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TRIPLE FACTORIZATIONS AND SUPERSOLUBILITY OF FINITE GROUPS

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Abstract In this paper we analyse the structure of a finite group of minimal order among the finite non-supersoluble groups possessing a triple factorization by supersoluble subgroups of pairwise relatively prime indices. As an application we obtain some sufficient conditions for a triple factorized group by supersoluble subgroups of pairwise relatively prime indices to be supersoluble. Many results appear as consequences of our analysis.

Keywords: finite group; supersoluble group; factorization

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1. Introduction

All groups considered are finite.

A group G is said to be the product of its subgroups H and K, or G is *factorized* by H and K, if $G = HK$. Factorizations are quite natural in group theory. For example, a group is often expressible as a product of two non-conjugate maximal subgroups. In this context, a natural line of investigation opens up when one asks how the structure of the factors H and K affects the structure of G.

In the study of factorized groups the concept of the factorizer of a normal subgroup is frequently useful. Let N be a normal subgroup of a factorized group $G = HK$. Then the factorizer of N is the subgroup $HN \cap KN$. It is rather easy to see that $HN \cap KN = (H \cap KN)N = (HN \cap K)N = (H \cap KN)(HN \cap K)$, that is, the factorizer of N is a triply factorized group. The concept of *triple factorization* was introduced by Kegel in 1961. This is a factorization of a group G involving three subgroups H, K and L of the type $G = HK = HL = KL$.

The evidence is that the existence of a triple factorization can have greater consequences for the group structure than does a single factorization. For example, a non-nilpotent group can be factorized as a product of two nilpotent subgroups. However, we have the following theorem.

Theorem 1.1 (Kegel [8, Folgerung 2]). *A group that has a triple factorization by nilpotent subgroups is nilpotent.*

Kegel's result is definitely not true for supersoluble triple factorizations, that is, one can find a non-supersoluble group that has a triple factorization by supersoluble subgroups, even in the particular case when the subgroups have pairwise relatively prime indices in the group.

In [**9**], Wang proves that for a group G whose order has at least three different prime divisors, G is supersoluble if and only if there exist three maximal supersoluble subgroups of G whose indices are three different primes. A recent paper of Flowers and Wakefield [**5**] explores some other circumstances in which to derive the supersolubility of a group G that is triply factorized by supersoluble subgroups that have pairwise relatively prime indices in G.

In this paper we analyse the structure of a group G of minimal order among the non-supersoluble groups that have a triple factorization by supersoluble subgroups of pairwise relatively prime indices (see Theorem 2.3), and give a method to construct such a minimal configuration (see Example 2.5). All classical results we know due to Doerk [**3**], Friesen [**6**] and Wang [**9**] appear as direct consequences of our approach. Furthermore, we will be able to weaken the sufficient conditions given in [**5**].

Our starting point is a classical result due to Wielandt.

Theorem 1.2 (Wielandt [4, Theorem 1.3.4]). *If a group G possesses three soluble subgroups H, K and L whose indices $|G:H|$, $|G:K|$, $|G:L|$ are pairwise coprime, then G is itself soluble.*

2. The results

Our first result shows that if a group has a triple factorization by subgroups with a Sylow tower of a fixed linear ordering of all primes, then the group has the same property.

Definition 2.1 (see [4, pp. 358 and 359]). Let $\prec$ be an arbitrary linear ordering on the set $\mathbb{P}$ of all primes. We say that a group G *possesses a Sylow tower of type* $\prec$ if G possesses a normal Hall π_p-subgroup for each subset $\pi_p = \{q \colon q \prec p\}$.

Obviously, a group with a Sylow tower is soluble.

Consider the linear ordering $\prec$ of the set $\mathbb{P}$ opposite to the natural ordering:

$$p \prec q \quad \iff \quad p \geqslant q, \quad p, q \in \mathbb{P}.$$

It is well known that a supersoluble group possesses a Sylow tower of type $\prec$ (see [**7**, Satz VI.9.1 (c)]). For this reason it is usually said that a group G *possesses a Sylow tower of supersoluble type* if G possesses a normal Hall π_p-subgroup for each subset $\pi_p = \{q \colon q \geqslant p\}$.

