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AN EMERGENCY FACILITY WITH MINIMAL ADMISSION

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We study an $M/M/\infty$ queuing system in which each arrival has a random urgency and is admitted if and only if it is more urgent than all individuals currently receiving service. The system represents, for example, a less-than-magnanimous emergency facility. For this system, and also for a closely related "Parody," we study the busy period distribution and, to a lesser extent, occupancy. Both exact and heavy traffic results are given.

1. INTRODUCTION

A favorite interpretation of the venerable $M/M/\infty$ queue is that it models an emergency facility treating randomly arriving urgent cases. No case is turned away; each receives immediate attention; and the length of stay is unaffected by facility congestion. In seeking to reduce the strain on such a facility, while retaining the accolade "emergency," it is arguable that the last two of these features are inviolable; furthermore, the system should never be declared full—potentially, service must be unlimited. The only recourse, then, is to alter the definition of "urgent." One approach is to turn away all but very urgent cases. This thins the arrival stream and creates a new $M/M/\infty$ queue. An alternative and more interesting scheme, possessing some political finesse, is to admit only the cases that are more urgent than all those presently receiving attention. It is this parsimonious scheme that is our object of study.

Specifically, we consider a Minimal Admission Emergency Facility (MAEF) with the following characteristics:

A1. Initially, the system is empty; cases arrive in a rate λ Poisson stream.

A2. Each case has a real-valued urgency with continuous distribution $\mathcal{U}$.

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A3. A case is admitted to the facility if its urgency exceeds those of every case currently in service; otherwise, it is lost forever.

A4. The sojourn time for each admitted case is unit exponential.

A5. The arrival stream, the case urgencies, and the sojourns are mutually independent.

Little is foregone through taking $\mathcal{U}$ to be the uniform distribution on (0,1), and this we do. (Being a rudimentary model for the notion of "transient records," the MAEF has a number of other perspectives; see [4].)

We analyze the MAEF both for fixed λ and in heavy traffic ($\lambda \to \infty$). For the latter, it is fruitful to introduce a second system, named the Parody, which emulates the MAEF in a rather oblique way. In the Parody system the service facility has a countable stack of slots—the top one labeled S_1, the next S_2, and so on. At any given time, a slot may be either vacant or occupied by a single admitted case. Let S^* denote the first slot below S_1 that is occupied. The rules for the Parody are as follows:

B1. Initially, every slot is occupied.

B2. Whenever the case in slot S_1 completes its service, there is an influx of new cases into the system, instantaneously filling all of the slots above S^*.

B3. The sojourn times for admitted cases are independent unit exponentials.

In view of rule B2, rule B1 is tantamount to the system being initially empty. A realization of the Parody is depicted in Figure 1. The bars indicate slot occupancy and the arrows indicate sojourn completion times for S_1. A snapshot of the occupancy pattern and the associated slot S^* are shown at time t.

As well as being a tool for studying the MAEF, the parameter-free Parody would seem to be of intrinsic interest. Here, we shall be content to examine developments

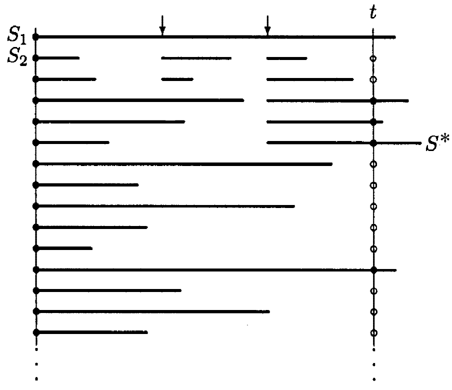

FIGURE 1. A realization of the Parody.

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in slots S_1 through S_n, for fixed but arbitrary n, and this obviates an interactive particle system construction.

There are many questions of substance concerning the respective busy periods, occupancy processes, urgency profiles of occupants, streams of lost and departing cases, and so forth. This article is largely concerned with busy periods for the two systems; it is organized as follows. One probability distribution plays a distinguished role in the proceedings, and we devote Section 2 to this. The main work begins in Section 3 with a study of the busy period for the MAEF; Section 4 is a parallel for the rather more tractable Parody. Section 5 deals with asymptotics for (a generalization of) the Parody with a large, finite number of slots; and Section 6 discusses the MAEF in heavy traffic, with progress being made by welding the two systems together. Much information on occupancy is a consequence of the busy period material: a few simple properties for the MAEF are recorded in Section 3, and some more far-reaching ideas are indicated very briefly in Section 7.

