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JANTE'S LAW PROCESS

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Abstract

Consider the process which starts with $N \geq 3$ distinct points on $\mathbb{R}^d$, and fix a positive integer $K < N$. Of the total N points keep those $N - K$ which minimize the energy amongst all the possible subsets of size $N - K$, and then replace the removed points by K independent and identically distributed points sampled according to some fixed distribution ζ. Repeat this process ad infinitum. We obtain various quite nonrestrictive conditions under which the set of points converges to a certain limit. This is a very substantial generalization of the 'Keynesian beauty contest process' introduced in Grinfeld *et al.* (2015), where $K = 1$ and the distribution ζ was uniform on the unit cube.

Keywords: Keynesian beauty contest; rank-driven process; interacting particle system

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1. Introduction and auxiliary results

We study a generalization of the model presented in Grinfeld *et al.* [2]. Fix an integer $N \geq 3$ and some d-dimensional random variable ζ. Now arbitrarily choose N distinct points on $\mathbb{R}^d$, $d \geq 1$. The process in [2], called the Keynesian beauty contest process, is a discrete-time process with the following dynamics: given the configuration of N points we compute its centre of mass μ and discard the point most distant from μ; if there is more than one, we choose each one with equal probability. Then this point is replaced with a new point drawn independently each time from the distribution of ζ. In [2] it was shown that when ζ has a uniform distribution on a unit cube, then the configuration converges to some random point on $\mathbb{R}^d$, with the exception of the most distant point.

The aim of this paper is to remove the assumption on the uniformity of ζ and obtain some general sufficient conditions under which a similar convergence takes place. Additionally, it turns out that we can naturally generalize the process by removing not just one but $K \geq 2$ points at the same time, and then replacing them with K new independent and identically distributed (i.i.d.) points sampled from ζ. We also give the process we introduce a different name which we believe describes its essence much better. The 'Law of Jante' is the concept that describes a pattern of group behaviour towards individuals within Scandinavian countries that criticises individual success and achievement as unworthy and inappropriate, in other words, it is better to be 'like everyone else'. The concept was created by Sandemose [5], in which the author identified the Law of Jante as ten rules. This has been a very popular concept in Nordic countries since then.

We use mostly the same notation as in [2]. Namely, let $\mathcal{X}_n = (x_1, x_2, \ldots, x_n)$ denote a vector of n points $x_i \in \mathbb{R}^d$, and let $\mu_n(\mathcal{X}_n) = n^{-1} \sum_{i=1}^n x_i$ be the barycentre of $\mathcal{X}_n$.

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Denote by $\mathrm{ord}(\mathcal{X}_n) = (x_{(1)}, x_{(2)}, \dots, x_{(n)})$ the barycentric order statistics of $x_1, \dots, x_n$, so that

$$\|x_{(1)} - \mu_n(\mathcal{X}_n)\| \le \|x_{(2)} - \mu_n(\mathcal{X}_n)\| \le \cdots \le \|x_{(n)} - \mu_n(\mathcal{X}_n)\|.$$

Here and throughout, $\|x\|$ denotes the Euclidean norm in $\mathbb{R}^d$, $x \cdot y$ is a dot product of two vectors $x, y \in \mathbb{R}^d$, and $B_r(x) = \{y \in \mathbb{R}^d : \|y - x\| < r\}$ is an open ball of radius r centred at x. As in [2], we also define, for $\mathcal{X}_n = (x_1, x_2, \dots, x_n) \in \mathbb{R}^{dn}$,

$$\begin{aligned} G_n(\mathcal{X}_n) &= G_n(x_1, \dots, x_n) \\ &= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{i-1} \|x_i - x_j\|^2 \\ &= \sum_{i=1}^{n} \|x_i - \mu_n(\mathcal{X}_n)\|^2 \\ &= \inf_{y \in \mathbb{R}^d} \sum_{i=1}^{n} \|x_i - y\|^2. \end{aligned}$$

We can think of $G_n(\mathcal{X}_n)$ as a measure of the 'diversity' among individuals with properties $x_1, \dots, x_n$. In physics, G_n often corresponds to the moment of inertia; however, it can be viewed as 'the energy' from the perspective of potential theory. For simplicity, we use this term in the current paper.

In [2], where $K = 1$, the authors called $x_{(n)}$ the *extreme* point of $\mathcal{X}_n$, that is, a point of $x_1, \dots, x_n$ farthest from the barycentre, and defined the *core* of $\mathcal{X}_n$ as $\mathcal{X}_n' = (x_{(1)}, \dots, x_{(n-1)})$, the vector of $x_1, \dots, x_n$ with (one of) the extreme point removed. They also defined $F_n(\mathcal{X}_n) = G_{n-1}(\mathcal{X}_n')$ and $F(t) = F_N(\mathcal{X}(t))$.

In our paper, when $K \ge 1$, we redefine the core as the subset of $x_1, \dots, x_N$ containing $N - K$ elements which minimizes the diversity of the remaining individuals, that is, the subset which minimizes

$$\min_{\{y_1, \dots, y_{N-K}\} \subset \{x_1, \dots, x_N\}} G_{N-K}(y_1, \dots, y_{N-K}).$$

We will show below that, in fact, when $K = 1$ both definitions coincide.

The process runs as follows. Let $\mathcal{X}(t) = \{X_1(t), \dots, X_N(t)\}$ be distinct points in $\mathbb{R}^d$. Given $\mathcal{X}(t)$, let $\mathcal{X}'(t)$ be the core of $\mathcal{X}(t)$ and replace $\mathcal{X}(t) \setminus \mathcal{X}'(t)$ by K i.i.d. ζ-distributed random variables so that

$$\mathcal{X}(t+1) = \mathcal{X}'(t) \cup \{\zeta_{t+1;1}, \dots, \zeta_{t+1;K}\},$$

where $\zeta_{t;j}$, $t = 1, 2, \dots$, $j = 1, 2, \dots, K$, are i.i.d. random variables with a common distribution ζ. In the case where there is more than one element in the core, that is, a few configurations which minimize diversity, we chose any element with equal probability, precisely as in [2]. Now let $F(t) = G_{n-K}(\mathcal{X}'(t))$.

