Proceedings of the Royal Society of Edinburgh, **151**, 1611–1641, *2021*
DOI:10.1017/prm.2020.75

Semilinear elliptic equations involving mixed local and nonlocal operators

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(MS received 23 June 2020; accepted 15 September 2020)

In this paper, we consider an elliptic operator obtained as the superposition of a classical second-order differential operator and a nonlocal operator of fractional type. Though the methods that we develop are quite general, for concreteness we focus on the case in which the operator takes the form $-\Delta + (-\Delta)^s$, with $s \in (0, 1)$. We focus here on symmetry properties of the solutions and we prove a radial symmetry result, based on the moving plane method, and a one-dimensional symmetry result, related to a classical conjecture by G.W. Gibbons.

Keywords: Operators of mixed order; existence; symmetry; moving plane; qualitative properties of solutions

2020 Mathematics subject classification: 35A01, 35B65, 35R11

1. Introduction

In this article we discuss some symmetry properties for the solutions of semilinear equations driven by a mixed operator. Specifically, we will consider operators that combine local and nonlocal features. For the sake of concreteness, we focus on operators of the form

$$\mathcal{L} := -\Delta + (-\Delta)^s \tag{1.1}$$

where $s \in (0, 1)$ and

$$(-\Delta)^s u(x) := \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy.$$

The study of mixed operators has a consolidated interest in the recent literature, both in terms of theoretical studies and in view of real-world applications. The development of the theory includes, among others, viscosity solutions methods (see [**2**–**4**, **12**, **23**, **41**, **42**]), parabolic equations (see [**31**]), Aubry-Mather theory (see [**25**]), Cahn-Hilliard equations (see [**19**]), porous medium equations (see [**26**]) phase transitions (see [17]), fractional damping effects (see [27]), Bernstein-type regularity results (see [16]), existence/non-existence results (see [1, 48]), regularity theory (see [9, 22]), estimates for the associated Green function (see [21]).

Concrete applications of mixed operators also arise naturally in plasma physics (see [14]) and population dynamics (see [29]), and numerical methods have been also developed to take into account the specifics of mixed operators (see [13]).

In this article, we provide two sets of symmetry results for solutions of semilinar equations driven by mixed operators: the first type of results deals with the radial symmetry of the solutions, and relies on the moving plane method; the second type of results is inspired by a classical conjecture by G.W. Gibbons and establishes the one-dimensional symmetry of the global solutions that attain uniformly their limit values at infinity.

In this spirit, the first symmetry result that we present is as follows:

THEOREM 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function, and let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with C^1 boundary. We assume that Ω is symmetric and convex with respect to the hyperplane $\{x_1 = 0\}$.*

If $u \in C(\mathbb{R}^N)$ is any non identically vanishing weak solution of

$$\begin{cases} \mathcal{L}u = f(u) & \text{in } \Omega, \\ u \equiv 0 & \text{in } \mathbb{R}^N \backslash \Omega, \\ u \geqslant 0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

then u is symmetric with respect to $\{x_1 = 0\}$ and strictly increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$.

Theorem 1.1 is a symmetry result in the spirit of Gidas, Ni, Nirenberg [40]. Since this milestone result, the literature concerning symmetry/monotonicity results has extensively grown, and it is beyond our scopes to give here an exhaustive list of references; we limit ourselves to mention the series of papers [7, 10, 20, 24, 32, 51], where analogues of theorem 1.1 are obtained for elliptic systems and for elliptic equations/systems in the presence of singularities. As usual, from theorem 1.1 one deduces that if Ω is a ball, then the solutions of (1.2) are necessarily radial and radially decreasing. We stress that in [9] we proved interior H^k estimates for the operator $\mathcal{L}$, and this seems to indicate that the addition of the local part $-\Delta$ to the fractional one push towards a 'local behaviour' of $\mathcal{L}$. To the contrary, in theorem 1.1 we are able to prove *strict monotonicity* for non-negative solutions, which is true as well in the purely nonlocal case (see [43]), but fails to hold in the local case (see, e.g., [47]).

The proof of theorem 1.1 that we present combines the integral formulation of the moving plane method (see [46, 49]) with suitable adaptations of some results in [44], where the case of integral equations was taken into account by introducing a new small-volume maximum principle and a strong maximum principle for anti-symmetric supersolutions. See also [6, 28, 30, 33, 43, 50] for related moving plane methods in the nonlocal setting.

As we shall see in § 2, the assumption that Ω has C^1 boundary allows us to introduce a 'good' functional setting in which carrying out the integral formulation

https://doi.org/10.1017/prm.2020.75 Published online by Cambridge University Press

of the moving plane method. Furthermore, by using the results in [**9**], one can prove that any weak solution $u \in H^1(\mathbb{R}^N)$ of (1.2) (see definition 2.2) is actually continuous on $\mathbb{R}^N$, provided that f is sufficiently regular and Ω is strictly convex.

In terms of one-dimensional symmetry for global solutions under uniform limit assumptions, we have the following result:

THEOREM 1.2. *Let $f \in C^1(\mathbb{R})$ be such that*

$$\sup_{|r|\geqslant 1} f'(r) < 0. \tag{1.3}$$

Let $u \in C^3(\mathbb{R}^N) \cap W^{4,\infty}(\mathbb{R}^N)$ be a classical solution of the problem

$$\begin{cases} \mathcal{L}u = f(u) & \textit{in } \mathbb{R}^N, \\ \lim\limits_{t\to\pm\infty} u(y,t) = \pm 1 & \textit{uniformly for } y \in \mathbb{R}^{N-1}. \end{cases} \tag{1.4}$$

Then, there exists $u_0 : \mathbb{R} \to \mathbb{R}$ such that

$$u(y,t) = u_0(t) \qquad \textit{for every } x = (y,t) \in \mathbb{R}^N. \tag{1.5}$$

The result in theorem 1.2 is inspired by a classical conjecture by G.W. Gibbons, formulated when $\mathcal{L}$ was the classical Laplace operator and motivated by the cosmological problem of detecting the shape of the interfaces which 'separate' the different regions of the universe after the big bang (see [**39**]).

The classical Gibbons conjecture was established, independently and with different methods, by [**5**, **8**, **34**]. See also [**35–37**] for related results.

The fractional version of Gibbons conjecture (i.e., the case in which the operator in (1.4) is the fractional Laplacian) has been established in [**18**, **38**]. As a matter of fact, the method developed in [**38**] is very general and comprises a number of different operators in a unified way: for this, our proof of theorem 1.2 will rely on the general structure provided in [**38**] by showing that the structural hypothesis of [**38**] are fulfilled in the case that we consider here.

In the rest of the paper we provide the proof of theorem 1.1, which is contained in § 2, and that of theorem 1.2, which is contained in § 3.

Though not explicitly used in this paper, we also remark that the methods developed here also lead to a Hopf-type result, that we state and prove in appendix A for the sake of completeness.

2. Radial symmetry and proof of theorem 1.1

In this section, we prove theorem 1.1. To this end, without loss of generality, we may assume that

$$\inf_{x\in\Omega} x_1 = -1. \tag{2.1}$$

We will combine the integral version of the moving plane method (see [**46**]) with a suitable generalization of a strong maximum principle for *antisymmetric supersolutions* (see [**44**]).

Let us now introduce and fix some notation needed in what follows. We define the bilinear form

$$B(u,v) := \int_{\mathbb{R}^N} \langle \nabla u, \nabla v \rangle \, \mathrm{d}x + \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y, \tag{2.2}$$

and the function space

$$\mathcal{D}(\Omega) := \left\{ u \in H^1(\mathbb{R}^N) \text{ s.t. } u \equiv 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}. \tag{2.3}$$

Remark 2.1. Since Ω has C^1 boundary, any function $u \in \mathcal{D}(\Omega)$ satisfies

$$u\big|_{\Omega} \in H_0^1(\Omega).$$

In this setting, we give the following definition of weak solution of (1.2):

Definition 2.2. We say that a function $u : \Omega \to \mathbb{R}$ is a weak solution of (1.2) if $u \in \mathcal{D}(\Omega)$ and it satisfies the following properties:

(i) $u > 0$ a.e. in Ω;

(ii) for any $\varphi \in \mathcal{D}(\Omega)$ one has

$$B(u,\varphi) = \int_{\mathbb{R}^N} f(u(x))\varphi(x) \, \mathrm{d}x. \tag{2.4}$$

Also, given a set $U \subset \mathbb{R}^N$, we let

$$\rho(v,U) := \int_U |\nabla v|^2 + [v]^2_{H^s(U)}, \tag{2.5}$$

where

$$[v]^2_{H^s(U)} := \iint_{U \times U} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}} \, \mathrm{d}x \, \mathrm{d}y,$$

and

$$\mathcal{H}(U) := \left\{ v \in L^2(\mathbb{R}^N) \text{ s.t. } v \in H^1(U) \right\}. \tag{2.6}$$

As customary, for any $v \in L^2(\mathbb{R}^N)$ we define the positive and negative parts of v as follows

$$v^+ := \max\{v, 0\} \quad \text{and} \quad v^- := \max\{-v, 0\}.$$

As it is well known,

$$v(x) = v^+(x) - v^-(x), \quad \text{for a.e. } x \in \mathbb{R}^N \tag{2.7}$$

and

$$v^+(x)v^-(x) = 0, \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{2.8}$$

It is useful to observe that the functional introduced in (2.5) is monotone with respect to the operation of taking the positive and negative parts, as pointed out in the following result:

LEMMA 2.3. *Let $U \subset \mathbb{R}^N$ be an open set and let $v \in \mathcal{H}(U)$. Then $v^\pm \in \mathcal{H}(U)$ and*

$$\rho(v^\pm, U) \leqslant \rho(v, U), \tag{2.9}$$

with strict inequality if v changes sign.

Proof. Since $v \in \mathcal{H}(U) = L^2(\mathbb{R}^N) \cap H^1(U)$, it is easy to see that $v^\pm \in \mathcal{H}(U)$, in light of (2.7) and (2.8). We then focus on the proof of (2.9).

