OPTIMUM INSURANCE CONTRACTS WITH BACKGROUND RISK AND HIGHER-ORDER RISK ATTITUDES

BY

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ABSTRACT

In this paper, we study an optimal insurance problem in the presence of background risk from the perspective of an insured with higher-order risk attitudes. We introduce several useful dependence notions to model positive dependence structures between the insurable risk and background risk. Under these dependence structures, we compare insurance contracts of different forms in higher-order risk attitudes and establish the optimality of stop-loss insurance form. We also explicitly derive the optimal retention level. Finally, we carry out a comparative analysis and investigate how the change in the insured's initial wealth or background risk affects the optimal retention level.

KEYWORDS

Background risk, higher-order risk attitudes, stochastically increasing, right tail increasing, stop-loss insurance.

1. INTRODUCTION

Since the seminal work of Arrow (1963), great attention has been drawn to study optimal insurance problems in the past half century. See, for example, Raviv (1979), Huberman *et al.* (1983), Young (1999), Kaluszka (2001), Bernard *et al.* (2015a), and references therein. While these papers enhance the understanding of the insurance demand, they are confined to the single-risk framework. On the other hand, in practice, an insured is usually confronted with multiple sources of risk. In addition to the major insurable risk under consideration, there could be investment risk, human capital risk and other uninsurable risks. This phenomenon calls for the development of the insured's risk transfer strategy from a holistic perspective, by taking multiple sources of risk into consideration.

The importance of comprehensive risk management has been recognized by many scholars in the past 30 years. For example, the optimal proportion rate of quota-share insurance is studied together with the demand of financial assets in

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Mayers and Smith (1983). Doherty and Schlesinger (1983) instead investigate the optimal retention level of stop-loss insurance by assuming that the insured's initial wealth is random. Assuming that the alternatives satisfy the *principle of indemnity,*[1] Gollier (1996) finds that under the expected value premium principle the optimal contract may contain a "disappearing deductible", which may result in the marginal indemnity strictly larger than one. Moreover, by treating other sources of risk as the background risk, Dana and Scarsini (2007) generalize Raviv (1979)'s optimal insurance model, and obtain some qualitative properties of optimal solutions under several special assumptions on the dependence between the insurable risk and background risk. Focusing on some special utility functions, Huang *et al.* (2013) find explicit optimal solutions. Recently, Chi and Tan (2015) find these optimal contracts may lead to ex post *moral hazard*, which would prevent their applications in practice. They further apply the mean-variance analysis to derive the optimal insurance that preclude ex post moral hazard in the sense of Huberman *et al.* (1983).

In many of the afore-mentioned studies, the optimization criterion is either to maximize the expected utility of the insured's final wealth or to minimize the insured's risk exposure under some risk measure. A common feature of these optimization criteria is to preserve the second degree stochastic dominance. In other words, the insured is often assumed to be risk averse. In addition to risk aversion, many other notions have also been developed to explain behaviors reflecting higher-order risk attitudes. For example, Kimball (1990) finds that the precautionary saving motive is closely related to the convex first derivative of utility function. This condition is referred to as prudence in the literature. Interestingly, Noussair *et al.* (2014) carry out an experiment with a large demographically representative sample of participants to test the risk attitudes of an individual. They observe that the majority of the participants' decisions are consistent with risk aversion, prudence and temperance, which is characterized by the concave second derivative of the individual's utility function. We refer to Eeckhoudt and Schlesinger (2013) and Denuit and Eeckhoudt (2013) for comprehensive reviews of higher-order risk attitudes.

In some other studies, higher-order risk attitudes of the insured have already been considered in the analysis of insurance demand with background risk. Specifically, assuming that the insured is risk-averse and prudent, Eeckhoudt and Kimball (1992) find that the existence of background risk will make the insured raise the coinsurance rate or reduce the retention level if the background risk increases with respect to the insurable risk in the sense of the third increasing convex order. Note that their studies focus only on two specific types of insurance contracts, namely quota-share insurance or stop-loss insurance, which limits their applications in practice. On the other hand, Mahul (2000) considers a quite general class of admissible insurance contracts that are assumed to follow the principle of indemnity. He obtains a similar result as Gollier (1996) in favor of disappearing deductible even if a more general positive dependence structure between background risk and the insurable risk is assumed. More precisely, when background risk is increasing with respect to the insurable loss in the sense of nth increasing convex order, the optimality of disappearing deductible is obtained for the insured with the risk preference preserving $(n+1)$th degree stochastic dominance. While this result sounds very interesting, it may be inappropriate to be used in practice because Huberman *et al.* (1983) point out that this optimal contract introduces an incentive for the insured to underreport the actual loss and benefit himself/herself. To make the optimal solutions applicable, it is necessary to impose some constraints to reduce the ex post moral hazard.

In this paper, we revisit the optimal insurance design with background risk and higher-order risk attitudes, assuming that the admissible insurance contract follows the principle of indemnity and has an increasing[2] retained loss function. As in the literature, we further assume that the insurance premium is calculated based only upon the expected indemnity. Once the retained insurable loss for two contracts can be ordered in convex order, it is shown that the insured's final wealth can also be ranked in the opposite order in the sense of $(n+1)$th degree stochastic dominance when the background risk increases with respect to the insurable loss in the nth increasing convex order. If the contracts under comparison are subject to more strict constraints, this result also holds even if the dependence assumption between the background risk and the insurable loss is relaxed to be right tail increasing. For both positive dependence structures, the stop-loss insurance is always a preferred choice of the insured with the risk preference preserving the $(n+1)$th degree stochastic dominance. Further, within the expected utility framework, we derive the optimal retention level of stop-loss insurance explicitly, which is found to heavily rely on the insured's initial wealth and background risk.

It should be emphasized that Lu *et al.* (2012) also investigate the comparison of insurance contracts with possibly different types in the presence of background risk. Specifically, using the results in Cai and Wei (2012), they show that the stop-loss insurance is an optimal choice of a risk-averse insured when the background risk is stochastically increasing in the insurable loss. In this paper, we extend their study to a more general stochastic dependence between the background risk and the insurable risk at the cost of a constraint imposed on the insured's higher-order risk attitudes. On the other hand, it is worth mentioning that this constraint is quite weak as it is naturally satisfied by the insured with mixed risk aversion. The concept of mixed risk aversion is introduced by Caballé and Pomansky (1996) and is met by most commonly used utility functions in finance and economics. Furthermore, it is necessary to point out that Mahul (2000) also considers the optimal insurance design with higher-order risk attitudes and a rather general dependence structure. In contrast, we impose an additional constraint on the admissible insurance contract such that it has an increasing retained loss function. By comparing Mahul's optimal insurance solutions with ours, it is easy to find that this constraint plays a critical role in optimal insurance design with background risk and makes the optimal solution change from disappearing deductible to the stop-loss insurance .

The rest of the paper is organized as follows. In Section 2, we introduce an optimal insurance model with background risk, where the insurance premium is calculated based on the expected indemnity. Some notions of positive dependence between the insurable risk and background risk are introduced and their relationships are discussed in Section 3. Under these dependence assumptions, Section 4 compares insurance contracts of different types, and finds that the stop-loss insurance is the optimal choice of the insured with higher-order risk attitudes. Section 5 gives an explicit expression for the optimal retention level and investigates how it is affected by the change in the insured's initial wealth or the background risk. Some concluding remarks are provided in Section 6. Finally, the appendix collects the proofs to propositions established in the paper.