2. A DISTRIBUTION

It will be shown in Section 5 that *there is a probability distribution* $\mathfrak{F}$ *on* $[0,\infty)$ *with Laplace transform given by*

$$\hat{\mathfrak{F}}(s) = \int_0^\infty e^{-sx}\mathfrak{F}\{dx\} = 1 - \exp\{-E_1(s)\}, \quad s > 0, \tag{1}$$

with E_1 being an exponential integral:

$$E_1(s) = \int_s^\infty x^{-1}e^{-x}\,dx, \quad s > 0.$$

By differentiating (1), it follows that $\mathfrak{F}$ satisfies (uniquely) the distributional equation

$$x\mathfrak{F}\{dx\} = \begin{cases} 0, & x \in [0,1) \\ (1 - \mathfrak{F}(x-1))\,dx, & x \in [1,\infty). \end{cases}$$

This, in turn, entails that $\mathfrak{F}$ is absolutely continuous with density $\mathfrak{f}$ specified by

$$\mathfrak{f}(x) = \begin{cases} 0, & x \in [0,1) \\ f_n(x), & x \in [n,n+1), n \in \mathbb{N}, \end{cases}$$

where $f_1(x) = x^{-1}$, $x \in [1,2)$, and for $n = 2,3,\ldots$,

$$f_n(x) = \frac{1}{x}\left\{nf_{n-1}(n) - \int_{n-1}^{x-1} f_{n-1}(u)\,du\right\}, \qquad x \in [n,n+1).$$

$\mathfrak{f}$ is continuous on $[1,\infty)$ and convex decreasing on both $[1,2]$ and $[2,\infty)$. It has a cusp at 2 and, more generally, a discontinuity in its $(n-1)$st derivative at n for $n = 2,3,\ldots$.

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All moments of $\mathfrak{F}$ exist and are given by

$$\mu_n(\mathfrak{F}) := \int_{[0,\infty)} x^n \mathfrak{F}\{dx\} = e^{\gamma} q_n, \qquad n \in \mathbb{N},$$

where γ is Euler's constant and each q_n is rational: in particular, $q_1 = 1$, $q_2 = 2$, $q_3 = 9/2$, and $q_4 = 34/3$.

3. THE MAEF

The MAEF can be constructed formally as a right-continuous Markov jump process $\xi = (\xi_t : 0 \le t < \infty)$ with state space $\mathcal{M}_p[0,1]$, the set of point measures on the unit interval, equipped with the vague topology (see [5, p. 124]). Each $\mu \in \mathcal{M}_p[0,1]$ has the representation $\mu = \sum_i \delta_{u_i}$, where $0 \le u_1 \le \cdots \le u_{|\mu|}$ and $|\mu| = \mu\{[0,1]\} < \infty$, δ_u being the Dirac measure at u. $\xi_t = \mu$ will mean that there are $|\mu|$ cases present at time t with urgencies u_i. In harmony with characteristics A1–A5, we stipulate that the process ξ has rate 1 transitions $\mu \to \mu - \delta_{u_i}$, $1 \le i \le |\mu|$, and rate $\lambda\, du$ transitions $\mu \to \mu + \delta_u$, $u \in [u_{|\mu|},1]$), with obvious conventions when $|\mu| = 0$. Furthermore, according to characteristic A1, ξ_0 is the zero measure, unless stated otherwise. Alternatively, we may construct the sojourns of admitted cases directly, arriving at ξ as a byproduct.

A basic self-similarity property of ξ is that its restriction to $[u,1]$ is simply a rescaled version of ξ with retarded arrival rate $\lambda(1-u)$.

Let $T = \inf\{t \ge 0 : |\xi_t| \ne 0\}$ be the time of the first admission, and denote the MAEF *busy period* by

$$\theta_\lambda = \inf\{t > T : |\xi_t| = 0\} - T.$$

In this section, we are concerned with properties of θ_λ: in particular, we obtain, in Theorem 1, exact expressions for its Laplace transform, $\psi_s(\lambda) = \mathbb{E}e^{-s\theta_\lambda}$, $s > 0$.

