $\boldsymbol{u} = \mathcal{P}\boldsymbol{u} \oplus \mathcal{P}_0\boldsymbol{u}$ (which extend to $\boldsymbol{\mathcal{H}}_{\uparrow}$ by (2.58)), we have

$$\boldsymbol{w}^{\pm}_{\lambda,k} = (\mathcal{P} - \mathcal{R}^{\mp}_{\mu}\mathcal{P}\boldsymbol{\mathcal{D}})\tilde{\boldsymbol{w}}_{\lambda,k} \oplus (\mathcal{P}_0 + \mu^{-1}\mathcal{P}_0\boldsymbol{\mathcal{D}})\tilde{\boldsymbol{w}}_{\lambda,k},$$

where the component in $\mathcal{H}_0$ vanishes since

$$\mathcal{P}_0\boldsymbol{\mathcal{D}}\tilde{\boldsymbol{w}}_{\lambda,k} = -\mathcal{P}_0\tilde{\boldsymbol{\mathcal{A}}}\tilde{\boldsymbol{w}}_{\lambda,k} = -\mu\mathcal{P}_0\tilde{\boldsymbol{w}}_{\lambda,k}.$$

Hence, the families

$$w^{\pm}_{\lambda,k} := (\mathcal{P} - \mathcal{R}^{\mp}_{\mu}\mathcal{P}\boldsymbol{\mathcal{D}})\tilde{\boldsymbol{w}}_{\lambda,k} \in \mathcal{H}_{\uparrow} \tag{2.59}$$

are generalized spectral bases for the unbounded operator A.

3. Application to the two-dimensional sea-keeping problem

3.1. Abstract formulation

We show here that our water-wave problem (1.1)–(1.4) as well as the free version (1.5) can be expressed as abstract wave equations such as (2.16) involving unbounded self-adjoint operators.

https://doi.org/10.1017/S0308210506000138 Published online by Cambridge University Press

3.1.1. *The coupled problem*

Substituting (1.4) in (1.3) (recall that we have chosen $\mathbb{M} = \mathrm{Id}$), the sea-keeping problem reads as follows:

$$\Delta\Phi = 0 \quad \text{in } \Omega, \tag{3.1}$$

$$\partial_t^2\Phi + \partial_y\Phi = 0 \quad \text{on } F, \tag{3.2}$$

$$\partial_n\Phi + \left(\mathbb{K}p + \int_\Gamma \Phi\nu \,\mathrm{d}\gamma\right) \cdot \nu = 0 \quad \text{on } \Gamma, \tag{3.3}$$

$$\mathrm{d}_t^2 p + \mathbb{K}p + \int_\Gamma \Phi\nu \,\mathrm{d}\gamma = 0. \tag{3.4}$$

The key of the abstract formulation lies in the following remark: at every time t, if we know the acceleration potential only on the free surface F, say $\varphi := \Phi|_F$, as well as the position p of the floating body, then equations (3.1) and (3.3) determine Φ everywhere. This suggests the introduction of the operator Θ, which maps the pair (φ, p) onto the solution $\Phi = \Theta(\varphi, p)$ of the boundary-value problem

$$\left.\begin{aligned} \Delta\Phi &= 0 && \text{in } \Omega, \\ \Phi &= \varphi && \text{on } F, \\ \partial_n\Phi + \left(\int_\Gamma \Phi\nu \,\mathrm{d}\gamma\right) \cdot \nu &= -\mathbb{K}p \cdot \nu && \text{on } \Gamma. \end{aligned}\right\} \tag{3.5}$$

Hence, our system (3.1)–(3.4) amounts to

$$\partial_t^2\varphi + \partial_y\Theta(\varphi, p) = 0 \quad \text{on } F,$$

$$\mathrm{d}_t^2 p + \mathbb{K}p + \int_\Gamma \Theta(\varphi, p)\nu \,\mathrm{d}\gamma = 0,$$

which can be condensed as a single equation on $u := (\varphi, p)$:

$$\mathrm{d}_t^2 u + Au = 0, \quad \text{where } Au := \left(\partial_y\Theta u, \mathbb{K}p + \int_\Gamma \Theta u\nu \,\mathrm{d}\gamma\right). \tag{3.6}$$

To prove that the operator A is self-adjoint, we have to specify the proper functional framework in which it is defined. We first give a rigorous definition of Θ.

Lemma 3.1. *The operator Θ formally defined by (3.5) appears as a continuous operator from $H^{1/2}(F) \times \mathbb{C}^2$ (where $H^{1/2}(F)$ is the trace space of the standard Sobolev space $H^1(\Omega)$) to the weighted Sobolev space*

$$W^1(\Omega) := \{\Psi; \eta\Psi \in L^2(\Omega) \text{ and } \nabla\Psi \in L^2(\Omega)^3\},$$

where

$$\eta(X) := (1 + \|X\|^2)^{-1/2}(\ln(2 + \|X\|^2))^{-1}.$$

Proof. The basic property of $W^1(\Omega)$ is that the quantity $\int_\Omega \|\nabla\Psi\|^2$ defines a norm in $W^1(\Omega)/\mathbb{C}$ equivalent to the quotient norm [3]. Hence, the following variational

https://doi.org/10.1017/S0308210506000138 Published online by Cambridge University Press

formulation of (3.5):

find $\Phi \in W^1(\Omega)$ such that $\Phi|_F = \varphi$ and

$$\int_\Omega \nabla\Phi \cdot \overline{\nabla\Psi} + \left(\int_\Gamma \Phi\nu \,\mathrm{d}\gamma\right) \cdot \left(\int_\Gamma \bar{\Psi}\nu \,\mathrm{d}\gamma\right) = -\mathbb{K}p \cdot \int_\Gamma \bar{\Psi}\nu \,\mathrm{d}\gamma$$
$$\text{for all } \Psi \in W^1(\Omega) \text{ such that } \Psi|_F = 0,$$

which is easily derived from Green's formula, falls within the province of Lax–Milgram theorem, since the second sesquilinear term is positive. This means that this variational problem is well posed, provided that φ belongs to the trace space $W^{1/2}(F)$ of $W^1(\Omega)$. The statement of the lemma follows by noting that $H^{1/2}(F) \subset W^{1/2}(F)$, since $H^1(\Omega) \subset W^1(\Omega)$. More precisely, the following topological equality holds:

$$H^{1/2}(F) = W^{1/2}(F) \cap L^2(F), \tag{3.7}$$

which is easily deduced from the characterization of $W^{1/2}(F)$ given in [3]. □

Consider then the Hilbert space

$$\mathcal{H} := L^2(F) \times \mathbb{C}^2$$

equipped with the inner product

$$(u, v)_{\mathcal{H}} := \int_F \varphi\bar{\psi} \,\mathrm{d}x + \mathbb{K}p \cdot \bar{q} \quad \text{for } u = (\varphi, p) \text{ and } v = (\psi, q),$$

as well as its subspace

$$\mathcal{V} := H^{1/2}(F) \times \mathbb{C}^2.$$

Proposition 3.2. *The operator A formally defined by (3.6) appears as an unbounded positive self-adjoint injective operator in $\mathcal{H}$, given by*

$$(Au, v)_{\mathcal{H}} := a(u, v) \quad \text{for all } u \in \mathrm{D}(A) \text{ and all } v \in \mathcal{V}, \tag{3.8}$$
$$\mathrm{D}(A) := \{u \in \mathcal{V};\ \exists K_u > 0 \text{ and all } v \in \mathcal{V},\ |a(u, v)| \leqslant K_u \|v\|_{\mathcal{H}}\}, \tag{3.9}$$

where $a(\cdot, \cdot)$ is the Hermitian form defined in $\mathcal{V} \times \mathcal{V}$ by

$$a(u, v) := \int_\Omega \nabla\Theta u \cdot \overline{\nabla\Theta v} + \left(\mathbb{K}p + \int_\Gamma \Theta u\nu \,\mathrm{d}\gamma\right) \cdot \left(\mathbb{K}\bar{q} + \int_\Gamma \overline{\Theta v}\nu \,\mathrm{d}\gamma\right),$$

for all $u = (\varphi, p)$ and $v = (\psi, q)$ in $\mathcal{V}$.