2. The model

Suppose that in a fixed time period, an insured endowed with initial wealth w faces two sources of risk X and Y, where X is insurable and non-negative and Y is the background risk and may be negative. Both X and Y are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with finite means. To reduce the risk exposure, the decision-maker would seek to purchase an insurance contract, in which an amount of risk $f(X)$ is ceded to an insurer and the residual risk $I(X) = X - f(X)$ is retained. The function $I(x)$ is usually called the retained loss function. As in Cai and Wei (2012), we assume that the admissible insurance contract follows the principle of indemnity and has an increasing retained loss function. Specifically, the set of admissible retained loss functions is given by

$$\mathfrak{C} = \{0 \leq I(x) \leq x : I(x) \text{ is an increasing function}\}.$$

Obviously, the retained loss function of disappearing deductible does not belong to the above set. To cover the potential loss for the insured, the insurer will need to collect premium. As in most of the literature, we assume that the insurer is risk-neutral, and calculates the insurance premium based only on the expected indemnity. Using $\pi(.)$ to represent the premium principle, the insurance premium is calculated as

$$\pi(f(X)) = \mathcal{P}\left(\mathbb{E}[f(X)]\right), \tag{2.1}$$

for some differentiable function $\mathcal{P}(.)$ with $\mathcal{P}(0) = 0$ and $\mathcal{P}'(x) \geq 1$ for any $x \geq 0$. Particularly, if $\mathcal{P}(x) = (1+\theta)x$ for some $\theta > 0$, we recover the expected value premium principle. We remark that the assumption of the expected indemnity-based premium principle plays a critical role in determining the optimal contract in this paper. Notably, this premium principle may be not realistic from a practical perspective. However, it has been frequently used in the literature, mainly because of its mathematical tractability. Under other premium principles, optimal insurance problems, especially in the presence of the background risk, become very challenging, as evidenced in Young (1999). Therefore, we simply use the expected indemnity based premium principle in this paper. For a comprehensive review of different premium principles, readers are referred to Young (2004).

Notably, Bernard and Vanduffel (2014) and Bernard *et al.* (2015b) used a different approach to price insurance claims by taking financial market into consideration. They also proved that an insurance contract is not necessary when the payoff can be replicated through financial derivatives.

With an insurance arrangement, the insured's final wealth $W_I(X, Y)$ has the following expression:

$$W_I(X, Y) = w - Y - I(X) - \pi(X - I(X)).$$

Usually, the objective is to maximize the expected utility of the insured's final wealth. Mathematically, the optimization problem is formulated as

$$\max_{I \in \mathfrak{C}} \mathbb{E}[u(W_I(X, Y))]. \tag{2.2}$$

There are many choices for the utility function u. In the early literature, the utility function is assumed to be increasing and concave, reflecting the risk aversion of an individual. Such an optimization criterion aims to work for a general increasing concave utility function, or equivalently maximize the final wealth in the sense of second degree stochastic dominance (SSD, see Definition 2.1), see for example Gollier and Schlesinger (1996). Later, in order to explain certain behaviors of a risk averse agent, higher-order risk attitudes, such as prudence and temperance, have been considered. In fact, all these optimization criteria can be summarized in terms of higher-degree stochastic dominance, as defined below.

Definition 2.1 (Ekern 1980). *Random wealth W_1 is said to dominate random wealth W_2 in nth degree stochastic dominance, denoted as $W_1 \geq_{n\text{-}SD} W_2$, if $\mathbb{E}[u(W_1)] \geq \mathbb{E}[u(W_2)]$ for any $u(\cdot) \in \mathcal{U}_{n\text{-}icv}$ provided that the expectations exist, where*

$$\mathcal{U}_{n\text{-}icv} = \left\{ u(\cdot) : (-1)^{k-1} u^{(k)}(\cdot) \geq 0 \text{ for all } k = 1, 2, \ldots, n \right\}.$$

Here, $u^{(k)}(\cdot)$ is the kth derivative of function $u(\cdot)$.

The above definition introduces a sequence of notions about stochastic dominance and thus develops a general framework to study higher-order risk attitudes. The cases of $n = 2, 3, 4$ have already been extensively discussed in the literature. Specifically, $n = 2$ corresponds to the SSD and reflects general risk aversion. The third and fourth degree stochastic dominance rules, respectively, reflect prudence and temperance, see Eeckhoudt and Schlesinger (2013). It is worth mentioning that, the limiting case $\mathcal{U}_{\infty\text{-}icv}$ denotes the collection of completely monotone utility functions, which describe the mixed risk aversion attitude. For more details about mixed risk aversion, readers are referred to Caballé and Pomansky (1996).

Note that nth degree stochastic dominance is also called nth increasing concave order. Another closely related concept is nth increasing convex order, with its definition stated below.

Definition 2.2 (Shaked and Shanthikumar 2007). *Random variable Z_1 is said to be less than Z_2 in* nth *increasing convex order, denoted as $Z_1 \leq_{n\text{-}icx} Z_2$, if $\mathbb{E}[v(Z_1)] \leq \mathbb{E}[v(Z_2)]$ for any $v(\cdot) \in \mathcal{V}_{n\text{-}icx}$ provided that the expectations exist, where*

$$\mathcal{V}_{n\text{-}icx} = \left\{ v(\cdot) : v^{(k)} \geq 0 \textit{ for all } k = 1, 2, \ldots, n \right\}.$$

Note that $v(\cdot) \in \mathcal{V}_{n\text{-}icv}$ if and only if $-v(-\cdot) \in \mathcal{U}_{n\text{-}icx}$. In this sense, the nth increasing convex order and the nth increasing concave order (nth degree stochastic dominance) are dual concepts. For more detailed discussions of higher degree increasing convex/concave orders, readers are referred to Shaked and Shanthikumar (2007) or Müller and Stoyan (2002).

3. Positive dependence notions based on higher-degree stochastic orders

In insurance practice, positive dependence between risks commonly exists. For example, a driver is required to insure against the liability loss but not necessarily the collision and comprehensive damage (for his/her own vehicle). In this case, the collision and comprehensive damage could serve as a background risk. Both losses will depend on the severity of the accident and are likely to vary in the same direction. Readers are also referred to Dana and Scarsini (2007) for more examples in property insurance as well as health insurance. In these examples, one common feature is that one risk is likely to be large when the other is large, and vice versa. According to the literature, such a feature can be modeled by the notion of stochastic increasingness (see Definition 3.1), as seen in Dana and Scarsini (2007), Cai and Wei (2012), and references therein. In this paper, we aim to relax the assumption of stochastic increasingness and study the optimal insurance problem with more general positive dependence notions. To this end, we shall introduce the notions of right tail increasingness and stochastic increasingness based on high-degree stochastic orders ($\uparrow^{RTI}_{n\text{-}icx}$ and $\uparrow_{n\text{-}icx}$, respectively, see Definitions 3.1 and 3.2). Notably, some special cases of these notions have already been considered in Eeckhoudt and Kimball (1992) and Mahul (2000).

Below, we present the mathematical definitions of the dependence notions mentioned above.

Definition 3.1. *Y is said to be increasing with respect to X in the* nth *increasing convex order, denoted as $Y \uparrow_{n\text{-}icx} X$, if $\mathbb{E}[v(Y)|X = x]$ is increasing over the set $S(X)$ for any $v \in \mathcal{V}_{n\text{-}icx}$ such that $\mathbb{E}[|v(Y)|] < \infty$, where $S(X)$ is the support of X.*

Intuitively, $Y \uparrow_{n\text{-}icx} X$ means that $[Y|X = x_1] \leq_{n\text{-}icx} [Y|X = x_2]$ for any $x_1, x_2 \in S(X)$ with $x_1 \leq x_2$. It is easy to see that $Y \uparrow_{n\text{-}icx} X$ implies $Y \uparrow_{m\text{-}icx} X$ for any positive integer m larger than n. When $n = 1$, $Y \uparrow_{n\text{-}icx} X$ reduces to

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TABLE 1

JOINT PROBABILITY FUNCTION OF (X, Y).

y\x	0	1	2
0	2/15	1/6	1/30
1	1/5	1/6	3/10

TABLE 2

CONDITIONAL PROBABILITY MASS FUNCTION OF Y GIVEN $\{X > x - \delta\}$.

y\x	0	1	2
0	1/3	3/10	1/10
1	2/3	7/10	9/10

$Y \uparrow_{SI} X$, a dependence notion introduced by Lehmann (1966) and referred to as "stochastic increasingness".

Definition 3.2. *Y is said to be right tail increasing with respect to X in the nth increasing convex order, denoted as $Y \uparrow^{RTI}_{n\text{-}icx} X$, if $\mathbb{E}[v(Y)|X > x]$ is increasing in x for any $v \in \mathcal{V}_{n\text{-}icx}$ such that $\mathbb{E}[|v(Y)|] < \infty$.*

Similar to the notion of $\uparrow_{n\text{-}icx}$, $Y \uparrow^{RTI}_{n\text{-}icx} X$ means that $[Y|X > x_1] \leq_{n\text{-}icx} [Y|X > x_2]$ for any $x_1 \leq x_2$. It is easy to see that $Y \uparrow^{RTI}_{n\text{-}icx} X$ implies $Y \uparrow^{RTI}_{n+1\text{-}icx} X$ for any $n = 1, 2, \cdots$. In particular, $Y \uparrow^{RTI}_{n\text{-}icx} X$ for $n = 1$ is referred to as *Y right tail increasing* with respect to X, also denoted as $Y \uparrow_{RTI} X$. This concept was proposed and discussed by Barlow and Proschan (1975).