LEMMA 1: *The family of functions g_s, $s > 0$ defined by*

$$g_s(\lambda) = \{s + \lambda(1 - \psi_s(\lambda))\}^{-1}, \qquad \lambda > 0,$$

is interrelated according to the functional equation

$$g_s' = -g_s g_{s+1}, \qquad s > 0, \tag{2}$$

where the prime denotes the derivative with respect to λ.

PROOF: Fix $y > 0$. Consider the process ξ with initial condition $\xi_0 = \delta_{(1-y)}$, and let θ denote the associated busy period. During the sojourn of the initial case, a new case is admitted at rate λy. Suppose that such an admission does occur, at time t; then, by the self-similarity of ξ, the system first empties of cases having urgency in excess of $1 - y$ after a length of time equal in law to $t + \theta_{\lambda y}$. At this juncture, the initial case may or may not have departed. These deliberations lead to a distributional equation for θ: Let X_1 and T_1 be exponential variables of rates 1 and λy, respectively, and let $\theta_1 \stackrel{d}{=} \theta_{\lambda y}$, assuming mutual independence. Then,

$$\theta \stackrel{d}{=} \begin{cases} X_1, & X_1 < T_1 \\ T_1 + \theta_1 + \mathbf{1}(X_1 > T_1 + \theta_1)\theta', & X_1 \geq T_1, \end{cases}$$

where $\theta' \stackrel{d}{=} \theta$ is independent of X_1, T_1, and θ_1. Conditioning on T_1 and θ_1, we may take Laplace transforms and, after some routine calculation, obtain that

$$\mathbb{E}e^{-s\theta} = \frac{1 - \lambda y(\psi_{s+1}(\lambda y) - \psi_s(\lambda y))}{s + 1 + \lambda y(1 - \psi_{s+1}(\lambda y))} = 1 - \frac{g_{s+1}(\lambda y)}{g_s(\lambda y)}, \qquad s > 0. \tag{3}$$

Now, regard s as fixed, and observe that $\psi_\lambda(s) = \int_0^1 \mathbb{E}e^{-s\theta}\,dy$. Therefore, integration of (3) gives

$$\int_0^\lambda \left(1 - \frac{g_{s+1}(x)}{g_s(x)}\right) dx = \lambda\psi_\lambda(s) = s + \lambda - \frac{1}{g_s(\lambda)}, \qquad \lambda > 0. \tag{4}$$

The value of the integrand is confined to $[0,1]$, and, hence, (4) entails that g_s is continuous on $(0,\infty)$. This justifies differentiation of (4), which leads to (2). ■

THEOREM 1: *For each $\lambda > 0$,*

$$\psi_\lambda(s) = 1 - \frac{1}{\lambda}\left\{\left(\sum_{n\geq 0} R_n(s)\lambda^n\right)^{-1} - s\right\}, \qquad s > 0, \tag{5}$$

where the R_n are rational functions defined recursively by

$$R_0(s) = \frac{1}{s}, \qquad R_{n+1}(s) = -\frac{1}{n+1}\sum_{r=0}^{n} R_{n-r}(s)R_r(s+1), \quad n = 0,1,\ldots, s > 0, \tag{6}$$

so that

$$R_1(s) = -\frac{1}{s(s+1)}, \quad R_2(s) = \frac{2s+3}{2s(s+1)^2(s+2)},$$
$$R_3(s) = -\frac{6s^3 + 34s^2 + 61s + 35}{6s(s+1)^3(s+2)^2(s+3)},$$

and, furthermore,

$$\mathbb{E}\theta_\lambda = 1 + \frac{1}{4}\lambda - \frac{1}{72}\lambda^2 + \frac{37}{432}\lambda^3 + \cdots. \tag{7}$$

PROOF: By definition of g_s,

$$\psi_\lambda(s) = 1 - \lambda^{-1}(\{g_s(\lambda)\}^{-1} - s), \qquad s, \lambda > 0.$$

To develop the required expansion for g_s, first apply Leibniz's formula to (2) to obtain

$$g_s^{(n+1)} = -\sum_{r=0}^{n} \binom{n}{r} g_s^{(n-r)} g_{s+1}^{(r)}, \qquad n = 0,1,\ldots. \tag{8}$$

Let $R_0(s) := s^{-1} = g_s(0+)$ and use (8) to show that $R_n(s) := g_s^{(n)}(0+)/n!$, $n \in \mathbb{N}$ is well defined and satisfies (6). A straightforward induction argument using (8) verifies that

$$|g_s^{(n)}(\lambda)/n!| < s^{-(n+1)}, \qquad n \in \mathbb{N}, s, \lambda > 0.$$