For this, recalling (2.5) and using again (2.7) and (2.8), we get

$$\begin{aligned}
\rho(v,U) &= \int_U |\nabla v|^2 + \iint_{U\times U} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_U |\nabla (v^+ - v^-)|^2 + \iint_{U\times U} \frac{|(v^+-v^-)(x)-(v^+-v^-)(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_U |\nabla v^+|^2 + \int_U |\nabla v^-|^2 \\
&\quad + \iint_{U\times U} \frac{|v^+(x)-v^+(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,dy + \iint_{U\times U} \frac{|v^-(x)-v^-(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - 2\iint_{U\times U} \frac{\big(v^+(x)-v^+(y)\big)\big(v^-(x)-v^-(y)\big)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \rho(v^+,U) + \rho(v^-,U) + 2\iint_{U\times U} \frac{v^+(x)v^-(y)+v^+(y)v^-(x)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\geqslant \rho(v^+,U) + \rho(v^-,U),
\end{aligned}$$

which gives the desired result in (2.9). □

Inspired by [44], we now deal with a linear problem associated to the reflection with respect to a given hyperplane. For this, with the notation in (2.2) and (2.3), for every open and bounded set $\Omega \subset \mathbb{R}^N$, we define the first (variational) eigenvalue of the operator $\mathcal{L}$ introduced in (1.1) as

$$\Lambda_1(\Omega) := \inf_{u\in\mathcal{D}(\Omega)} \frac{B(u,u)}{\|u\|^2_{L^2(\Omega)}}. \tag{2.10}$$

On account of remark 2.1, we see that

$$\Lambda_1(\Omega) \geqslant \Lambda_{-\Delta}(\Omega), \tag{2.11}$$

where $\Lambda_{-\Delta}(\Omega)$ stands for the first eigenvalue of $-\Delta$ in Ω with homogeneous Dirichlet boundary conditions. Recalling that

$$\Lambda_{-\Delta}(\Omega) \to +\infty \quad \text{as } |\Omega| \to 0,$$

and setting

$$\Lambda_1(r) := \inf\big\{\Lambda_1(\Omega) \text{ with } \Omega \subset \mathbb{R}^n \text{ open with } |\Omega| = r\big\}, \quad r > 0,$$

it follows from (2.11) that

$$\Lambda_1(r) \to +\infty \quad \text{as } r \to 0^+. \tag{2.12}$$

Furthermore, let $H \subset \mathbb{R}^N$ be an open and affine halfspace. We denote by

$$Q : \mathbb{R}^N \to \mathbb{R}^N$$

the reflection with respect to ∂H. For convenience, we will sometimes denote with

$$\bar{x} := Q(x), \tag{2.13}$$

for every $x \in \mathbb{R}^N$. With this notation at hand, we say that a function $v : \mathbb{R}^N \to \mathbb{R}$ is *antisymmetric with respect to* Q if

$$v(\bar{x}) = -v(x), \quad \text{for every } x \in \mathbb{R}^N. \tag{2.14}$$

Moreover, we give the following definition of antisymmetric supersolutions:

Definition 2.4. Let $U \subset H$ be an open and bounded set. Let $c \in L^\infty(U)$. We say that a function $v : \mathbb{R}^N \to \mathbb{R}$ is an antisymmetric supersolution of

$$\begin{cases} \mathcal{L}v = cv & \text{in } U, \\ v \equiv 0 & \text{in } H \backslash U, \end{cases} \tag{2.15}$$

if it satisfies the following properties:

(i) v is antisymmetric,

(ii) $v \in \mathcal{H}(U')$ for some open set $U' \subset \mathbb{R}^N$ such that $Q(U') = U'$ and $\overline{U} \subset U'$,

(iii) $v \geqslant 0$ in $H \backslash U$ and, for every $\varphi \in \mathcal{D}(U)$ with $\varphi \geqslant 0$, one has

$$B(v, \varphi) \geqslant \int_U c(x) v(x) \varphi(x)\, dx. \tag{2.16}$$

The aim is now to provide a suitable maximum principle for antisymmetric supersolutions, as given in definition 2.4.

We start with the following observation on the bilinear form introduced in (2.2):

Lemma 2.5. *Let $U' \subset \mathbb{R}^N$ be an open set such that $Q(U') = U'$. Let $v \in \mathcal{H}(U')$ be an antisymmetric function such that*

$$v \geqslant 0 \quad \textit{in } H \backslash U, \tag{2.17}$$

for a certain open and bounded set $U \subset H$ with the property that

$$\overline{U} \subset H \cap U'. \tag{2.18}$$

Then, the function

$$w := \chi_H v^- \in \mathcal{D}(U) \tag{2.19}$$

and it holds that

$$B(w, w) \leqslant -B(v, w). \tag{2.20}$$

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Proof. We first prove (2.19). To this end we first observe that, since $v \in L^2(\mathbb{R}^N)$, one obviously has $w \in L^2(\mathbb{R}^N)$. Also, recalling (2.6), we know that $v \in H^1(U')$, and therefore it is easy to see that $v^- \in H^1(U')$. In addition, in light of (2.17), we have that $v^- \equiv 0$ in $H \setminus U$. As a consequence of these observations and of (2.18), we have that there exists an open set W such that

$$\overline{U} \subset W \subset \overline{W} \subset U' \cap H \quad \text{and} \quad v^- \in H_0^1(W).$$

Therefore, if we identify $w = \chi_H v^-$ with the zero extension of v^- outside of U, we get that $w \in H^1(\mathbb{R}^N)$. Moreover, we have that $w \equiv 0$ in $\mathbb{R}^N \setminus U$. These considerations imply (2.19).

Now we focus on the proof of (2.20). Recalling (2.2), we observe that

$$\begin{aligned} &B(w,w) + B(v,w) \\ &= \int_{\mathbb{R}^N} |\nabla w|^2 \,\mathrm{d}x + \iint_{\mathbb{R}^{2N}} \frac{(w(x)-w(y))^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y \\ &\quad + \int_{\mathbb{R}^N} \langle \nabla v, \nabla w \rangle \,\mathrm{d}x + \iint_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{2.21}$$

We notice that, thanks to (2.17),

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla w|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} \langle \nabla v, \nabla w \rangle \,\mathrm{d}x &= \int_U |\nabla v^-|^2 \,\mathrm{d}x + \int_U \langle \nabla v, \nabla v^- \rangle \,\mathrm{d}x \\ &= \int_U |\nabla v^-|^2 \,\mathrm{d}x - \int_U \langle \nabla v^-, \nabla v^- \rangle \,\mathrm{d}x = 0. \end{aligned} \tag{2.22}$$

Furthermore, we remark that, for any $x \in \mathbb{R}^N$,

$$\begin{aligned} &w(x)\big(w(x)+v(x)\big) \\ &= \chi_H(x) v^-(x) \big(\chi_H(x) v^-(x) + \chi_H(x) v(x) + \chi_{\mathbb{R}^N \setminus H}(x) v(x)\big) \\ &= \chi_H(x) v^-(x) \big(\chi_H(x) v^+(x) + \chi_{\mathbb{R}^N \setminus H}(x) v(x)\big) = 0, \end{aligned}$$

and therefore

$$\begin{aligned} &(w(x)-w(y))^2 + (v(x)-v(y))(w(x)-w(y)) \\ &= (w(x)-w(y))\big((w(x)+v(x)) - (w(y)+v(y))\big) \\ &= -w(x)(w(y)+v(y)) - w(y)(w(x)+v(x)). \end{aligned}$$

As a consequence, using (2.14) and the change of variable $Y := \bar{y}$ (also recall the notation in (2.13)), we obtain

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{(w(x)-w(y))^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y + \iint_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))(w(x)-w(y))}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y \\ &= \iint_{\mathbb{R}^{2N}} \frac{w(x)(w(y)+v(y)) + w(y)(w(x)+v(x))}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

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$$
\begin{aligned}
&= -2 \iint_{\mathbb{R}^{2N}} \frac{w(x)(w(y)+v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= -2 \iint_{H\times\mathbb{R}^N} \frac{v^-(x)\big(\chi_H(y)v^-(y)+v(y)\big)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= -2 \iint_{H\times\mathbb{R}^N} \frac{v^-(x)\big(\chi_H(y)v^+(y)+\chi_{\mathbb{R}^N\setminus H}(y)v(y)\big)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= -2 \iint_{H\times H} \frac{v^-(x)v^+(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y - 2\iint_{H\times(\mathbb{R}^N\setminus H)} \frac{v^-(x)v(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= -2 \iint_{H\times H} \frac{v^-(x)v^+(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + 2\iint_{H\times(\mathbb{R}^N\setminus H)} \frac{v^-(x)v(\bar{y})}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= -2 \iint_{H\times H} \frac{v^-(x)v^+(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + 2\iint_{H\times H} \frac{v^-(x)v(Y)}{|x-\bar{Y}|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}Y \\
&= -2 \iint_{H\times H} v^-(x)v^+(y)\left(\frac{1}{|x-y|^{N+2s}} - \frac{1}{|x-\bar{y}|^{N+2s}}\right)\mathrm{d}x\,\mathrm{d}y \\
&\quad - 2\iint_{H\times H} \frac{v^-(x)v^-(y)}{|x-\bar{y}|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant 0.
\end{aligned}
$$

Plugging this information and (2.22) into (2.21) we obtain (2.20), as desired. □

With the aid of lemma 2.5, we now prove the following maximum principle:

Proposition 2.6. *Let $U \subset \mathbb{R}^N$ be an open and bounded set with $\overline{U} \subset H$. Moreover, let $c \in L^\infty(U)$ be such that*

$$\|c^+\|_{L^\infty(U)} < \Lambda_1(U), \tag{2.23}$$

where the notation in (2.10) has been used.

Then, every antisymmetric supersolution v of (2.15) in U is nonnegative on the whole of H, that is, $v(x) \geqslant 0$ for a.e. $x \in H$.

Proof. We consider the function w introduced in (2.19) and we claim that

$$w \equiv 0. \tag{2.24}$$

To prove it, we argue towards a contradiction, supposing that $\|w\|_{L^2(U)} \neq 0$. By lemma 2.5, we know that $w \in \mathcal{D}(U)$, and hence it is an admissible test function in (2.16). Accordingly,

$$B(v, w) \geqslant \int_U c(x)v(x)w(x)\,\mathrm{d}x.$$

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From this, (2.10), (2.20) and (2.23), we conclude that

$$\Lambda_1(U)\|w\|^2_{L^2(U)} \leqslant B(w,w) \leqslant -B(v,w) \leqslant -\int_U c(x)v(x)w(x)\,\mathrm{d}x$$
$$= \int_U c(x)w^2(x)\,\mathrm{d}x \leqslant \|c^+\|_{L^\infty(U)}\|w\|^2_{L^2(U)} < \Lambda_1(U)\|w\|^2_{L^2(U)},$$

which is a contradiction. This proves (2.24), thus leading to the desired result. □

We are now in the position of establishing a strong maximum principle for antisymmetric supersolutions which is the counterpart in the setting of mixed local–nonlocal operators of [**44**, proposition 3.6] (a modification of these arguments will lead to a Hopf-type result, as pointed out in appendix A):

Proposition 2.7. *Let $U \subset H$ be an open and bounded set. Let $c \in L^\infty(U)$ and let v be an antisymmetric supersolution of (2.15) in U. Assume that*

$$v \geqslant 0 \quad \textit{a.e. in } H. \tag{2.25}$$

Then, either $v \equiv 0$ in $\mathbb{R}^N$ or

$$\operatorname{ess\,inf}_K v > 0, \quad \textit{for every compact set } K \subset U. \tag{2.26}$$

Proof. If $v \equiv 0$ in $\mathbb{R}^N$, there is nothing to prove, so we assume that

$$v \not\equiv 0 \quad \text{in } \mathbb{R}^N. \tag{2.27}$$

In this case, it suffices to show that, for a fixed $x_0 \in U$, one has

$$\operatorname{ess\,inf}_{B_r(x_0)} v > 0, \tag{2.28}$$

for some radius $r > 0$ small enough. We then prove (2.28).