Proposition 3.3. *The dependence notions defined above have the following implications:*

$$\begin{array}{ccc} Y \uparrow_{SI} X & \Longrightarrow & Y \uparrow_{n\text{-}icx} X \\ \Downarrow & & \Downarrow \\ Y \uparrow_{RTI} X & \Longrightarrow & Y \uparrow^{RTI}_{n\text{-}icx} X, \end{array}$$

for any positive integer n.

We remark that all these implications are strict, as demonstrated by the following examples.

Example 3.4. *$Y \uparrow_{RTI} X$ does not imply $Y \uparrow_{SI} X$.*

Let (X, Y) be a discrete bivariate random vector with probability function $\mathbb{P}\{X = x, Y = y\}$ given in Table 1. Simple calculations yield that $\mathbb{P}\{Y > 0|X = 1\} = 0.5 < 0.6 = \mathbb{P}\{Y > 0|X = 0\}$, which implies that Y is not stochastically increasing with respect to X.

On the other hand, the conditional probability mass function $\mathbb{P}\{Y = y|X > x - \delta\}$ for $x = 0, 1, 2$ is given in Table 2, where δ is an arbitrarily small positive

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TABLE 3

CONDITIONAL PROBABILITIES: $\mathbb{P}\{Y = y|X = x\}$.

x\y	0	1	2
0	0.2	0.5	0.3
1	0.3	0.3	0.4

TABLE 4

CONDITIONAL EXPECTATIONS: $\mathbb{E}[(Y - y)_+|X = x]$.

x\y	0	1	2
0	1.1	0.3	0
1	1.1	0.4	0

number. Noting that for a fixed y, $\mathbb{P}\{Y = y|X > x\}$ takes only three possible values as listed in Table 2, it is easy to verify that $Y \uparrow_{RTI} X$.

Example 3.5. *$Y \uparrow_{2\text{-}icx} X$ does not imply $Y \uparrow_{SI} X$.*

Let (X, Y) be a discrete bivariate random vector with conditional probability function $\mathbb{P}\{Y = y|X = x\}$ given in Table 3. Further, assume that $\mathbb{P}\{X = 0\} = \mathbb{P}\{X = 1\} = 0.5$. Note that $\mathbb{P}\{Y > 0|X = 0\} = 0.8 > 0.7 = \mathbb{P}\{Y > 0|X = 1\}$, which disproves that Y is stochastically increasing with respect to X.

On the other hand, the conditional expectations $\mathbb{E}[(Y-y)_+|X = x]$ are given by Table 4, where $(z)_+ = \max\{z, 0\}$. As functions of t, both $\mathbb{E}[(Y - t)_+|X = 0]$ and $\mathbb{E}[(Y - t)_+|X = 1]$ are piecewise linear continuous functions with changing points at 0, 1 and 2. Therefore, it is easy to see that $\mathbb{E}[(Y - t)_+|X = 0] \le \mathbb{E}[(Y - t)_+|X = 1]$ for all t, which means $Y \uparrow_{2\text{-}icx} X$.

4. COMPARISON OF INSURANCE CONTRACTS

In this section, we will identify the optimal insurance form by comparing admissible insurance contracts for an insured with higher-order risk attitudes. To proceed, we first introduce the convex order to compare the indemnities.

Definition 4.1. *Random variable Z_1 is said to be smaller than Z_2 in the* convex order*, denoted as $Z_1 \le_{cx} Z_2$, if $\mathbb{E}[h(Z_1)] \le \mathbb{E}[h(Z_2)]$ for any convex function $h(z)$ provided that the expectations exist.*

By comparing the notions of convex order and the nth increasing convex order, it is easy to see that $Z_1 \le_{cx} Z_2$ if and only if $Z_1 \le_{2\text{-}icx} Z_2$ and $\mathbb{E}[Z_1] = \mathbb{E}[Z_2]$. If two indemnities are able to be ranked according to this stochastic order, then the corresponding final wealth levels of the insured can be ordered in some sense, as stated in the following proposition.

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Proposition 4.2. *Let $I_1(x)$ and $I_2(x)$ be two retained loss functions in $\mathfrak{C}$ such that $I_1(X) \le_{cx} I_2(X)$. If $Y \uparrow_{n\text{-}icx} X$, then $W_{I_1}(X, Y) \ge_{n+1\text{-}SD} W_{I_2}(X, Y)$.*

Remark 4.3. *It is worth mentioning the special case of Proposition 4.2 with $n = 1$. Specifically, if $Y \uparrow_{SI} X$, then $I_1(X) \le_{cx} I_2(X)$ implies $W_{I_1}(X, Y) \ge_{2\text{-}SD} W_{I_2}(X, Y)$. Note that $\mathbb{E}[W_{I_1}(X, Y)] = \mathbb{E}[W_{I_2}(X, Y)]$. From an economic perspective, this result indicates that the change of insurance strategy from I_1 to I_2 results in an increase in risk of the insured's final wealth, see Rothschild and Stiglitz (1971). This special case also recovers Lemma 3.3 of Cai and Wei (2012).*

It is well-known from the literature that for any $I \in \mathfrak{C}$, there exists a non-negative d such that $I_d(X) \le_{cx} I(X)$, where

$$I_d(x) = \min\{x, d\}. \tag{4.1}$$

Therefore, Proposition 4.2 implies the optimality of stop-loss insurance, as stated in the following corollary.

Corollary 4.4. *If $Y \uparrow_{n\text{-}icx} X$, then a solution to the optimization problem (2.2) with $u(\cdot) \in \mathcal{U}_{n+1\text{-}icv}$ is given by*

$$I^*(x) = \min\{x, d^*\}$$

for some non-negative d^*.

We remark that Corollary 4.4 generalizes Theorem 3.1 of Lu *et al.* (2012), which proves the optimality of stop-loss insurance under the assumption of $Y \uparrow_{SI} X$. Furthermore, by comparing the above optimal insurance form with that in Mahul (2000), it is easy to find that the increasing constraint on the retained loss function has an important influence on the optimal insurance solution, which is changed from the disappearing deductible to the stop-loss insurance. On the other hand, when this monotonic condition is imposed, we know from the above corollary and Arrow (1963) that the introduction of background risk Y satisfying $Y \uparrow_{n\text{-}icx} X$ does not change the optimality of stop-loss insurance. Intuitively, if $Y \uparrow_{n\text{-}icx} X$, both sources of risk are positively dependent and have no internal hedges, then the risk-averse insured will choose to cede all the tail risk under the premium principle based on the expected indemnity.

Even if the dependence assumption of $Y \uparrow_{n\text{-}icx} X$ is weakened to $Y \uparrow^{RTI}_{n\text{-}icx} X$, the same approach can be applied to rank some admissible insurance contracts, as illustrated in the following proposition.

Proposition 4.5. *Let $I_1(\cdot)$ and $I_2(\cdot)$ be two differentiable retained loss functions in $\mathfrak{C}$ satisfying the following conditions:*

(i) $\mathbb{E}[I_1(X)] = \mathbb{E}[I_2(X)]$, *and*
(ii) *there exists an $x_0 \ge 0$ such that $(I_2'(x) - I_1'(x))(x - x_0) \ge 0$ for all $x \ge 0$.*

If $Y \uparrow^{RTI}_{n\text{-}icx} X$, then $W_{I_1}(X, Y) \ge_{n+1\text{-}SD} W_{I_2}(X, Y)$.

Remark 4.6. *For $I_1(x)$ and $I_2(x)$ satisfying conditions in Proposition 4.5, it is easy to see that the function $I_2(x)$ up-crosses the function $I_1(x)$, then Lemma 3 in Ohlin (1969) implies $I_1(X) \leq_{cx} I_2(X)$. Therefore, comparing Proposition 4.2 with Proposition 4.5, we find that the cost to relax the assumption of stochastic dependence between the insurable risk and background risk is to impose a more stringent constraint on the insurance contracts to be compared.*

For any admissible retained loss function $I(x)$ with $0 \leq I'(x) \leq 1$, there exists an $I_d(x)$ in (4.1) such that $I_d(x)$ and $I(x)$ satisfy conditions in Proposition 4.5, and hence $I(x)$ is inferior to $I_d(x)$ for an insured with risk preference preserving $(n+1)$th degree stochastic dominance. Note that, however, not all retained loss functions in $\mathfrak{C}$ satisfy the condition $0 \leq I'(x) \leq 1$. To establish the optimality of stop-loss insurance over the set $\mathfrak{C}$ under the assumption of $Y \uparrow^{RTI}_{n\text{-}icx} X$, we need the following proposition.