Theorem 2.2. *Let $\prec$ be an arbitrary linear ordering on the set $\mathbb{P}$ of all primes. Let G be a group. Assume that H, K and L are subgroups of G whose indices in G are pairwise relatively prime. If H, K and L have Sylow towers of type $\prec$, then G has a Sylow tower of type $\prec$.*

Proof. Since H, K and L are soluble, the group G is soluble by Theorem 1.2. Moreover, by [**4**, Lemma A.1.6 (b)], $G = HK = HL = KL$.

We prove the theorem by induction on the order of G. Let p be the least prime dividing the order of G in the linear ordering $\prec$. Then p divides at most one of the indices $\{|G : H|, |G : K|, |G : L|\}$ or, equivalently, at least two of the subgroups $\{H, K, L\}$ contain some Sylow p-subgroup of G. Assume that K contains a Sylow p-subgroup P_K of G. Since K has a Sylow tower of type $\prec$ and p is the least prime dividing the order of G in the ordering $\prec$, P_K is normal in K. Similarly, we can assume that there exists a Sylow p-subgroup of G, P_L, say, such that $L \leqslant \mathrm{N}_G(P_L)$. Note that $|G : \mathrm{N}_G(P_K)|$ divides $|G : K|$ and $|G : \mathrm{N}_G(P_L)|$ divides $|G : L|$, and so $|G : \mathrm{N}_G(P_K)|$ and $|G : \mathrm{N}_G(P_L)|$ are relatively prime. Since $\mathrm{N}_G(P_K)$ and $\mathrm{N}_G(P_L)$ are conjugate in G, we have that $\mathrm{N}_G(P_L) = \mathrm{N}_G(P_K) = G$. Since $G = KL$, we have that $P = P_L = P_K$ is a normal Sylow p-subgroup of G. Consider the quotient group $\bar{G} = G/P$. Then $\bar{H} = HP/P$, $\bar{K} = K/P$ and $\bar{L} = L/P$ are subgroups of $\bar{G}$ with a Sylow tower of type $\prec$. If $\bar{H}$ is a proper subgroup of $\bar{G}$, then $\bar{H}$, $\bar{K}$, $\bar{L}$ are subgroups of pairwise relatively prime indices in $\bar{G}$. By induction, the group G/P has a Sylow tower of type $\prec$. If $\bar{H} = \bar{G}$, then $G = HP$ and then $G/P \cong H/(H \cap P)$ is a group with Sylow tower of type $\prec$. Thus, in any case, the group $\bar{G}$ possesses a Sylow tower of type $\prec$ and so does G. □

Theorem 2.3. *Let $\mathfrak{U}$ be the class of all supersoluble groups. Let $\mathfrak{Y}$ and $\mathfrak{T}$ be two classes of groups that are closed under taking epimorphic images and subgroups and such that $\mathfrak{T} \subseteq \mathfrak{U}$. Let n be an integer $n \geqslant 3$ and let $\mathcal{X}(\mathfrak{Y})$ be the class composed of all groups $X \in \mathfrak{Y}$ with n supersoluble proper subgroups Y_i, $i = 1, 2, \ldots, n$, such that $Y_1 \in \mathfrak{T}$ and $\gcd(|X : Y_i|, |X : Y_j|) = 1$ whenever $i \neq j$.*

Suppose that $\mathcal{X}(\mathfrak{Y}) \setminus \mathfrak{U}$ is non-empty. Then $n = 3$ and if G is a group of minimal order in $\mathcal{X}(\mathfrak{Y}) \setminus \mathfrak{U}$, then G is a soluble group with the following properties.

(1) *Every proper epimorphic image of G is a supersoluble group. If p is the greatest prime dividing the order of G and P is a Sylow p-subgroup of G, then $P = \mathrm{Soc}(G) = \mathrm{F}(G)$ is a self-centralizing minimal normal subgroup of G that is complemented in G by a core-free maximal subgroup of G. Write $|P| = p^m$. Then, $m > 1$.*

    Let M denote a core-free maximal subgroup M of G that complements P in G.