First of all, in light of (2.25), (2.27) and the fact that v is antisymmetric, we can find a bounded set $M \subset H$, with positive measure, which does not contain a small neighbourhood of x_0 and such that

$$\delta := \inf_M v > 0. \tag{2.29}$$

In addition, by (2.12), we find a radius

$$r \in \left(0, \frac{\operatorname{dist}(x_0; (\mathbb{R}^N \setminus H) \cup M)}{4}\right) \tag{2.30}$$

such that

$$\Lambda_1(B_{2r}(x_0)) > \|c\|_{L^\infty(U)}. \tag{2.31}$$

We now pick a function $g \in C^2_0(\mathbb{R}^N, [0,1])$ such that

$$g(x) := \begin{cases} 1, & \text{if } x \in B_r(x_0), \\ 0, & \text{if } x \in \mathbb{R}^N \setminus B_{2r}(x_0). \end{cases}$$

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Moreover, for a given $a > 0$ to be chosen later, we define the function

$$h : \mathbb{R}^N \to \mathbb{R}, \quad h(x) := g(x) - g(\bar{x}) + a\,(\chi_M(x) - \chi_M(\bar{x}))\,, \tag{2.32}$$

where we are using the notation in (2.13). We also define the sets $U_0 := B_{2r}(x_0)$ and $U_0' := B_{3r}(x_0) \cup Q(B_{3r}(x_0))$.

We observe that h is antisymmetric, and moreover

$$h \equiv 0 \;\text{ on } H\backslash(U_0 \cup M) \quad \text{ and } \quad h \equiv a \;\text{ on } M, \tag{2.33}$$

thanks to (2.30). From (2.30) we also deduce that

$$(M \cup Q(M)) \cap U_0' = \varnothing. \tag{2.34}$$

This and the fact that M is bounded give that $h \in \mathcal{H}(U_0')$. We now claim that there exists a constant $C_1 > 0$, depending on g, such that

$$B(g, \varphi) \leqslant C_1 \int_{U_0} \varphi(x)\,\mathrm{d}x, \quad \text{for every } \varphi \in \mathcal{D}(U_0) \text{ with } \varphi \geqslant 0. \tag{2.35}$$

Indeed, for any $\varphi \in \mathcal{D}(U_0)$ with $\varphi \geqslant 0$, by an integration by parts,

$$\begin{aligned} \int_{\mathbb{R}^N} \langle \nabla g, \nabla \varphi \rangle \,\mathrm{d}x &= \int_{U_0} \langle \nabla g, \nabla \varphi \rangle \,\mathrm{d}x \\ &= -\int_{U_0} \Delta g\, \varphi \,\mathrm{d}x \leqslant \|g\|_{C^2(\mathbb{R}^N)} \int_{U_0} \varphi(x)\,\mathrm{d}x. \end{aligned} \tag{2.36}$$

Moreover, by proposition 2.3-(ii) in [44] (applied here with $v := g$ and $u := \varphi$), we have that

$$\begin{aligned} \frac{1}{2} \iint_{\mathbb{R}^{2N}} \frac{(g(x) - g(y))(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y &= \int_{\mathbb{R}^N} (-\Delta)^s g(x)\, \varphi(x)\,\mathrm{d}x \\ = \int_{U_0} (-\Delta)^s g(x)\, \varphi(x)\,\mathrm{d}x &\leqslant \|(-\Delta)^s g\|_{L^\infty(U_0)} \int_{U_0} \varphi(x)\,\mathrm{d}x. \end{aligned}$$

Recalling (2.2), this and (2.36) imply (2.35). Similarly, one has that

$$B(g \circ Q, \varphi) \leqslant C_2 \int_{U_0} \varphi(x)\,\mathrm{d}x, \quad \text{for every } \varphi \in \mathcal{D}(U_0) \text{ with } \varphi \geqslant 0, \tag{2.37}$$

for some $C_2 > 0$. In addition, we see that, for any $\varphi \in \mathcal{D}(U_0)$ and any $x \in \mathbb{R}^N$, from (2.30) we infer that

$$(\chi_M(x) - \chi_M(\bar{x}))\varphi(x) = 0;$$

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as a consequence,

$$
\begin{aligned}
&\frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{\big((\chi_M(x)-\chi_M(\bar{x}))-(\chi_M(y)-\chi_M(\bar{y}))\big)(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad=-\frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{(\chi_M(x)-\chi_M(\bar{x}))\varphi(y)+(\chi_M(y)-\chi_M(\bar{y}))\varphi(x)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad=-\iint_{U_0\times\mathbb{R}^N} \frac{(\chi_M(y)-\chi_M(\bar{y}))\varphi(x)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad=-\iint_{U_0\times\mathbb{R}^N} \frac{(\chi_M(y)-\chi_M(\bar{y}))\varphi(x)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad=-\int_{U_0}\varphi(x)\left(\int_M \frac{\mathrm{d}y}{|x-y|^{N+2s}}-\int_{Q(M)}\frac{\mathrm{d}y}{|x-y|^{N+2s}}\right)\mathrm{d}x\\
&\quad=-\int_{U_0}\varphi(x)\left(\int_M \frac{dy}{|x-y|^{N+2s}}-\int_{M}\frac{dy}{|x-\bar{y}|^{N+2s}}\right)\mathrm{d}x\\
&\quad\leqslant -C_0\int_{U_0}\varphi(x)\,\mathrm{d}x,
\end{aligned}
\tag{2.38}
$$

where

$$
C_0 := \inf_{x\in U_0}\left(\int_M \frac{\mathrm{d}y}{|x-y|^{N+2s}}-\int_M \frac{\mathrm{d}y}{|x-\bar{y}|^{N+2s}}\right).
$$

We stress on the fact that the constant C_0 is finite, thanks to (2.30).

Now, recalling (2.32), and using (2.35), (2.37) and (2.38), we conclude that, for any $\varphi\in\mathcal{D}(U_0)$, one has

$$
\begin{aligned}
B(h,\varphi) &= B(g,\varphi)+B(g\circ Q,\varphi)\\
&\qquad + a\iint_{\mathbb{R}^{2N}} \frac{\big((\chi_M(x)-\chi_M(\bar{x}))-(\chi_M(y)-\chi_M(\bar{y}))\big)(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant C_a\int_{U_0}\varphi(x)\,\mathrm{d}x,
\end{aligned}
\tag{2.39}
$$

where

$$
C_a := C_1 + C_2 - 2a\,C_0.
$$

Now we perform our choice of the parameter a: we choose $a>0$ such that

$$
C_a < -\|c\|_{L^\infty(U_0)}.
$$

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In particular, with this choice, (2.39) yields that

$$\begin{aligned} B(h,\varphi) &\leqslant -\|c\|_{L^\infty(U_0)} \int_{U_0} \varphi(x)\,\mathrm{d}x \leqslant -\|c^-\|_{L^\infty(U_0)} \int_{U_0} \varphi(x)\,\mathrm{d}x \\ &\leqslant -\int_{U_0} c^-(x)\varphi(x)\,\mathrm{d}x \leqslant -\int_{U_0} c^-(x)h(x)\varphi(x)\,\mathrm{d}x \\ &\leqslant \int_{U_0} c^+(x)h(x)\varphi(x)\,\mathrm{d}x - \int_{U_0} c^-(x)h(x)\varphi(x)\,\mathrm{d}x \\ &= \int_{U_0} c(x)h(x)\varphi(x)\,\mathrm{d}x, \end{aligned} \tag{2.40}$$

since $h(x) = g(x) \in [0,1]$ for every $x \in U_0$. Now, we recall (2.29), we define the function $\tilde{v}$ as

$$\tilde{v}(x) := v(x) - \frac{\delta}{a} h(x), \tag{2.41}$$

and we notice that $\tilde{v} \in \mathcal{H}(U_0')$ and it is antisymmetric, since both v and h are so. Furthermore, by (2.25), (2.29) and (2.33), we have that

$$\tilde{v} \geqslant 0 \quad \text{on } H\backslash U_0.$$

In addition, for any $\varphi \in \mathcal{D}(U_0)$ with $\varphi \geqslant 0$,

$$\begin{aligned} B(\tilde{v},\varphi) &= B(v,\varphi) - \frac{\delta}{a} B(h,\varphi) \\ &\geqslant \int_{U_0} c(x)v(x)\varphi(x)\,\mathrm{d}x - \frac{\delta}{a}\int_{U_0} c(x)h(x)\varphi(x)\,\mathrm{d}x \\ &= \int_{U_0} c(x)\tilde{v}(x)\varphi(x)\,\mathrm{d}x, \end{aligned}$$

thanks to (2.16) and (2.40).

As a consequence, we have that $\tilde{v}$ is an antisymmetric supersolution of

$$\begin{cases} \mathcal{L}\tilde{v} = c\tilde{v} & \text{in } U_0, \\ \tilde{v} \equiv 0 & \text{in } H\backslash U_0. \end{cases}$$

Since $\|c\|_{L^\infty(U_0)} < \Lambda_1(U_0)$, thanks to (2.31), we are in the position to apply proposition 2.6 to conclude that $\tilde{v} \geqslant 0$ a.e. on U_0. Recalling (2.41), this gives

$$v \geqslant \frac{\delta}{a} > 0 \quad \text{a.e. on } B_r(x_0).$$

This establishes (2.28), and the proof of proposition 2.7 is thereby complete. □

With this preliminary work, we now prove theorem 1.1. For this, let $u \in C(\Omega)$ be a weak solution of (1.2). We fix the usual notation needed to implement the moving plane method. For every $\lambda \in (-1, 1)$ we define the following:

$$\begin{aligned} \Sigma_\lambda &:= \begin{cases} \{x \in \mathbb{R}^N : x_1 < \lambda\}, & \text{if } \lambda < 0 \\ \{x \in \mathbb{R}^N : x_1 > \lambda\}, & \text{if } \lambda \geqslant 0, \end{cases} \\ \Omega_\lambda &:= \Omega \cap \Sigma_\lambda, \\ Q_\lambda(x) &= x_\lambda := (2\lambda - x_1, x_2, \ldots, x_N), \\ \text{and} \quad u_\lambda(x) &:= u(x_\lambda). \end{aligned} \tag{2.42}$$

We also define the function

$$c(x) := \begin{cases} \dfrac{f(u_\lambda(x)) - f(u(x)))}{u_\lambda(x) - u(x)}, & \text{if } u_\lambda(x) \neq u(x), \\ 0, & \text{if } u_\lambda(x) = u(x). \end{cases} \tag{2.43}$$

We observe that $c \in L^\infty(\Omega_\lambda)$, thanks to the Lipschitz assumption on f.

Furthermore, setting

$$v_\lambda := u_\lambda - u, \tag{2.44}$$

we point out the following simple yet important observations.

