Canad. Math. Bull. Vol. **61** (3), 2018 pp. 458–463
http://dx.doi.org/10.4153/CMB-2017-044-4

Global Holomorphic Functions in Several Non-Commuting Variables II

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Abstract. We give a new proof that bounded non-commutative functions on polynomial polyhedra can be represented by a realization formula, a generalization of the transfer function realization formula for bounded analytic functions on the unit disk.

1 Introduction

Let $\mathbb{M}_n$ denote the $n \times n$ matrices with complex entries, and let $\mathbb{M}^d = \bigcup_{n=1}^{\infty} \mathbb{M}_n^d$ be the set of all d-tuples of matrices of the same size. A *non-commutative function* (nc-function) on a set $E \subseteq \mathbb{M}^d$ is a function $\phi: E \to \mathbb{M}^1$ that satisfies

- ϕ is graded, which means that if $x \in E \cap \mathbb{M}_n^d$, then $\phi(x) \in \mathbb{M}_n$;
- ϕ is intertwining preserving, which means that if $x, y \in E$ and S is a linear operator satisfying $Sx = yS$, then $S\phi(x) = \phi(y)S$.

The points x and y are d-tuples, so we write $x = (x^1, \dots, x^d)$ and $y = (y^1, \dots, y^d)$. By $Sx = yS$, we mean that $Sx^r = y^r S$ for each $1 \le r \le d$. See [9] for a general reference to nc-functions.

The principal result of [2] was a realization formula for nc-functions that are bounded on polynomial polyhedra; the object of this note is to give a simpler proof of this formula, (see Theorem 1.2).

Let δ be an $I \times J$ matrix whose entries are non-commutative polynomials in d variables. If $x \in \mathbb{M}_n^d$, then $\delta(x)$ can be naturally thought of as an element of $\mathcal{B}(\mathbb{C}^J \otimes \mathbb{C}^n, \mathbb{C}^I \otimes \mathbb{C}^n)$, where $\mathcal{B}$ denotes the bounded linear operators, and all norms we use are operator norms on the appropriate spaces. We define

$$B_\delta := \{x \in \mathbb{M}^d : \|\delta(x)\| < 1\}. \tag{1.1}$$

Any set of the form (1.1) is called a *polynomial polyhedron*. Let $H^\infty(B_\delta)$ denote the nc-functions on B_δ that are bounded, and let $H_1^\infty(B_\delta)$ denote the closed unit ball, those nc-functions that are bounded by 1 for every $x \in B_\delta$.

Received by the editors March 29, 2017; revised June 29, 2017.
Published electronically June 6, 2018.
Author J. A. was partially supported by National Science Foundation Grants DMS 1361720 and DMS 1665260. Author J. M. was partially supported by National Science Foundation Grant DMS 1565243.
AMS subject classification: **15A54.**
Keywords: non-commutative function, realization formula.

Definition 1.1 A *free realization* for ϕ consists of an auxiliary Hilbert space $\mathcal{M}$ and an isometry

$$\begin{array}{c c} & \begin{array}{cc} \mathbb{C} & \mathcal{M}\otimes\mathbb{C}^I \end{array} \\ \begin{array}{c} \mathbb{C} \\ \mathcal{M}\otimes\mathbb{C}^J \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}$$

such that for all $x \in B_\delta$, we have

$$\phi(x) = \begin{array}{c} A \\ \otimes \\ 1 \end{array} + \begin{array}{c} B \\ \otimes \\ 1 \end{array} \begin{array}{c} 1 \\ \otimes \\ \delta(x) \end{array} \left[1 - \begin{array}{c} D \\ \otimes \\ 1 \end{array} \begin{array}{c} 1 \\ \otimes \\ \delta(x) \end{array} \right]^{-1} \begin{array}{c} C \\ \otimes \\ 1 \end{array}. \tag{1.2}$$

The 1s need to be interpreted appropriately. If $x \in \mathbb{M}_n^d$, then (1.2) means

$$\phi(x) = \begin{array}{c} A \\ \otimes \\ \mathrm{id}_{\mathbb{C}^n} \end{array} + \begin{array}{c} B \\ \otimes \\ \mathrm{id}_{\mathbb{C}^n} \end{array} \begin{array}{c} \mathrm{id}_{\mathcal{M}} \\ \otimes \\ \delta(x) \end{array} \left[\begin{array}{c} \mathrm{id}_{\mathcal{M}} \\ \otimes \\ \mathrm{id}_{\mathbb{C}^I} \\ \otimes \\ \mathrm{id}_{\mathbb{C}^n} \end{array} - \begin{array}{c} D \\ \otimes \\ \mathrm{id}_{\mathbb{C}^n} \end{array} \begin{array}{c} \mathrm{id}_{\mathcal{M}} \\ \otimes \\ \delta(x) \end{array} \right]^{-1} \begin{array}{c} C \\ \otimes \\ \mathrm{id}_{\mathbb{C}^n} \end{array}.$$

We adopt the convention of [11] and write tensors vertically to enhance legibility. The bottom-most entry corresponds to the space on which x originally acts; the top corresponds to the intrinsic part of the model on $\mathcal{M}$.