(2) *Every proper subgroup of G is a supersoluble group.*

(3) *M is a non-nilpotent group and every proper subgroup of M is cyclic.*

(4) *$|M| = qr^k$, where q and r are primes such that r divides $q-1$ and qr^k divides $p-1$. Moreover, if Q is a Sylow q-subgroup of M, then Q is normal in M.*

(5) *$G' = QP$. In particular, G' is non-nilpotent.*

Proof. Since the class $\mathfrak{Y}$ is fixed and well understood, we write in the proof simply $\mathcal{X}$ instead of $\mathcal{X}(\mathfrak{Y})$.

Suppose that $H_1, \dots, H_n$ are supersoluble subgroups of G such that $H_1 \in \mathfrak{T}$ and $\gcd(|G : H_i|, |G : H_j|) = 1$ whenever $i \neq j$. By [**4**, Lemma A.1.6 (b)], we have $G = H_i H_j$ if $i \neq j$. Since $n \geqslant 3$, by Theorem 2.2, G is soluble group that possesses a Sylow tower of supersoluble type. The order of G is divided by at least three different primes and if p is the greatest prime dividing the order of G, then G has a normal Sylow p-subgroup P.

Step 1. Let N be a minimal normal subgroup of G. If one of the subgroups H_i supplements N, then G/N is supersoluble. Otherwise, the quotient group G/N belongs to the class $\mathcal{X}$ and, by minimality of G, we have that G/N is supersoluble. This implies that G is a soluble primitive group (see [**4**, Corollary IV.2.13 and Theorem III.2.7]). Hence, $P = \mathrm{Soc}(G) = \mathrm{F}(G)$ is a minimal normal subgroup of G. Moreover, $\mathrm{C}_G(P) = P$ and there exists a core-free maximal subgroup of G, M, say, that complements P in G and M is supersoluble. Write $|P| = p^m$. Then $m > 1$, since G is a non-supersoluble group.

Step 2. Let U be a maximal subgroup of G. Then, either $G = UP$ or $|G : U| = q$ is a prime such that $p \neq q$. If $G = UP$, then $U \cong G/P$ and then U is supersoluble. Assume that $|G : U|$ is a prime q and that $p \neq q$. Then q divides at most one of the indices $\{|G : H_i|\colon i = 1, 2, \dots, n\}$. If q divides none of the indices $\{|G : H_i|\colon i = 1, 2, \dots, n\}$, then $U \cap H_i$, $i = 1, 2, \dots, n$, are proper supersoluble subgroups of U such that $U \cap H_1 \in \mathfrak{T}$ and $\gcd(|U : U \cap H_i|, |U : U \cap H_j|) = 1$ whenever $i \neq j$. Then $U \in \mathcal{X}$. By minimality of G, U is supersoluble. So, assume that q divides one (and only one) of the indices $\{|G : H_i|\colon i = 1, 2, \dots, n\}$. Assume that q divides $|G : H_j|$. Note that for any $i \in \{1, 2, \dots, n\} \setminus \{j\}$, we have that $U \cap H_i$ is a supersoluble proper subgroup of U and $|U : U \cap H_i| = |G : H_i|$. Let $U_{q'}$ denote a Hall q'-subgroup of G contained in U. If $(H_j)_{q'}$ is a Hall q'-subgroup of H_j, then there exists a conjugate $(H_j)^*_{q'}$ of $(H_j)_{q'}$ contained in $U_{q'}$. Note that

$$|G : (H_j)^*_{q'}| = \begin{cases} |G : U|\,|U : (H_j)^*_{q'}| = q|U : (H_j)^*_{q'}| \\ |G : H_j|\,|H_j : (H_j)_{q'}| = |G : H_j|q^{\alpha} & \text{for some integer } \alpha \geqslant 0. \end{cases}$$

Therefore, $\{(H_j)^*_{q'}, U \cap H_i\colon i \in \{1, 2, \dots, n\} \setminus \{j\}\}$ are supersoluble proper subgroups of pairwise coprime indices in U. Moreover, if $j \neq 1$, then $U \cap H_1 \in \mathfrak{T}$, and if $j = 1$, then $(H_1)^*_{q'} \in \mathfrak{T}$. Thus, in any case, $U \in \mathcal{X}$. By minimality of G, we have that U is supersoluble.