LEMMA 2.8. *Let u be a weak solution of (1.2) according to definition 2.2.*

Then, the function v_λ in (2.44) is an antisymmetric supersolution of (2.15) in Ω_λ, according to definition 2.4, with c as in (2.43).

Proof. We notice that $v_\lambda \in H^1(\mathbb{R}^N) \subset \mathcal{H}(U')$, for every open set $U' \subset \mathbb{R}^N$ such that $Q(U') = U'$ and $\overline{\Omega_\lambda} \subset U'$. Moreover, since $u \geqslant 0$ in $\mathbb{R}^N$ and $u \equiv 0$ on $\Sigma_\lambda \backslash \Omega_\lambda$, we have that $v_\lambda \geqslant 0$ on $\Sigma_\lambda \backslash \Omega_\lambda$. In addition, for any $\varphi \in \mathcal{D}(\Omega_\lambda)$ and for any $x \in \mathbb{R}^N$, we have

$$\begin{aligned} \langle \nabla u_\lambda(x), \nabla \varphi(x) \rangle &= \big(-\partial_1 u, \partial_2 u, \ldots, \partial_N u \big)(\bar{x}) \cdot \big(\partial_1 \varphi, \partial_2 \varphi, \ldots, \partial_N \varphi \big)(x) \\ &= \big(\partial_1 u, \partial_2 u, \ldots, \partial_N u \big)(X) \cdot \big(-\partial_1 \varphi, \partial_2 \varphi, \ldots, \partial_N \varphi \big)(\bar{X}) \\ &= \langle \nabla u(X), \nabla \varphi_\lambda(X) \rangle, \end{aligned} \tag{2.45}$$

where $X := \bar{x}$. Similarly, setting also $Y := \bar{y}$,

$$\begin{aligned} &\frac{(u_\lambda(x) - u_\lambda(y))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} = \frac{(u(\bar{x}) - u(\bar{y}))(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}} \\ &\quad = \frac{(u(X) - u(Y))(\varphi(\bar{X}) - \varphi(\bar{Y}))}{|\bar{X} - \bar{Y}|^{N+2s}} = \frac{(u(X) - u(Y))(\varphi(\bar{X}) - \varphi(\bar{Y}))}{|X - Y|^{N+2s}} \\ &\quad = \frac{(u(X) - u(Y))(\varphi_\lambda(X) - \varphi_\lambda(Y))}{|X-Y|^{N+2s}}. \end{aligned}$$

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From this and (2.45), we obtain that

$$\begin{aligned}
&B(u_\lambda,\varphi)\\
&\quad=\int_{\mathbb{R}^N}\langle\nabla u_\lambda(x),\nabla\varphi(x)\rangle\,\mathrm{d}x+\iint_{\mathbb{R}^{2N}}\frac{(u_\lambda(x)-u_\lambda(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&\quad=\int_{\mathbb{R}^N}\langle\nabla u(X),\nabla\varphi_\lambda(X)\rangle\,\mathrm{d}X+\iint_{\mathbb{R}^{2N}}\frac{(u(X)-u(Y))(\varphi_\lambda(X)-\varphi_\lambda(Y))}{|X-Y|^{N+2s}}\,\mathrm{d}X\,\mathrm{d}Y\\
&\quad=B(u,\varphi_\lambda).
\end{aligned}$$

As a consequence, since $\varphi_\lambda\in\mathcal{D}(Q_\lambda(\Omega_\lambda))\subset\mathcal{D}(\Omega)$, we can use definition 2.2 to find that

$$\begin{aligned}
B(u_\lambda,\varphi)&=\int_{\mathbb{R}^N}f(u(x))\varphi_\lambda(x)\,\mathrm{d}x\\
&=\int_{\mathbb{R}^N}f(u(\bar{X}))\varphi(X)\,\mathrm{d}X=\int_{\mathbb{R}^N}f(u_\lambda(X))\varphi(X)\,\mathrm{d}X.
\end{aligned}$$

Therefore,

$$\begin{aligned}
B(v_\lambda,\varphi)&=B(u_\lambda,\varphi)-B(u,\varphi)\\
&=\int_{\mathbb{R}^N}f(u_\lambda(x))\varphi(x)\,\mathrm{d}x-\int_{\mathbb{R}^N}f(u(x))\varphi(x)\,\mathrm{d}x\\
&=\int_{\mathbb{R}^N}c(x)v_\lambda(x)\varphi(x)\,\mathrm{d}x,
\end{aligned}$$

which proves (2.16), and thereby completes the proof of lemma 2.8. □

LEMMA 2.9. *Let u be a weak solution of (1.2), and let v_λ be as in (2.44). If there exists some $\lambda\in(-1,0)$ such that $v_\lambda\equiv 0$ in $\mathbb{R}^N$, then*

$$u\equiv 0 \text{ in } \Omega \qquad (\text{hence, } u\equiv 0 \text{ in } \mathbb{R}^N). \tag{2.46}$$

Proof. We proceed essentially as in [44]: to begin with, since $\lambda\in(-1,0)$, we have $\ell:=1+\lambda\in(0,1)$; hence, following the notation in (2.42), we consider the set

$$\Omega_\ell=\{x\in\Omega:\ x_1>\ell\}$$

and we notice that, by definition, one has $Q_\lambda(\Omega_\ell)\cap\Omega=\varnothing$ (see also (2.1)). As a consequence, since $u\equiv 0$ outside Ω and $v_\lambda=u-u_\lambda=0$ in $\mathbb{R}^N$, we get

$$u\equiv u_\lambda\equiv 0 \quad \text{on } \Omega_\ell. \tag{2.47}$$

In particular, setting $\eta:=1+\lambda/2$ and

$$\Omega_\eta=\{x\in\Omega:\ x_1>\eta\},$$

from (2.47) we obtain (notice that $\Omega_\eta\cup Q_\eta(\Omega_\eta)\subseteq\Omega_\ell$)

$$v_\eta=u-u_\eta=0 \quad \text{on } \Omega_\eta.$$

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Using once again proposition 2.7 (with the choice $H := \Sigma_\eta$, $U := \Omega_\eta$ and $v := v_\eta$), we then deduce that

$$v_\eta \equiv 0 \quad \text{on } \mathbb{R}^N.$$

Gathering together these facts, we conclude that u has two different parallel symmetry hyperplanes, namely $\partial\Sigma_\lambda = \{x_1 = \lambda\}$ and $\partial\Sigma_\eta = \{x_1 = \eta\}$. Using this last fact, it is easy to derive the claim in (2.46). Indeed, since $\partial\Sigma_\lambda$ is a symmetry hyperplane for u, since $u \equiv 0$ out of Ω and since

$$Q_\lambda\Big(\Omega\backslash\big(\Omega_\lambda \cup Q_\lambda(\Omega_\lambda)\big)\Big) \cap \Omega = \varnothing,$$

we derive that

$$u \equiv 0 \text{ on } \mathcal{O} := \Omega\backslash(\Omega_\lambda \cup Q_\lambda(\Omega_\lambda)); \tag{2.48}$$

on the other hand, since also $\partial\Sigma_\eta$ is a symmetry hyperplane for u and since

$$Q_\eta\big(\Omega_\lambda \cup Q_\lambda(\Omega_\lambda)\big) \subseteq \mathcal{O},$$

we infer that

$$u \equiv 0 \text{ on } \Omega_\lambda \cup Q_\lambda(\Omega_\lambda). \tag{2.49}$$

By combining (2.48) and (2.49), we immediately obtain (2.46). □

With these considerations, we are now ready to prove theorem 1.1.

Proof of theorem 1.1. Let $u \in C(\mathbb{R}^N)$ be any *non identically vanishing weak solution of* (1.2). For every $\lambda \in (-1, 0)$, we define the function

$$w_\lambda : \mathbb{R}^N \to \mathbb{R}, \quad w_\lambda(x) := \begin{cases} (u-u_\lambda)^+(x) & \text{in } \Sigma_\lambda, \\ (u-u_\lambda)^-(x) & \text{in } \mathbb{R}^N\backslash\Sigma_\lambda, \end{cases} \tag{2.50}$$

where, *differently from before*, we have set

$$(u-u_\lambda)^- := \min\{u-u_\lambda, 0\},$$

which is *nonpositive*. We claim that

$$w_\lambda \in H^1(\mathbb{R}^N). \tag{2.51}$$

Indeed, we know that $u \in H^1(\mathbb{R}^N)$ and thus $u - u_\lambda \in H^1(\mathbb{R}^N)$. Accordingly, we have that (see e.g. the Chain Rule on page 296 of [**45**])

$$(u-u_\lambda)^+ \in H^1(\mathbb{R}^N). \tag{2.52}$$

Moreover, $u \in C(\mathbb{R}^N)$, and consequently

$$(u-u_\lambda)^+ \in C(\mathbb{R}^N). \tag{2.53}$$

In addition, $u = u_\lambda$ along $\partial\Sigma_\lambda$. From this fact, (2.52) and (2.53), we obtain that

$$(u-u_\lambda)^+\chi_{\Sigma_\lambda} \in H^1_0(\Sigma_\lambda) \subset H^1(\mathbb{R}^N), \tag{2.54}$$

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see e.g. [15, theorem 9.17]. Similarly,

$$(u - u_\lambda)^- \chi_{\mathbb{R}^N \setminus \Sigma_\lambda} \in H^1(\mathbb{R}^N). \tag{2.55}$$

We also observe that

$$w_\lambda = (u - u_\lambda)^+ \chi_{\Sigma_\lambda} + (u - u_\lambda)^- \chi_{\mathbb{R}^N \setminus \Sigma_\lambda}.$$

From this, (2.54) and (2.55), we obtain (2.51), as desired.

Furthermore, we claim that

$$w_\lambda \equiv 0 \text{ in } \mathbb{R}^N \setminus (\Omega_\lambda \cup Q_\lambda(\Omega_\lambda)) \subset \mathbb{R}^N \setminus \Omega. \tag{2.56}$$

Indeed, if $x \in \Sigma_\lambda \setminus \Omega_\lambda$, then $w_\lambda(x) = (0 - u_\lambda(x))^+ = 0$. If instead $x \in Q_\lambda(\Sigma_\lambda \setminus \Omega_\lambda)$, then $\bar{x} \in \Sigma_\lambda \setminus \Omega_\lambda$ and accordingly

$$0 = w_\lambda(\bar{x}) = (u(\bar{x}) - u_\lambda(\bar{x}))^+ = (u_\lambda(x) - u(x))^+.$$

This gives that $u_\lambda(x) \leqslant u(x)$, and therefore $w_\lambda(x) = (u(x) - u_\lambda(x))^- = 0$.