Finally, to complete the specification of the process, we allow the initial configuration $\mathcal{X}(0)$ to be arbitrary or random, with the only requirement that all the points of $\mathcal{X}(0)$ must lie in the support of ζ.

The following statement links the $K = 1$ case with the general $K \ge 1$.

Lemma 1. *If $K = 1$ then the only point not in the core is the one which is furthermost from the centre of mass of $\mathcal{X}$.*

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Proof. Let $\mathcal{X} = (x_1, \dots, x_N)$. Without loss of generality (w.l.o.g.) assume that $\sum_{i=1}^N x_i = 0 \in \mathbb{R}^d$ and, thus, the centre of mass of $\mathcal{X}$ is located at 0. Here, L consists of all subsets of $\{1, \dots, N\}$ containing just one element. If we discard the lth point, denoted by $\mu_l = 1/(N-1)\sum_{i\neq l} x_i = -x_l/(N-1)$, we obtain

$$\begin{aligned}
G(l, \mathcal{X}) &= \sum_{i=1}^N \|x_i - \mu_l\|^2 - \|x_l - \mu_l\|^2 \\
&= \sum_{i=1}^N \|x_i\|^2 + N\|\mu_l\|^2 - 2\mu_l \cdot \sum_{i=1}^N x_i - \|x_l - \mu_l\|^2 \\
&= \sum_{i=1}^N \|x_i\|^2 + N\frac{\|x_l\|^2}{(N-1)^2} - \frac{\|x_l N\|^2}{(N-1)^2} \\
&= -\|x_l\|^2 \frac{N}{(N-1)^2} + \sum_{i=1}^N \|x_i\|^2.
\end{aligned}$$

Therefore, the minimum of $G(l, \mathcal{X})$ is achieved by choosing an x_l with the largest $\|x_l\|$, that is, the furthermost from the centre of mass. □

Corollary 1. *If $K = 1$, Jante's law process coincides with the process studied in [2].*

The following statement is a trivial consequence of the definition of F.

Lemma 2. *For any $1 \le K \le N-2$ and any distribution of ζ, we have $F(t+1) \le F(t)$.*

In the $K = 1$ case, the above statement coincides with Corollary 2.1 of [2].

Remark 1. It is worth noting that discarding $\mathcal{X}^*$, in general, does not mean necessarily discarding the K furthest points from the centre of mass of $\mathcal{X}$, unlike in the $K = 1$ case. For example, let $d = 1$, $N = 5$, $K = 3$, and set $\mathcal{X} = (-24, -19, -14, 28, 29)$. Then the centre of mass is at $\mu = 0$ and, thus, points 28 and 29 have the largest and the second largest distance from μ, while it is clear that the energy is minimized by keeping exactly these two points in the core and discarding the rest.

Finally, define *the range* of the configuration: for $n \ge 2$ and $x_1, \dots, x_n \in \mathbb{R}^d$, write

$$D_n(x_1, \dots, x_n) = \max_{1\le i,j\le n} \|x_i - x_j\|.$$

The following statement is taken from [2, Lemma 2.2].

Lemma 3. *Let $n \ge 2$ and $x_1, \dots, x_n \in \mathbb{R}^d$. Then*

$$\tfrac{1}{2} D_n(x_1, \dots, x_n)^2 \le G_n(x_1, \dots, x_n) \le \tfrac{1}{2}(n-1) D_n(x_1, \dots, x_n)^2.$$

Let $D(t) = D_{N-K}(\mathcal{X}'(t))$. From Lemma 3, we have

$$\sqrt{\frac{2}{N-K-1} F(t)} \le D(t) \le \sqrt{2F(t)}. \tag{1}$$

From Lemmas 2 and 3, it also follows immediately that

$$D(t+1) \le \sqrt{2F(t)} \le D(t)\sqrt{N-K-1}. \tag{2}$$

In addition, let $\mu'(t) = \mu_{N-K}(\mathcal{X}'(t))$ be the centre of mass of the core.

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Assumption 1. *We assume that* $2K < N$.

Observe that if Assumption 1 is not fulfilled then all the points of the core can migrate large distances and that $F = 0$ does not necessarily imply that the configuration stops moving. For example, one can take $N = 4$, $K = 2$, and $\zeta \sim \text{Bernoulli}(p)$ to see that the core jumps from 0 to 1 and back infinitely often almost surely (a.s.).

In the other case, the new core must contain at least one point of the old core, and in the following lemma we show that if newly sampled points are far from the core, they immediately get rejected.

Lemma 4. *Under Assumption 1, if all the distances between* K *newly sampled points and the points of the core are more than* $C = D\sqrt{N - K - 1}$, *then* $\mathcal{X}'(t + 1) = \mathcal{X}'(t)$.

Proof. Since $N - 2K \geq 1$, the new core $\mathcal{X}'(t + 1)$ must contain at least one point of the old core $\mathcal{X}'(t)$. By (2), $D(t + 1) \leq D(t)\sqrt{N - K - 1}$ and, therefore, if one of the new points is in the new core, it should be no further than $D(t)\sqrt{N - K - 1}$ from one of the points of the old core. □

Finally, we use the following notation. For any two sets $A, B \subset \mathbb{R}^d$, let

$$\text{dist}(A, B) = \inf_{x \in A,\, y \in B} \|x - y\|.$$

We write $\mathcal{X}'(t) \to \infty$ if $\min\{\|x\|,\ x \in \mathcal{X}'(t)\} = \text{dist}(\mathcal{X}'(t), 0) \to \infty$, otherwise we write $\mathcal{X}'(t) \not\to \infty$. Observe that the 'convergence to infinity' is equivalent to the Kuratowski convergence (or convergence in Fell topology) to the empty set. We will also write $\mathcal{X}'(t) \to \phi \in \mathbb{R}^d$ if all the elements of the set of $\mathcal{X}'(t)$ converge to ϕ as $t \to \infty$.