The following theorem was proved in [2]; another proof appears in [6].

Theorem 1.2 *The function ϕ is in $H_1^\infty(B_\delta)$ if and only if it has a free realization.*

It is a straightforward calculation that any function of the form (1.2) is in $H_1^\infty(B_\delta)$. We wish to prove the converse. We shall use two other results: Theorems 1.4 and 1.5.

If $E \subset \mathbb{M}^d$, we let E_n denote $E \cap \mathbb{M}_n^d$. If $\mathcal{K}$ and $\mathcal{L}$ are Hilbert spaces, a $\mathcal{B}(\mathcal{K},\mathcal{L})$-valued nc function on a set $E \subseteq \mathbb{M}^d$ is a function ϕ such that

- ϕ is $\mathcal{B}(\mathcal{K},\mathcal{L})$ graded, which means if $x \in E_n$, then $\phi(x) \in \mathcal{B}(\mathcal{K}\otimes\mathbb{C}^n, \mathcal{L}\otimes\mathbb{C}^n)$;
- ϕ is intertwining preserving, which means if $x, y \in E$ and S is a linear operator satisfying $Sx = yS$, then

$$\begin{array}{c} \mathrm{id}_{\mathcal{L}} \\ \otimes \\ S \end{array} \phi(x) = \phi(y) \begin{array}{c} \mathrm{id}_{\mathcal{K}} \\ \otimes \\ S \end{array}.$$

Definition 1.3 An nc-model for $\phi \in H_1^\infty(B_\delta)$ consists of an auxiliary Hilbert space $\mathcal{M}$ and a $\mathcal{B}(\mathbb{C}, \mathcal{M}\otimes\mathbb{C}^J)$-valued nc-function u on B_δ such that, for all pairs $x, y \in B_\delta$ that are on the same level, *i.e.*, both in $B_\delta \cap \mathbb{M}_n^d$ for some n,

$$1 - \phi(y)^*\phi(x) = u(y)^* \left[\begin{array}{c} 1 \\ \otimes \\ 1-\delta(y)^*\delta(x) \end{array} \right] u(x). \tag{1.3}$$

Again, the 1s have to be interpreted appropriately. If $x, y \in B_\delta \cap \mathbb{M}_n^d$, then (1.3) means

$$\mathrm{id}_{\mathbb{C}^n} - \phi(y)^*\phi(x) = u(y)^* \left[\begin{array}{c} \mathrm{id}_{\mathcal{M}} \\ \otimes \\ \mathrm{id}_{\mathbb{C}^J\otimes\mathbb{C}^n} - \delta(y)^*\delta(x) \end{array} \right] u(x).$$

Theorem 1.4 *A graded function on B_δ has an nc-model if and only if it has a free realization.*

Theorem 1.4 was proved in [2], but a simpler proof was given by Balasubramanian [5]. Let us note for future reference that the functions u in (1.3) are locally bounded, and therefore holomorphic [2, Theorem. 4.6].

The finite topology on $\mathbb{M}^d$ (also called the disjoint union topology) is the topology in which a set Ω is open if and only if for every n, Ω_n is open in the Euclidean topology on $\mathbb{M}_n^d$. If $\mathcal{H}$ is a Hilbert space, and Ω is finitely open, we shall let $\mathrm{Hol}_{\mathcal{H}}^{\mathrm{nc}}(\Omega)$ denote the $\mathcal{B}(\mathbb{C}, \mathcal{H})$ graded nc-functions on Ω that are holomorphic on each Ω_n. (A function u is holomorphic in this context if for each n, each $x \in \Omega_n$, and each $h \in \mathbb{M}_n^d$, the limit $\lim_{t\to 0} 1/t(u(x+th)-u(x))$ exists.) A sequence of functions u^k on Ω is finitely locally uniformly bounded if for each point $\lambda \in \Omega$, there is a finitely open neighborhood of λ inside Ω on which the sequence is uniformly bounded.