From these observations, we obtain (2.56). Then, (2.51) and (2.56) give that we can take w_λ as an admissibile test function in (2.4). In this way, we obtain

$$B(u, w_\lambda) = \int_{\mathbb{R}^N} f(u(x)) w_\lambda(x)\, \mathrm{d}x. \tag{2.57}$$

Similarly,

$$B(u_\lambda, w_\lambda) = \int_{\mathbb{R}^N} f(u_\lambda(x)) w_\lambda(x)\, \mathrm{d}x. \tag{2.58}$$

Subtracting (2.58) to (2.57), and recalling (2.2), we get

$$\begin{aligned} &\int_{\mathbb{R}^N} \langle \nabla(u - u_\lambda), \nabla w_\lambda \rangle \,\mathrm{d}x \\ &\qquad + \iint_{\mathbb{R}^{2N}} \frac{((u(x) - u_\lambda(x)) - (u(y) - u_\lambda(y)))(w_\lambda(x) - w_\lambda(y))}{|x-y|^{N+2s}} \,\mathrm{d}x \,\mathrm{d}y \\ &\quad = \int_{\mathbb{R}^N} \big(f(u(x)) - f(u_\lambda(x))\big) w_\lambda(x) \,\mathrm{d}x. \end{aligned} \tag{2.59}$$

Now, we use formula (3.9) in [46], which gives that

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{\big((u(x) - u_\lambda(x)) - (u(y) - u_\lambda(y))\big)(w_\lambda(x) - w_\lambda(y))}{|x-y|^{N+2s}} \,\mathrm{d}x \,\mathrm{d}y \\ &\quad \geqslant \iint_{\mathbb{R}^{2N}} \frac{|w_\lambda(x) - w_\lambda(y)|^2}{|x-y|^{N+2s}} \,\mathrm{d}x \,\mathrm{d}y \geqslant 0. \end{aligned}$$

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Using this information into (2.59), and recalling (2.56), we obtain that

$$\begin{aligned}\int_{\mathbb{R}^N}\langle\nabla(u-u_\lambda),\nabla w_\lambda\rangle\,dx &\leqslant \int_{\mathbb{R}^N}\big(f(u(x))-f(u_\lambda(x))\big)w_\lambda(x)\,\mathrm{d}x\\ &=\int_{\mathbb{R}^N}\frac{f(u(x))-f(u_\lambda(x))}{u(x)-u_\lambda(x)}\big(u(x)-u_\lambda(x)\big)w_\lambda(x)\,\mathrm{d}x\\ &=\int_{\mathbb{R}^N}\frac{f(u(x))-f(u_\lambda(x))}{u(x)-u_\lambda(x)}w_\lambda^2(x)\,\mathrm{d}x\\ &=\int_{\Omega_\lambda\cup Q_\lambda(\Omega_\lambda)}\frac{f(u(x))-f(u_\lambda(x))}{u(x)-u_\lambda(x)}w_\lambda^2(x)\,\mathrm{d}x.\end{aligned}\tag{2.60}$$

We also notice that, thanks to (2.56),

$$\int_{\mathbb{R}^N}\langle\nabla(u-u_\lambda),\nabla w_\lambda\rangle\,\mathrm{d}x=\int_{\mathbb{R}^N}|\nabla w_\lambda|^2\,\mathrm{d}x=\int_{\Omega_\lambda\cup Q_\lambda(\Omega_\lambda)}|\nabla w_\lambda|^2\,\mathrm{d}x.$$

From this and (2.60), we deduce that

$$\begin{aligned}\int_{\Omega_\lambda\cup Q_\lambda(\Omega_\lambda)}|\nabla w_\lambda|^2\,\mathrm{d}x &\leqslant \int_{\Omega_\lambda\cup Q_\lambda(\Omega_\lambda)}\frac{f(u(x))-f(u_\lambda(x))}{u(x)-u_\lambda(x)}w_\lambda^2(x)\,\mathrm{d}x\\ &\leqslant C\int_{\Omega_\lambda\cup Q_\lambda(\Omega_\lambda)}|w_\lambda|^2\,\mathrm{d}x,\end{aligned}\tag{2.61}$$

for some constant $C>0$, depending on f and $\|u\|_{L^\infty(\Omega)}$.

Now, using lemma 2.10 in [11], we obtain that

$$\int_{\Omega_\lambda\cup Q_\lambda(\Omega_\lambda)}|\nabla w_\lambda|^2\,\mathrm{d}x\leqslant C|\Omega_\lambda\cup Q_\lambda(\Omega_\lambda)|^{1/N}\int_{\Omega_\lambda\cup Q_\lambda(\Omega_\lambda)}|\nabla w_\lambda|^2\,\mathrm{d}x,\tag{2.62}$$

to renaming C, which possibly depends also on N. As a consequence, if λ is sufficiently close to -1, we see that

$$C|\Omega_\lambda\cup Q_\lambda(\Omega_\lambda)|^{1/N}<\frac{1}{2},$$

which, combined with (2.62), gives that

$$\int_{\Omega_\lambda\cup Q_\lambda(\Omega_\lambda)}|\nabla w_\lambda|^2\,\mathrm{d}x=0,$$

provided that λ is sufficiently close to -1. From this and the Poincaré inequality we get that $w_\lambda\equiv 0$ in $\Omega_\lambda\cup Q_\lambda(\Omega_\lambda)$ if λ is sufficiently close to -1, which,

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recalling (2.50), implies that

$$u \leqslant u_\lambda \text{ in } \Omega_\lambda \quad \text{if } \lambda \text{ is sufficiently close to } -1. \tag{2.63}$$

Now, we define the set

$$\Lambda_0 := \big\{\lambda \in (-1,0): \ u \leqslant u_t \text{ in } \Omega_t \text{ for every } t \in (-1,\lambda]\big\},$$

and we explicitly notice that, since $0 = u \leqslant u_\lambda$ on $\Sigma_\lambda \backslash \Omega_\lambda$, one also has

$$\Lambda_0 = \big\{\lambda \in (-1,0): \ u \leqslant u_t \text{ in } \Sigma_t \text{ for every } t \in (-1,\lambda]\big\}. \tag{2.64}$$

In light of (2.63), the following quantity is well defined:

$$\overline{\lambda} := \sup \Lambda_0. \tag{2.65}$$

The goal is now to prove that

$$\overline{\lambda} = 0. \tag{2.66}$$

For this, we argue by contradiction and assume that

$$\overline{\lambda} < 0.$$

We then recall the definition of v_λ in (2.44) and we observe that, since u is continuous in Ω, $v_{\overline{\lambda}} \geqslant 0$ in $\Omega_{\overline{\lambda}}$. Actually, in view of (2.64) we have

$$v_{\overline{\lambda}} \geqslant 0, \quad \text{in } \Sigma_{\overline{\lambda}}.$$

Since, by assumption, u is not identically vanishing, from lemma 2.9 we derive that $v_{\overline{\lambda}} \not\equiv 0$ in $\mathbb{R}^N$ as well; as a consequence, lemma 2.8 and proposition 2.7 (applied here with the choice $H := \Sigma_{\overline{\lambda}}$, $U := \Omega_{\overline{\lambda}}$ and $v := v_{\overline{\lambda}}$) ensure that

$$v_{\overline{\lambda}} > 0, \quad \text{in } \Omega_{\overline{\lambda}}. \tag{2.67}$$

Let then $K \subseteq \Omega_{\overline{\lambda}}$ be a given compact set, to be chosen later on. Since the map $(\lambda, x) \mapsto v_\lambda(x)$ is continuous, we can find a suitable $\overline{\tau} = \overline{\tau}(K) > 0$ such that

$$v_{\overline{\lambda}+\tau} > 0 \text{ in } K \qquad (\text{for all } \tau \in (0,\overline{\tau})). \tag{2.68}$$

We then consider, for every fixed $\tau \in (0,\overline{\tau})$, the function $w_{\overline{\lambda}+\tau}$ defined as in (2.50) (with $\lambda := \overline{\lambda} + \tau$). We notice that, thanks to (2.51) and (2.56), we can take $w_{\overline{\lambda}+\tau}$ as an admissible test function in (2.4), obtaining that

$$B(u, w_{\overline{\lambda}+\tau}) = \int_{\mathbb{R}^N} f(u(x)) w_{\overline{\lambda}+\tau}(x)\, \mathrm{d}x \quad \text{and}$$
$$B(u_{\overline{\lambda}+\tau}, w_{\overline{\lambda}+\tau}) = \int_{\mathbb{R}^N} f(u_{\overline{\lambda}+\tau}(x)) w_{\overline{\lambda}+\tau}(x)\, \mathrm{d}x.$$

From here, we repeat the same argument in (2.59)–(2.61) to find that

$$\int_{\Omega_{\overline{\lambda}+\tau} \cup Q(\Omega_{\overline{\lambda}+\tau})} |\nabla w_{\overline{\lambda}+\tau}|^2 \, \mathrm{d}x \leqslant C \int_{\Omega_{\overline{\lambda}+\tau} \cup Q(\Omega_{\overline{\lambda}+\tau})} |w_{\overline{\lambda}+\tau}|^2 \, \mathrm{d}x,$$

where $Q = Q_{\overline{\lambda}+\tau}$ and $C > 0$ is a constant depending on f and on $\|u\|_{L^\infty(\Omega)}$.

From this, recalling (2.68), we obtain that

$$\int_{\Omega_{\overline{\lambda}+\tau}\cup Q(\Omega_{\overline{\lambda}+\tau})} |\nabla w_{\overline{\lambda}+\tau}|^2\,\mathrm{d}x \leqslant C\int_{(\Omega_{\overline{\lambda}+\tau}\setminus K)\cup Q(\Omega_{\overline{\lambda}+\tau}\setminus K)} |w_{\overline{\lambda}+\tau}|^2\,\mathrm{d}x.$$

Hence, making again use of lemma 2.10 in [**11**], we get

$$\begin{aligned}&\int_{\Omega_{\overline{\lambda}+\tau}\cup Q(\Omega_{\overline{\lambda}+\tau})} |\nabla w_{\overline{\lambda}+\tau}|^2\,\mathrm{d}x\\ &\quad\leqslant C\,|(\Omega_{\overline{\lambda}+\tau}\setminus K)\cup Q(\Omega_{\overline{\lambda}+\tau}\setminus K)|^{1/N}\int_{(\Omega_{\overline{\lambda}+\tau}\setminus K)\cup Q(\Omega_{\overline{\lambda}+\tau}\setminus K)} |\nabla v_{\overline{\lambda}+\tau}|^2\,\mathrm{d}x,\end{aligned} \tag{2.69}$$

up to relabelling $C > 0$ (which may also depend on N). Now we choose the compact K big enough and the number $\overline{\tau}$ small enough such that

$$C\,|(\Omega_{\overline{\lambda}+\tau}\setminus K)\cup Q(\Omega_{\overline{\lambda}+\tau}\setminus K)|^{1/N} < 1.$$

Using this information into (2.69), we conclude that

$$\int_{\Omega_{\overline{\lambda}+\tau}\cup Q(\Omega_{\overline{\lambda}+\tau})} |\nabla w_{\overline{\lambda}+\tau}|^2\,\mathrm{d}x = 0.$$

From this and the Poincaré inequality, we find that $w_{\overline{\lambda}+\tau} \equiv 0$ in $\Omega_{\overline{\lambda}+\tau}$, hence

$$u \leqslant u_{\overline{\lambda}+\tau} \quad \text{in } \Omega_{\overline{\lambda}+\tau},$$

for every $\tau \in (0, \overline{\tau})$, provided $\overline{\tau} > 0$ is small enough. This yields a contradiction with (2.65), from which we conclude that (2.66) holds true, as desired.