Proposition 4.7. *If $Y \uparrow^{RTI}_{n\text{-}icx} X$, then $W_{I_d}(X, Y) \geq_{n+1\text{-}SD} W_I(X, Y)$ for any $I(x), I_d(x) \in \mathfrak{C}$ such that $\mathbb{E}[I(X)] = \mathbb{E}[I_d(X)]$.*

We remark that Proposition 4.7 implies Corollary 4.4. That means, when the dependence assumption is weakened from $Y \uparrow_{n\text{-}icx} X$ to $Y \uparrow^{RTI}_{n\text{-}icx} X$, the stop-loss insurance still keeps its optimality under an optimization criterion preserving $(n+1)$th degree stochastic dominance. Therefore, for any utility function $u \in \mathcal{U}_{n+1\text{-}icv}$, if $Y \uparrow^{RTI}_{n\text{-}icx} X$, then the study of the optimal insurance model (2.2) reduces to deriving the optimal retention level of stop-loss insurance. Mathematically, it is equivalent to solving the following maximization problem

$$\max_{d \geq 0} \mathbb{E}\left[u\left(w - Y - \min\{X, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+])\right)\right]. \tag{4.2}$$

The solution to the above optimization problem will be discussed in detail in the next section.

5. Optimal retention level and comparative analysis

In this section, we focus on stop-loss insurance contracts. We shall first find the optimal retention level to solve the problem (4.2), and then investigate how the insured's initial wealth and background risk affect the optimal stop-loss insurance by carrying out a comparative analysis.

To avoid tedious discussions, we assume X is continuously distributed throughout this section. We further make the following assumption:

Assumption 5.1.

(i) The insured has a utility function $u \in \mathcal{U}_{n+1\text{-}icv}$;
(ii) Y is right tail increasing in X with respect to the nth increasing convex order, i.e., $Y \uparrow^{RTI}_{n\text{-}icx} X$;

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(*iii*) *The insurance premium is calculated by* (2.1)*, where* $\mathcal{P}$ *is convex and twice differentiable with* $\mathcal{P}(0) = 0$ *and* $\mathcal{P}^{'}(x) \geq 1$ *for any* $x \geq 0$*;*

Define

$$\phi(d) = \mathbb{P}\{X > d\} \times \mathcal{P}'(\mathbb{E}[(X-d)_+]),\ 0 \leq d < ess\sup X, \tag{5.1}$$

where $ess \sup X = \inf\{y \geq 0 : \mathbb{P}\{X > y\} = 0\}$. Here, $\inf \emptyset = \infty$ by convention. Under Assumption 5.1, $\phi(d)$ is a decreasing and continuous function with

$$\begin{cases} \phi(d) > 1, & 0 \leq d < d_s; \\ \phi(d) \leq 1, & \text{otherwise,} \end{cases}$$

where $d_s = \inf\{d \geq 0 : \phi(d) \leq 1\}$. Under the stop-loss insurance contract, the insured's final wealth varies with the retention level. To emphasize this dependence, we rewrite the insured's final wealth as

$$W_d(X, Y) = w - Y - \min\{X, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+]).$$

Now, an optimal choice of the retention level of stop-loss insurance can be given in the following proposition.

Proposition 5.1. *Under Assumption 5.1, define*

$$\Psi(d) = \mathcal{P}'(\mathbb{E}[(X-d)_+]) \times \frac{\mathbb{E}[u'(W_d(X, Y))]}{\mathbb{E}[u'(W_d(X, Y))|X > d]},\ \forall\, 0 \leq d < ess\sup X, \tag{5.2}$$

then $\Psi(d)$ *is decreasing over the interval* $[d_s, ess\sup X)$*, and the optimal retention level that solves Problem* (4.2) *is given by*

$$d^* = \inf\{d \in [d_s, ess\sup X) : \Psi(d) \leq 1\}. \tag{5.3}$$

Different from Lu *et al.* (2012), the above proposition derives the optimal retention level of stop-loss insurance explicitly, which appears to rely heavily on the insured's risk preference, the stochastic dependence between the background risk and the insurable risk, and the insurance price. In particular, when the insurance price is actuarially fair, i.e., $\mathcal{P}(x) = x$, it is of interest to investigate the demand for insurance in the literature. In that case, we have $\phi(d) \leq 1$ for any d and thus $d_s = 0$. Furthermore, it is easy to see that

$$\Psi(0) = \frac{\mathbb{E}\left[u'(w - Y - \mathbb{E}[X])\right]}{\mathbb{E}\left[u'(w - Y - \mathbb{E}[X])|X > 0\right]} \leq 1,$$

where the inequality follows from the assumptions of $u(\cdot) \in \mathcal{U}_{n+1\text{-}icv}$ and $Y \uparrow^{RTI}_{n\text{-}icx} X$. Therefore, we conclude that $d^* = 0$. In other words, the optimal strategy for the insured is to cede all the insurable risk. Formally, this result is stated in the following corollary.

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Corollary 5.2. *Under Assumption 5.1 with $\mathcal{P}(x) = x$, full insurance is the optimal strategy for the insured.*

The above corollary indicates that full coverage is optimal for the insured if the insurance contract is fairly priced. This result extends Mossin's theorem by taking background risk into consideration. It also generalizes Proposition 3 of Doherty and Schlesinger (1983) and Proposition 3.5 of Lu *et al.* (2012), whose assumptions about the dependence structure are special cases of Assumption 5.1(ii).

In the following, we shall investigate how the optimal retention level is affected by the change of the insured's initial wealth or background risk. This is a very challenging task under a general dependence structure. In order to carry out the comparative analysis, we shall assume some special dependence structures characterized by the hazard rate order, which is formally defined below:

Definition 5.3 (Müller and Stoyan 2002). *Random variable Z_1 is said to be less than Z_2 in the hazard rate order, denoted as $Z_1 \le_{hr} Z_2$, if $\frac{\mathbb{P}\{Z_2>z\}}{\mathbb{P}\{Z_1>z\}}$ is increasing in z.*

In insurance economics, it is of great interest to analyze how the optimal retention level of stop-loss insurance is affected by the insured's initial wealth. Schlesinger (1981) is the first to study this problem in the absence of background risk, and finds that the insured with a lower level of initial wealth will choose a smaller retention level if his/her preference exhibits decreasing absolute risk aversion (DARA).[3] This result can be easily extended to the case with a background risk independent of the insurable risk, because Corollary 3 in Gollier (2001) indicates that the DARA is preserved by the introduction of an independent background risk. In fact, this result also holds in the presence of some positive dependence structures between the background risk and the insurable risk, as shown in the following proposition.

Proposition 5.4. *Assume that the preference of the risk-averse insured exhibits DARA and that $\mathcal{P}(x)$ is convex and twice differentiable with $\mathcal{P}(0) = 0$ and $\mathcal{P}'(x) \ge 1$ for any $x \ge 0$. The optimal retention level d^* to Problem* (4.2) *is increasing in the initial wealth w if (X, Y) satisfies the following conditions:*

(i) $[Y|X = x_1] \le_{hr} [Y|X = x_2]$ *for any $x_1, x_2 \in S(X)$ such that $x_1 \le x_2$;*
(ii) $[Y + a|X = x] \le_{hr} [Y + b|X = x]$ *for any $x \in S(X)$ and $a \le b$.*

Remark 5.5. *Among the two conditions on (X, Y) in the above proposition, the first condition concerns more about the interdependence between Y and X, while the second condition emphasizes the marginal distribution of Y conditional on $X = x$. Intuitively, Condition (i) means that Y increases in X in the sense of hazard rate order, which implies $Y \uparrow_{SI} X$ and thus $Y \uparrow^{RTI}_{n\text{-}icx} X$ for any positive integer n. On the other hand, Condition (ii) implies that the hazard rate function of $[Y|X = x]$ is increasing, given that $[Y|X = x]$ has a continuous distribution for any fixed $x \in S(X)$.*

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If Y has an increasing hazard rate function and is independent of X, then it is easy to verify that the random vector (X, Y) satisfies Conditions (i) and (ii) in Proposition 5.4. Another special dependence structure satisfying these two conditions is comonotonicity. Specifically, if there exists an increasing function B such that $Y = B(X)$, then Conditions (i) and (ii) are satisfied by (X, Y). Notably, the two conditions are also satisfied by a structure combining comonotonicity and independence, namely, $Y = Z + B(X)$, where random variable Z is independent of X and has an increasing hazard rate function.