This bound controls the error term in the Taylor expansion of g_s and, consequently,

$$g_s(\lambda) = g_s(\lambda+) = \sum_{n\geq 0} \frac{g_s^{(n)}(0+)}{n!} \lambda^n = \sum_{n\geq 0} R_n(s)\lambda^n, \qquad 0 < \lambda < s, s > 0. \tag{9}$$

Next, integrate (2) to produce

$$g_s = s^{-1} \exp\left\{-\int_0^{\lambda} g_{s+1}(x)\, dx\right\}, \qquad s, \lambda > 0, \tag{10}$$

and use this, together with analytic continuation, to extend the domain of validity in (9): first to $0 < \lambda < s + 1$, and then, by iteration, to $\lambda > 0$. Thus, we have proved (5).

Because θ_λ is stochastically dominated by the busy period of the $M(\lambda)/M(1)/\infty$ queue, all moments exist. We calculate

$$\begin{aligned}\mathbb{E}\theta_\lambda = \lim_{s\to 0+} s^{-1}(1-\psi_\lambda(s)) &= \lim_{s\to 0+} \lambda^{-1}(\{sg_s(\lambda)\}^{-1} - 1)\\ &= \lambda^{-1}\left(\exp\left\{\int_0^{\lambda} g_1(x)\, dx\right\} - 1\right)\\ &= \lambda^{-1}\left(\exp\left\{\sum_{n\geq 0} \frac{\lambda^{n+1} R_n(1)}{n+1}\right\} - 1\right),\end{aligned}$$

where the third equality uses (10) and monotone convergence. The final expression yields the expansion in (7). ■

It is natural to inquire what happens in heavy traffic; however, we have been unable to tackle large λ asymptotics directly using (2). Instead, an entirely fresh route via the Parody system leads eventually to the following result, an outline proof of which is given in Section 6.

THEOREM 2: *As* $\lambda \to \infty$,

$$\frac{\theta_\lambda}{\log\log\lambda} \xrightarrow{d} \mathfrak{F}.$$

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That the limit law is supported by $[1,\infty)$ is plausible: roughly $\log\lambda$ cases are admitted up to time 1, and θ_λ must exceed the greatest of their sojourns, which is about $\log\log\lambda$. There is also a glimmer of an explanation here for the diminishing lack of smoothness of $\mathfrak{f}$ at successive positive integers.

We now turn to occupancy. Successive MAEF emptying times are regeneration epochs for the process ξ; among these times are the emptying instants for the $M/M/\infty$ queue sharing the same arrival stream and (potential) sojourns. It follows that ξ_t converges in distribution to a random element of $\mathcal{M}_p([0,1])$, say ξ_∞, which is a still of the MAEF in equilibrium.

A few properties of ξ_∞ follow promptly from our knowledge of θ_λ. Let the maximal urgency be given by

$$u_\infty^* = \inf\{u \geq 0 : \xi_\infty([u,1]) = 0\}.$$

Then, using (7), we have

$$\mathbb{P}(u_\infty^* = 0) = \mathbb{P}(|\xi_\infty| = 0) = (1 + \lambda\mathbb{E}\theta_\lambda)^{-1} = 1 - \lambda + \frac{3}{4}\lambda^2 - \frac{35}{72}\lambda^3 + \cdots =: p(\lambda), \qquad \lambda > 0.$$

Thus, by the self-similarity of ξ, the distribution of u_∞^* is given by

$$\mathbb{P}(u_\infty^* \leq u) = p(\lambda(1-u)), \qquad \lambda > 0,\ 0 \leq u \leq 1.$$

Furthermore, when the MAEF contains m cases and the maximal urgency is u, the instantaneous admission rate is $\lambda(1-u)$ and the departure rate is m. Balancing these rates and exploiting ergodicity yields the equilibrium mean occupancy

$$\mathbb{E}|\xi_\infty| = \lambda\mathbb{E}(1-u_\infty^*) = \int_0^\lambda p(x)\,dx = \lambda - \frac{1}{2}\lambda^2 + \frac{1}{4}\lambda^3 - \frac{35}{288}\lambda^4 + \cdots, \qquad \lambda > 0.$$

Finally, replacing λ with $\lambda(1-u)$ in the last expansion gives the equilibrium mean urgency profile $\mathbb{E}\xi_\infty([u,1])$, $0 \leq u \leq 1$.