Proof. The link between the formal definition (3.6) of A and the form $a(\cdot, \cdot)$ follows from Green's formula and the definition (3.5) of Θ, which yield

$$a(u, v) = \int_F \partial_y \Theta u \bar{v} \,\mathrm{d}x + \left(\mathbb{K}p + \int_\Gamma \Theta u\nu \,\mathrm{d}\gamma\right) \cdot \overline{\mathbb{K}q} = (Au, v)_{\mathcal{H}}.$$

By lemma 3.1, we may infer that $a(\cdot, \cdot)$ is continuous on $\mathcal{V} \times \mathcal{V}$. It is clearly positive and Hermitian. Moreover, we show below that, for some positive constants λ_0 and m, we have

$$a(u, u) + \lambda_0 \|u\|_{\mathcal{H}}^2 \geqslant m\|u\|_{\mathcal{V}}^2 \quad \text{for all } u \in \mathcal{V}. \tag{3.10}$$

For such a form, we know that (3.8)–(3.2) defines a positive self-adjoint operator [27]. The fact that it is injective is obvious: $a(u,u)=0$ implies that $\Theta u=0$, and thus $u=0$ (Θ is clearly injective).

Property (3.10) can be proved by contradiction. Suppose that there exists a sequence $u_n=(\varphi_n,p_n)\in\mathcal{V}$ such that $\|u_n\|_{\mathcal{V}}=1$ and

$$a(u_n,u_n)+\lambda_0\|u_n\|^2_{\mathcal{H}}\to 0 \quad \text{as } n\to+\infty.$$

The fact that $\|u_n\|_{\mathcal{H}}\to 0$ tells us that $\varphi_n\to 0$ in $L^2(F)$ and $p_n\to 0$. On the other hand, $a(u_n,u_n)\to 0$ shows that $\Theta u_n\to 0$ in $W^1(\Omega)$. Hence, $\varphi_n=(\Theta u_n)|_F\to 0$ in $W^{1/2}(F)$. From (3.7) we deduce that $\varphi_n\to 0$ in $H^{1/2}(F)$, and thus $u_n\to 0$ in $\mathcal{V}$, which contradicts the assumption that $\|u_n\|_{\mathcal{V}}=1$. □

3.1.2. *The free problem*

In the absence of the floating body, the propagation of water waves is described by the system (1.5), which leads to an abstract wave equation similar to (3.6):

$$\mathrm{d}_t^2\tilde{\phi}+\tilde{A}\tilde{\phi}=0, \quad \text{where } \tilde{A}\tilde{\phi}:=\partial_y\tilde{\Theta}\tilde{\phi}, \tag{3.11}$$

and where $\tilde{\Theta}$ denotes the harmonic lifting that maps a function $\tilde{\phi}$ defined on $\tilde{F}$ to the solution $\tilde{\Phi}$ to

$$\left.\begin{aligned} \Delta\tilde{\Phi}&=0 \quad \text{in } \tilde{\Omega},\\ \tilde{\Phi}&=\tilde{\phi} \quad \text{on } \tilde{F}. \end{aligned}\right\} \tag{3.12}$$

As in lemma 3.1, $\tilde{\Theta}$ defines a continuous operator from $H^{1/2}(\tilde{F})$ to $W^1(\tilde{\Omega})$, and proposition 3.2 can be transposed to the operator $\tilde{A}$, which is self-adjoint in $L^2(\tilde{F})$: it is now associated with the Hermitian form

$$a(\tilde{\phi},\tilde{\psi}):=\int_{\tilde{\Omega}}\nabla\tilde{\Theta}\tilde{\phi}\cdot\overline{\nabla\tilde{\Theta}\tilde{\psi}} \quad \text{for all } \tilde{\phi},\tilde{\psi}\in H^{1/2}(\tilde{F}).$$

3.2. Spectral analysis of the free problem

3.2.1. *Construction of a generalized spectral basis*

We first exhibit a *diagonal representation* of the free operator $\tilde{A}=\partial_y\tilde{\Theta}$ defined in (3.11). We proceed as in [21] using the horizontal Fourier transform:

$$\mathcal{F}\varphi(\kappa):=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathrm{e}^{-\mathrm{i}\kappa x}\varphi(x)\,\mathrm{d}x, \quad \kappa\in\mathbb{R},$$

which appears as a unitary transformation from $L^2(\mathbb{R}_x)$ to $L^2(\mathbb{R}_\kappa)$. Recall that its inverse $\mathcal{F}^{-1}=\mathcal{F}^*$ is given by

$$\mathcal{F}^*\hat{\varphi}(x)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathrm{e}^{+\mathrm{i}\kappa x}\hat{\varphi}(\kappa)\,\mathrm{d}\kappa.$$

Applying $\mathcal{F}$ to the equations (3.12) satisfied by $\tilde{\Phi}=\tilde{\Theta}\tilde{\phi}$ and solving the resulting differential equation in the y direction yields

$$\mathcal{F}\tilde{\Phi}(\kappa,y)=\exp(|\kappa|y)\mathcal{F}\tilde{\phi}(\kappa).$$

Taking the y-derivative of this expression at $y = 0$, we deduce that

$$\partial_y \tilde{\Theta} = \mathcal{F}^* |\kappa| \mathcal{F}. \tag{3.13}$$

Consider then the change of variables

$$\kappa \in \mathbb{R} \to (\lambda, k) := (|\kappa|, \operatorname{sgn} \kappa) \in \mathbb{R}^+ \times \{\pm 1\}.$$

The corresponding transformation

$$\hat{\varphi}(\kappa) \xrightarrow{\mathcal{C}} \mathcal{C}\hat{\varphi}(\lambda, k) := \hat{\varphi}(k\lambda)$$

is unitary from $L^2(\mathbb{R}_\kappa)$ to $L^2(\mathbb{R}^+ \times \{\pm 1\})$ since

$$\int_{\mathbb{R}} \left| \hat{\varphi}(\kappa) \right|^2 \mathrm{d}\kappa = \int_{\mathbb{R}^+} \sum_{k=\pm 1} |\mathcal{C}\hat{\varphi}(\lambda, k)|^2 \, \mathrm{d}\lambda.$$

As a consequence, formula (3.13) converts to the required form (2.7), i.e.

$$\tilde{A} = \tilde{U}^* \lambda \tilde{U},$$

where $\tilde{U} := \mathcal{C}\mathcal{F}$ appears as a product of unitary operators; it is then unitary from $L^2(\mathbb{R})$ to the *spectral* space $L^2(\mathbb{R}^+ \times \{\pm 1\})$.

This diagonal representation of $\tilde{A}$ can now be interpreted by its generalized eigenfunction expansion. Indeed, by introducing the functions

$$\tilde{\phi}_{\lambda,k}(x) := \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}k\lambda x} \quad \text{for } \lambda \in \mathbb{R}^+ \text{ and } k = \pm 1, \tag{3.14}$$

we see that $\tilde{U}$ reads

$$(\tilde{U}\tilde{\phi})_{\lambda,k} = \int_{\mathbb{R}} \tilde{\phi}(x) \overline{\tilde{\phi}_{\lambda,k}(x)} \, \mathrm{d}x.$$

We can interpret this integral as a duality product between two weighted L^2-spaces of the form

$$L^2_s(\mathbb{R}) := \{\tilde{\phi}; \eta_s \tilde{\phi} \in L^2(\mathbb{R})\}, \quad \text{where } \eta_s(x) := (1 + x^2)^{s/2}. \tag{3.15}$$

Indeed, $L^2_s(\mathbb{R})$ and $L^2_{-s}(\mathbb{R})$ are obviously dual to each other when $L^2(\mathbb{R})$ is identified with its own dual. Noting that $\tilde{\phi}_{\lambda,k} \in L^2_{-s}(\mathbb{R})$, provided that $s > \frac{1}{2}$, the above expression of $\tilde{U}$ becomes

$$(\tilde{U}\tilde{\phi})_{\lambda,k} = \langle \tilde{\phi}, \tilde{\phi}_{\lambda,k} \rangle \quad \text{for all } \tilde{\phi} \in L^2_s(\mathbb{R}), \tag{3.16}$$

where $\langle \cdot, \cdot \rangle$ denotes the integral over $\mathbb{R}$ in the rest of this section. To sum up, we have almost proved the following proposition.