Proposition 5.6. *Assume that the preference of the risk-averse insured exhibits DARA and that $\mathcal{P}(x)$ is convex and twice differentiable with $\mathcal{P}(0) = 0$ and $\mathcal{P}'(x) \geq 1$ for any $x \geq 0$. The optimal retention level to Problem* (4.2) *is reduced if the background risk Y_1 is replaced by Y_2, where Y_1, Y_2 and X satisfy the following two conditions:*

(i) $[Y_1 | X = x] \leq_{hr} [Y_2 | X = x]$ *for any $x \in S(X)$;*
(ii) *The function* $\frac{\mathbb{E}[u'(w-Y_2)|X=x]}{\mathbb{E}[u'(w-Y_1)|X=x]}$ *is increasing in x for any w.*

Remark 5.7. *Intuitively, Conditions (i) and (ii) indicate that Y_2 is "more risky" than Y_1. When Y_1 is replaced by Y_2, the insured has a more risky portfolio and will choose to reduce the retention level and thus transfer more risk.*

Although these two conditions, especially Condition (ii), look complicated, two special dependence structures can be easily verified to fall into this category: independence and comonotonicity. Specifically, if (i) Y_1, Y_2 are independent of X and $Y_1 \leq_{hr} Y_2$, or (ii) there exist increasing functions $B_1(\cdot)$ and $B_2(\cdot)$ such that $B_1(0) \leq B_2(0)$, $B_1'(x) \leq B_2'(x)$ for any $x \geq 0$, and $Y_1 = B_1(X)$, $Y_2 = B_2(X)$, then Y_1, Y_2 and X satisfy Conditions (i) and (ii).

It is worth noting that the effect of background risk on the optimal retention level of stop-loss insurance has been analyzed in Eeckhoudt and Kimball (1992). They conclude that the introduction of a zero-mean background risk satisfying $Y \uparrow_{3-icx} X$ would reduce the optimal retention level for a risk-averse and prudent insured. Essentially, their result concerns the shift from a zero background risk (i.e., no background risk) to a positively dependent background risk with zero mean. Proposition 5.6 studies a different type of shift between background risks and thus complements the study in Eeckhoudt and Kimball (1992).

6. Concluding remarks

In this paper, we employ a few useful notions to model positive dependence and study the optimal insurance problems with background risk. We manage to conduct comparison between different insurance contracts concerning high-order risk attitudes (Propositions 4.2 and 4.5), and thereby establish the optimality of the stop-loss insurance form (Corollary 4.4 and Proposition 4.7). These results significantly contribute to the literature since they deal with more general positive dependence structures. They also enhance the applicability of the stop-loss insurance. Specifically, the stop-loss insurance form has been proved to be optimal in the literature, but with relatively restrictive assumptions on the dependence structure (such as independence or stochastic increasingness). Our results extend this conclusion to a more general setting, which enable practitioners to use the stop-loss insurance with more confidence.

With the focus on the stop-loss insurance form, we obtain an expression for the optimal retention level in Proposition 5.1 and thus completely solve the the optimal insurance problem. In the case when the information about the utility function or the dependence structure is incomplete or unavailable, Proposition 5.1 gives a lower bound independent of these information and thus provides a useful guideline for practice. Based on the result derived in Proposition 5.1, we further conduct a comparative analysis to investigate how a change in the initial wealth level or a shift in the background risk affects the optimal insurance design. The results presented in Propositions 5.4 and 5.6 are reasonable and consistent with intuitions. Specifically, a decrease in the initial wealth or a shift to a more "dangerous" background risk will increase the insurance demand. Our contribution is to theoretically justify these intuitions in a relatively general framework and identify what specific assumptions are needed.

Admittedly, there are unsolved problems. As mentioned in Section 2, the assumption of the expected indemnity-based premium principle is crucial yet not quite realistic. It is of practical importance to study the optimal insurance problems under other premium principles. It will be a challenging task, especially in the presence of background risk. The comparative analysis in the paper reveals what specific role the insurance premium plays in determining the optimal insurance contract and shall shed some light on further studies. Another remaining problem is to conduct comparative analysis on other factors, such as the utility function or the premium principle. Such kind of analysis is of interest to both academics and practitioners. We leave these problems as future research.

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Notes

1. The principle of indemnity, which is widely accepted in the insurance practice, requires that the indemnity is non-negative and less than the insurable loss.

2. Throughout the paper, "increasing" and "decreasing" mean "non-decreasing" and "non-increasing", respectively.

3. The preference of an individual with the utility function $u(\cdot)$ is said to exhibit DARA if Arrow–Pratt measure of absolute risk aversion $A_u(z) = -\frac{u''(z)}{u'(z)}$ is decreasing.

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APPENDIX

A. PROOF

Proof of Proposition 3.3. The "horizontal" implications are based on the relationship between stochastic orders with different degrees and are easy to verify. The implication from $\uparrow_{SI}$ to $\uparrow_{RTI}$ has been established in Barlow and Proschan (1975). Another "vertical" implication can be proved similarly. ■

Proof of Proposition 4.2. According to Definition 2.1, it suffices to show that

$$\mathbb{E}\left[u(w - I_1(X) - Y - \pi(X - I_1(X)))\right] \geq \mathbb{E}\left[u(w - I_2(X) - Y - \pi(X - I_2(X)))\right]$$

for any $u \in \mathcal{U}_{n+1\text{-}icv}$, provided that the expectations exist. Noting that $I_1(X) \leq_{cx} I_2(X)$ implies $\mathbb{E}[I_1(X)] = \mathbb{E}[I_2(X)]$, we have $\pi(X - I_1(X)) = \pi(X - I_2(X))$ because the insurance premium is calculated based only on the expected indemnity. Therefore, it is equivalent to prove $\mathbb{E}[v(I_2(X) + Y)] \geq \mathbb{E}[v(I_1(X) + Y)]$ for any $v \in \mathcal{V}_{n+1\text{-}icx}$.

For any $v \in \mathcal{V}_{n+1\text{-}icx}$, its convexity yields that

$$v(I_2(X) + Y) - v(I_1(X) + Y) \geq v'(I_1(X) + Y)(I_2(X) - I_1(X)),$$

which in turn implies

$$\begin{aligned}\mathbb{E}[v(I_2(X) + Y)] - \mathbb{E}[v(I_1(X) + Y)] &\geq \mathbb{E}[v'(I_1(X) + Y)(I_2(X) - I_1(X))] \\ &= \mathbb{E}\left[p(X)(I_2(X) - I_1(X))\right],\end{aligned}$$

where $p(X) = \mathbb{E}[v'(I_1(X)+Y)|X]$. Since $Y \uparrow_{n\text{-}icx} X$ and $v'(\cdot) \in \mathcal{V}_{n\text{-}icx}$, then $\mathbb{E}[v'(I_1(x)+Y)|X]$ is an increasing function of X. Noting that $I_1(x)$ is increasing, $p(X)$ is also increasing in X. Using a similar argument as in the proof of Lemma 3.3 of Cai and Wei (2012), one concludes that $\mathbb{E}[p(X)(I_2(X) - I_1(X))] \geq 0$, which implies $\mathbb{E}[v(I_2(X) + Y)] \geq \mathbb{E}[v(I_1(X) + Y)]$. This completes the proof. ■

Proof of Proposition 4.5. Similar to Proposition 4.2, it suffices to show

$$\mathbb{E}[v(I_2(X) + Y)] \geq \mathbb{E}[v(I_1(X) + Y)] \quad \text{for any } v \in \mathcal{V}_{n+1\text{-}icx}, \tag{A.1}$$

provided that the expectations exist.