Thus, any maximal subgroup of G is supersoluble.

Step 3. By [**3**, Satz 2 (b)], M is not abelian and every proper subgroup of M is abelian. Moreover, $|M|$ is divided by at least two different primes. Let U be a proper subgroup of M. Then P, regarded as a module for U, is completely reducible by [**4**, Theorem A.11.5]. Since $\mathrm{C}_U(P) = 1$, P possesses a faithful and irreducible component. Since U is abelian, we apply [**4**, Theorem B.9.8] to conclude that U is cyclic. If M were nilpotent, then M would be cyclic. This is not possible. Thus, M is not nilpotent. By [**3**, Satz 2 (c)], $|M|$ is divided by exactly two different primes. Hence, $|G|$ has exactly three prime divisors and, in particular, $n = 3$.

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Write $|M| = q^a r^k$ with q, r primes such that $q > r$, $a, k \geqslant 1$. If Q is a Sylow q-subgroup of M, then Q is a cyclic normal subgroup of M since M has a Sylow tower of supersoluble type. If $\Phi(Q) \neq 1$, then $\Phi(Q)R$ is abelian since $\Phi(Q)R$ is a proper subgroup of M. Hence, $\Phi(Q) \leqslant \mathrm{Z}(M)$. On the other hand, note that $M' \leqslant Q$. If $M' \leqslant \Phi(Q) \leqslant \Phi(M)$, then $M/\Phi(M)$ would be abelian. This would imply the nilpotence of M, a contradiction that shows that $M' = Q$. Then, $\Phi(Q) \leqslant M' \cap \mathrm{Z}(M)$. Since all Sylow subgroups of M are abelian, we can apply [**7**, Satz VI.14.3] to conclude that $M' \cap \mathrm{Z}(M) = 1$. Thus, $\Phi(Q) = 1$. Therefore, Q is a cyclic group of order q. In other words, $a = 1$.

Finally, note that $\mathrm{C}_M(Q) = Q\Phi(R)$. Then $M/\mathrm{C}_M(Q) \cong R/\Phi(R)$ is a cyclic group of order r. Since $M/\mathrm{C}_M(Q)$ is isomorphic to a subgroup of $\mathrm{Aut}(Q)$ that is a cyclic group of order $q-1$, we conclude that $r \mid q-1$.

Step 4. Let C be a non-trivial proper subgroup of M. Then C is a cyclic p'-group. Therefore, P, regarded as a module for C, is completely reducible by [**4**, Theorem A.11.5]. Since CP is supersoluble, every irreducible C-submodule of P is one dimensional. Moreover, $P = [P, C] \times \mathrm{C}_P(C)$ by [**4**, Theorem A.12.15]. If $[P, C] = 1$, then $C \leqslant \mathrm{C}_G(P) = P$ and then $C = 1$, contrary to our supposition. Hence, $[P, C] \neq 1$. This means that P contains an irreducible and faithful C-submodule. Then $|C|$ divides $p-1$ by [**4**, Theorem B.9.8]. In particular, $q \mid p-1$ and $r^k \mid p-1$.

Step 5. We have that QP is a normal subgroup of G and $G/QP \cong R$. Hence, $G' \leqslant QP$. Since $P \leqslant G'$ and $G/P \cong M$ is non-abelian, we conclude that $G' = QP$. Since P is self-centralizing in G, we have that G' is non-nilpotent.

This completes the proof of the theorem. □

Remark 2.4. Note that if G is a group of minimal order in $\mathcal{X}(\mathfrak{Y}) \setminus \mathfrak{U}$ such that H_i, $i = 1, 2, 3$, are supersoluble subgroups of G whose indices in G are pairwise relatively prime, then $|G| = p^m q r^k$, where p, q, r are primes such that $p > q > r$, $k \geqslant 1$ and $m > 1$. Moreover, r divides $q-1$ and qr^k divides $p-1$

Note also that necessarily each of the three indices $\{|G : H_i|, i = 1, 2, 3\}$ is a power of a different prime. We can assume (with the notation of Theorem 2.3) the following.