Proposition 3.3. *In the functional scheme $L^2_s(\mathbb{R}) \subset L^2(\mathbb{R}) \subset L^2_{-s}(\mathbb{R})$ with $s > \frac{1}{2}$, the family $\{\tilde{\phi}_{\lambda,k} \in L^2_{-s}(\mathbb{R});\ (\lambda, k) \in \mathbb{R}^+ \times \{\pm 1\}\}$ associated with the spectral space $L^2(\mathbb{R}^+ \times \{\pm 1\})$ is a generalized spectral basis of $\tilde{A}$ (in the sense of definition 2.1). Moreover, if $s > 1$, the stability condition (2.29) holds.*

https://doi.org/10.1017/S0308210506000138 Published online by Cambridge University Press

Proof. It only remains to verify the local Hölder continuity of $\tilde{\phi}_{\lambda,k}$, i.e. property (2.5). Here this is a consequence of the inequality

$$|e^{i\tau} - e^{i\tau'}| \leqslant 2^{1-\alpha}|\tau - \tau'|^{\alpha} \quad \text{for all } \tau, \tau' \in \mathbb{R} \text{ and all } \alpha \in [0, 1]. \tag{3.17}$$

which is easily proved by noting that

$$|e^{i\tau} - e^{i\tau'}| \leqslant 2 \quad \text{and} \quad |e^{i\tau} - e^{i\tau'}| = \left| \int_{\tau}^{\tau'} e^{it}\, dt \right| \leqslant |\tau - \tau'|,$$

and taking the product of these inequalities raised to the powers $1 - \alpha$ and α, respectively. Applying (3.17) here yields

$$\|\tilde{\phi}_{\lambda,k} - \tilde{\phi}_{\lambda',k}\|^2_{L^2_{-s}(\mathbb{R})} \leqslant C|\lambda - \lambda'|^{2\alpha} \int_{\mathbb{R}} \frac{|x|^{2\alpha}}{(1 + x^2)^s}\, dx,$$

where the integral on the right-hand side is bounded if $\alpha < s - \frac{1}{2}$. As $s > \frac{1}{2}$, property (2.5) is proved. Moreover, if $s > 1$, one can choose α such that $\frac{1}{2} < \alpha < s - \frac{1}{2}$: the stability condition then follows from proposition 2.8. □

We are thus in the context of § 2.1. In particular we know that every function of $\tilde{A}$ has the generalized eigenfunction expansion

$$f(\tilde{A})\tilde{\phi} = L^2(\mathbb{R})\text{-PV} \int_{\mathbb{R}^+} f(\lambda) \sum_{k=\pm 1} \langle \tilde{\phi}, \tilde{\phi}_{\lambda,k} \rangle \tilde{\phi}_{\lambda,k}\, d\lambda \quad \text{for all } \tilde{\phi} \in D(f(\tilde{A})) \cap L^2_s(\mathbb{R}). \tag{3.18}$$

3.2.2. *Green's function and integral representation*

Our aim here is to justify formula (2.15) for the resolvent $\tilde{R}_\zeta$ of the free operator $\tilde{A}$. The spectral representation yields in this case

$$\langle \tilde{R}_\zeta \tilde{\phi}, \tilde{\psi} \rangle = \int_{\mathbb{R}^+} \frac{\langle\!\langle \Upsilon_\lambda, \overline{\tilde{\phi}} \otimes \tilde{\psi} \rangle\!\rangle}{\lambda - \zeta}\, d\lambda \tag{3.19}$$

where

$$\Upsilon_\lambda(x, x') := \sum_{k=\pm 1} \overline{\tilde{\phi}_{\lambda,k}(x)} \otimes \tilde{\phi}_{\lambda,k}(x') = \frac{1}{\pi} \cos(\lambda|x - x'|), \tag{3.20}$$

and $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ stands for a double integral over $\mathbb{R}_x \times \mathbb{R}_{x'}$, more precisely for the duality product between $L^2_{-s}(\mathbb{R}) \otimes L^2_{-s}(\mathbb{R})$ and $L^2_s(\mathbb{R}) \otimes L^2_s(\mathbb{R})$: the kernel Υ_λ clearly belongs to the former.

Permuting $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ with the integral on $\mathbb{R}^+$, we have

$$\langle \tilde{R}_\zeta \tilde{\phi}, \tilde{\psi} \rangle = \langle\!\langle g_\zeta, \overline{\tilde{\phi}} \otimes \tilde{\psi} \rangle\!\rangle = \int_{\mathbb{R}_x \times \mathbb{R}_{x'}} g_\zeta(x, x') \tilde{\phi}(x) \overline{\tilde{\psi}(x')}\, dx\, dx',$$

where g_ζ is the *Green function* of $\tilde{A}$, i.e. the kernel of its resolvent, which is formally given by

$$g_\zeta(x, x') := \text{PV} \int_{\mathbb{R}^+} \frac{\Upsilon_\lambda(x, x')}{\lambda - \zeta}\, d\lambda = \frac{1}{\pi}\, \text{PV} \int_{\mathbb{R}^+} \frac{\cos(\lambda|x - x'|)}{\lambda - \zeta}\, d\lambda. \tag{3.21}$$

The function ψ can be eliminated from the above formula, which then takes the usual form of the integral representation of $\tilde{R}_\zeta$:

$$(\tilde{R}_\zeta \tilde{\phi})(x') = \int_{\mathbb{R}} g_\zeta(x, x') \tilde{\phi}(x) \, \mathrm{d}x \quad \text{for all } x' \in \mathbb{R}. \tag{3.22}$$

We have to specify the proper interpretation of the principal value in (3.21) which allows the permutation leading to this integral representation. Here the Green function has a weak singularity, for the problem is one-dimensional.

Proposition 3.4. *The Green function of the free water-wave problem belongs to* $L^2_{-s}(\mathbb{R}) \otimes L^2_{-s}(\mathbb{R})$ *with* $s > \frac{1}{2}$*: it is given by (3.21), where the principal value is understood in* $L^2_{-s}(\mathbb{R}) \otimes L^2_{-s}(\mathbb{R})$ *at* $\lambda = +\infty$*. And the integral representation (3.22) holds for every* $\tilde{\phi} \in L^2_s(\mathbb{R})$.

Proof. First note that, for fixed x and x' such that $x \neq x'$, the principal value in (3.21) exists in $\mathbb{C}$. Indeed, integrating by parts yields

$$\int_0^M \frac{\cos(\lambda r)}{\lambda - \zeta} \, \mathrm{d}\lambda = \int_0^M \frac{\sin(\lambda r)}{r(\lambda - \zeta)^2} \, \mathrm{d}\lambda + \frac{\sin(Mr)}{r(M - \zeta)}, \quad r = |x - x'|,$$

which clearly admits a limit when $M \to +\infty$. This formula can be written

$$\int_0^M \frac{\Upsilon_\lambda}{\lambda - \zeta} \, \mathrm{d}\lambda = \int_0^M \frac{\Upsilon_\lambda^{(-1)}}{(\lambda - \zeta)^2} \, \mathrm{d}\lambda + \frac{\Upsilon_M^{(-1)}}{M - \zeta},$$

where

$$\Upsilon_\lambda^{(-1)}(x, x') := \int_0^\lambda \Upsilon_\mu(x, x') \, \mathrm{d}\mu = \frac{1}{\pi} \frac{\sin(\lambda |x - x'|)}{|x - x'|}.$$

The kernel $\Upsilon_\lambda^{(-1)}$ is bounded, and thus belongs to $L^2_{-s}(\mathbb{R}) \otimes L^2_{-s}(\mathbb{R})$ for every $\lambda > 0$. Moreover,

$$\|\Upsilon_\lambda^{(-1)}\|_{L^2_{-s}(\mathbb{R}) \otimes L^2_{-s}(\mathbb{R})} \leqslant C \lambda^{2/3}$$

for some positive constant C, since

$$\begin{aligned} \int_{\mathbb{R}} \left| \frac{\sin(\lambda |x - x'|)}{x - x'} \right|^2 \eta^2_{-s}(x) \, \mathrm{d}x & \\ &\leqslant \lambda^2 \int_{|x - x'| < \lambda^{-2/3}} \mathrm{d}x + \lambda^{4/3} \int_{|x - x'| > \lambda^{-2/3}} \eta^2_{-s}(x) \, \mathrm{d}x \\ &\leqslant C \lambda^{4/3}. \end{aligned}$$