Note that $v(X+Y) = \int_0^X v'(x+Y)dx + v(Y) = \int_0^\infty v'(x+Y)\,\mathbb{I}\{X > x\}dx + v(Y)$, where $\mathbb{I}\{A\}$ is the indicator function of an event A. Therefore, for $i = 1, 2$, we have

$$\begin{aligned}\mathbb{E}[v(I_i(X) + Y)] &= \mathbb{E}\left[\int_0^\infty v'(z + Y)\,\mathbb{I}\{I_i(X) > z\}dz\right] + \mathbb{E}[v(Y)] \\ &= \int_0^\infty \mathbb{E}\left[v'(z + Y)\,\mathbb{I}\{I_i(X) > z\}\right]dz + \mathbb{E}[v(Y)],\end{aligned} \tag{A.2}$$

where the interchangeability of the expectation and the integration of the first term on the right-hand side is due to Tonelli's theorem, because the expectation exists (implied by the existence of two other expectations) and the integrand is non-negative. Therefore, (A.1) reduces to

$$\int_0^\infty \mathbb{E}[v'(Y+z)\,\mathbb{I}\{I_2(X) > z\}]\,dz \geq \int_0^\infty \mathbb{E}[v'(Y+z)\,\mathbb{I}\{I_1(X) > z\}]\,dz. \tag{A.3}$$

Since $(I_2'(x) - I_1'(x))(x - x_0) \geq 0$ for all $x \geq 0$ and $I_1(0) = I_2(0) = 0$, there are two possible positions between $I_1(x)$ and $I_2(x)$:

(i) $I_1(x) \geq I_2(x)$ for all $x \geq 0$, or
(ii) there exists an $x_e \geq x_0$ such that $(I_2(x) - I_1(x))(x - x_e) \geq 0$ for any non-negative x.

Since $\mathbb{E}[I_1(X)] = \mathbb{E}[I_2(X)]$, if (i) is true, then $I_1(X)$ and $I_2(X)$ are equal almost surely and (A.3) immediately follows. Otherwise, if (ii) is true, then $\{I_1(X) > z\} \subset \{I_2(X) > z\}$ for any $z > I_1(x_e)$, and $\{I_2(X) > z\} \subset \{I_1(X) > z\}$ for any $z < I_1(x_e)$. It is easy to verify that

$$\begin{aligned}&v'(Y+z)(\mathbb{I}\{I_2(X) > z\} - \mathbb{I}\{I_1(X) > z\}) \\ &\geq v'(Y+I_1(x_e))(\mathbb{I}\{I_2(X) > z\} - \mathbb{I}\{I_1(X) > z\}) \qquad \text{for any } z.\end{aligned} \tag{A.4}$$

Again, using Tonelli's theorem to interchange the order of expectations and integrations, one obtains that, for $i = 1, 2$

$$\begin{aligned}
&\int_0^\infty \mathbb{E}[v'(Y+I_1(x_e))\,\mathbb{I}\{I_i(X)>z\}]dz \\
&\quad = \mathbb{E}\left[v'(Y+I_1(x_e))\int_0^\infty \mathbb{I}\{I_i(X)>z\}\,dz\right] \\
&\quad = \mathbb{E}[v'(Y+I_1(x_e))I_i(X)] \\
&\quad = \mathbb{E}\left[v'(Y+I_1(x_e))\int_0^\infty I_i'(t)\mathbb{I}\{X>t\}\,dt\right] \\
&\quad = \int_0^\infty \mathbb{E}\left[v'(Y+I_1(x_e))\,I_i'(t)\mathbb{I}\{X>t\}\right]dt \\
&\quad = \int_0^\infty \mathbb{E}\left[v'(Y+I_1(x_e))|X>t\right]\,I_i'(t)\mathbb{P}\{X>t\}\,dt \\
&\quad = \int_0^\infty \psi(t)\,I_i'(t)\mathbb{P}\{X>t\}\,dt,
\end{aligned} \tag{A.5}$$

where $\psi(t) = \mathbb{E}[v'(Y+I_1(x_e))|X>t]$. Note that $\psi(t)$ is increasing since $Y \uparrow_{n\text{-}icx}^{RTI} X$. Recalling that $(I_2'(x) - I_1'(x))(x-x_0) \geq 0$, one gets

$$\begin{aligned}
&\int_0^\infty \psi(t)\,(I_2'(t) - I_1'(t))\mathbb{P}\{X>t\}\,dt \\
&\quad \geq \int_0^\infty \psi(x_0)\,(I_2'(t) - I_1'(t))\mathbb{P}\{X>t\}\,dt \\
&\quad = \psi(x_0)(\mathbb{E}[I_2(X) - I_1(X)]) = 0.
\end{aligned} \tag{A.6}$$

The required inequality (A.3) follows from (A.4), (A.5) and (A.6). ∎

Proof of Proposition 4.7. Similar to Proposition 4.2, it suffices to show

$$\mathbb{E}[v(I(X)+Y)] \geq \mathbb{E}[v(I_d(X)+Y)] \quad \text{for any } v \in \mathcal{V}_{n+1\text{-}icx},$$

provided that the expectations exist.

If $\mathbb{P}\{I(X) > d\} = 0$, then it follows from the facts $0 \leq I(x) \leq x$ and $\mathbb{E}[I(X)] = \mathbb{E}[I_d(X)]$ that $I(X) = I_d(X)$ almost surely, which immediately implies the desired result.

Otherwise, if $\mathbb{P}\{I(X) > d\} > 0$, noting that $0 \leq I(x) \leq x$, we get

$$\{I_d(X) > z\} = \emptyset \subset \{I(X) > z\},\ \forall z \geq d$$

and

$$\{I_d(X) > z\} = \{X > z\} \supset \{I(X) > z\},\ \forall z < d.$$

Therefore, for any $v \in \mathcal{V}_{n+1\text{-}icx}$, we have

$$\begin{aligned}
&v'(Y+z)\,(\mathbb{I}\{I(X)>z\} - \mathbb{I}\{I_d(X)>z\}) \\
&\quad \geq v'(Y+d)\,(\mathbb{I}\{I(X)>z\} - \mathbb{I}\{I_d(X)>z\}) \qquad \text{for any } z.
\end{aligned} \tag{A.7}$$

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Define

$$I^{-1}(x) = \inf\{y \geq 0 : I(y) > x\},\ \forall x \geq 0.$$

Since $I(x)$ is an increasing function with $0 \leq I(x) \leq x$, then we have

(i) $I^{-1}(x)$ is an increasing function with $I^{-1}(x) \geq x$;
(ii) $\{I(X) > z\}$ is equivalent to either $\{X > I^{-1}(z)\}$ or $\{X \geq I^{-1}(z)\}$.

Since $Y \uparrow_{n\text{-}icx}^{RTI} X$, then it is easy to get

$$\mathbb{E}\left[v'(Y+d)|I(X) > z_1\right] \geq \mathbb{E}\left[v'(Y+d)|I(X) > z_2\right]$$

and

$$\mathbb{E}\left[v'(Y+d)|I(X) > z_2\right] \geq \mathbb{E}\left[v'(Y+d)|X > z_2\right],$$

for any $0 \leq z_2 < z_1$ with $\mathbb{P}\{I(X) > z_1\} > 0$. Therefore, for any $z \in [0, d]$, we have

$$\begin{aligned}
&\mathbb{E}\left[v'(Y+d)(\mathbb{I}\{I(X) > z\} - \mathbb{I}\{I_d(X) > z\})\right] \\
&\quad = \mathbb{E}\left[v'(Y+d)|I(X) > z\right]\mathbb{P}\{I(X) > z\} - \mathbb{E}\left[v'(Y+d)|X > z\right]\mathbb{P}\{X > z\} \\
&\quad \geq \mathbb{E}\left[v'(Y+d)|I(X) > z\right](\mathbb{P}\{I(X) > z\} - \mathbb{P}\{X > z\}) \\
&\quad \geq \mathbb{E}\left[v'(Y+d)|I(X) > d\right](\mathbb{P}\{I(X) > z\} - \mathbb{P}\{X > z\}).
\end{aligned}$$

On the other hand, for each $z > d$, it holds that

$$\mathbb{E}\left[v'(Y+d)\mathbb{I}\{I(X) > z\}\right] \geq \mathbb{E}\left[v'(Y+d)|I(X) > d\right]\mathbb{P}\{I(X) > z\}. \tag{A.8}$$