With (2.66) at hand, we are in the position to complete the proof. Indeed, since $\overline{\lambda} = 0$, we see that, for all $\lambda \in (-1, 0)$ and all $x \in \Omega_\lambda$, one has

$$u(x) \leqslant u_\lambda(x) = u(2\lambda - x_1, x_2, \ldots, x_N).$$

Consequently,

$$u(x) \leqslant u(-x_1, x_2, \ldots, x_N), \tag{2.70}$$

for all $x \in \Omega \cap \{x_1 < 0\}$. In the same way, sliding the moving plane from right to left, one sees that, for all $x \in \Omega \cap \{x_1 > 0\}$, one has

$$u(x) \leqslant u(-x_1, x_2, \ldots, x_N).$$

This implies that

$$u(-x_1, x_2, \ldots, x_N) \leqslant u(x),$$

for all $x \in \Omega \cap \{x_1 < 0\}$. From this and (2.70), we conclude that

$$u(x) = u(-x_1, x_2, \ldots, x_N),$$

for all $x \in \Omega$, which says that u is symmetric with respect to $\{x_1 = 0\}$.

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Furthermore, since $u \not\equiv 0$ in $\mathbb{R}^N$, it follows from lemma 2.9 that $v_\lambda \not\equiv 0$ for every $\lambda \in (-1, 0)$; hence, (2.66) and proposition 2.7 give

$$u(x) < u_\lambda(x) = u(2\lambda - x_1, x_2, \dots, x_N) \qquad (\text{for all } x \in \Omega_\lambda). \tag{2.71}$$

From (2.71) it plainly follows that u is strictly increasing in the x_1-direction in $\Omega \cap \{x_1 < 0\}$, and the proof of theorem 1.1 is thereby complete. □

3. One-dimensional symmetry and proof of theorem 1.2

In this section we provide the proof of theorem 1.2. For this, we indicate the points $x \in \mathbb{R}^N$ by

$$(y, t), \text{ with } y \in \mathbb{R}^{N-1} \text{ and } t \in \mathbb{R}.$$

Moreover, since we are interested in classical solutions to (1.4), we define

$$\mathbb{X} := C^3(\mathbb{R}^N) \cap W^{4,\infty}(\mathbb{R}^N). \tag{3.1}$$

Remark 3.1. We notice that, if $u \in \mathbb{X}$, it is possible to compute $\mathcal{L}u$ in the classical sense, i.e., $\mathcal{L}u(x)$ is well-defined for all $x \in \mathbb{R}^N$. As a matter of fact, to give a pointwise meaning to $\mathcal{L}u$ it suffices to have $u \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

We shall derive theorem 1.2 from the abstract approach developed in [**38**]. To this end, we check that the assumptions introduced in [**38**] are satisfied in our setting. We list these assumptions here for the convenience of the reader:

(H1) if $\varphi \in \mathbb{X}$ satisfies $\mathcal{L}\varphi = f(\varphi)$ in $\mathbb{R}^N$, then there exists an operator $\tilde{\mathcal{L}}$, acting on a suitable space of functions $\tilde{\mathbb{X}} \subseteq C(\mathbb{R}^N)$ which is translation-invariant[1], such that $\partial_\nu \varphi \in \tilde{\mathbb{X}}$ for any unit vector $\nu \in \mathbb{R}^N$ and

$$\tilde{\mathcal{L}}(\partial_\nu \varphi) = f'(\varphi)\, \partial_\nu \varphi \quad \text{on } \mathbb{R}^N;$$

(H2) if $\varphi \in \mathbb{X}$ is a solution of (1.4), if $\{z_k\}_{k=1}^\infty$ is an arbitrary sequence of points in $\mathbb{R}^N$ (possibly unbounded) and if

$$\varphi_k := \varphi(\cdot + z_k) \quad \text{for any } k \in \mathbb{N},$$

then there exists a function $\varphi_0 \in \mathbb{X}$ such that, up to a sub-sequence,

$$\begin{aligned} &\lim_{k\to\infty} \varphi_k(x) = \varphi_0(x), \\ &\lim_{k\to\infty} \nabla\varphi_k(x) = \nabla\varphi_0(x) \\ \text{and} \quad &\lim_{k\to\infty} \mathcal{L}\varphi_k(x) = \varphi_0(x), \end{aligned}$$

for all $x \in \mathbb{R}^N$;

[1] A (non-void) set $V \subseteq C(\mathbb{R}^N)$ is *translation-invariant* if, for every function $\varphi \in V$ and every point $y \in \mathbb{R}^N$, the 'translated' function $x \mapsto \varphi(x + y)$ belongs to V.

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(H3) if $w \in \tilde{\mathbb{X}}$ satisfies $\tilde{\mathcal{L}}w + c(x)w = 0$ in $\mathbb{R}^N$, with

$$w(y,t) \geqslant 0 \text{ if } |t| \leqslant M \text{ and } c(y,t) \geqslant \kappa \text{ if } |t| \geqslant M$$

for some constants M, $\kappa > 0$, then

$$w(x) \geqslant 0 \qquad \text{for all } x \in \mathbb{R}^N;$$

(H4) if $\varphi \in \mathbb{X}$ and if $w \in \tilde{\mathbb{X}}$ satisfies $\tilde{\mathcal{L}}w = f'(\varphi)w$ in $\mathbb{R}^N$, then

$$\begin{cases} w \geqslant 0 \text{ in } \mathbb{R}^N, \\ w(0) = 0, \end{cases} \implies w \equiv 0 \text{ on } \mathbb{R}^N;$$

(H5) given $\mu_- < \mu_+ \in \mathbb{R}$, if $U \subseteq \mathbb{R}^N$ is an open set contained in

$$\mathcal{S} := \{x = (y,t) \in \mathbb{R}^N : t \leqslant \mu_- \text{ or } t \geqslant \mu_+\}$$

and if $v \in \mathbb{X}$ satisfies $\mathcal{L}v + c(x)v = 0$ in $\mathbb{R}^N$, with

$$v(x) \geqslant 0 \text{ in } \mathbb{R}^N \backslash U \text{ and } c(x) \geqslant \kappa \text{ on } U$$

for some constant $\kappa > 0$, then

$$v(x) \geqslant 0 \qquad \text{for all } x \in \mathbb{R}^N;$$

(H6) if $\varphi \in \mathbb{X}$ and if $v \in \mathbb{X}$ satisfies $\mathcal{L}v = f(v + \varphi) - f(v)$ in $\mathbb{R}^N$, then

$$\begin{cases} v \geqslant 0 \text{ in } \mathbb{R}^N, \\ v(0) = 0, \end{cases} \implies v \equiv 0 \text{ on } \mathbb{R}^N.$$

The next lemmata establish the validity of (H1)–(H6) in our setting.

LEMMA 3.2 Validity of (H1). *For every $\varphi \in \mathbb{X}$ and every unit vector $\nu \in \mathbb{R}^N$, one has*

$$\mathcal{L}(\partial_\nu \varphi) = \partial_\nu \big(\mathcal{L}\varphi\big). \tag{3.2}$$

In particular, assumption (H1) is fulfilled with the choices

$$\tilde{\mathcal{L}} := \mathcal{L} \tag{3.3}$$

and

$$\tilde{\mathbb{X}} := C^2(\mathbb{R}^N) \cap W^{3,\infty}(\mathbb{R}^N). \tag{3.4}$$

Proof. First of all, if $\mathbb{X}$ is as in (3.1) and $\tilde{\mathbb{X}}$ is as in (3.4), we obviously have that, for every $\varphi \in \mathbb{X}$ and every unit vector $\nu \in \mathbb{R}^N$,

$$\partial_\nu \varphi \in \tilde{\mathbb{X}} \quad \text{and} \quad -\Delta(\partial_\nu \varphi) = \partial_\nu(-\Delta\varphi).$$

Moreover, since $\mathbb{X} \subseteq W^{3,\infty}(\mathbb{R}^N)$, we can use formula (4.1) in [**38**], obtaining that

$$(-\Delta)^s(\partial_\nu \varphi) = \partial_\nu \big((-\Delta)^s \varphi\big).$$

Gathering together these facts, we obtain (3.2), as desired. As a result, with the choices in (3.3) and (3.4), assumption (H1) is obviously satisfied. □

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Remark 3.3. On account of remark 3.1, if $u \in \tilde{\mathbb{X}}$ it is possible to compute $\mathcal{L}u$ pointwise in $\mathbb{R}^N$. As a consequence, since in this section we shall always deal with functions belonging to $\tilde{\mathbb{X}}$ (or to $\mathbb{X} \subset \tilde{\mathbb{X}}$), the solvability of any equation involving $\mathcal{L}$ is *always meant in the pointwise sense.*

Lemma 3.4 Validity of (H2). *Let $\mathbb{X}$ be as in (3.1). Let $\varphi \in \mathbb{X}$ and $\{z_k\}_{k=1}^{\infty}$ be a sequence of points in $\mathbb{R}^N$ (possibly unbounded). Let also*

$$\varphi_k := \varphi(\cdot + z_k) \quad \textit{for any } k \in \mathbb{N}. \tag{3.5}$$

Then, there exists a function $\varphi_0 \in \mathbb{X}$ such that, up to a sub-sequence,

$$\lim_{k\to\infty} \varphi_k(x) = \varphi_0(x), \tag{3.6}$$

$$\lim_{k\to\infty} \nabla\varphi_k(x) = \nabla\varphi_0(x) \tag{3.7}$$

$$\textit{and} \quad \lim_{k\to\infty} \mathcal{L}\varphi_k(x) = \mathcal{L}\varphi_0(x), \tag{3.8}$$

for all $x \in \mathbb{R}^N$. In particular, assumption (H2) is fulfilled.

A less regular version of lemma 3.4 will be given in remark 3.5.