As a consequence $(\lambda - \zeta)^{-2} \Upsilon_\lambda^{(-1)}$ is integrable over $\mathbb{R}^+$ with values in $L^2_{-s}(\mathbb{R}) \otimes L^2_{-s}(\mathbb{R})$ and

$$g_\zeta = \mathrm{PV} \int_{\mathbb{R}^+} \frac{\Upsilon_\lambda}{\lambda - \zeta} \, \mathrm{d}\lambda = \int_{\mathbb{R}^+} \frac{\Upsilon_\lambda^{(-1)}}{(\lambda - \zeta)^2} \, \mathrm{d}\lambda.$$

The integral representation is then easily justified:

$$\begin{aligned}\langle \tilde{R}_\zeta \tilde{\phi}, \tilde{\psi} \rangle &= \lim_{M\to+\infty} \left\langle\!\!\left\langle \int_0^M \frac{\Upsilon_\lambda}{\lambda-\zeta}\,\mathrm{d}\lambda, \overline{\tilde{\phi}} \otimes \tilde{\psi} \right\rangle\!\!\right\rangle \\ &= \left\langle\!\!\left\langle \lim_{M\to+\infty} \int_0^M \frac{\Upsilon_\lambda}{\lambda-\zeta}\,\mathrm{d}\lambda, \overline{\tilde{\phi}} \otimes \tilde{\psi} \right\rangle\!\!\right\rangle,\end{aligned}$$

where the first equality follows from (3.19) by Fubini's theorem, and the second follows from the above property of g_ζ. □

Proposition 3.4 is not optimal: formula (3.22) actually is valid for every $\tilde{\phi} \in L^2(\mathbb{R})$, but this is not used here.

3.2.3. *The limiting absorption principle*

Since we are in the context of definition 2.1, the statement of proposition 2.3 holds for the free problem. And the generalized eigenfunction expansion (2.24) can be translated into the limit form of the integral representation (3.22). This is the object of the following proposition.

PROPOSITION 3.5. *For every $\lambda > 0$, the one-sided limits $\tilde{R}^\pm_\lambda$ of the free resolvent $\tilde{R}_\zeta$ exist in $\mathcal{B}(L^2_s(\mathbb{R}), L^2_{-s}(\mathbb{R}))$ for $s > \frac{1}{2}$. They satisfy the integral representation*

$$(\tilde{R}^\pm_\lambda \tilde{\phi})(x') = \int_{\mathbb{R}} g^\pm_\lambda(x,x')\tilde{\phi}(x)\,\mathrm{d}x \quad \textit{for all } x' \in \mathbb{R}, \tag{3.23}$$

where $g^\pm_\lambda \in L^2_{-s}(\mathbb{R}) \otimes L^2_{-s}(\mathbb{R})$ are the one-sided limits of g_ζ, which can be expressed in the form

$$g^\pm_\lambda(x,x') = \pm \mathrm{i}\mathrm{e}^{\pm\mathrm{i}\lambda r} + \frac{1}{\pi}\operatorname{Re}\{\mathrm{e}^{\mathrm{i}\lambda r}E_1(\mathrm{i}\lambda r)\} \quad \textit{with } r := |x-x'|, \tag{3.24}$$

where E_1 denotes the exponential integral function [1].

Proof. Here formula (2.24) yields

$$\langle \tilde{R}^\pm_{\lambda_0} \tilde{\phi}, \tilde{\psi} \rangle = \mathrm{PV}\int_{\mathbb{R}^+\setminus\{\lambda_0\}} \frac{\langle\!\langle \Upsilon_\lambda, \overline{\tilde{\phi}} \otimes \tilde{\psi} \rangle\!\rangle}{\lambda-\lambda_0}\,\mathrm{d}\lambda \pm \mathrm{i}\pi \langle\!\langle \Upsilon_{\lambda_0}, \overline{\tilde{\phi}} \otimes \tilde{\psi} \rangle\!\rangle,$$

for $\tilde{\phi}$ and $\tilde{\psi}$ in $L^2_s(\mathbb{R})$. The existence of the principal value at λ_0 follows from the local Hölder continuity of Υ_λ (which derives from that of $\tilde{\phi}_{\lambda,k}$). Using the same arguments as in the proof of proposition 3.4, this formula amounts to the integral representation (3.23), where

$$g^\pm_{\lambda_0} = \mathrm{PV}\int_{\mathbb{R}^+\setminus\{\lambda_0\}} \frac{\Upsilon_\lambda}{\lambda-\lambda_0}\,\mathrm{d}\lambda \pm \mathrm{i}\pi\Upsilon_{\lambda_0}. \tag{3.25}$$

The equivalent expression (3.24) is deduced from (3.21) written in the form

$$g_\zeta(x,x') = \frac{1}{2\pi}\left\{ \mathrm{PV}\int_{\mathbb{R}^+} \frac{\mathrm{e}^{+\mathrm{i}\lambda r}}{\lambda-\zeta}\,\mathrm{d}\lambda + \mathrm{PV}\int_{\mathbb{R}^+} \frac{\mathrm{e}^{-\mathrm{i}\lambda r}}{\lambda-\zeta}\,\mathrm{d}\lambda \right\}.$$

Indeed, considering each integral as a Cauchy integral in the complex λ-plane, the residue theorem allows us to move the integration path to the half-imaginary axis located in the half-λ-plane where $\exp(\pm i\lambda r)$ is exponentially decreasing. For instance, if $\operatorname{Re}\zeta > 0$ and $\operatorname{Im}\zeta > 0$, we have

$$\mathrm{PV}\int_{\mathbb{R}^+} \frac{e^{+i\lambda r}}{\lambda - \zeta}\,d\lambda = 2i\pi e^{i\zeta r} + \int_{\mathbb{R}^+} \frac{e^{-tr}}{t + i\zeta}\,dt$$
$$= 2i\pi e^{i\zeta r} + e^{i\zeta r}\int_{i\zeta r + \mathbb{R}^+} \frac{e^{-z}}{z}\,dz,$$

where the last integral is merely $E_1(i\zeta r)$. Similarly, the second Cauchy integral is equal to $\exp(-i\zeta r)E_1(-i\zeta r)$, which yields

$$g_\zeta(x, x') = ie^{i\zeta r} + \frac{1}{2\pi}\{e^{+i\zeta r}E_1(+i\zeta r) + e^{-i\zeta r}E_1(-i\zeta r)\}.$$

The case in which $\operatorname{Re}\zeta > 0$ and $\operatorname{Im}\zeta < 0$ lead to the same expression, where the first term $i\exp(i\zeta r)$ must be replaced by $-i\exp(-i\zeta r)$. Taking the limit as $\zeta \in \mathbb{C}^\pm$ tends to $\lambda > 0$ gives (3.24). □

3.3. Compactness of the perturbation

We prove here that the sea-keeping problem (3.6) corresponds to a compact perturbation of the free problem (3.11) using the trick described in § 2.5, i.e. by comparing the resolvents $\mathcal{A} := R_{-\alpha}$ and $\tilde{\mathcal{A}} := \tilde{R}_{-\alpha}$ of A and $\tilde{A}$ for some $\alpha > 0$ in a cumulative functional framework.

3.3.1. *The comparison operator*

The free and perturbed energy spaces $\tilde{\mathcal{H}} := L^2(\tilde{F})$ and $\mathcal{H} := L^2(F) \times \mathbb{C}^2$ appear naturally as subspaces of $\boldsymbol{\mathcal{H}} := L^2(\tilde{F}) \times \mathbb{C}^2$, since (2.56) holds with complementary subspaces given by $\mathcal{H}_0 := L^2(F_0)$, where $F_0 := \tilde{F} \setminus F$ and $\tilde{\mathcal{H}}_0 := \mathbb{C}^2$. We denote by $\mathcal{P} : \boldsymbol{\mathcal{H}} \to \mathcal{H}$ and $\mathcal{P}_0 : \boldsymbol{\mathcal{H}} \to \mathcal{H}_0$ the canonical restrictions associated with the perturbed decomposition, that is,

$$\mathcal{P}(\tilde{\phi}, p) = (\tilde{\phi}|_F, p) \quad \text{and} \quad \mathcal{P}_0(\tilde{\phi}, p) = \tilde{\phi}|_{F_0} \quad \text{for all } (\tilde{\phi}, p) \in \boldsymbol{\mathcal{H}}.$$

Consider then the comparison operator (2.57) for some given $\alpha > 0$,

$$\boldsymbol{\mathcal{D}} := \boldsymbol{\mathcal{A}} - \tilde{\boldsymbol{\mathcal{A}}}, \quad \text{where } \boldsymbol{\mathcal{A}} := R_{-\alpha} \oplus 0 \text{ and } \tilde{\boldsymbol{\mathcal{A}}} := \tilde{R}_{-\alpha} \tilde{\oplus} 0,$$

which can be written

$$\boldsymbol{\mathcal{D}}\boldsymbol{u} = \{R_{-\alpha}\mathcal{P}\boldsymbol{u} - \mathcal{P}(\tilde{R}_{-\alpha}\tilde{\phi}, 0)\} \oplus \{-\mathcal{P}_0(\tilde{R}_{-\alpha}\tilde{\phi}, 0)\} \quad \text{for all } \boldsymbol{u} = (\tilde{\phi}, p) \in \boldsymbol{\mathcal{H}}.$$

The aim of this section is to prove the following proposition.