A word on deeper aspects of occupancy is given in Section 7.

4. THE PARODY

In this section, we study the Parody, carrying out a program similar to that in Section 3.

Formalizing rules B1–B3, we define the occupancy pattern for the Parody on n slots to be a continuous-time Markov chain $\zeta = (\zeta_t : 0 \leq t < \infty)$ with state space $\{0,1\}^n$, where $\zeta_t = (\zeta_t^{(1)}, \ldots, \zeta_t^{(n)})$, the understanding being that

$$\zeta_t^{(i)} = \begin{cases} 1 & \text{if slot } S_i \text{ is occupied at time } t \\ 0 & \text{otherwise.} \end{cases}$$

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ζ has unit rate transitions as follows:

$$(1,0,\ldots,0) \to (1,1,\ldots,1),$$
$$(1,0,\ldots,0,1,x_{j+1},\ldots,x_n) \to (1,1,\ldots,1,x_{j+1},\ldots,x_n), \qquad 2 \le j \le n,$$
$$(x_1,\ldots,1,\ldots,x_n) \to (x_1,\ldots,0,\ldots,x_n),$$

where each $x_i \in \{0,1\}$. In line with rule B1, we assume that $\zeta_0 \equiv (1,\ldots,1)$. The paths of ζ are taken to be right-continuous.

For $1 \le j \le n$, let Θ_j be the first time that an influx prescribed by rule B2 fills at least each of the slots S_1 through S_j:

$$\Theta_j = \inf\{t > 0 : \zeta_{t-}^{(i)} = 0,\ \zeta_t^{(i)} = 1,\ 2 \le i \le j\};$$

in particular, we shall regard Θ_n as the *Parody busy period*.

The following result provides exact information on Θ_n.

THEOREM 3: *Let $n \ge 1$. The busy period for the Parody on n slots has Laplace transform given by*

$$\Psi_n(s) := \mathbb{E}e^{-s\Theta_n} = 1 - \prod_{j=0}^{n} (s+j)^{(-1)^j \binom{n}{j}}, \qquad s > 0. \tag{11}$$

Furthermore,

$$\mathbb{E}\Theta_n = \prod_{j=1}^{n} j^{(-1)^j \binom{n}{j}}. \tag{12}$$

PROOF: By considering the respective events that S_j is occupied and vacant at time Θ_{j-1}, we may observe that Θ satisfies the distributional equation

$$\Theta_1 \stackrel{d}{=} X_1, \qquad \Theta_j \stackrel{d}{=} \Theta_{j-1} + \mathbf{1}(X_j > \Theta_{j-1})\Theta_j', \quad 2 \le j \le n, \tag{13}$$

where $\Theta_j' \stackrel{d}{=} \Theta_j$ is an independent variable and X_j is the exponential sojourn time for the initial individual in slot S_j, which is thereby independent of Θ_{j-1}. Taking Laplace transforms by first conditioning on Θ_{j-1} on the right yields, after a little rearrangement, the following recursion for Ψ:

$$\Psi_1(s) = (s+1)^{-1}, \qquad 1 - \Psi_j(s) = \frac{1 - \Psi_{j-1}(s)}{1 - \Psi_{j-1}(s+1)}, \quad 2 \le j \le n,\ s > 0.$$

Define $\phi_j(s) = \log(1 - \Psi_j(s))$ and take logarithms in the above recursion to obtain

$$\begin{aligned}\phi_n(s) &= -\Delta\phi_{n-1}(s) = (-\Delta)^{n-1}\phi_1(s)\\ &= (-\Delta)^{n-1}\log(s(s+1)^{-1}) = (-\Delta)^n \log s,\ s > 0,\end{aligned}$$

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where Δ is the forward difference operator with respect to s. Hence,

$$1 - \Psi_n(s) = \exp\{(-\Delta)^n \log s\}, \qquad s > 0,$$

which is equivalent to (11). To prove (12), divide the last display by s and let $s \to 0+$. ■

The next result is the companion to Theorem 2; it shows the likeness between the Parody on $\lceil \log \lambda \rceil$ slots and the MAEF, when λ is large. The proof is given at the end of Section 5.

THEOREM 4: *As* $n \to \infty$,

$$\frac{\Theta_n}{\log n} \xrightarrow{d} \mathfrak{F}.$$

Results for the equilibrium slot occupancy pattern may be obtained in a manner similar to those for MAEF occupancy in Section 3.