The following wandering Montel theorem was proved in [1]. If u is in $\mathrm{Hol}_{\mathcal{H}}^{\mathrm{nc}}(\Omega)$ and V is a unitary operator on $\mathcal{H}$, define $V * u$ by $(V * u)|_{\Omega_n} = \overset{V}{\underset{\mathrm{id}_{\mathbb{C}^n}}{\otimes}} u|_{\Omega_n} \quad \forall_n$.

Theorem 1.5 *Let Ω be finitely open, $\mathcal{H}$ a Hilbert space, and $\{u^k\}$ a finitely locally uniformly bounded sequence in $\mathrm{Hol}_{\mathcal{H}}^{\mathrm{nc}}(\Omega)$. Then there exists a sequence $\{U^k\}$ of unitary operators on $\mathcal{H}$ such that $\{U^k * u^k\}$ has a subsequence that converges finitely locally uniformly to a function in $\mathrm{Hol}_{\mathcal{H}}^{\mathrm{nc}}(B_\delta)$.*

Let $\phi \in H_1^\infty(B_\delta)$. We shall prove Theorem 1.2 in the following steps.

I For every $z \in B_\delta$, show that $\phi(z)$ is in $\mathrm{Alg}(z)$, the unital algebra generated by the elements of z.
II Prove that for every finite set $F \subseteq B_\delta$, there is an nc-model for a function ψ that agrees with ϕ on F.
III Show that these nc-models have a cluster point that gives an nc-model for ϕ.
IV Use Theorem 1.4 to get a free realization for ϕ.

Remarks 1.6 Step I is noted in [2] as a corollary of Theorem 1.2; proving it independently allows us to streamline the proof of Theorem 1.2.

To prove Step II, we use one direction of [3, Theorem 1.3] that gives necessary and sufficient conditions to solve a finite interpolation problem on B_δ. The proof of necessity of this theorem used Theorem 1.2, but for Step II we only need the sufficiency of the condition, and the proof of this in [3] did not use Theorem 1.2.

All three known proofs of Theorem 1.2 start by proving a realization on finite sets, and then somehow taking a limit. In [2], this was done by considering partial nc-functions; in [6], it was done by using non-commutative kernels to get a compact set in which limit points must exist. In the current paper, we use the wandering Montel theorem.

2 Step I

Let $\{e_j\}_{j=1}^n$ be the standard basis for $\mathbb{C}^n$. For x in $\mathbb{M}_n$ or $\mathbb{M}_n^d$, let $x^{(k)}$ denote the direct sum of k copies of x. If $x \in \mathbb{M}_n^d$ and s is invertible in $\mathbb{M}_n$, then $s^{-1}xs$ denotes the d-tuple $(s^{-1}x^1s, \dots, s^{-1}x^ds)$.

Lemma 2.1 *Let $z \in \mathbb{M}_n^d$, with $\|z\| < 1$. Assume $w \notin \operatorname{Alg}(z)$. Then there is an invertible $s \in \mathbb{M}_{n^2}$ such that $\|s^{-1}z^{(n)}s\| < 1$ and $\|s^{-1}w^{(n)}s\| > 1$.*

Proof Let $\mathcal{A} = \operatorname{Alg}(z)$. Since $w \notin \mathcal{A}$, and $\mathcal{A}$ is finite dimensional and therefore closed, the Hahn–Banach theorem says that there is a matrix $K \in \mathbb{M}_n$ such that $\operatorname{tr}(aK) = 0$ for all $a \in \mathcal{A}$ and $\operatorname{tr}(wK) \neq 0$. Let $u \in \mathbb{C}^n \otimes \mathbb{C}^n$ be the direct sum of the columns of K, and $v = e_1 \oplus e_2 \oplus \cdots e_n$. Then for any $b \in \mathbb{M}_n$ we have

$$\operatorname{tr}(bK) = \langle b^{(n)}u, v\rangle.$$

Let $\mathcal{A} \otimes \operatorname{id}$ denote $\{a^{(n)} : a \in \mathcal{A}\}$. We have $\langle a^{(n)}u, v\rangle = 0$, for all $a \in \mathcal{A}$ and $\langle w^{(n)}u, v\rangle \neq 0$.

Let $\mathcal{N} = (\mathcal{A} \otimes \operatorname{id})u$. This is an $\mathcal{A} \otimes \operatorname{id}$-invariant subspace, but it is not $w^{(n)}$ invariant (since $v \perp \mathcal{N}$, but v is not perpendicular to $w^{(n)}u$). So decomposing $\mathbb{C}^n \otimes \mathbb{C}^n$ as $\mathcal{N} \oplus \mathcal{N}^\perp$, every matrix in $\mathcal{A} \otimes \operatorname{id}$ has 0 in the $(2,1)$ entry, and $w^{(n)}$ does not.