PROPOSITION 3.6. *For $s, s' < \frac{3}{2}$, the operator $\boldsymbol{\mathcal{D}}$ is compact from $L^2_{-s}(\tilde{F}) \times \mathbb{C}^2$ to $L^2_{s'}(\tilde{F}) \times \mathbb{C}^2$.*

Let us first make explicit the above definition of $\boldsymbol{\mathcal{D}}$. For $\boldsymbol{u} = (\tilde{\phi}, p) \in \boldsymbol{\mathcal{H}}$, define $\tilde{\phi}_\alpha := \tilde{R}_{-\alpha}\tilde{\phi} \in \tilde{\mathcal{H}}$ and $u_\alpha := R_{-\alpha}\mathcal{P}\boldsymbol{u} = (\varphi_\alpha, p_\alpha) \in \mathcal{H}$, so that

$$\boldsymbol{\mathcal{D}}\boldsymbol{u} = (\varphi_\alpha - (\tilde{\phi}_\alpha)|_F, p_\alpha) \oplus (-(\tilde{\phi}_\alpha)|_{F_0}). \tag{3.26}$$

By virtue of (3.6), u_α is the solution to the following coupled problem:

$$\partial_y \Theta u_\alpha + \alpha\varphi_\alpha = \tilde{\phi}|_F,$$
$$\mathbb{K} p_\alpha + \int_\Gamma \Theta u_\alpha \nu \, \mathrm{d}\gamma + \alpha p_\alpha = p.$$

From the definition (3.5) of Θ, this amounts to solving the boundary-value problem

$$\begin{aligned} \Delta\Phi_\alpha &= 0 \quad \text{in } \Omega, \\ \partial_y\Phi_\alpha + \alpha\Phi_\alpha &= \tilde{\phi} \quad \text{on } F, \\ \partial_n\Phi_\alpha + \left(\mathbb{K}p_\alpha + \int_\Gamma \Phi_\alpha\nu\,\mathrm{d}\gamma\right)\cdot\nu &= 0 \quad \text{on } \Gamma, \\ \mathbb{K}p_\alpha + \int_\Gamma \Phi_\alpha\nu\,\mathrm{d}\gamma + \alpha p_\alpha &= p, \end{aligned}$$

where $\Phi_\alpha := \Theta u_\alpha$, so $\varphi_\alpha = (\Phi_\alpha)|_F$. Similarly, for the free problem we have $\tilde{\phi}_\alpha = (\tilde{\Phi}_\alpha)|_{\tilde{F}}$, where

$$\left.\begin{aligned} \Delta\tilde{\Phi}_\alpha &= 0 \quad \text{in } \tilde{\Omega}, \\ \partial_y\tilde{\Phi}_\alpha + \alpha\tilde{\Phi}_\alpha &= \tilde{\phi} \quad \text{on } \tilde{F}. \end{aligned}\right\} \tag{3.27}$$

As a consequence, to find $\boldsymbol{\mathcal{D}u}$, we must solve first (3.27) and then the following coupled system, where $\Psi := \Phi_\alpha - (\tilde{\Phi}_\alpha)|_\Omega$:

$$\left.\begin{aligned} \Delta\Psi &= 0 && \text{in } \Omega, \\ \partial_y\Psi + \alpha\Psi &= 0 && \text{on } F, \\ \partial_n\Psi + \left(\mathbb{K}q + \int_\Gamma \Psi\nu\,\mathrm{d}\gamma\right)\cdot\nu &= -\partial_n\tilde{\Phi}_\alpha - \left(\int_\Gamma \tilde{\Phi}_\alpha\nu\,\mathrm{d}\gamma\right)\cdot\nu && \text{on } \Gamma, \\ \mathbb{K}q + \int_\Gamma \Psi\nu\,\mathrm{d}\gamma + \alpha q &= p - \int_\Gamma \tilde{\Phi}_\alpha\nu\,\mathrm{d}\gamma. \end{aligned}\right\} \tag{3.28}$$

Relation (3.26) thus becomes

$$\boldsymbol{\mathcal{D}u} = (\Psi|_F, q) \oplus (-(\tilde{\Phi}_\alpha)|_{F_0}). \tag{3.29}$$

In order to prove that $\boldsymbol{\mathcal{D}}$ enters the framework of definition 2.9, we need precise information about the solutions of problems (3.27) and (3.28), which will be obtained with the help of the 'immersed' Green function. The latter provides an explicit integral representation of $\tilde{\Phi}_\alpha$ and an implicit one for Ψ.

3.3.2. *The 'immersed' Green function*

The Green function g_ζ introduced in § 3.2 (see proposition 3.4) is not adapted to study the solution to (3.28) because the perturbation extends under the free surface. This function represents the response of the free surface to a point source located on this free surface. Here we need the response of the whole sea to an immersed excitation, represented by the 'immersed' Green function, which is merely the function generally used in the literature for integral equations (see, for example,

https://doi.org/10.1017/S0308210506000138 Published online by Cambridge University Press

[28, 42]). We denote it by $G_\zeta = G_\zeta(X, X')$, where $X = (x, y)$ and $X' = (x', y')$ belong to $\tilde{\Omega}$, and $\zeta \in \mathbb{C} \setminus \mathbb{R}^+$. It satisfies the following equations:

$$\begin{aligned} \Delta_X G_\zeta(X, X') &= \delta_{X'}(X) \quad \text{for } X \in \tilde{\Omega}, \\ \partial_y G_\zeta(X, X') - \zeta G_\zeta(X, X') &= 0 \qquad\quad \text{for } X \in \tilde{F}, \end{aligned}$$

where $\delta_{X'}$ denotes the Dirac measure at point X'. It is given by

$$G_\zeta(X, X') = \frac{1}{2\pi}\left\{ \ln\left|\frac{z - \overline{z'}}{z - z'}\right| + 2\int_{\mathbb{R}^+} \frac{\operatorname{Re}(e^{i\lambda(\bar{z} - z')})}{\lambda - \zeta}\,d\lambda \right\}, \tag{3.30}$$

where $z = z(X) := x + iy$ and $z' = z'(X) := x' + iy'$. Of course, this expression coincides with (3.21) when $X, X' \in \tilde{F}$, that is, $y = y' = 0$: the free surface Green function is simply the trace on $\tilde{F} \times \tilde{F}$ of the immersed one.

In the following we are interested only in the particular case $\zeta = -\alpha$ with $\alpha > 0$ for which G_ζ has a stronger decay at infinity than for $\operatorname{Im}\zeta \neq 0$.