The rest of the paper is organized as follows. First, in Section 2, we show that a.s. $F(t) \to 0$ or $\mathcal{X}'(t)$ goes to ∞ (Theorem 1). Next, in Section 3, we show that under some conditions either all elements of $\mathcal{X}'(t)$ converge to a point, or $\mathcal{X}'(t) \to \infty$ (Theorem 2). In Section 4 we deal with the case $d = 1$ and $K = 1$, where we obtain, in particular, that $\mathcal{X}'(t)$ converges a.s. to a finite point for many distributions, as well as strengthen Theorem 2 (see Theorems 3 and 4).

2. Shrinking

Let ζ be *any* random variable on $\mathbb{R}^d$. As usual, define the support of this random variable as

$$\text{supp}\,\zeta = \{A \in \mathbb{R}^d : \mathbb{P}(\zeta \in A) > 0\} = \{x \in \mathbb{R}^d : \text{for all } \varepsilon > 0, \mathbb{P}\,(\zeta \in B_\varepsilon(x)) > 0\};$$

see, for example, [4]. It turns out that the following statement, which is probably known, is true.

Proposition 1. *It holds that* $\text{supp}\,\zeta$ *is bounded if and only if there exists some function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *such that, for any* $x \in \text{supp}\,\zeta$,

$$\mathbb{P}(\zeta \in B_\delta(x)) \geq f(\delta) \quad \textit{for all } \delta > 0.$$

Proof. Suppose that such a function exists, but the support of ζ is not bounded. Fix any $\Delta > 0$. Then there must exist a infinite sequence of points $\{x_n\}_{n=1}^\infty \subseteq \text{supp}\,\zeta$, such that $\|x_i - x_j\| > 2\Delta$, whenever $i \neq j$. Since the sets $\{B_\Delta(x_n)\}$ are disjoint, this would imply that

$$\mathbb{P}(\zeta \in \mathbb{R}^d) \geq \mathbb{P}\bigg(\bigcup_{n=1}^\infty \{\zeta \in B_\Delta(x_n)\}\bigg) \geq \sum_{n=1}^\infty f(\Delta) = \infty,$$

which is impossible.

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Conversely, assume that $\operatorname{supp}\zeta$ is bounded. For all $\delta > 0$, define

$$f(\delta) = \inf_{x \in \operatorname{supp}\zeta} \mathbb{P}(\|\zeta - x\| \le \delta).$$

We will show that $f(\delta) > 0$. Indeed, if not, there exists a sequence $\{x_n\}$ such that $\mathbb{P}(\|\zeta - x_n\| \le \delta) \to 0$ as $n \to \infty$. Since the support of ζ is compact, $\{x_n\}$ must have a convergent subsequence; w.l.o.g. we can assume that it is $\{x_n\}$ itself and, thus, there is an x such that $x_n \to x$, and there exists N such that $\|x_n - x\| < \delta/2$ for all $n \ge N$. On the other hand, for these n,

$$\mathbb{P}\bigg(\|\zeta - x\| \le \frac{\delta}{2}\bigg) \le \mathbb{P}(\|\zeta - x_n\| \le \delta)$$

by the triangle inequality. Since the right-hand side converges to 0, this implies $\mathbb{P}(\|\zeta - x\| \le \delta/2) = 0$ so $x \notin \operatorname{supp}\zeta$, which contradicts the fact that $x = \lim_{n\to\infty} x_n \in \operatorname{supp}\zeta$ by the definition of the support. □

Theorem 1. *Given any distribution ζ on $\mathbb{R}^d$, for any $N \ge 3$ and $1 \le K \le N-2$, we have*

$$\mathbb{P}(\{F(t) \to 0\} \cup \{\mathcal{X}'(t) \to \infty\}) = 1.$$

In particular, if ζ has compact support then $F(t) \to 0$ a.s.

Note that $F(t) \to 0$ is equivalent to $D(t) \to 0$.

Proof of Theorem 1. We first make use of the following lemma.

Lemma 5. *Suppose we are given a bounded set $S \in \mathbb{R}^d$ such that $\mathbb{P}(\zeta \in S) > 0$ and $N-K$ points $x_1, \dots, x_{N-K}$ in $(\operatorname{supp}\zeta) \cap S$ satisfying $F(\{x_1, \dots, x_{N-K}\}) > \varepsilon_1$. Let $\varepsilon_2 = \varepsilon_1/2(N-K)^2$. Then there exists a positive constant σ, depending only on ε_1, S, K, and N, such that*

$$\mathbb{P}(F(\{\zeta_1, \dots, \zeta_K, x_1, \dots, x_{N-K}\}') < F(\{x_1, \dots, x_{N-K}\}) - \varepsilon_2) \ge \sigma.$$

Proof. We start with the $K = 1$ case. Denote

$$D = \max_{1 \le i, j \le N-K} \|x_i - x_j\| \quad \text{and} \quad S_* = \{x \colon \operatorname{dist}(x, S) < D\sqrt{N-K-1}\},$$

then the set $\overline{S_*}$ is a compact set such that $\{\zeta, x_1, \dots, x_{N-1}\}' \in \overline{S_*}$ regardless of where the point ζ is sampled, using Lemma 4. Since $\overline{S_*}$ is compact, it follows from Proposition 1 applied to $\zeta \cdot \mathbf{1}_{\{\zeta \in S\}}$ that there is an $f \colon \mathbb{R}^+ \to \mathbb{R}^+$ such that, for any $x \in \operatorname{supp}\zeta \cap \overline{S_*}$, we have $\mathbb{P}(\zeta \in B_\delta(x)) \ge f(\delta)$. Assume that the core centre of mass is $\mu' = 0$, and that (w.l.o.g.) $\|x_1\| \ge \|x_l\|$ for all $1 \le l \le N-1$. Let $\mu' = (y + x_2 + \dots + x_{N-1})/(N-1)$ and consider the function

$$h(y) = \sum_{i=2}^{N-1} \|x_i - \mu'\|^2 + \|y - \mu'\|^2,$$

continuous in y. Pick a point x_j from $\{x_2, \dots, x_{N-1}\}$ such that $\|x_1 - x_j\| \ge D/2$, otherwise $\|x_i - x_j\| \le \|x_1 - x_j\| + \|x_1 - x_i\| < D$ for all indices i, j contradicting the definition of D.