The above two equations, together with (A.2) and (A.7), lead to

$$\begin{aligned}
&\mathbb{E}\left[v(Y+I(X))\right] - \mathbb{E}\left[v(Y+I_d(X))\right] \\
&= \int_0^\infty \mathbb{E}\left[v'(Y+z)(\mathbb{I}\{I(X) > z\} - \mathbb{I}\{I_d(X) > z\})\right]dz \\
&\geq \int_0^\infty \mathbb{E}\left[v'(Y+d)(\mathbb{I}\{I(X) > z\} - \mathbb{I}\{I_d(X) > z\})\right]dz \\
&\geq \int_d^\infty \mathbb{E}\left[v'(Y+d)\mathbb{I}\{I(X) > z\}\right]dz + \int_0^d \mathbb{E}\left[v'(Y+d)(\mathbb{I}\{I(X) > z\} - \mathbb{I}\{I_d(X) > z\})\right]dz \\
&\geq \mathbb{E}\left[v'(Y+d)|I(X) > d\right]\left(\int_d^\infty \mathbb{P}\{I(X) > z\}dz + \int_0^d \mathbb{P}\{I(X) > z\} - \mathbb{P}\{X > z\}dz\right) \\
&= \mathbb{E}\left[v'(Y+d)|I(X) > d\right](\mathbb{E}[I(X)] - \mathbb{E}[I_d(X)]) = 0,
\end{aligned}$$

which completes the proof. ■

Proof of Proposition 5.1. The proof of this proposition is a slight modification to that of Theorem 4.2 in Chi (2017).

First, we show that $\Psi(d)$ defined in (5.2) is decreasing over the interval $[d_s, \mathit{ess}\sup X)$. More specifically, denote $g_1(d) = \mathcal{P}'(\mathbb{E}[(X-d)_+])\mathbb{E}[u'(W_d(X,Y))]$ and $g_2(d) = \mathbb{E}[u'(W_d(X,Y))|X > d]$, then we have

$$\Psi(d) = \frac{g_1(d)}{g_2(d)} \quad \text{and} \quad \Psi'(d) \times (g_2(d))^2 = g_1'(d)g_2(d) - g_1(d)g_2'(d).$$

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Note that

$$\begin{aligned}
g_1'(d) &= -S_X(d)\mathcal{P}''(\mathbb{E}[(X-d)_+])\mathbb{E}[u'(W_d(X,Y))] \\
&\quad + S_X(d)\,(\mathcal{P}'(\mathbb{E}[(X-d)_+]))^2\mathbb{E}[u''(W_d(X,Y))] \\
&\quad - \mathcal{P}'(\mathbb{E}[(X-d)_+])\mathbb{E}[u''(W_d(X,Y))\mathbb{I}\{X>d\}] \\
&\le (\mathcal{P}'(\mathbb{E}[(X-d)_+]))^2 S_X(d)\mathbb{E}[u''(W_d(X,Y))] \\
&\quad - \mathcal{P}'(\mathbb{E}[(X-d)_+])\mathbb{E}[u''(W_d(X,Y))\mathbb{I}\{X>d\}] \\
&\le \phi(d)\mathcal{P}'(\mathbb{E}[(X-d)_+])\mathbb{E}[u''(W_d(X,Y))\mathbb{I}\{X>d\}] \\
&\quad - \mathcal{P}'(\mathbb{E}[(X-d)_+])\mathbb{E}[u''(W_d(X,Y))\mathbb{I}\{X>d\}] \\
&= (\phi(d)-1)\mathcal{P}'(\mathbb{E}[(X-d)_+])\mathbb{E}[u''(W_d(X,Y))\mathbb{I}\{X>d\}],
\end{aligned} \tag{A.9}$$

where $S_X(d)$ is the survival function of X and $\phi(d)$ is defined in (5.1). The first inequality holds because of $\mathcal{P}''(\cdot)\ge 0$ and $u'(\cdot)\ge 0$, while the second inequality is due to the assumption $u''(\cdot)\le 0$. In order to analyze the derivative of $g_2(d)$, we introduce an auxiliary function

$$\ell(x,y) = \mathbb{E}[u'(w - Y - x - \mathcal{P}(\mathbb{E}[(X-x)_+]))\,|\,X>y].$$

Clearly, $g_2(d) = \ell(d,d)$ and $g_2'(d) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\ell(x,y)\Big|_{(x,y)=(d,d)}$. Since $Y \uparrow_{n\text{-}icx}^{RTI} X$ and $u \in \mathcal{U}_{n+1\text{-}icv}$, then it is easy to conclude that $\ell(x,y)$ is increasing in y. Therefore, we have

$$g_2'(d) \ge \frac{\partial}{\partial x}\ell(x,y)\Big|_{(x,y)=(d,d)} = (\phi(d)-1)\mathbb{E}[u''(W_d(X,Y))\,|\,X>d]. \tag{A.10}$$

Combining (A.9) and (A.10), one obtains that for any $d \ge d_s$,

$$\begin{aligned}
&\Psi'(d)\times(g_2(d))^2 \\
&= g_1'(d)g_2(d) - g_1(d)g_2'(d) \\
&\le (\phi(d)-1)\mathcal{P}'(\mathbb{E}[(X-d)_+])\mathbb{E}[u''(W_d(X,Y))\,|\,X>d]\times\mathbb{E}\left[u'(W_d(X,Y))\mathbb{I}\{X>d\}\right] \\
&\quad - (\phi(d)-1)\mathbb{E}[u''(W_d(X,Y))\,|\,X>d]\times\mathcal{P}'(\mathbb{E}[(X-d)_+])\mathbb{E}[u'(W_d(X,Y))] \\
&= -(\phi(d)-1)\mathcal{P}'(\mathbb{E}[(X-d)_+])\mathbb{E}[u''(W_d(X,Y))\,|\,X>d]\mathbb{E}[u'(W_d(X,Y))\mathbb{I}\{X\le d\}] \\
&\le 0,
\end{aligned}$$

where the last inequality holds because $\mathcal{P}'(\cdot)\ge 0$, $u'(\cdot)\ge 0$, $u''(\cdot)\le 0$ and $\phi(d)\le 1$ for any $d\ge d_s$. This implies $\Psi'(d)\le 0$ for any $d\in[d_s, ess\sup X)$.

Following, we denote the objective function of optimization problem (4.2) by

$$L(d) = \mathbb{E}[u(W_d(X,Y))] = \mathbb{E}[u(w - Y - \min\{X,d\} - \mathcal{P}(\mathbb{E}[(X-d)_+]))].$$

Note that

$$\begin{aligned}
L'(d) &= \mathcal{P}'(\mathbb{E}[(X-d)_+])\,S_X(d)\mathbb{E}[u'(W_d(X,Y))] - \mathbb{E}[u'(W_d(X,Y))\mathbb{I}\{X>d\}] \\
&\ge \phi(d)\mathbb{E}[u'(W_d(X,Y))\mathbb{I}\{X>d\}] - \mathbb{E}[u'(W_d(X,Y))\mathbb{I}\{X>d\}] \\
&= (\phi(d)-1)\mathbb{E}[u'(W_d(X,Y))\mathbb{I}\{X>d\}],
\end{aligned}$$

which in turn implies $L'(d) \geq 0$ for any $d < d_s$. Therefore, $L(d)$ is increasing over the interval $[0, d_s]$. Further, noting that $L'(d) = \mathbb{E}[u'(W_d(X, Y))\mathbb{I}\{X > d\}](\Psi(d) - 1)$, where $\Psi(d)$ is shown to be decreasing over $[d_s, ess\sup X)$, we conclude that the maximum value of $L(d)$ is attainable at d^*, where d^* is defined in (5.3). The proof is thus completed. ■

Before presenting the proofs to Propositions 5.4 and 5.6, we need to introduce several auxiliary lemmas. The first lemma, which is reproduced from Theorem 3.4 in Shanthikumar and Yao (1991), provides a useful characterization of hazard rate order by bivariate functions.

Lemma A.1. *Assume that Z_1 and Z_2 are independent. $Z_1 \leq_{hr} Z_2$ if and only if $\mathbb{E}[g(Z_1, Z_2)] \geq \mathbb{E}[g(Z_2, Z_1)]$ for any bivariate function g such that $\Delta g(x, y)$ is increasing in y for any $y \geq x$, where $\Delta g(x, y) = g(x, y) - g(y, x)$.*

With the help of the above characterization of hazard rate order, we have the following lemma.