LEMMA 3.7. *When $r := |x - x'| \to +\infty$, we have $G_{-\alpha}(X, X') = O(r^{-2})$ uniformly in every strip $y_0 < y, y' \leqslant 0$. And the same behaviour holds for every spatial derivative of $G_{-\alpha}$.*

Proof. Under the assumption that $y + y'$ remains bounded, we have

$$\ln\left|\frac{z - \overline{z'}}{z - z'}\right| = \frac{(y + y')^2 - (y - y')^2}{2r^2} + O(r^{-4}),$$

which shows that the first term in (3.30) is $O(r^{-2})$. Following the same idea as in the proof of (3.24), the second term can be written as

$$\int_{\mathbb{R}^+} \frac{\operatorname{Re}(e^{i\lambda(\bar{z} - z')})}{\lambda + \alpha}\,d\lambda = \operatorname{Re}\int_{\mathbb{R}^+} \frac{e^{-t\rho}}{t - i\alpha}\,dt, \quad \text{where } \rho := r - i(y + y').$$

Then, integrating by parts twice yields

$$\int_{\mathbb{R}^+} \frac{e^{-t\rho}}{t - i\alpha}\,dt = \frac{1}{\rho^2}\int_{\mathbb{R}^+} \frac{2e^{-t\rho}}{(t - i\alpha)^3}\,dt + \frac{1}{\rho^2\alpha^2} + \frac{i}{\rho\alpha}.$$

The two first terms of the right-hand side are obviously $O(r^{-2})$. As α is real, the real part of the third term is also $O(r^{-2})$. Hence, the lemma is proved for $G_{-\alpha}$. For its spatial derivatives, the same arguments apply: the logarithmic term is dealt with by a direct derivation, and the latter formula can be derived with respect to ρ. □

3.3.3. *The free problem*

For $\tilde{\phi} \in L^2(\tilde{F})$, problem (3.27) is well posed in

$$W^1_{\tilde{F}}(\tilde{\Omega}) := \{\tilde{\Phi} \in W^1(\tilde{\Omega});\ \tilde{\Phi}|_{\tilde{F}} \in L^2(\tilde{F})\},$$

where $W^1(\tilde{\Omega})$ is given in lemma 3.1. Indeed, its variational formulation reads

$$\int_{\tilde{\Omega}} \nabla\tilde{\Phi}_\alpha \cdot \overline{\nabla\tilde{\Phi}'} + \alpha\int_F \tilde{\Phi}_\alpha\overline{\tilde{\Phi}'}\,dx = \int_F \tilde{\phi}\overline{\tilde{\Phi}'}\,dx \quad \text{for all } \tilde{\Phi}' \in W^1_{\tilde{F}}(\tilde{\Omega}),$$

where the left-hand side defines an inner product in $W^1_{\tilde F}(\tilde\Omega)$. Hence, the operator $\tilde\Theta_\alpha$ which maps $\tilde\phi$ to the solution $\tilde\Phi_\alpha$ of (3.27) is continuous from $L^2(\tilde F)$ to $W^1_{\tilde F}(\tilde\Omega)$. The knowledge of $G_{-\alpha}$ yields an explicit integral representation of this operator:

$$(\tilde\Theta_\alpha\tilde\phi)(X') = \tilde\Phi_\alpha(X') = \int_{\tilde F} G_{-\alpha}((x,0),X')\tilde\phi(x)\,\mathrm{d}x \quad \text{for all } X'\in\tilde\Omega \tag{3.31}$$

which is classically obtained by Green's formula. This relation is merely the harmonic lifting in $\tilde\Omega$ of the integral representation (3.22) since $\tilde\Theta_\alpha = \tilde\Theta R_{-\alpha}$ and $G_{-\alpha}((x,0),\cdot) = \tilde\Theta g_{-\alpha}(x,\cdot)$, where $\tilde\Theta$ is defined in (3.12).

We want to extend formula (3.31) to every $\tilde\phi\in L^2_{-s}(\tilde F)$. We are interested only in the local behaviour of $\tilde\Phi_\alpha$, say in $\tilde\Omega_R := \{X\in\tilde\Omega;\ \|X\|<R\}$, since the perturbed problem (3.28) involves only $(\tilde\Phi_\alpha)|_\Gamma$ and $(\partial_n\tilde\Phi_\alpha)|_\Gamma$.

LEMMA 3.8. *For $s<\frac{3}{2}$ and $R>0$, the mapping $\tilde\Theta_\alpha : \tilde\phi\to\tilde\Phi_\alpha$ extends to a bounded operator from $L^2_{-s}(\tilde F)$ to $H^1_\Delta(\tilde\Omega_R) := \{\tilde\Phi\in H^1(\tilde\Omega_R);\ \Delta\tilde\Phi = 0$ in $\tilde\Omega_R\}$.*

Proof. Choose $R'>R$ and denote by $\chi_{R'}$ the characteristic function of the set $\{|x|<R'\}$. Then $\tilde\Theta_\alpha$ can be split as follows:

$$\tilde\Theta_\alpha = \tilde\Theta_\alpha^{R'} + \tilde\Theta_\alpha^{\infty}, \quad \text{where } \tilde\Theta_\alpha^{R'}\tilde\phi := \tilde\Theta_\alpha(\chi_{R'}\tilde\phi) \text{ and } \tilde\Theta_\alpha^{\infty}\tilde\phi := \tilde\Theta_\alpha((1-\chi_{R'})\tilde\phi).$$

For every $s>0$, the operator $\tilde\Theta_\alpha^{R'}$ is clearly continuous from $L^2_{-s}(\tilde F)$ to $W^1_{\tilde F}(\tilde\Omega)$, and thus also to $H^1_\Delta(\tilde\Omega_R)$ by restriction. To deal with $\tilde\Theta_\alpha^\infty$, note that, by Schwarz's inequality, (3.31) yields

$$|(\tilde\Theta_\alpha^\infty\tilde\phi)(X')|^2 \leqslant \|\tilde\phi\|^2_{L^2_{-s}(\tilde F)} \int_{|x|>R'} |\eta_s(x)G_{-\alpha}((x,0),X')|^2\,\mathrm{d}x.$$

Lemma 3.7 shows that if $s<\frac{3}{2}$, the above integral is a bounded function of $X'\in\tilde\Omega_R$. And a similar inequality holds for every spatial derivative of $\tilde\Theta_\alpha^\infty\tilde\phi$. Hence, $\tilde\Theta_\alpha^\infty$ is continuous from $L^2_{-s}(\tilde F)$ to $H^m(\tilde\Omega_R)$ for every $m>0$. The conclusion follows. □

3.3.4. *The perturbed problem*

As above, problem (3.28) is well posed in the space $W^1_F(\Omega)\times\mathbb{C}^2$. Indeed, its variational formulation can be written

$$\begin{aligned}\int_\Omega \nabla\Psi\cdot\overline{\nabla\Psi'} + \alpha\int_F \Psi\overline{\Psi'}\,\mathrm{d}x + \alpha\mathbb{K}q\cdot\overline{q'} + \left(\mathbb{K}q + \int_\Gamma \Psi\nu\,\mathrm{d}\gamma\right)\cdot\left(\mathbb{K}\overline{q'} + \int_\Gamma \overline{\Psi'}\nu\,\mathrm{d}\gamma\right) \\ = -\int_\Gamma \partial_n\tilde\Phi_\alpha\overline{\Psi'}\,\mathrm{d}\gamma - \left(\int_\Gamma \tilde\Phi_\alpha\nu\,\mathrm{d}\gamma\right)\cdot\left(\mathbb{K}\overline{q'} + \int_\Gamma \overline{\Psi'}\nu\,\mathrm{d}\gamma\right) + \mathbb{K}p\cdot\overline{q'},\end{aligned}$$

for all pairs $(\Psi',q')\in W^1_F(\Omega)\times\mathbb{C}^2$. The left-hand side is clearly a coercive sesquilinear form in this space, and the right-hand side depends continuously on $(\tilde\Phi_\alpha,p)\in H^1_\Delta(\tilde\Omega_R)\times\mathbb{C}^2$ (where R must be chosen so that $\Gamma\subset\tilde\Omega_R$). This means that the operator that maps $(\tilde\Phi_\alpha,p)$ to the solution (Ψ,q) of (3.28) is continuous from $H^1_\Delta(\tilde\Omega_R)\times\mathbb{C}^2$ to $W^1_F(\Omega)\times\mathbb{C}^2$. Moreover, Ψ satisfies the following usual integral representation [28, 42] for all $X'\in\Omega$:

$$\Psi(X') = \int_\Gamma (\Psi(X)\partial_{n_X}G_{-\alpha}(X,X') - \partial_n\Psi(X)G_{-\alpha}(X,X'))\,\mathrm{d}\gamma_X. \tag{3.32}$$

Lemma 3.9. *For $s' < \frac{3}{2}$, the mapping*

$$(\tilde{\Phi}_\alpha, p) \to (\Psi|_F, q)$$

is continuous from $H^1_\Delta(\tilde{\Omega}_R) \times \mathbb{C}^2$ to $H^{1/2}_{s'}(F) \times \mathbb{C}^2$, where $H^{1/2}_{s'}(F) := H^{1/2}(F) \cap L^2_{s'}(F)$.