- $|G : H_1| = p^\alpha$. Then necessarily $|G : H_1| = |P| = p^m$ and H_1 is a core-free maximal subgroup of G.
- $|G : H_2| = q$. Then $H_2 = PR^g$ for some $g \in G$.
- $|G : H_3| = r^\gamma$. Then $L = PQR_0$ for some $R_0 \leqslant R$ such that $|G : H_3| = r^\gamma = |R : R_0|$. Moreover, L is normal in G, since $G' = QP$.

We finish this section with a general construction of a non-supersoluble group G with three supersoluble subgroups H_i, $i = 1, 2, 3$, such that $\gcd(|G : H_i|, |G : H_j|) = 1$ if $i \neq j$.

Example 2.5. Let p, q, r be three primes, $p > q > r$, and let k be an integer such that $k \geqslant 1$, such that r divides $q-1$ and qr^k divides $p-1$. (An example of such numbers are $p = 13$, $q = 3$, $r = 2$ and $k = 2$.)

If R is a cyclic group of order r^k, then there exists an irreducible R-module, Q, say, over the field $\mathrm{GF}(q)$ such that $\mathrm{C}_R(Q) = \Phi(R)$ (by [**4**, Theorem B.10.3]) and, since r

divides $q-1$, then $\dim_{\mathrm{GF}(q)} Q = 1$ (by [**4**, Theorem B.9.8]). The semi-direct product $H_1 = [Q]R$ is a non-nilpotent supersoluble group of order qr^k such that the subgroup $\mathrm{F}(H_1) = \mathrm{C}_{H_1}(Q) = Q \times \Phi(R)$ is isomorphic to a cyclic group of order qr^{k-1}. Now, there exists a faithful and irreducible H_1-module P over the field $\mathrm{GF}(p)$, again by [**4**, Theorem B.10.3]. If $\dim_{\mathrm{GF}(p)} P = 1$, then H_1 would be isomorphic to a cyclic group of order dividing $p-1$, by [**4**, Theorem A.21.1]. Therefore, $\dim_{\mathrm{GF}(p)} P = m > 1$ and then the semidirect product $G = [P]H_1$ is a non-supersoluble group of order $p^m qr^k$ with $m > 1$.

Consider the subgroups $H_1 = QR$, $H_2 = PR$ and $H_3 = P\,\mathrm{C}_{H_1}(Q)$. Note that $|G : H_3| = |R : \Phi(R)| = r$, $|G : H_2| = |Q| = q$ and $|G : H_1| = |P| = p^m$, and so H_1, H_2 and H_3 have pairwise coprime indices in G. We know that H_1 is supersoluble. To see that H_3 is supersoluble we write $C = \mathrm{C}_{H_1}(Q)$ and write P_C to denote the $\mathrm{GF}(p)$-vector space P regarded as a C-module. By [**4**, Theorem A.11.5], P_C is completely reducible. Since C is a cyclic group of order qr^{k-1} and qr^{k-1} divides $p-1$, every irreducible submodule of P_C is one dimensional by [**4**, Theorem B.9.8]. This means that H_3 is supersoluble. By analogous reasoning, since r^k divides $p-1$, the subgroup $H_2 = PR$ is also supersoluble.

3. Corollaries

Theorem 2.3 and Example 2.5 show that the supersolubility of a group with three supersoluble subgroups whose indices are pairwise relatively prime cannot be deduced unless some new condition is imposed.

For instance, in [**3**] Doerk assumes that the group G has four supersoluble subgroups whose indices are pairwise relatively prime. His result is a consequence of Theorem 2.3.

Corollary 3.1 (Doerk [3, Satz 4]). *If a group G has four supersoluble subgroups whose indices are pairwise relatively prime, then G is supersoluble.*

Next we analyse the conditions given in [**5**]. Our next result improves Theorem 1.2 of [**5**] and it is also a consequence of Theorem 2.3.