Consider the configuration $\{x_j, x_2, \dots, x_{N-1}\}$, where we have removed the point x_1 and replaced it with x_j. This configuration has centre of mass $\mu' = (x_2+\cdots+x_{N-1}+x_j)/(N-1) = (x_j - x_1)/(N-1)$. The Lyapunov function evaluated for this configuration is precisely $h(x_j)$. Denote $F_{\text{old}} = F(\{x_1, \dots, x_{N-1}\})$. Then

$$
\begin{aligned}
h(x_j) &= \sum_{i=2}^{N-1} \|x_i - \mu'\|^2 + \|x_j - \mu'\|^2 \\
&= \sum_{i=1}^{N-1} \|x_i - \mu'\|^2 + \|x_j - \mu'\|^2 - \|x_1 - \mu'\|^2 \\
&= \sum_{i=1}^{N-1} (\|x_i\|^2 + \|\mu'\|^2 - 2x_i\mu') + \|x_j\|^2 + \|\mu'\|^2 - 2x_j \cdot \mu' - \|x_1\|^2 - \|\mu'\|^2 \\
&\quad + 2x_1\mu' \\
&= \sum_{i=1}^{N-1} \|x_i\|^2 + (N-1)\|\mu'\|^2 + \|x_j\|^2 - \|x_1\|^2 - 2(x_j - x_1) \cdot \left(\frac{x_j - x_1}{N-1}\right) \\
&\le F_{\text{old}} + \frac{\|x_j - x_1\|^2}{N-1} - 2\frac{\|x_j - x_1\|^2}{N-1} \\
&\le F_{\text{old}} - \frac{D^2}{4(N-1)} \\
&\le \left(1 - \frac{1}{2(N-1)^2}\right) F_{\text{old}},
\end{aligned}
$$

where the last inequality follows from (1). Hence, for some $\delta > 0$, if $\|y - x_j\| \le \delta$ then $h(y) < (1 - 1/4(N-1)^2)F_{\text{old}}$. So if ζ is sampled in $B_\delta(x_j)$ then we have a substantial decrease and this is with probability bounded below by $f(\delta)$, the result is thus proved for the $K = 1$ case with $\sigma = f(\delta)$.

The general case can be reduced to the $K = 1$ case as follows. Set $N' = N - K + 1$ and replace all N by N' in the arguments above. The decrease of F in this case will be at least $\varepsilon_2(N')$. Indeed, since, if at least one particle falls in the ball $\{y \colon \|y - x_j\| \le \delta\}$, we could choose the sub-configuration, where exactly one point falls in this ball while x_1 is removed, and since we are taking the minimum over all available configurations, the decrease has to be greater than or equal to for this specific choice. □

Assume that $\mathbb{P}(\mathcal{X}'(t) \to \infty) < 1$, otherwise Theorem 1 follows immediately. Recall that $B_r(0)$ is a ball of radius r centred at the origin and note that

$$
\{\mathcal{X}'(t) \not\to \infty\} = \bigcup_{r=1}^{\infty} \{\mathcal{X}'(t) \in B_r(0) \text{ i.o.}\} = \bigcup_{r=1}^{\infty} G_r, \tag{3}
$$

where

$$
G_r = \bigcap_{k \ge 0} \{\tau_{k,r} < \infty\}, \qquad \tau_{k,r} = \inf\{t \colon t > \tau_{k-1,r}, \mathcal{X}'(t) \in B_r(0)\}, \quad k = 1, 2, \dots,
$$

with the convention that $\tau_{0,r} = 0$, $\inf \varnothing = +\infty$, and that if $\tau_{k,r} = +\infty$ then $\tau_{k',r} = +\infty$ for all $k' \ge k$. We use the abbreviation 'i.o.' for infinitely often.

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By the monotonicity of F, we have $F(t) \downarrow F_\infty \geq 0$. We show that, in fact,

$$\mathbb{P}(\{\mathcal{X}'(t) \not\to \infty\} \cap \{F_\infty > 0\}) = 0, \tag{4}$$

which is equivalent to the statement of the theorem.

Let n_0 be some integer larger than $4(N-K)^2$, this quantity being related to ε_2 from Lemma 5. Since

$$\{F_\infty > 0\} = \bigcup_{n=n_0}^{\infty} \left\{F_\infty > \frac{1}{n}\right\} = \bigcup_{n=n_0}^{\infty} \bigcup_{m=0}^{\infty} \{F_\infty \in I_{n,m}\},$$

where $I_{n,m} = [1/n + M/n^2, 1/n + (m+1)/n^2)$ are disjoint sets for each fixed n. Consequently, taking into account (3), to establish (4) it suffices to show that, for each fixed n, m, and r,

$$\mathbb{P}(G_r \cap \{F_\infty \in I_{n,m}\}) = 0.$$

Let $A_k = \{F(\tau_{k,r} + 1) \in I_{n,m}\} \cap \{\tau_{k,r} < \infty\}$. Then, obviously,

$$G_r \cap \{F_\infty \in I_{n,m}\} \subset \bigcup_{k_0 \geq 0} \bigcap_{k \geq k_0} A_k. \tag{5}$$

We will now show that, for all k_0, $\mathbb{P}(\bigcap_{k \geq k_0} A_k) = 0$, which will imply that the probability of the right-hand side and, hence, that of the left-hand side of (5) is 0. Indeed, for any positive integer L,

$$\mathbb{P}\biggl(\bigcap_{k \geq k_0} A_k\biggr) \leq \mathbb{P}\biggl(\bigcap_{k=k_0}^{k_0+L} A_k\biggr) = \mathbb{P}(A_{k_0}) \prod_{k=k_0+1}^{t_0+L} \mathbb{P}\biggl(A_k \biggm| \bigcap_{s=k_0}^{k-1} A_s\biggr).$$

We now proceed to calculate the conditional probabilities, $\mathbb{P}(A_k \mid \bigcap_{s=k_0}^{k-1} A_s)$. Setting $\varepsilon_1 = 1/n$, letting S be the ball of radius $\sqrt{2(1/n + (m+1)/n^2)}(1 + \sqrt{N-K-1})$ centred at 0 in Lemma 5, and using the bound (1), we obtain

$$\varepsilon_2 = \frac{\varepsilon_1}{4(N-K)^2} = \frac{1}{4n(N-K)^2} > \frac{1}{n^2}$$

and, thus, with probability at least σ, given by Lemma 5, F will exit $I_{n,m}$, that is,

$$\mathbb{P}(F(\tau_{k,r} + 1) \in I_{n,m} \mid F(\tau_{k_0,r} + 1), F(\tau_{k_0+1,r} + 1), \dots, F(\tau_{k-1,r} + 1) \in I_{n,m}, \tau_{k,k} < \infty)$$
$$\leq 1 - \sigma,$$

since $\zeta_{\tau_{k,r}+1;\,j}$ are all independent from $\mathcal{F}_{\tau_{k,r}}$ for $1 \leq j \leq K$.