Let $s = \alpha I_{\mathcal{N}} + \beta I_{\mathcal{N}^\perp}$, with $\alpha \gg \beta > 0$. Then

$$s^{-1}\begin{bmatrix} A & B \\ C & D \end{bmatrix} s = \begin{bmatrix} A & \frac{\beta}{\alpha}B \\ \frac{\alpha}{\beta}C & D \end{bmatrix}.$$

If the ratio α/β is large enough, then for each of the d matrices z^r, the corresponding $s^{-1}(z^r \otimes \operatorname{id})s$ will have strict contractions in the (1,1) and (2,2) slots, and each $(1,2)$ entry will be small enough so that the whole thing is a contraction.

For w, however, as the $(2,1)$ entry is non-zero, the norm of $s^{-1}w^{(n)}s$ can be made arbitrarily large. ∎

Lemma 2.2 *Let $z \in B_\delta \cap \mathbb{M}_n^d$, and $w \in \mathbb{M}_n$ not be in $\mathcal{A} := \operatorname{Alg}(z)$. Then there is an invertible $s \in \mathbb{M}_{n^2}$ such that $s^{-1}z^{(n)}s \in B_\delta$ and $\|s^{-1}w^{(n)}s\| > 1$.*

Proof As in the proof of Lemma 2.1, we can find an invariant subspace $\mathcal{N}$ for $\mathcal{A} \otimes \operatorname{id}$ that is not w-invariant. Decompose $\delta(z^{(n)})$ as a map from $(\mathcal{N} \otimes \mathbb{C}^J) \oplus (\mathcal{N}^\perp \otimes \mathbb{C}^J)$ into $(\mathcal{N} \otimes \mathbb{C}^I) \oplus (\mathcal{N}^\perp \otimes \mathbb{C}^I)$. With s as in Lemma 2.1, and $\alpha \gg \beta > 0$, and P the projection from $\mathbb{C}^n \otimes \mathbb{C}^n$ onto $\mathcal{N}$, we get

$$\delta(s^{-1}z^{(n)}s) = \begin{bmatrix} \overset{P}{\underset{\operatorname{id}}{\otimes}}\ \delta(z^{(n)})\ \overset{P}{\underset{\operatorname{id}}{\otimes}} & \frac{\beta}{\alpha}\ \overset{P}{\underset{\operatorname{id}}{\otimes}}\ \delta(z^{(n)})\ \overset{P^\perp}{\underset{\operatorname{id}}{\otimes}} \\ 0 & \overset{P^\perp}{\underset{\operatorname{id}}{\otimes}}\ \delta(z^{(n)})\ \overset{P^\perp}{\underset{\operatorname{id}}{\otimes}} \end{bmatrix}. \tag{2.1}$$

The matrix is upper triangular because every entry of δ is a polynomial, and $\mathcal{N}$ is $\mathcal{A}$-invariant. For α/β large enough, every matrix of the form (2.1) with $z \in B_\delta$ is a contraction, so $s^{-1}z^{(n)}s \in B_\delta$. But $s^{-1}w^{(n)}s$ will contain a non-zero entry multiplied by $\frac{\alpha}{\beta}$, so we achieve the claim. ∎

Theorem 2.3 *If ϕ is in $H^\infty(B_\delta)$, then for all $z \in B_\delta$, we have $\phi(z) \in \operatorname{Alg}(z)$.*

Proof We can assume that $z \in B_\delta$ and that $\|\phi\| \leq 1$ on B_δ. Let $w = \phi(z)$. If $w \notin \operatorname{Alg}(z)$, then by Lemma 2.2, there is an s such that $s^{-1}z^{(n)}s \in B_\delta$ and $\|\phi(s^{-1}z^{(n)}s)\| = \|s^{-1}w^{(n)}s\| > 1$, a contradiction. ∎

https://doi.org/10.4153/CMB-2017-044-4 Published online by Cambridge University Press

Note that Theorem 2.3 does not hold for all nc-functions. In [4] it was shown that there is a class of nc functions, called fat functions, for which the implicit function theorem holds, but Theorem 2.3 fails.