Lemma A.2. *Let $u(\cdot)$ be an increasing concave utility function with DARA. For any $Y_1 \leq_{hr} Y_2$ and $w_1 \leq w_2$, it holds that*

$$\frac{\mathbb{E}[u'(w_1 - Y_1)]}{\mathbb{E}[u'(w_2 - Y_1)]} \leq \frac{\mathbb{E}[u'(w_1 - Y_2)]}{\mathbb{E}[u'(w_2 - Y_2)]}. \tag{A.11}$$

Proof. Without loss of generality, assume Y_1 and Y_2 are independent. Inequality (A.11) is equivalent to

$$\mathbb{E}[u'(w_1 - Y_1)u'(w_2 - Y_2)] \leq \mathbb{E}[u'(w_1 - Y_2)u'(w_2 - Y_1)]. \tag{A.12}$$

Let $g(x, y) = u'(w_1 - y)u'(w_2 - x)$, then inequality (A.12) can be rewritten as $\mathbb{E}[g(Y_1, Y_2)] \geq \mathbb{E}[g(Y_2, Y_1)]$. According to Lemma A.1, it suffices to prove that $\frac{\partial}{\partial y}\Delta g(x, y) \geq 0$ for any $x \leq y$.

Since $u(\cdot)$ is an increasing concave function, then we have $u'(z) \geq 0$ and $A_u(z) = -\frac{u''(z)}{u'(z)} \geq 0$ for any z. Furthermore, the DARA property of $u(\cdot)$ implies that $A_u(z)$ is decreasing, and hence $u'(z)$ is log-convex. Therefore, for any $x \leq y$, it holds that

$$\begin{aligned}
\frac{\partial}{\partial y}\Delta g(x, y) &= -u''(w_1 - y)u'(w_2 - x) + u'(w_1 - x)u''(w_2 - y) \\
&= A_u(w_1 - y)u'(w_1 - y)u'(w_2 - x) - A_u(w_2 - y)u'(w_1 - x)u'(w_2 - y) \\
&\geq A_u(w_2 - y)(u'(w_1 - y)u'(w_2 - x) - u'(w_1 - x)u'(w_2 - y)) \geq 0,
\end{aligned}$$

where the first inequality is derived by the decreasing property of $A_u(z)$ and the second inequality is due to log-convexity of $u'(\cdot)$. This completes the proof. ■

In addition, we have to introduce another useful lemma below.

Lemma A.3. *Let $h_1(\cdot)$ and $h_2(\cdot)$ be two non-negative functions. If $h_1(x)/h_2(x)$ is increasing in x, then*

$$\frac{\mathbb{E}[h_1(X)\mathbb{I}\{X \leq d\}]}{\mathbb{E}[h_2(X)\mathbb{I}\{X \leq d\}]} \leq \frac{\mathbb{E}[h_1(X)\mathbb{I}\{X > d\}]}{\mathbb{E}[h_2(X)\mathbb{I}\{X > d\}]}, \ \forall d \geq 0.$$

Proof. The proof is straightforward by noting that $h_1(X) \leq \frac{h_1(d)}{h_2(d)}h_2(X)$ for $X \leq d$ and $h_1(X) \geq \frac{h_1(d)}{h_2(d)}h_2(X)$ for $X > d$. ■

Proof of Proposition 5.4. Consider two initial wealth levels $w_1 \le w_2$. Let d_i^* be the optimal retention level corresponding to the initial wealth w_i for $i = 1, 2$. Our task is to show $d_1^* \le d_2^*$.

Define

$$\Psi_i(d) = \mathcal{P}'(\mathbb{E}[(X-d)_+]) \times \frac{\mathbb{E}[u'(W_d^i(X, Y))]}{\mathbb{E}[u'(W_d^i(X, Y))|X > d]}$$

for $i = 1, 2$, where $W_d^i(X, Y) = w_i - Y - \min\{X, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+])$. Since the hazard rate order is more strict than the usual stochastic order, then it follows from Condition (i) that $Y \uparrow_{SI} X$, which in turn implies that Y is right tail increasing with respect to X. Recalling from Proposition 5.1 that $d_i^* = \inf\{d \in [d_s, ess \sup X) : \Psi_i(d) \le 1\}$, it suffices to show that $\Psi_1(d) \le \Psi_2(d)$. Simple algebra yields that $\Psi_1(d) \le \Psi_2(d)$ is equivalent to

$$\frac{\mathbb{E}[u'(W_d^1(X, Y))\mathbb{I}\{X \le d\}]}{\mathbb{E}[u'(W_d^2(X, Y))\mathbb{I}\{X \le d\}]} \le \frac{\mathbb{E}[u'(W_d^1(X, Y))\mathbb{I}\{X > d\}]}{\mathbb{E}[u'(W_d^2(X, Y))\mathbb{I}\{X > d\}]}.$$

Let $h_i(x) = \mathbb{E}[u'(W_d^i(X, Y))|X = x]$ for $i = 1, 2$ and $x \in S(X)$. According to Lemma A.3, the proof of the above inequality reduces to verifying that $\frac{h_1(x)}{h_2(x)}$ is increasing in x.

Note that for any $0 \le x_1 < x_2$, we have

$$\begin{aligned}
\frac{h_1(x_1)}{h_2(x_1)} &= \frac{\mathbb{E}[u'(w_1 - Y - \min\{x_1, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+]))|X = x_1]}{\mathbb{E}[u'(w_2 - Y - \min\{x_1, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+]))|X = x_1]} \\
&\le \frac{\mathbb{E}[u'(w_1 - Y - \min\{x_1, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+]))|X = x_2]}{\mathbb{E}[u'(w_2 - Y - \min\{x_1, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+]))|X = x_2]} \\
&\le \frac{\mathbb{E}[u'(w_1 - Y - \min\{x_2, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+]))|X = x_2]}{\mathbb{E}[u'(w_2 - Y - \min\{x_2, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+]))|X = x_2]} = \frac{h_1(x_2)}{h_2(x_2)},
\end{aligned}$$

where the first inequality follows from Proposition A.2 and Conditions (i), and the second inequality follows from Lemma A.2 and Conditions (ii). The proof is thus completed. ∎

Proof of Proposition 5.6. The proof is a slight modification to that of Proposition 5.4. More specifically, define

$$\begin{aligned}
\tilde{\Psi}_i(d) &= \mathcal{P}'(\mathbb{E}[(X-d)_+]) \\
&\quad \times \frac{\mathbb{E}[u'(w - Y_i - \min\{X, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+]))]}{\mathbb{E}[u'(w - Y_i - \min\{X, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+]))|X > d]} \qquad \text{for } i = 1, 2,
\end{aligned}$$

and we need to show that $\tilde{\Psi}_2(d) \le \tilde{\Psi}_1(d)$. According to Lemma A.3, it suffices to verify that $\frac{\tilde{h}_2(x)}{\tilde{h}_1(x)}$ is increasing in x, where

$$\tilde{h}_i(x) = \mathbb{E}[u'(w - Y_i - \min\{x, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+]))|X = x], \; x \in S(X)$$

for $i = 1, 2$.

Note that for any $0 \leq x_1 < x_2$, it holds that

$$\begin{aligned}\frac{\tilde{h}_2(x_1)}{\tilde{h}_1(x_1)} &= \frac{\mathbb{E}[u'(w - Y_2 - \min\{x_1, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+]))|X = x_1]}{\mathbb{E}[u'(w - Y_1 - \min\{x_1, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+]))|X = x_1]} \\ &\leq \frac{\mathbb{E}[u'(w - Y_2 - \min\{x_1, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+]))|X = x_2]}{\mathbb{E}[u'(w - Y_1 - \min\{x_1, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+]))|X = x_2]} \\ &\leq \frac{\mathbb{E}[u'(w - Y_2 - \min\{x_2, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+]))|X = x_2]}{\mathbb{E}[u'(w - Y_1 - \min\{x_2, d\} - \mathcal{P}(\mathbb{E}[(X-d)_+]))|X = x_2]} = \frac{\tilde{h}_2(x_2)}{\tilde{h}_1(x_2)},\end{aligned}$$

where the first inequality is due to Condition (ii) and the second inequality follows from Condition (i) and Lemma A.2. The proof is thus completed. ■