Proof of lemma 3.4. We observe that, since $\varphi \in \tilde{\mathbb{X}}$, the sequences

$$\{D^\alpha \varphi_k\}_{k=1}^{\infty}$$

are equi-continuous and equi-bounded on $\mathbb{R}^N$, for every multi-index $\alpha \in \mathbb{N}^N$ satisfying $0 \leqslant |\alpha| \leqslant 3$. As a consequence, Arzelà–Ascoli's Theorem ensures the existence of some function $\varphi_0 \in \mathbb{X}$ such that (up to a sub-sequence)

$$\lim_{k\to\infty} D^\alpha \varphi_k = D^\alpha \varphi_0 \quad \text{locally uniformly in } \mathbb{R}^N, \tag{3.9}$$

for every $\alpha \in \mathbb{N}^N$ with $|\alpha| \leqslant 3$. Hence, (3.6) and (3.7) plainly follows from (3.9). We also deduce from (3.9) that

$$\lim_{k\to\infty} \Delta\varphi_k(x) = \Delta\varphi_0(x) \quad \text{locally uniformly in } \mathbb{R}^N. \tag{3.10}$$

We now claim that

$$\lim_{k\to\infty} (-\Delta)^s \varphi_k(x) = (-\Delta)^s \varphi_0(x) \qquad \text{for every } x \in \mathbb{R}^N. \tag{3.11}$$

To prove it, for any $x \in \mathbb{R}^N$ and for any $k \in \mathbb{N}$, we set

$$\mathcal{I}_k(z) := \frac{\varphi_k(x+z) - \varphi_k(x-z) - 2\varphi_k(x)}{|z|^{N+2s}} \qquad \text{for any } z \neq 0.$$

On account of (3.9), we have that

$$\lim_{k\to\infty} \mathcal{I}_k(z) = \frac{\varphi_0(x+z) - \varphi_0(x-z) - 2\varphi_0(x)}{|z|^{N+2s}} \qquad \text{for all } z \neq 0. \tag{3.12}$$

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Moreover, recalling the definition of φ_k in (3.5), we see that, for every $z \neq 0$,

$$
\begin{aligned}
|\mathcal{I}_k(z)| &= \frac{\big|\varphi_k(x+z)+\varphi_k(x-z)| - 2\varphi_k(x)\big|}{|z|^{N+2s}} \\
&\leqslant \max_{|\alpha|=2} \|D^\alpha \varphi_k\|_{L^\infty(\mathbb{R}^N)} \frac{1}{|z|^{N+2s-2}} \chi_{\{0<|z|\leqslant 1\}} \\
&\quad + 4\|\varphi_k\|_{L^\infty(\mathbb{R}^N)} \frac{1}{|z|^{N+2s}} \chi_{\{|z|>1\}} \\
&= \max_{|\alpha|=2} \|D^\alpha \varphi\|_{L^\infty(\mathbb{R}^N)} \frac{1}{|z|^{N+2s-2}} \chi_{\{0<|z|\leqslant 1\}} \\
&\quad + 4\|\varphi\|_{L^\infty(\mathbb{R}^N)} \frac{1}{|z|^{N+2s}} \chi_{\{|z|>1\}}.
\end{aligned} \tag{3.13}
$$

Now, since $\varphi \in \mathbb{X}$, we have that

$$
\begin{aligned}
g(z) := &\max_{|\alpha|=2} \|D^\alpha \varphi\|_{L^\infty(\mathbb{R}^N)} \frac{1}{|z|^{N+2s-2}} \chi_{\{0<|z|\leqslant 1\}} \\
&+ 4\|\varphi\|_{L^\infty(\mathbb{R}^N)} \frac{1}{|z|^{N+2s}} \chi_{\{|z|>1\}} \in L^1(\mathbb{R}^N).
\end{aligned}
$$

From this, (3.12) and (3.13) we deduce that we can apply the Dominated Convergence Theorem to conclude that, for any $x \in \mathbb{R}^N$,

$$
\begin{aligned}
&\lim_{k\to\infty} \int_{\mathbb{R}^N} \frac{\varphi_k(x+z) - \varphi_k(x-z) - 2\varphi_k(x)}{|z|^{N+2s}} \, \mathrm{d}z \\
&\quad = \int_{\mathbb{R}^N} \frac{\varphi_0(x+z) - \varphi_0(x-z) - 2\varphi_0(x)}{|z|^{N+2s}} \, \mathrm{d}z.
\end{aligned}
$$

This proves (3.11). From (3.10) and (3.11), recalling (1.1), we obtain (3.8). □

REMARK 3.5. By taking a closer inspection to the proof of lemma 3.4, one can easily recognize that the following result holds: *if $\varphi \in \tilde{\mathbb{X}}$ and if $\{z_k\}_{k\in\mathbb{N}} \subseteq \mathbb{R}^N$, there exists a function $\varphi_0 \in \tilde{\mathbb{X}}$ such that, up to a sub-sequence,*

$$
\lim_{k\to\infty} \varphi_k(x) = \varphi_0(x), \qquad \lim_{k\to\infty} \nabla\varphi_k(x) = \nabla\varphi_0(x)
$$
$$
\text{and} \quad \lim_{k\to\infty} \mathcal{L}\varphi_k(x) = \mathcal{L}\varphi_0(x),
$$

for every $x \in \mathbb{R}^N$, where $\varphi_k := \varphi(\cdot + z_k)$.

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LEMMA 3.6 Validity of (H3) and (H5). *Let $\tilde{\mathbb{X}}$ be the space defined in (3.4). Moreover, let $c : \mathbb{R}^N \to \mathbb{R}$ be any function and let $w \in \tilde{\mathbb{X}}$ satisfy*

$$\mathcal{L}w + c(x)w = 0 \quad \textit{in } \mathbb{R}^N. \tag{3.14}$$

We assume that

$$w(x) \geqslant 0 \textit{ in } \mathbb{R}^N \setminus U \textit{ and } c(x) \geqslant \kappa \textit{ on } U \tag{3.15}$$

for some open set $U \subseteq \mathbb{R}^N$ and some constant $\kappa > 0$. Then

$$w(x) \geqslant 0 \qquad \textit{for all } x \in \mathbb{R}^N. \tag{3.16}$$

In particular, assumptions (H3) and (H5) are fulfilled with the choice in (3.3).

Proof. Arguing by contradiction, we suppose that $m := \inf_{\mathbb{R}^N} w < 0$, and we choose a sequence of points $\{z_k\}_{k=1}^{\infty}$ in $\mathbb{R}^N$ satisfying

$$\lim_{k\to\infty} w(z_k) = m. \tag{3.17}$$

Since $m < 0$, it is not restrictive to assume that

$$w(z_k) \leqslant \frac{m}{2} < 0 \qquad \text{for all } k \in \mathbb{N}. \tag{3.18}$$

As a consequence, also in light of (3.15), for every $k \in \mathbb{N}$ we have

$$z_k \in U \quad \text{and} \quad c(z_k) \geqslant \kappa > 0. \tag{3.19}$$

Now, thanks to (3.14), from (3.18) and (3.19) we deduce that

$$\mathcal{L}w(z_k) = -c(z_k)w(z_k) \geqslant -\frac{m\,\kappa}{2} > 0, \qquad \text{for all } k \in \mathbb{N}.$$

In particular, setting $w_k := w(\cdot + z_k)$, we obtain

$$\mathcal{L}w_k(0) \geqslant -\frac{m\,\kappa}{2} > 0, \qquad \text{for all } k \in \mathbb{N}. \tag{3.20}$$

On the other hand, since $w \in \tilde{\mathbb{X}}$, from remark 3.5 we infer the existence of some function $w_0 \in \tilde{\mathbb{X}}$ such that (up to a sub-sequence)

$$\lim_{k\to\infty} w_k(x) = w_0(x) \quad \text{and} \quad \lim_{k\to\infty} \mathcal{L}w_k(x) = \mathcal{L}w_0(x), \tag{3.21}$$

for every fixed $x \in \mathbb{R}^N$. By taking the limit as $k \to \infty$ in (3.20), we then get

$$\mathcal{L}w_0(0) \geqslant -\frac{m\,\kappa}{2} > 0. \tag{3.22}$$

Now, we observe that, on account of (3.17) and (3.21), one has

$$w_0(0) = \lim_{k\to\infty} w_k(0) = \lim_{k\to\infty} w(z_k) = m = \inf_{\mathbb{R}^N} w \leqslant w(x + z_k) = w_k(x),$$

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for every $x \in \mathbb{R}^N$ and every $k \in \mathbb{N}$. As a consequence,

$$w_0(0) \leqslant w_0(x) \qquad \text{for every } x \in \mathbb{R}^N,$$

and thus $x = 0$ is a minimum point for w_0 in $\mathbb{R}^N$. In particular,

$$\Delta w_0(0) \geqslant 0 \quad \text{and} \quad -(-\Delta)^s w_0(0) = \text{P.V.} \int_{\mathbb{R}^N} \frac{w_0(x) - w_0(0)}{|x|^{N+2s}}\, dx \geqslant 0.$$

Therefore, recalling (1.1), this implies that $\mathcal{L}w_0(0) \leqslant 0$, which is in contradiction with (3.22). This completes the proof of (3.16).

We point out that, with the choice in (3.3), from the first part of lemma 3.6 we obtain the validity of assumption (H3). Indeed, for this, it is enough to apply the first part of lemma 3.6 with

$$U := \big\{x = (y,t) \in \mathbb{R}^N \text{ s.t. } |t| \geqslant M\big\},$$

for some $M > 0$. Furthermore, from the first part of lemma 3.6 we also obtain the validity of assumption (H5), by simply observing that $\mathbb{X} \subset \tilde{\mathbb{X}}$. □

Lemma 3.7 Validity of (H4) and (H6). *Let $\tilde{\mathbb{X}}$ be as in (3.4). Let $c : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ be any function satisfying*

$$c(x,0) = 0 \qquad \textit{for every } x \in \mathbb{R}^N. \tag{3.23}$$

Let $w \in \tilde{\mathbb{X}}$ satisfy

$$\mathcal{L}w + c(x,w) = 0 \quad \textit{in } \mathbb{R}^N. \tag{3.24}$$

Then

$$\begin{cases} w \geqslant 0 \ \textit{in } \mathbb{R}^N, \\ w(0) = 0, \end{cases} \quad \Longrightarrow \ w \equiv 0 \ \textit{on } \mathbb{R}^N. \tag{3.25}$$

In particular, assumptions (H4) and (H6) are fulfilled with the choices in (3.3) and (3.4).

Proof. We observe that, thanks to the assumptions in (3.25), $x = 0$ is a minimum point for w in $\mathbb{R}^N$. As a consequence, we have that

$$\Delta w(0) \geqslant 0 \qquad \text{and} \qquad -(-\Delta)^s w(0) = \text{P.V.} \int_{\mathbb{R}^N} \frac{w(x)}{|x|^{N+2s}}\, dx \geqslant 0. \tag{3.26}$$

On the other hand, by (3.23) and (3.24), and recalling also that $w(0) = 0$, we get

$$0 = c(0,0) = c(0,w(0)) = -\mathcal{L}w(0) = \Delta w(0) - (-\Delta)^s w(0) \geqslant -(-\Delta)^s w(0).$$

Gathering together this and (3.26), we conclude that

$$0 = -(-\Delta)^s w(0) = \text{P.V.} \int_{\mathbb{R}^N} \frac{w(x)}{|x|^{N+2s}}\, dx.$$

Since $w \geqslant 0$ in $\mathbb{R}^N$, we deduce that $w \equiv 0$ on the whole of $\mathbb{R}^N$, which completes the proof of the claim in (3.25).