3 Step II

Let $F = \{x_1, \dots, x_N\}$. Define $\lambda = x_1 \oplus \cdots \oplus x_N$, and define $w = \phi(x_1) \oplus \cdots \oplus \phi(x_N)$. As nc functions preserve direct sums (a consequence of being intertwining preserving) we need to find a function ψ in $H_1^\infty(B_\delta)$ that has an nc model, and satisfies $\psi(\lambda) = w$.

Let $\mathcal{P}_d$ denote the nc polynomials in d variables, and define

$$I_\lambda = \{q \in \mathcal{P}_d : q(\lambda) = 0\}.$$

Let $V_\lambda = \{x \in \mathbb{M}^d : q(x) = 0 \text{ whenever } q \in I_\lambda\}$. We will need the following theorem from [3].

Theorem 3.1 *Let $\lambda \in B_\delta \cap \mathbb{M}_n^d$ and $w \in \mathbb{M}_n$. There exists a function ψ in the closed unit ball of $H^\infty(B_\delta)$ such that $\psi(\lambda) = w$ if*

(i) *$w \in \mathrm{Alg}(\lambda)$, so there exists $p \in \mathcal{P}_d$ such that $p(\lambda) = w$.*

(ii) $\sup\{\|p(x)\| : x \in V_\lambda \cap B_\delta\} \le 1$.

Moreover, if the conditions are satisfied, ψ can be chosen to have a free realization.

Since $\phi(\lambda) = w$, by Theorem 2.3, there is a free polynomial p such that $p(\lambda) = w$; so condition (i) is satisfied. To see condition (ii), note that for all $x \in V_\lambda \cap B_\delta$, we have $p(x) = \phi(x)$. Indeed, by Theorem 2.3, there is a polynomial q so that $q(\lambda \oplus x) = \phi(\lambda \oplus x)$. Therefore $q(\lambda) = p(\lambda)$, so, since $x \in V_\lambda$, we also have $q(x) = p(x)$, and hence $p(x) = \phi(x)$. But ϕ is in the unit ball of $H_1^\infty(B_\delta)$, so $\|\phi(x)\| \le 1$ for every x in B_δ.

So we can apply Theorem 3.1 to conclude that there is a function ψ in $H^\infty(B_\delta)$ that has a free realization, and that agrees with ϕ on the finite set F.

We note that the converse of Theorem 3.1 is also true. Given Theorem 2.3, the converse is almost immediate.

4 Steps III and IV

Let $\Lambda = \{x_j\}_{j=1}^\infty$ be a countable dense set in B_δ. For each k, let $F_k = \{x_1, \dots, x_k\}$. By Step II, there is a function $\psi^k \in H_1^\infty(B_\delta)$ that has a free realization and agrees with ϕ on F_k. By Theorem 1.4, there exists a Hilbert space $\mathcal{M}^k$ and a $\mathcal{B}(\mathbb{C}, \mathcal{M}^k \otimes \mathbb{C}^J)$ valued nc function u^k on B_δ so that, for all n, for all $x, y \in B_\delta \cap \mathbb{M}_n^d$, we have

$$1 - \psi^k(y)^* \psi^k(x) = u^k(y)^* \left[\overset{1}{\underset{1-\delta(y)^*\delta(x)}{\otimes}} \right] u^k(x). \tag{4.1}$$

Embed each $\mathcal{M}^k$ in a common Hilbert space $\mathcal{H}$. Since the left-hand side of (4.1) is bounded, it follows that u^k are locally bounded, so we can apply Theorem 1.5 to find a sequence of unitaries U^k such that, after passing to a subsequence, $U^k * u^k$ converges to a function v in $\mathrm{Hol}^{\mathrm{nc}}_{\mathcal{H}}(\Omega)$. We have therefore that

$$1 - \phi(y)^*\phi(x) = v(y)^* \left[\underset{1-\delta(y)^*\delta(x)}{\overset{1}{\otimes}} \right] v(x) \tag{4.2}$$

holds for all pairs (x, y) that are both in $\Lambda \cap \mathbb{M}_n^d$ for any n. So by continuity, we get that (4.2) is an nc model for ϕ on all B_δ, completing Step III.

Finally, Step IV follows by applying Theorem 1.4.

5 Closing Remarks

One can modify the argument to get a realization formula for $\mathcal{B}(\mathcal{K}, \mathcal{L})$-valued bounded nc functions on B_δ, or to prove Leech theorems (also called Toeplitz-corona theorems [8, 10]. For finite-dimensional $\mathcal{K}$ and $\mathcal{L}$, this was done in [2]; for infinite-dimensional $\mathcal{K}$ and $\mathcal{L}$, the formula was proved in [6] using results from [7].

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