Proof. From (3.7) we know that $H^{1/2}(F)$ is the trace space of $W^1_F(\Omega)$. On the other hand, the above integral representation and lemma 3.7 show that $\Psi(x, 0) = O(|x|^{-2})$, and hence $\Psi|_F \in L^2_{s'}(F)$ for every $s' < \frac{3}{2}$. □

3.3.5. *Proof of proposition 3.6 and consequences*

The characterization (3.29) of $\boldsymbol{\mathcal{D}}$ together with lemmas 3.8 and 3.9 show that $\boldsymbol{\mathcal{D}}$ is continuous from $L^2_{-s}(\tilde{F}) \times \mathbb{C}^2$ to $\{H^{1/2}_{s'}(F) \oplus H^{1/2}(F_0)\} \times \mathbb{C}^2$ for $s, s' < \frac{3}{2}$. Moreover, by the Rellich theorem, the canonical injections $H^{1/2}(F_0) \subset L^2(F_0)$ and $H^{1/2}_{s'}(F) \subset L^2_{s''}(F)$ are compact if $s' > s''$. So proposition 3.6 is proved.

This guarantees the compactness of the perturbation in the sense of definition 2.9.

Corollary 3.10. *For $\frac{1}{2} < s < \frac{3}{2}$, the operators $\boldsymbol{\mathcal{D}}\widetilde{\boldsymbol{\mathcal{R}}}^{\pm}_{\mu}$ are compact in*

$$\boldsymbol{\mathcal{H}}_{\downarrow} := L^2_s(\tilde{F}) \times \mathbb{C}^2 \quad \textit{for every } \mu \in]0, \alpha^{-1}[.$$

Proof. From proposition 3.5, we know that $\widetilde{\boldsymbol{\mathcal{R}}}^{\pm}_{\mu} = \tilde{\mathcal{R}}^{\pm}_{\mu} \tilde{\oplus} (-\mu^{-1}\mathrm{Id}_{\mathbb{C}^2})$ is continuous from $L^2_s(\tilde{F}) \times \mathbb{C}^2$ to $L^2_{-s}(\tilde{F}) \times \mathbb{C}^2$ if $s > \frac{1}{2}$. Choosing $s' = s$ in proposition 3.6 yields the result. □

3.4. Conclusion

The previous sections show that our wave problem comes within the field of application of theorem 2.11. Indeed, we know a generalized spectral basis for the free problem, which satisfies the stability condition (proposition 3.3). And the perturbation we consider is compact (corollary 3.10). As a consequence, for $1 < s < \frac{3}{2}$, both families in (2.59), i.e.

$$w^{\pm}_{\lambda,k} := (\mathcal{P} - \mathcal{R}^{\mp}_{\mu}\mathcal{P}\boldsymbol{\mathcal{D}})(\tilde{w}_{\lambda,k}, 0) \in L^2_{-s}(\tilde{F}) \times \mathbb{C}^2, \quad \text{where } \mu := (\lambda + \alpha)^{-1},$$

define generalized spectral bases for A. It remains to show a more explicit characterization of these eigenfunctions, by a proper interpretation of $\mathcal{R}^{\mp}_{\mu}\mathcal{P}\boldsymbol{\mathcal{D}}$.

For $\boldsymbol{u} = (\tilde{\phi}, p) \in \boldsymbol{\mathcal{H}}_{\uparrow}$, define $v := \mathcal{P}\boldsymbol{\mathcal{D}}\boldsymbol{u} = (\psi, q)$ and $\dot{u} := \mathcal{R}_{\xi}v = (\dot{\varphi}, \dot{p})$, where $\xi \in \mathbb{C}^{\pm}$. Formula (3.29) tells us that $v = (\Psi|_F, q)$, where (Ψ, q) is obtained by successively solving (3.27) and (3.28). Moreover, by (2.54), we have $\dot{u} = -\xi^{-1}(\mathrm{Id} + \xi^{-1}R_{\zeta})v$, where $\xi = (\zeta + \alpha)^{-1}$ or, equivalently,

$$(A - \zeta)(\xi\dot{u} + v) = -\xi^{-1}v.$$

By virtue of the definition (3.6) of A, this means that

$$\partial_y \Theta(\xi\dot{u} + v) - \zeta(\xi\dot{\varphi} + \psi) = -\xi^{-1}\psi,$$
$$\mathbb{K}(\xi\dot{p} + q) + \int_{\Gamma} \Theta(\xi\dot{u} + v)\nu \, \mathrm{d}\gamma - \zeta(\xi\dot{p} + q) = -\xi^{-1}q.$$

By defining $\dot{\Phi}$ such that

$$\Theta(\xi\dot{u}+v)=\xi\dot{\Phi}+\Psi,$$

and using the equations (3.28), this system amounts to solving

$$\begin{aligned}
\Delta\dot{\Phi}&=0 && \text{in } \Omega,\\
\partial_y\dot{\Phi}-\zeta\dot{\Phi}&=0 && \text{on } F,\\
\partial_n\dot{\Phi}+\biggl(\mathbb{K}\dot{p}+\int_\Gamma\dot{\Phi}\nu\,\mathrm{d}\gamma\biggr)\cdot\nu&=\xi^{-1}\biggl(\partial_n\tilde{\Phi}_\alpha+\nu\cdot\int_\Gamma\tilde{\Phi}_\alpha\nu\,\mathrm{d}\gamma\biggr) && \text{on } \Gamma,\\
\mathbb{K}\dot{p}+\int_\Gamma\dot{\Phi}\nu\,\mathrm{d}\gamma-\zeta\dot{p}&=\xi^{-1}\biggl(-p+\int_\Gamma\tilde{\Phi}_\alpha\nu\,\mathrm{d}\gamma\biggr).
\end{aligned}$$

This means that, to find $\dot{u}:=\mathcal{R}_\xi\mathcal{P}\boldsymbol{\mathcal{D}}\boldsymbol{u}$, we must solve first (3.27) and then the above system, which yields $\dot{u}=(\dot{\Phi}|_F,\dot{p})$.

When $\xi\in\mathbb{C}^\mp\to\mu\in\,]0,\alpha^{-1}[$, that is, when $\zeta\in\mathbb{C}^\pm$ tends to $\lambda\in\mathbb{R}^+$, the limiting absorption principle tells us that the limit of $(\dot{\Phi},\dot{p})$ exists and satisfies the above equations with ζ replaced by λ (and ξ by μ). In the particular case $\boldsymbol{u}=-(\tilde{w}_{\lambda,k},0)$, we rediscover equations (1.10)–(1.13) satisfied by the scattered wave $(\dot{\Phi}^\pm_{\lambda,k},p^\pm_{\lambda,k})$, since in this situation $\tilde{\Phi}_\alpha=\mu\tilde{\Phi}_{\lambda,k}$, where $\tilde{\Phi}_{\lambda,k}$ is given in (1.6).

The radiation condition (1.14) that characterizes the outgoing or incoming solution is then easily justified using the immersed Green function (3.30) again. Indeed, the solution $(\dot{\Phi},\dot{p})$ of the above equations satisfies the integral representation (3.32) with $G_{-\alpha}$ replaced by G_ζ. Hence,

$$\dot{\Phi}^\pm_{\lambda,k}(X')=\int_\Gamma(\dot{\Phi}^\pm_{\lambda,k}(X)\partial_{n_X}G^\pm_\lambda(X,X')-\partial_n\dot{\Phi}^\pm_{\lambda,k}(X)G^\pm_\lambda(X,X'))\,\mathrm{d}\gamma_X\quad\forall X'\in\Omega,$$

where $G^\pm_\lambda$ are the one-sided limits of G_ζ, which shows that $\dot{\Phi}^\pm_{\lambda,k}$ satisfies the radiation condition (1.14) if the same holds for $G^\pm_\lambda(X,\cdot)$ and $\partial_{n_X}G^\pm_\lambda(X,\cdot)$ uniformly in $X\in\Gamma$. Following the same idea as in proposition 3.5, we see that

$$G^\pm_\lambda(X,X')=\pm\mathrm{i}\mathrm{e}^{\lambda(\pm\mathrm{i}|x-x'|+(y+y'))}+E^\pm_\lambda(X,X'),$$

where $E^\pm_\lambda(X,X')$ represents the evanescent component of $G^\pm_\lambda(X,X')$ whose contribution vanishes in the radiation condition. And the dominant term obviously satisfies (1.14). As a conclusion, we have justified the intuitive construction of § 1.2.