Corollary 3.2. *Let G be a group and, for $i = 1, 2, 3$, let H_i be supersoluble subgroups of G such that $\gcd(|G : H_i|, |G : H_j|) = 1$ if $i \neq j$. Let p_i, $i = 1, 2, 3$, be three different primes such that, for each i, the prime p_i divides $|G : H_i|$. Assume that $p_1 > p_2 > p_3$.*

If either p_2p_3 does not divide $p_1 - 1$ or p_3 does not divide $p_2 - 1$, then G is supersoluble.

Corollary 3.3 (Flowers and Wakefield [5, Theorem 1.2]). *Let G be a group and let H, K and L be supersoluble subgroups of G with pairwise relatively prime indices in G. Suppose that p is the greatest prime dividing the order of G, H is a Hall p'-subgroup of G, q is the greatest prime dividing the order of H, and q does not divide $p-1$. Then G is supersoluble.*

In [**5**, Theorem 1.1], the authors assume that the derived subgroup is nilpotent. We see that this result is a consequence of Theorem 2.3 if we take in this theorem the class $\mathfrak{Y} = \mathfrak{N}\mathfrak{A}$ composed of all groups with nilpotent derived subgroup.

Corollary 3.4 (Flowers and Wakefield [5, Theorem 1.1]). *Let G be a group and let H, K and L be supersoluble subgroups of G with pairwise relatively prime indices in G. Suppose that G' is nilpotent. Then G is supersoluble.*

The fact that the three subgroups H_1, H_2 and H_3 of the minimal counterexample of Theorem 2.3 are non-nilpotent motivates the next corollary. Imposing the nilpotence of one of the subgroups, i.e. taking $\mathfrak{X} = \mathfrak{N}$, the class of all nilpotent groups, in Theorem 2.3, we have the following corollary.

Corollary 3.5. *Let G be a group, let H and K be supersoluble subgroups of G, and let L be a nilpotent subgroup of G. Suppose that the indices $\{|G:H|, |G:K|, |G:L|\}$ are pairwise relatively prime. Then G is supersoluble.*

We bring the paper to a close with three results whose proofs cannot be deduced directly as corollaries of Theorem 2.3, although some of the above arguments and corollaries contribute to give short proofs of them. Two of them are known. The third is new.

If G is a supersoluble group whose order has at least three different prime divisors, then there exist three maximal supersoluble subgroups of G whose indices are three different primes. The converse is a result of Wang.

Theorem 3.6 (Wang [9]). *A group G is a supersoluble group whose order has at least three different prime divisors if and only if there exist three maximal supersoluble subgroups of G whose indices are three different primes.*

Proof. Only the sufficiency of the condition is in doubt. Let G be a non-supersoluble group with three maximal supersoluble subgroups H_i, $i = 1, 2, 3$, whose indices are three different primes, and assume that G has minimal order with such conditions. Obviously, the order of G has at least three different prime divisors and Theorem 2.2 and Step 1 of Theorem 2.3 apply. If p is the greatest prime dividing the order of G, then G has a core-free maximal subgroup of G, M, say, such that $|G:M| = p^m$, where $m > 1$, and no maximal subgroup of G has index p in G. Hence, $\gcd(|G:H_i|, |G:M|) = 1$ for any $i = 1, 2, 3$. This contradicts Corollary 3.1. The proof of the theorem is now complete. □

Next we deal with the most classical result on this subject. It is due to Friesen.

Lemma 3.7. *Let G be a group. Assume that H, K and L are supersoluble subgroups of G such that the indices $\{|G:H|, |G:K|, |G:L|\}$ are pairwise relatively prime. If K and L are normal in G, then G is supersoluble.*