5. PARODY ASYMPTOTICS

Here, we verify Theorem 4 and prepare the ground for proving Theorem 2. First, we introduce a Modified Parody system by replacing rule B2 with the following:

B2′. At the epochs $R_1, R_2, \ldots$ of an independent renewal process with associated distribution F, there is an influx of new cases, provided S_1 is vacant, filling all slots above S^*.

Rules B1 and B3 remain intact. Note that the developments in the Parody for slots below S_j constitute an instance of the Modified Parody with F the law of Θ_j.

Again, we shall deal with the top n slots, calling the Modified Parody occupancy pattern $\tilde{\zeta} = (\tilde{\zeta}_t : 0 \le t < \infty)$, with state space $\{0,1\}^n$, where $\tilde{\zeta}_t = (\tilde{\zeta}_t^{(1)}, \ldots, \tilde{\zeta}_t^{(n)})$ has an analogous interpretation to ζ_t. On each interval $[0, R_1), [R_1, R_2), \ldots,$ the right-continuous process $\tilde{\zeta}$ evolves as a continuous-time Markov chain with unit rate transitions of the form

$$(x_1, \ldots, 1, \ldots, x_n) \to (x_1, \ldots, 0, \ldots, x_n);$$

in addition, at the epoch R_k when, and only when, $\tilde{\zeta}_{R_k-}^{(1)} = 0$, there is a jump

$$(0, \ldots, 0) \to (1, \ldots, 1),$$
$$(0, \ldots, 0, 1, x_{j+1}, \ldots, x_n) \to (1, \ldots, 1, x_{j+1}, \ldots, x_n), \qquad 2 \le j \le n,$$

as appropriate; here, $x_i \in \{0,1\}$.

For $1 \le j \le n$, let

$$\tilde{\Theta}_j = \inf\{t > 0 : \tilde{\zeta}_{t-}^{(i)} = 0,\ \tilde{\zeta}_t^{(i)} = 1,\ 1 \le i \le j\},$$

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with $\widetilde{\Theta}_n = \widetilde{\Theta}_n(F)$ being the *Modified Parody busy period* (although "cycle" would be more accurate now). As a final twist, we allow F to depend on n.

The following complement to Theorem 4 shows that the asymptotic busy period length is robust in the face of the new influx rule.

THEOREM 5: *Let F_n, $n \in \mathbb{N}$, be probability distributions on $(0,\infty)$ for which*

$$\mu_2(F_n) < \infty \quad \textit{and} \quad \frac{\mu_2(F_n)}{\mu_1(F_n)\log n} \to 0; \tag{14}$$

then,

$$\frac{\widetilde{\Theta}_n(F_n)}{\log n} \xrightarrow{d} \mathfrak{F}. \tag{15}$$

Before proceeding with the proof, we need a little renewal theory. Let F and F_n, $n \in \mathbb{N}$, be probability distributions on $(0,\infty)$. The harmonic renewal measure on $(0,\infty)$ allied to F is given by $\nu_F = \sum_{j\geq 1} j^{-1} F^{*j}$, where F^{*j} is the j-fold convolution of F (see, e.g., [2]). Clearly,

$$\hat{\nu}_F(s) = -\log(1 - \hat{F}(s)), \qquad s > 0.$$

Define further a measure ν_n on $(0,\infty)$ by

$$\nu_n\{du\} = (1 - e^{-u})\nu_{F_n}\{du \log n\}, \qquad n \in \mathbb{N}.$$

LEMMA 2: *Let F_n, $n \in \mathbb{N}$, satisfy condition (14). Then,*

$$\nu_n\{du\} \to u^{-1}(1 - e^{-u})\, du \tag{16}$$

in the sense of weak convergence on bounded intervals of $(0,\infty)$.