From this we can conclude that $\mathbb{P}(A_k \mid \bigcap_{s=k_0}^{k-1} A_s) \leq 1 - \sigma$ yielding $\mathbb{P}(\bigcap_{k \geq k_0} A_k) \leq (1-\sigma)^L$ for all $L \geq 1$. Letting $L \to \infty$, we see that $\mathbb{P}(\bigcap_{k \geq k_0} A_k) = 0$ which, in turn, proves (4). □

Corollary 2. *Suppose that Assumption 1 holds, $d = 1$, and ζ has a support which is nowhere dense. Then*

$$\mathbb{P}(\{\mathcal{X}'(t) \to \phi \text{ for some } \phi\} \cup \{\mathcal{X}'(t) \to \infty\}) = 1.$$

Proof. Assume that $\mathcal{X}'(t) \not\to \infty$ occurs and for $a < b$ define

$$E_{a,b} = \Big\{\liminf_{t\to\infty} x_{(k)}(t) < a\Big\} \cap \Big\{\limsup_{t\to\infty} x_{(k)}(t) > b\Big\},$$

where $k \in \{1, 2, \dots, N-K\}$ and $x_{(k)}$ is the kth point of the core. By Theorem 1, $F(t) \to 0$, implying, in turn, that $D(t) \to 0$ and, hence, by Lemma 4,

$$\operatorname{dist}(\mathcal{X}'(t), \mathcal{X}'(t+1)) = \max_{1\le i,\, j\le N-K} |x_{(i)}(t) - x_{(j)}(t+1)| \to 0 \quad \text{as } t \to \infty. \tag{6}$$

Since $\operatorname{supp}\zeta$ is nowhere dense, there exist $x \in (a, b)$ and $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subseteq (\operatorname{supp}\zeta)^c$. However,

$$E_{a,b} \subseteq \operatorname{dist}(\mathcal{X}'(t), \mathcal{X}'(t+1)) > 2\varepsilon \text{ i.o.}\},$$

implying, from (6), that $\mathbb{P}(E_{a,b}) = 0$. Since this holds for all a and b, $\mathcal{X}'(t)$ must converge. This completes the proof. □

3. Convergence of the core

Definition 1. A subset $A \subseteq \operatorname{supp}\zeta$ is regular with parameters $\delta_A \in (0, 1), \sigma_A > 0$, and $r_A > 0$ if

$$\mathbb{P}(\zeta \in B_{r\delta_A}(x) \mid \zeta \in B_r(x)) \ge \sigma_A \quad \text{for any } x \in A,\ r \le r_A. \tag{7}$$

Assumption 2. *For any $x \in \operatorname{supp}\zeta$, there exists some $\gamma = \gamma(x)$ such that the set $B_\gamma(x) \cap (\operatorname{supp}\zeta)$ is regular.*

Remark 2. We can iterate (7) in order to establish

$$\mathbb{P}(\zeta \in B_{r\delta_A^k}(x) \mid \zeta \in B_r(x)) \ge \sigma_A^k, \qquad k \ge 2.$$

Hence, it is not difficult to check that if Definition 1 holds for some $\delta_A \in (0, 1)$ it holds for all $\delta \in (0, 1)$, albeit possibly with a smaller σ_A.

Lemma 6. *Under Assumption 2, for any compact subset $A \subset \operatorname{supp}\zeta$ and $\delta \in (0, 1)$, there exist r_A and σ_A such that A is regular with parameters δ, σ_A, and r_A.*

Proof. The union $\bigcup_{x\in A} B_{\gamma(x)}(x)$ is an open covering of A, where $B_{\gamma_x}(x)$ is the regular ball centred in x using Assumption 2. Since A is compact, it follows that there is a finite subcover of A. In other words, there exist sequences

$$\{x_k\}_{k=1}^M \subseteq A, \qquad \left.\begin{matrix}\{\sigma_k\}_{k=1}^M \\ \{r_k\}_{k=1}^M \\ \{\delta_k\}_{k=1}^M \\ \{\gamma_k\}_{k=1}^M\end{matrix}\right\} \subseteq \mathbb{R}^+$$

such that $A \subseteq \bigcup_{k=1}^M B_{\gamma_k}(x_k)$, and $\mathbb{P}(\zeta \in B_{r\delta_k}(x) \mid \zeta \in B_r(x)) \ge \sigma_k$ for $x \in B_{\gamma_k}(x_k)$ and $r \le r_k$. Now let $\sigma' = \min_{1\le k\le M} \sigma_k$, $\delta' = \max_{1\le k\le M} \delta_k$, and $r' = \min_{1\le k\le M} r_k$. It follows that, for any $x \in A$,

$$\mathbb{P}(\zeta \in B_{r\delta'}(x) \mid \zeta \in B_r(x)) \ge \sigma',$$

when $r \le r'$. We conclude by noting that, by Remark 2, there exists σ_A such that, for each $x \in A$,

$$\mathbb{P}(\zeta \in B_{r\delta}(x) \mid \zeta \in B_r(x)) \ge \sigma_A. \qquad \square$$

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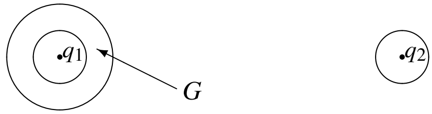

FIGURE 1: The shell G.