THEOREM 3.11. *Under the assumption that the operator A, defined in proposition 3.2, has no eigenvalues, both families*

$$w^\pm_{\lambda,k}:=((\tilde{\Phi}_{\lambda,k}+\dot{\Phi}^\pm_{\lambda,k})|_F,p^\pm_{\lambda,k}),\quad\lambda\in\mathbb{R}^+\text{ and }k=\pm1,$$

given by (1.6) and (1.10)–(1.14), and associated with the spectral space $L^2(\mathbb{R}^+\times\{\pm1\})$, define generalized spectral bases for A in the functional scheme

$$L^2_s(F)\times\mathbb{C}^2\subset L^2(F)\times\mathbb{C}^2\subset L^2_{-s}(F)\times\mathbb{C}^2\text{ with }s>1.$$

In particular the announced generalized eigenfunction expansion (1.15) of the transient coupled motions is now established: it has to be understood in the sense of (2.19).

Note that the absence of eigenvalues of A, which amounts to the uniqueness of the solution to (1.10)–(1.14), seems to be an open question. The case of the scattering by a fixed obstacle has been extensively studied, from the uniqueness proof pioneered by John [26] to the discovery of trapped waves by McIver [32] (see [28] for a review of these results). But very little is known about the coupled problem.

Acknowledgments

The authors gratefully acknowledge the anonymous referee for careful reading of the manuscript, comments and helpful suggestions, particularly regarding the statement and the proof of proposition 2.8.

References

1 M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions* (New York: Dover, 1970).

2 S. Agmon. Spectral properties of Schrödinger operators and scattering theory. *Ann. Scuola Norm. Sup. Pisa IV* **2** (1975), 151–218.

3 C. Amrouche. The Neumann problem in the half-space. *C. R. Acad. Sci. Paris Sér. I* **335** (2002), 151–156.

4 C. Athanasiadis, K. Kiriaki and I. G. Stratis. A spectral-theoretic approach for the solution of Maxwell's equations in stratified media. *Math. Meth. Appl. Sci.* **21** (1998), 685–700.

5 J.-P. Aubin. *Applied functional analysis* (Wiley, 2000).

6 J. T. Beale. Eigenfunction expansions for objects floating in an open sea. *Commun. Pure Appl. Math.* **30** (1977), 283–313.

7 Y. M. Berezanskii. *Expansions in eigenfunctions of selfadjoint operators.* Translations of Mathematical Monographs, vol. 17 (Providence, RI: American Mathematical Society, 1968).

8 Y. M. Berezansky, Z. G. Sheftel and G. F. Us. *Functional analysis, II*, Operator Theory Advances and Applications, vol. 86 (Basel: Birkhäuser, 1996).

9 M. S. Birman and M. Z. Solomjak. *Spectral theory of self-adjoint operators in Hilbert space* (Dordrecht: Reidel, 1972).

10 T. Bouhennache. Spectral analysis of a perturbed multistratified isotropic electric strip: new method. *Math. Meth. Appl. Sci.* **22** (1999), 689–716.

11 E. Croc and Y. Dermenjian. A perturbative method for the spectral analysis of an acoustic multistratified strip. *Math. Meth. Appl. Sci.* **21** (1998), 1681–1704.

12 Y. Dermenjian and P. Gaitan. Study of generalized eigenfunctions of a perturbed isotropic elastic half-space. *Math. Meth. Appl. Sci.* **23** (2000), 685–708.

13 Y. Dermenjian and J. C. Guillot. Théorie spectrale de la propagation des ondes acoustiques dans un milieu stratifié perturbé. *J. Diff. Eqns* **62** (1986), 357–409.

14 D. M. Eidus. The principle of limiting absorption. *Am. Math. Soc. Transl. 1* **47** (1965), 157–191.

15 I. M. Gel'fand and G. E. Shilov. *Generalized eigenfunctions*, vol. 3 (Academic, 1967).

16 C. Golstein. Eigenfunction expansions associated with the Laplacian for certain domains with infinite boundaries. I. *Trans. Am. Math. Soc.* **135** (1969), 1–31.

17 C. Golstein. Eigenfunction expansions associated with the Laplacian for certain domains with infinite boundaries. II. *Trans. Am. Math. Soc.* **135** (1969), 33–50.

18 C. Golstein. Eigenfunction expansions associated with the Laplacian for certain domains with infinite boundaries. III. *Trans. Am. Math. Soc.* **143** (1969), 283–301.

19 C. Hazard. Etude des résonances pour le problème linéarisé des mouvements d'un navire sur la houle. PhD thesis, University Paris VI (1991). (Reprinted as ENSTA technical report 257, 1991.)

20 C. Hazard. Analyse modale de la propagation des ondes. Habilitation thesis, University Paris VI (2001).

21 C. Hazard and M. Lenoir. Surface water waves. In *Scattering* (ed. R. Pike and P. Sabatier) (Academic, 2002).

22 C. Hazard and M. Meylan. Spectral theory for an elastic thin plate floating on water of finite depth. *SIAM J. Appl. Math.* (In the press.)

23 P. Henrici. *Applied and computational complex analysis*, vol. 3 (Wiley, 1986).

24 T. Ikebe. Eigenfunction expansions associated with the Schrödinger operators and their applications to scattering theory. *Arch. Ration. Mech. Analysis* **5** (1960), 1–34.

25 F. John. On the motion of floating bodies. I. *Commun. Pure Appl. Math.* **2** (1949), 13–57.

26 F. John. On the motion of floating bodies. II. *Commun. Pure Appl. Math.* **3** (1950), 45–101.

27 T. Kato. *Perturbation theory for linear operators* (Springer, 1976).

28 N. Kuznetsov, V. Maz'ya and B. Vainberg. *Linear water waves: a mathematical approach* (Cambridge University Press, 2002).

29 W. C. Lyford. Spectral analysis of the Laplacian in domains with cylinders. *Math. Ann.* **218** (1975), 213–251.

30 W. C. Lyford. Spectral analysis of the Laplacian in distorted periodic waveguides. *Math. Ann.* **236** (1978), 255–284.

31 M. Mabrouk and Z. Helali. The scattering theory of C. Wilcox in elasticity. *Math. Meth. Appl. Sci.* **25** (2002), 997–1044.

32 M. McIver. An example of non-uniqueness in the two-dimensional linear water wave problem. *J. Fluid Mech.* **315** (1996), 257–266.

33 A. Majda. Outgoing solutions for perturbations of $-\Delta$ with applications to spectral and scattering theory. *J. Diff. Eqns* **16** (1974), 515–547.

34 F. Menéndez-Conde. Eigenfunction expansions and spectral projections for isotropic elasticity outside an obstacle. *J. Math. Analysis Applic.* **299** (2004), 676–689.

35 M. Reed and B. Simon. *Methods of modern mathematical physics, vol. IV: analysis of operators* (Academic, 1978).

36 J. Sanchez Hubert and E. Sanchez Palencia. *Vibration and coupling of continuous systems* (Springer, 1989).

37 N. Shenk and D. Thoe. Eigenfunction expansions and scattering theory for perturbations of $-\Delta$. *J. Math. Analysis Applic.* **36** (1971), 313–351.

38 E. C. Titchmarsh. *Eigenfunction expansions associated with second-order differential equations* (Oxford University Press, 1946).

39 M. Vullierme-Ledard. The limiting amplitude principle applied to the motion of floating bodies. *Math. Model. Numer. Analysis* **21** (1987), 125–170.

40 R. Weder. Spectral analysis of strongly propagative systems. *J. Reine Angew. Math.* **354** (1984), 95–122.

41 R. Weder. *Spectral and scattering theory for wave propagation in perturbed stratified media* (Springer, 1991).

42 J. V. Wehausen and E. V. Laitone. Surface waves. In *Handbuch der Physik*, vol. 9 (Springer, 1960).

43 C. H. Wilcox. *Scattering theory for the d'Alembert equation in exterior domains* (Springer, 1975).

44 C. H. Wilcox. *Scattering theory for diffraction gratings* (Springer, 1984).

45 C. H. Wilcox. *Sound propagation in stratified fluids* (Springer, 1984).

46 K. Yosida. *Functional analysis* (Springer, 1974).

(*Issued 12 October 2007*)