PROOF: Take $s > 0$. Employing the inequality $|\hat{F}(s) - 1 + s\mu_1(F)| \leq \frac{1}{2}s^2\mu_2(F)$ and condition (14), we obtain, for large n,

$$1 - \hat{F}_n\left(\frac{s}{\log n}\right) = \frac{\mu_1(F_n)s}{\log n} + O\left(\frac{\mu_2(F_n)s^2}{\log^2 n}\right) \sim \frac{\mu_1(F_n)s}{\log n}.$$

Hence,

$$\begin{aligned}\hat{\nu}_n(s) &= \hat{\nu}_{F_n}\left(\frac{s}{\log n}\right) - \hat{\nu}_{F_n}\left(\frac{s+1}{\log n}\right) \\ &= \log\left\{\frac{1 - \hat{F}_n((s+1)/\log n)}{1 - \hat{F}_n(s/\log n)}\right\} \to \log\left(\frac{s+1}{s}\right).\end{aligned}$$

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The convergence claimed in (16) then follows from the continuity theorem for Laplace transforms of measures (see [3, p. 433]) and the formula

$$\int_0^\infty e^{-su}u^{-1}(1-e^{-u})\,du = \log\left(\frac{s+1}{s}\right). \qquad \blacksquare$$

PROOF OF THEOREM 5: Letting $\tilde{\Theta}_0 \stackrel{d}{=} R_1$, we have the distributional equation

$$\tilde{\Theta}_j \stackrel{d}{=} \tilde{\Theta}_{j-1} + \mathbf{1}(X_j > \tilde{\Theta}_{j-1})\tilde{\Theta}_j, \qquad 1 \le j \le n,$$

with the same conventions as in (13). Mimicking the proof of Theorem 3, we obtain

$$\tilde{\Psi}_n(s) := \mathbb{E}e^{-s\tilde{\Theta}_n} = 1 - \exp\{-\tilde{\phi}_n(s)\},$$

where

$$\tilde{\phi}_n(s) = (-\Delta)^n \hat{\nu}_{F_n}(s) = \int_0^\infty e^{-su}(1-e^{-u})^n \nu_{F_n}\{du\},$$

the integral arising from the linearity of Δ and the formula $(-\Delta)^n e^{-su} = e^{-su}(1-e^{-u})^n$.

Thus, for $t > 0$ and $K > 1$,

$$\tilde{\phi}_n\left(\frac{t}{\log n}\right) = \int_0^\infty e^{-tx}(1-n^{-x})^n(1-e^{-x})^{-1}\nu_n\{dx\} = \int_0^K + \int_K^\infty =: I_1 + I_2. \tag{17}$$

The integrand in (17) is bounded and so, by Lemma 2 above and Theorem 5.5 of [1], we have $I_1 \to \int_1^K e^{-tx}x^{-1}\,dx$. Moreover, using the convergence of $\hat{\nu}_n$ and Lemma 2, again, we have, for large n,

$$(1-e^{-1})I_2 \le \int_K^\infty e^{-tx}\nu_n\{dx\} = \int_0^\infty - \int_0^K \to \int_K^\infty e^{-tx}(1-e^{-x})x^{-1}\,dx.$$

Because K is arbitrary, we conclude from (17) that $\tilde{\phi}_n(t/\log n) \to \int_1^\infty e^{-tx}x^{-1}\,dx = E_1(t)$. Therefore,

$$\tilde{\Psi}_n\left(\frac{t}{\log n}\right) \to 1 - \exp\{-E_1(t)\}, \qquad t > 0.$$

The continuity theorem for Laplace transforms justifies the opening claim of Section 2 regarding the existence of $\mathfrak{F}$, and then (15). $\blacksquare$

PROOF OF THEOREM 4: Slots S_2 through S_n of the (standard) Parody behave like slots S_1 through S_{n-1} of a Modified Parody system, with F the unit exponential distribution. Thus, Theorem 4 is an immediate consequence of Theorem 5. $\blacksquare$

6. MAEF ASYMPTOTICS

This section indicates how Theorem 2 may be established by linking the MAEF to a certain Modified Parody.

For each $\lambda > e^e$, let MAEF$^{(\lambda)}$ be an MAEF with arrival rate λ. Consider admissions to MAEF$^{(\lambda)}$ that have urgency exceeding $1 - \epsilon_\lambda$, where $0 < \epsilon_\lambda < 1$. The successive epochs at which this system becomes empty of all such extreme cases form a renewal process with associated distribution G_λ, say. Next, for each $\lambda > 1$, create a Modified Parody process MP$^{(\lambda)}$ as follows: Take a Parody process on slots S_1 through $S_{\lceil \log \lambda \rceil}$; remove the top $\lceil \log \lambda\epsilon_\lambda \rceil$ slots; and let the aforementioned renewal epochs be the case influx times for the remaining stack, as per rule B2′. Finally, let $\tilde{\tau}_\lambda$ be the busy period for MP$^{(\lambda)}$ and let τ_λ be the first emptying time of MAEF$^{(\lambda)}$; thus, $\tilde{\tau}_\lambda \stackrel{d}{=} \widetilde{\Theta}_{\lceil \log \lambda \rceil - \lceil \log \lambda\epsilon_\lambda \rceil}(G_\lambda)$ and $\tau_\lambda \stackrel{d}{=} Y_\lambda + \theta_\lambda$, where Y_λ is an independent rate λ exponential variable.