Proof. Let G be a minimal counterexample to the lemma. As in the above theorem, the order of G has at least three different prime divisors and Theorem 2.2 and Step 1 of Theorem 2.3 apply. We use the notation of Theorem 2.3. Let U be a maximal subgroup of G. Then $|G:U| = q$ for some prime $q \neq p$. Moreover, $U \cap K$ and $U \cap L$ are normal in U and $|U:U \cap K|$ is either $|G:K|$ or $|U:K|$, and $|U:U \cap L|$ is either $|G:L|$ or $|U:L|$. As in Step 2, we have that U satisfies the hypothesis of the lemma. The minimal choice of G implies that U is supersoluble. Hence, every maximal subgroup of G is supersoluble and Steps 2–5 of Theorem 2.3 apply as well. Consequently, the structure of G coincides with the one described in Theorem 2.3. In this case exactly one of the subgroups H, K, L is normal in G. This contradiction proves the lemma. □

Theorem 3.8 (Friesen [6]). *If G is a group with A and B normal supersoluble subgroups of G with relatively prime indices in G, then G is supersoluble.*

Proof. Note that $G = AB$ and then $G/(A \cap B)$ is a supersoluble group. Hence, we can assume that $A \cap B \neq 1$.

Let N be a minimal normal subgroup of G. Applying the arguments of Step 1 of Theorem 2.3, we can prove that every proper epimorphic image of G is supersoluble. So, we can assume that G has a unique minimal normal subgroup N in G and $N \leqslant A \cap B$. Moreover, there exists a core-free maximal subgroup M of G such that $G = MN$. If p is the greatest prime dividing the order of G, then at least one of the subgroups A or B contains a Sylow p-subgroup P of G. Suppose that $P \leqslant A$. Since A is supersoluble, P is normal in A. Hence, P is normal in G and then $N = P$. Hence, $\gcd(|G : M|, |G : A|) = \gcd(|G : M|, |G : B|) = 1$. The group G has three supersoluble subgroups M, A and B whose indices $\{|G : M|, |G : A|, |G : B|\}$ are pairwise relatively prime and A and B are normal in G. Then G is supersoluble by the previous lemma. □

We say that two subgroups H and K of a group G are *mutually permutable* if H permutes with every subgroup of K and K permutes with every subgroup of H. Mutually permutable products have been extensively investigated and there are many interesting results available. The reader is referred to [**2**] for a complete account of this theory.

Asaad and Shaalan proved in [**1**] that if $G = HK$ is a mutually permutable product of two supersoluble subgroups H and K, then G is supersoluble provided that G' is nilpotent. This is true also for any number of factors (see [**2**, Theorem 5.2.21]). Our result affirms that if G has three supersoluble subgroups H_i, $i = 1, 2, 3$, with pairwise relatively prime indices in G and one of them, H_1, say, forms mutually permutable products with H_2 and H_3, then G is supersoluble.

Theorem 3.9. *Let G be a group with three supersoluble subgroups H_i, $i = 1, 2, 3$, with pairwise relatively prime indices in G. Assume that H_1 and H_j are mutually permutable subgroups for every $j \neq 1$. Then G is supersoluble.*

Proof. Let G be a minimal counterexample to the theorem. Let N be a minimal normal subgroup of G. If $G = H_iN$ for some $i = 1, 2, 3$, then G/N is supersoluble. Assume that H_iN/N, $i = 1, 2, 3$, are proper supersoluble subgroups of G/N. By [**2**, Lemma 4.1.10], H_1N/N and H_jN/N are mutually permutable supersoluble subgroups of G/N for any $j \neq 1$. Then G/N is supersoluble by minimality of G. Thus, as in Step 1 of Theorem 2.3, we can assume that there exists only one minimal normal subgroup N. By [**2**, Corollary 4.5.9], we have that $N \leqslant H_1 \cap H_2$ and $N \leqslant H_1 \cap H_3$. Moreover, there exists a core-free maximal subgroup M of G such that $G = MN$. If p is the greatest prime dividing the order of G, then at least two of the subgroups H_i, $i = 1, 2, 3$, contain a Sylow p-subgroup P of G. Note that if $P \leqslant H_i$ for some i, then P is normal in H_i. Hence, P is normal in G and then $P = N$. Therefore, $\gcd(|G : H_i|, p) = \gcd(|G : H_i|, |G : M|) = 1$ for all $i = 1, 2, 3$. By Corollary 3.1, the group G is supersoluble. But this contradicts our choice of G. Hence, the minimal counterexample does not exist and the theorem holds. □

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