Theorem 2. *Under Assumptions 1 and 2,*

$$\mathbb{P}(\{\mathcal{X}'(t) \to \phi \text{ for some } \phi \in \mathbb{R}^d\} \cup \{\mathcal{X}'(t) \to \infty\}) = 1.$$

Proof. First, $\mathbb{P}(\{\text{there exists } \lim_t \mu'(t)\} \triangle \{\mathcal{X}'(t) \to \phi \text{ for some } \phi\}) = 0$, since, if $\mu'(t)$ converges, then $\mathcal{X}'(t) \not\to \infty$, which implies $D(t) \to 0$ by Theorem 1, yielding convergence of the core to the same point.

From elementary calculus, it follows that if neither the centre of mass converges to a finite point nor the configurations goes to ∞, then there must exist two arbitrarily small nonoverlapping balls (w.l.o.g. centred at rational points) which are visited by μ' i.o., that is,

$$\Big\{\lim_t \mu'(t) \text{ does not exist}\Big\} \cap \{\mathcal{X}'(t) \not\to \infty\} = \bigcup_{n=1}^{\infty} \bigcup_{q_1,q_2 \in \mathbb{Q}^d,\ \|q_1-q_2\| \geq 6/n} E_{q_1,q_2,n}, \tag{8}$$

where

$$E_{q_1,q_2,n} = \{\mu'(t) \in B_{2/n}(q_1) \text{ i.o.}\} \cap \{\mu'(t) \in B_{2/n}(q_2) \text{ i.o.}\}.$$

To show (8), note that $\{\lim_t \mu'(t) \text{ does not exist}\} \cap \{\mathcal{X}'(t) \not\to \infty\}$ is equivalent to the existence of at least two distinct points in the set of accumulation points of $\{\mu'(t)\}_{t=1}^{\infty}$, say x_1 and x_2. Now take $q_1, q_2 \in \mathbb{Q}^d$ such that $\|q_j - x_j\| \leq 1/n,\ j = 1, 2$, then $\mu' \in B_{1/n}(x_j) \subseteq B_{2/n}(q_j),\ j = 1, 2$, i.o.; moreover, $\|q_1 - q_2\| \geq 8/n - 1/n - 1/n = 6/n$, as required. Thus, it suffices to prove that $\mathbb{P}(E_{q_1,q_2,n}) = 0$ for all $n \in \mathbb{N}$ and $q_1, q_2 \in \mathbb{Q}^d$ such that $\|q_1 - q_2\| \geq 6/n$ in order to show that the left-hand side of (8) has measure zero, and then the theorem will follow.

For simplicity, w.l.o.g. assume that $q_1 = 0$ and denote $E = E_{0,q_2,n}$, $R = 2/n$, and $G = (\operatorname{supp} \zeta) \cap (B_{2R}(0) \setminus B_R(0))$. Since every path from $B_{2/n}(0)$ to $B_{2/n}(q_2)$ must cross G, on E the shell G must be crossed i.o. (by this we mean that $\|\mu'(t)\| > 2R$ i.o. and $\|\mu'(t)\| < R$ i.o.); see Figure 1.

Since $\mathcal{X}'(t) \not\to \infty$ on E, it follows from Theorem 1 that $F(t) \to 0$ a.s. on E and therefore, additionally, $\mathcal{X}'(t) \subset G$ i.o. (the core points cannot jump over the set G once the spread is sufficiently small); moreover, the set G is regular by Lemma 6. We also have the following result.

Lemma 7. *Under Assumption 2, given $N - K$ points $x_1, \ldots, x_{N-K}$ in G, there are constants $a, \sigma \in (0, 1)$ depending on N, K, and σ_G only, such that*

$$\mathbb{P}(\{F(\{\zeta_1, \ldots, \zeta_K, x_1, \ldots, x_{N-K}\}') \leq aF(\{x_1, \ldots, x_{N-K}\})\}) \geq \sigma.$$

Remark 3. Note the similarity of this statement with Lemma 5; the difference here, however, comes from the fact that the probability of decay σ does not depend on the value of F, thanks to Assumption 2.

Proof. We start with the $K = 1$ case. Due to the translation invariance of F we can assume w.l.o.g. that $\sum_{i=1}^{N-1} x_i = 0$. Let $D = \max_{i,j \in \{1,\ldots,N-1\}} \|x_i - x_j\|$, assume, additionally,

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that $\|x_1\| \geq \|x_k\|$ for all k, and take x_j such that $\|x_1 - x_j\| \geq D/2$. Let

$$\mu' = \frac{x_2 + \cdots + x_{N-1} + \zeta}{N-1} = \frac{\zeta - x_1}{N-1} \quad \text{and} \quad F_{\text{old}} = F(\{x_1, \ldots, x_{N-1}\}).$$

If we take $\zeta \in B_{(1/8)\sqrt{F_{\text{old}}/N}}(x_1)$ then

$$\|\zeta - x_1\| \geq \|x_1 - x_j\| - \|\zeta - x_j\| \geq \frac{D}{2} - \frac{1}{8}\sqrt{\frac{F_{\text{old}}}{N}}.$$

From this we can deduce that $\|\zeta - x_1\|^2 \geq D^2/8 \geq F_{\text{old}}/4(N-1)$ for some fixed $a \in (0, 1)$ (which is only a function of N and K). By Lemma 4, the event $\{\zeta \notin B_{H\sqrt{2F_{\text{old}}}}(x_j)\}$, where $H = \sqrt{N-K-1}$, implies that $\{\zeta_1, x_1, \ldots, x_{N-1}\}' = \{x_1, \ldots, x_{N-1}\}$ (that is, ζ is eliminated) and by Lemma 6 we can assume that δ and σ are chosen such that

$$\mathbb{P}(\zeta \in B_{(1/8)\sqrt{F_{\text{old}}N}}(x_j) \mid \zeta \in B_{H\sqrt{2F_{\text{old}}}}(x_j)) \geq \sigma.$$