Theorem 2 is clearly equivalent to

$$\frac{\tau_\lambda}{\log\log\lambda} \xrightarrow{d} \mathfrak{F}. \qquad \textbf{(18)}$$

To prove (18), our strategy is to show that Theorem 5 can be applied to give

$$\frac{\tilde{\tau}_\lambda}{\log\log\lambda} \xrightarrow{d} \mathfrak{F} \qquad \textbf{(19)}$$

and that the processes MAEF$^{(\lambda)}$ and MP$^{(\lambda)}$ can be coupled to admit (18) as a consequence of (19). Here, $\epsilon_\lambda \to 0$ must be chosen with care: there is a tension between the demands of (19) and the coupling.

We now summarize the details.

Proof of Theorem 2: The main task is to obtain the following lemma, the proof of which is too lengthy to be included here; it is contained in [4]. The coupling is achieved essentially by metamorphosing MAEF$^{(\lambda)}$ (with $\mathcal{U}$ taken propitiously to be unit exponential) into MP$^{(\lambda)}$ via immaterial adjustments to case sojourns.

Lemma 3: *Let $K > 0$ and take*

$$\epsilon_\lambda = \lambda^{-1} \exp\{(\log\log\lambda)^3\}.$$

Then, the following conditions hold:

(a) The laws $F_n := G_{e^n}$, $n \in \mathbb{N}$, satisfy (14).
(b) The pairs MAEF$^{(\lambda)}$ and MP$^{(\lambda)}$, $\lambda > e^e$, can be coupled in such a way that

$$\lim_{\lambda\to\infty} \mathbb{P}[\tau_\lambda \mathbf{1}(\tau_\lambda \le K\log\log\lambda) = \tilde{\tau}_\lambda \mathbf{1}(\tilde{\tau}_\lambda \le K\log\log\lambda)] = 1.$$

This granted, let $x > 1$ and choose $K > x$. Suppressing dependence on λ, denote by A the event in the probability in (b) of Lemma 3, and let B be the event that

$\tau_\lambda/\log\log\lambda \le x$, with $\tilde{B}$ being defined similarly but with $\tilde{\tau}_\lambda$ in place of τ_λ. Since $B \cap A = \tilde{B} \cap A$, we have

$$\mathbb{P}(B) - \mathbb{P}(B \cap \bar{A}) = \mathbb{P}(B \cap A) = \mathbb{P}(\tilde{B} \cap A) = \mathbb{P}(\tilde{B}) - \mathbb{P}(\tilde{B} \cap \bar{A}). \quad \textbf{(20)}$$

However, by Theorem 5 and condition (a) of Lemma 3, $\lim_{\lambda\to\infty}\mathbb{P}(\tilde{B}) = \mathfrak{F}(x)$, and by condition (b) of Lemma 3, $\lim_{\lambda\to\infty}\mathbb{P}(\bar{A}) = 0$. Consequently, letting $\lambda \to \infty$ in (20), we obtain $\lim_{\lambda\to\infty}\mathbb{P}(B) = \mathfrak{F}(x)$. Thus, (18) is verified, and with it, Theorem 2. ■

Remark: We claim that $\mathbb{E}\Theta_n \sim e^\gamma \log n$ is a matter of direct calculation using (12) and that, with appreciably more effort, one can show

$$\mathbb{E}\theta_\lambda \sim e^\gamma \log\log\lambda. \quad \textbf{(21)}$$

We leave a *tidy* verification of (21) as an unsolved problem. ■

7. CLOSING COMMENTS

Many aspects of the two systems remain to be examined. For example, we have not obtained the equilibrium distribution of total occupancy for either system. Nevertheless, the busy period propositions do lay a foundation for a more detailed description of occupancy. In fact, it is possible to obtain space-time scaling-limit theorems for the occupancy profile of both the MAEF and the Parody; a synopsis of some results in this direction is given in [4].

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