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Now, we check the validity of assumption (H4). For this, recalling (3.3) and (3.4), we take $\varphi \in \mathbb{X}$ and we define

$$c(x, w) := -f'(\varphi(x))\, w.$$

We observe that c satisfies (3.23). Hence, we can apply the first part of lemma 3.7 to obtain that (H4) is satisfied. Finally, in order to show the validity of assumption (H6), given $\varphi \in \mathbb{X}$, we define

$$c(x, w) := f(\varphi(x) + w) - f(\varphi(x)).$$

This function satisfies (3.23). As a consequence of this and of the inclusion $\mathbb{X} \subseteq \tilde{\mathbb{X}}$, we deduce (H6) from the first part of lemma 3.7. □

Thanks to these statements, we can now prove theorem 1.2:

Proof of theorem 1.2. On account of lemmata 3.2, 3.4, 3.6 and 3.7, we know that the assumptions in (H1)–(H6) are fulfilled in the setting of theorem 1.2. Moreover, since $u \in \mathbb{X}$, we have that

$$\|u\|_{C^{1,\beta}(\mathbb{R}^N)} \text{ is finite for all } \beta \in (0, 1).$$

From these considerations and (1.3), we have that the assumptions of theorem 1.1 in [**38**] are satisfied. Hence, from theorem 1.1 in [**38**] we have that there exists some function $u_0 : \mathbb{R} \to \mathbb{R}$ such that (1.5) holds true. □

Appendix A. Hopf-type Lemma

We provide here a variation of proposition 2.7 leading to a linear growth from the boundary for antisymmetric solutions which can be seen as a Hopf Lemma in this setting.

Lemma A.1. *Let $x_0 \in H$ and $d > \operatorname{dist}(x_0, \partial H)$. Let p_0 be the projection of x_0 onto ∂H. Let v be nonnegative in H and suppose that v is an antisymmetric supersolution v of $\mathcal{L}v = 0$ in $B_d(x_0) \cap H$.*

Then, either $v \equiv 0$ or

$$\liminf_{\varepsilon \searrow 0} \frac{v(p_0 + \varepsilon\nu) - v(p_0)}{\varepsilon} > 0,$$

where $\nu := (x_0 - p_0)/|x_0 - p_0|$.

Proof. Up to a rigid motion, we suppose that $H = (0, +\infty) \times \mathbb{R}^{N-1}$ and $x_0 = (t_0, 0, \ldots, 0)$ with $t_0 > 0$. In this way, we have that p_0 is the origin, $\nu = (1, 0, \ldots, 0)$ and

$$t_0 = \operatorname{dist}(x_0, \partial H) < d.$$

We suppose that $v \not\equiv 0$ (otherwise we are done) and we exploit proposition 2.7 to deduce that

$$\delta := \operatorname{ess\,inf}_{B_d(p_1)} v > 0, \tag{A.1}$$

where $p_1 := p_0 + 8d\nu = (8d, 0, \ldots, 0)$.

We let $\rho \in (0, ((d - t_0)/4N))$ and consider an even function $\psi \in C_0^\infty((-\rho, \rho))$ with $\psi = 1$ in $(-\rho/2, \rho/2)$. Using the notation $x = (x_1, \ldots, x_N)$, we set

$$g(x) = x_1\, \psi(x_1) \ldots \psi(x_N).$$

We remark that $g \in C_0^\infty((-\rho, \rho)^N)$ and, in particular,

$$\|\mathcal{L}g\|_{L^\infty(\mathbb{R}^N)} \leqslant \bar{C},$$

for some $\bar{C} \in (0, +\infty)$. Furthermore, since ψ is even, we see that g is antisymmetric. Now, for every $x \in \mathbb{R}^N$, we define

$$w(x) := v(x) - \frac{\delta}{a}\Big(g(x) + a\big(\chi_{B_d(p_1)}(x) - \chi_{B_d(p_1)}(\bar{x})\big)\Big),$$

where $a > 0$ is a constant to be conveniently chosen in what follows. We remark that w is antisymmetric. We claim that

$$(-\rho, \rho)^N \subseteq B_d(x_0). \tag{A.2}$$

Indeed, if $\zeta = (\zeta_1, \ldots, \zeta_N) \in (-\rho, \rho)^N$ we have that

$$\begin{aligned} |\zeta - x_0| &= |\zeta - (t_0, 0, \ldots, 0)| \leqslant |\zeta_1 - t_0| + |\zeta_2| + \cdots + |\zeta_N| \\ &\leqslant t_0 + N\rho < t_0 + \frac{d - t_0}{4} = d - \frac{3(d - t_0)}{4} < d, \end{aligned}$$

thus proving (A.2)

Accordingly, if $x \in H \backslash B_d(x_0)$, then x lies outside $(-\rho, \rho)^N$, due to (A.2), whence $g(x) = 0$. This says that if $x \in H \backslash B_d(x_0)$, then

$$w(x) = v(x) - \delta\Big(\chi_{B_d(p_1)}(x) - \chi_{B_d(p_1)}(\bar{x})\Big) = v(x) - \delta\chi_{B_d(p_1)}(x) \geqslant 0, \tag{A.3}$$

thanks to (A.1).

Furthermore, if $\varphi \in \mathcal{D}(B_d(x_0))$,

$$(\chi_{B_d(p_1)}(x) - \chi_{B_d(p_1)}(\bar{x}))\varphi(x) = (\chi_{B_d(p_1)}(x) - \chi_{B_d(p_1)}(\bar{x}))\varphi(x)\chi_{B_d(x_0)}(x) = 0$$

for every $x \in \mathbb{R}^N$ and, as a consequence,

$$\begin{aligned} &\frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{\big((\chi_{B_d(p_1)}(x) - \chi_{B_d(p_1)}(\bar{x})) - (\chi_{B_d(p_1)}(y) - \chi_{B_d(p_1)}(\bar{y}))\big)(\varphi(x) - \varphi(y))}{|x - y|^{N+2s}}\, \mathrm{d}x\, \mathrm{d}y \\ &\quad = -\frac{1}{2}\iint_{\mathbb{R}^{2N}} \frac{(\chi_{B_d(p_1)}(x) - \chi_{B_d(p_1)}(\bar{x}))\varphi(y) + (\chi_{B_d(p_1)}(y) - \chi_{B_d(p_1)}(\bar{y}))\varphi(x)}{|x - y|^{N+2s}}\, \mathrm{d}x\, \mathrm{d}y \end{aligned}$$

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$$
\begin{aligned}
&= -\iint_{\mathbb{R}^{2N}} \frac{(\chi_{B_d(p_1)}(y) - \chi_{B_d(p_1)}(\bar{y}))\varphi(x)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= -\iint_{B_d(x_0)\times\mathbb{R}^N} \frac{(\chi_{B_d(p_1)}(y) - \chi_{B_d(p_1)}(\bar{y}))\varphi(x)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= -\iint_{B_d(x_0)\times\mathbb{R}^N} \frac{\chi_{B_d(p_1)}(y)\varphi(x)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \iint_{B_d(x_0)\times\mathbb{R}^N} \frac{\chi_{B_d(p_1)}(\bar{y})\varphi(x)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&= -\iint_{B_d(x_0)\times\mathbb{R}^N} \frac{\chi_{B_d(p_1)}(y)\varphi(x)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \iint_{B_d(x_0)\times\mathbb{R}^N} \frac{\chi_{B_d(p_1)}(y)\varphi(x)}{|x-\bar{y}|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\leqslant -C_0 \int_{B_d(x_0)} \varphi(x)\,\mathrm{d}x,
\end{aligned}
$$

where

$$
C_0 := \inf_{x\in B_d(x_0)} \left(\int_{B_d(p_1)} \frac{\mathrm{d}y}{|x-y|^{N+2s}} - \int_{B_d(p_1)} \frac{\mathrm{d}y}{|x-\bar{y}|^{N+2s}} \right) \in (0,+\infty).
$$

As a result, in $U := B_d(x_0)\cap H$, we have that

$$
\mathcal{L}w = \mathcal{L}v - \frac{\delta}{a}\Big(\mathcal{L}g + a\mathcal{L}\big(\chi_{B_d(p_1)}(x) - \chi_{B_d(p_1)}(\bar{x})\big)\Big) \geqslant 0 - \frac{\delta}{a}\big(\bar{C} - aC_0\big) \geqslant 0,
$$

as long as $a \geqslant \bar{C}/C_0$. This and (A.3) entail that w is an antisymmetric supersolution of $\mathcal{L}w = 0$ in U and therefore, by proposition 2.7 (used here with $c := 0$), we deduce that $w \geqslant 0$. In particular, if $x \in B_d(x_0)\cap H$,

$$
0 \leqslant w(x) = v(x) - \frac{\delta}{a} g(x),
$$

thus, we conclude that

$$
\begin{aligned}
\liminf_{\varepsilon\searrow 0} \frac{v(p_0+\varepsilon\nu)}{\varepsilon} &\geqslant \frac{\delta}{a} \liminf_{\varepsilon\searrow 0} \frac{g(p_0+\varepsilon\nu)}{\varepsilon} = \frac{\delta}{a} \liminf_{\varepsilon\searrow 0} \frac{g(\varepsilon,0,\ldots,0)}{\varepsilon} \\
&= \frac{\delta}{a} \liminf_{\varepsilon\searrow 0} \frac{\varepsilon\,\psi(\varepsilon)}{\varepsilon} = \frac{\delta}{a},
\end{aligned}
$$

which yields the desired result. □

It would be interesting to obtain lemma A.1 with $d = \mathrm{dist}(x_0, \partial H)$. When $s \in (0, \frac{1}{2})$ this can be achieved by following the proof of lemma A.1 and replacing the function g used there with the solution of $\mathcal{L}g_0 = 1$ in $B_d(x_0)$ and $g_0 = 0$ outside $B_d(x_0)$, and then considering g as the antisymmetric extension of g_0 outside H (in this setting, the Lipschitz growth of g_0 that follows from the results in [**21**] suffices for having a bounded $\mathcal{L}g$ and the desired linear growth from the boundary of H).

Acknowledgements
We would like to express our gratitude to the anonymous referee for the careful reading of the paper and for his/her valuable suggestions and comments, leading

to an undisputed improvement of the manuscript. The authors are members of INdAM. S. Biagi is partially supported by the INdAM-GNAMPA project *Metodi topologici per problemi al contorno associati a certe classi di equazioni alle derivate parziali*. S. Dipierro and E. Valdinoci are members of AustMS and are supported by the Australian Research Council Discovery Project DP170104880 NEW 'Nonlocal Equations at Work'. S. Dipierro is supported by the Australian Research Council DECRA DE180100957 'PDEs, free boundaries and applications'. E. Valdinoci is supported by the Australian Laureate Fellowship FL190100081 'Minimal surfaces, free boundaries and partial differential equations'. E. Vecchi is partially supported by the INdAM-GNAMPA project *Convergenze variazionali per funzionali e operatori dipendenti da campi vettoriali*.

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