Skipping the first few steps that are identical to those in Lemma 5, we obtain the following bound:

$$F(\{\zeta, x_2, \ldots, x_{N-K}\}) = \sum_{i=2}^{N-1} \|x_i - \mu'\|^2 + \|\zeta - \mu'\|^2 \leq \left(1 - \frac{1}{4(N-1)^2}\right) F_{\text{old}}.$$

Since $F(\{\zeta, x_2, \ldots, x_{N-K}\}) < F_{\text{old}}$, one of the points $x_1, \ldots, x_{N-1}$ must be discarded. Thus, in the $K = 1$ case, we can pick $a = 1 - 1/4(N-1)^2$. For general K, one can make an argument analogous to the one made at the end of the proof of Lemma 5. □

We return to the proof of Theorem 2. Define, for $t \geq 0$,

$$\eta(t) = \inf\{s \geq t + 1 \colon \mathcal{X}'(s) \neq \mathcal{X}'(s-1) \text{ or } F(s) = 0\}.$$

(Note that, by definition, if $F(\eta(t)) = 0$, that is, all the points of the core have converged to a single point, then $\eta(t+1) = \eta(t) + 1$. So from now on we assume that this is not the case.) Fix some large $M \geq 5$ such that

$$a^{\sigma M} \leq \tfrac{1}{16},$$

define $\tau_0 = \tau_0^{(M)}$ such that

$$\mathcal{X}'(\tau_0) \subseteq B_{7R/4}(0) \setminus B_{5R/4}(0), \qquad F(\tau_0) \leq \frac{R^2}{M^2 4^M},$$

and set $\tau_i = \eta(\tau_{i-1})$, $i = 1, 2, \ldots$ (that is, the next time the core changes). Since $F(t) \to 0$ on E, and we cross G i.o., we must visit the region $B_{7R/4}(0) \setminus B_{5R/4}(0)$ i.o. as well, therefore, $E \subseteq A_M = \{\tau_0^{(M)} < \infty\}$ for all $M \in \mathbb{N}$.

For $m \geq 0$, define

$$\begin{aligned}
A'_m &= A'_{m,M} = \left\{F(\tau_{(m+M)^2}) \leq \frac{R^2}{M^2 4^{2m+M}}\right\}, \\
A''_m &= A''_{m,M} = \{\mathcal{X}'(\tau_{(m+M)^2}) \subseteq B_{[2-2^{-m-2}]R}(0) \setminus B_{[1+2^{-m-2}]R}(0)\}, \qquad (9) \\
A_M &= A_{m,M} = A_{m-1} \cap (A'_M \cap A''_M).
\end{aligned}$$

Note that the definition is even consistent for $m = 0$ if we define $A_{-1} = \{\tau_0 < \infty\}$ and that, in principle, A_m, A'_m, and A''_m also depend on M, but we omit the second index where this does not create a confusion.

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Lemma 8. *It holds that* $\mathbb{P}(A_{m+1} \mid A_M) \geq 1 - \mathrm{e}^{-\sigma^2(m+M)}$, $m = 0, 1, 2, \ldots$.

Proof. First, note that $A_m \subseteq A''_{m+1}$. Indeed, since $2K < N$, we must have, in the core of the new configuration, at least one point from the previous core (this is not true, in general, if $2K \geq N$), so

$$\min_{x \in \mathcal{X}'(t+1)} \|x\| \geq \min_{x \in \mathcal{X}'(t)} \|x\| - D(t+1)$$

and, as a result on A_m, we have

$$\begin{aligned}
\operatorname{dist}(\mathcal{X}'(\tau_{(m+M+1)^2}), B_R(0)) &= \min_{x \in \mathcal{X}'(\tau_{(m+M+1)^2})} \|x\| - R \\
&\geq \min_{x \in \mathcal{X}'(\tau_{(m+M)^2})} \|x\| - R - \sum_{t=\tau_{(m+M)^2+1}}^{\tau_{(m+M+1)^2}} D(t) \\
&\geq \min_{x \in \mathcal{X}'(\tau_{(m+M)^2})} \|x\| - R - [2(m+M)+1]\sqrt{2F(\tau_{(m+M)^2})} \\
&\geq \bigg(1 + \frac{1}{2^{m+2}} - 1 - \frac{2(m+M)+1}{\sqrt{M^2 4^{2m+M}}}\bigg) R \\
&\geq \bigg(\frac{1}{2^{m+2}} - \frac{1}{2^{m+3}} \frac{2(m+M)+1}{M 2^{M+m-3}}\bigg) R \\
&\geq \frac{R}{2^{m+3}}
\end{aligned}$$

since, for all $j \geq 0$, we have $D(t+j) \leq \sqrt{2F(t)}$ by Lemmas 2 and 3, and $(2(m+M)+1)/M2^{M+m-3} < 1$ for all $m \geq 0$ as long as $M \geq 5$. By a similar argument

$$\operatorname{dist}(\mathcal{X}'(\tau_{(m+M+1)^2}), (B_{2R}(0))^c) = 2R - \max_{x \in \mathcal{X}'(\tau_{(m+M+1)^2})} \|x\| \geq \frac{R}{2^{m+3}}$$

and, hence, A''_{m+1} occurs.

Consequently, when A_M occurs, $\mathcal{X}'(t) \subseteq G$ for all $t \in (\tau_{(m+M)^2}, \tau_{(m+1+M)^2})$. At the same time the core undergoes $N = 2(m+M)+1$ changes between the times $\tau_{(m+M)^2}$ and $\tau_{(m+M+1)^2}$. During each of these changes the function F does not increase, and with probability at least σ decreases by a factor at least $a < 1$ regardless of the past, by Lemma 7. Hence,

$$\mathbb{P}(F(\tau_{(m+M+1)^2}) > a^{\sigma N/2} F(\tau_{(m+M)^2})) \leq \mathbb{P}\bigg(Y_1 + \cdots + Y_N < \frac{\sigma N}{2}\bigg),$$

where Y_i are i.i.d. Bernoulli(σ) random variables. It suffices now to obtain a bound on the right-hand side, since $a^{\sigma N/2} \leq a^{\sigma(m+M)} \leq a^{\sigma M} \leq \frac{1}{16}$. However, the bound for the sum of Y_i follows from the Hoeffding inequality [3]:

$$\mathbb{P}\bigg(Y_1 + \cdots + Y_N < \frac{\sigma N}{2}\bigg) \leq \exp\bigg(-\frac{\sigma^2 N}{2}\bigg) \leq \exp(-\sigma^2(m+M)).$$

Consequently, A'_{m+1} and, hence, A_{m+1} also occur with probability at least $\exp(-\sigma^2(m+M))$. This completes the proof. □

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We continue with the proof of Theorem 2. Note that, for a fixed M, $A_{m,M}$ is a decreasing sequence of events. Let $\bar{A}_M = \bigcap_{m=0}^{\infty} A_{m,M}$. Lemma 8 implies, by induction on m, that

$$\begin{aligned}
\mathbb{P}(\bar{A}_M) &= \mathbb{P}(A_{0,M}) \prod_{m=1}^{\infty} \mathbb{P}(A_{m,M} \mid A_{m-1,M}) \\
&\geq \mathbb{P}(A_{0,M}) \prod_{m=1}^{\infty} (1 - \mathrm{e}^{-\sigma^2(M+m)}) \\
&\geq \mathbb{P}(A_{0,M}) \bigg[1 - \sum_{m=1}^{\infty} \mathrm{e}^{-\sigma^2(M+m)}\bigg] \\
&\geq \mathbb{P}(A_{0,M})[1 - \sigma^{-2}\mathrm{e}^{-\sigma^2 M}].
\end{aligned}$$

It is straightforward to see that on $\bar{A}_M$ the points of the core $\mathcal{X}'(t)$ do not ever leave the set G after time τ_0, hence, $\sup_{t>\tau_0} \|\mu'(t)\| < 3R/4$ on $\bar{A}_M$. At the same time on E, we must visit $B_{2/n}(q_2)$ which lies outside of the convex hull of G, thus, $\sup_{t>\tau_0} \|\mu'(t)\| > 3R/4$ and, therefore, $E \cap \bar{A}_M = \varnothing$. Since $E \subseteq A_{0,M}$ and $\bar{A}_M \subseteq A_{0,M}$, we have

$$\begin{aligned}
\mathbb{P}(E) &= \mathbb{P}(E \setminus \bar{A}_M) \\
&\leq \mathbb{P}(A_{0,M} \setminus \bar{A}_M) \\
&= \mathbb{P}(A_{0,M}) - \mathbb{P}(\bar{A}_M) \\
&\leq \sigma^{-2}\mathrm{e}^{-\sigma^2 M}\mathbb{P}(A_{0,M}) \\
&\leq \sigma^{-2}\mathrm{e}^{-\sigma^2 M} \quad \text{for any } M \geq 0.
\end{aligned}$$

Since M can be arbitrarily large, we see that $\mathbb{P}(E) = 0$. □

4. The $K = d = 1$ case

In the $K = 1$ case and the space $\mathbb{R}^1$, we can obtain some more detailed results, given some further assumptions. If $d = 1$, we also write $\mathcal{X}'(t) \to +\infty$ whenever $\lim_{t\to\infty} \min\{x,\ x \in \mathcal{X}'(t)\} = \infty$; similarly, write $\mathcal{X}'(t) \to -\infty$ whenever $\lim_{t\to\infty} \max\{x,\ x \in \mathcal{X}'(t)\} = -\infty$.

Assumption 3. (At most exponential oscillations in the tail.) *Suppose that there exist some $R_+, R_- \in \mathbb{R}$, and a constant $C \geq 0$ such that, for any $a \geq R_+$ and $u > 0$, we have*

$$\mathbb{P}(a + u < \zeta \leq a + 2u) \leq C\mathbb{P}(a < \zeta \leq a + u).$$

Similarly, for all $a \leq R_-$ and $u < 0$, we have

$$\mathbb{P}(a + 2u < \zeta \leq a + u) \leq C\mathbb{P}(a + u < \zeta \leq a).$$

Remark 4. Observe that nearly all common continuous distributions satisfy this assumption (exponential, normal, Pareto, and so on). An example of a distribution for which the assumption is not fulfilled is, for example, one with the density

$$f(x) = \begin{cases} \frac{1}{2}\mathrm{e}^{-|x|}, & \lfloor x \rfloor \text{ is even,} \\ \mathrm{e}^{-2|x|} & \text{otherwise,} \end{cases}$$

which has support on the whole $\mathbb{R}$.

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By iterating the property in Assumption 3 for $a \geq R_+$, we find that, for $k = 1, 2, \dots,$

$$\mathbb{P}(\zeta \in (a + (k-1)u, a + ku]) \leq C^{k-1}\mathbb{P}(\zeta \in (a, a+u]).$$

It also follows that if we take $R_+ < a < b < c$ then

$$\begin{aligned}\mathbb{P}(\zeta \in (b, c]) &\leq \mathbb{P}\Big(\zeta \in \bigcup_{k=1}^{\lceil (c-a)/(b-a)\rceil} (a + (k-1)(b-a), k(b-a)]\Big) \\ &\leq \sum_{k=1}^{\lceil (c-a)/(b-a)\rceil} C^{k-1}\mathbb{P}(\zeta \in (a, b]).\end{aligned} \tag{10}$$

Using (10) one can compare the probabilities of selecting a new point in the intervals of different length and/or that are not consecutive; we see that in this case the upper bound we obtain is a polynomial in C.

Remark 5. The assumption is somewhat related to the concept of O-regular variation (see [1, p. 65]) in the following sense. If we let $g(x) = \mathbb{P}(R_+ < \zeta \leq R_+ + x)$ for $x > 0$, then we see from (10) that $\limsup_{x\to\infty} g(tx)/g(x) \leq \sum_{k=1}^{\lceil t\rceil} C^{k-1}$ for $t \geq 1$. Therefore, g is an O-regularly varying function; moreover, if the support of ζ is $\mathbb{R}^+$ and $R_+ = 0$, then the distribution function of ζ itself is an O-regularly varying function.

Assumption 3 immediately implies that the tail region is free of isolated atoms; moreover, it turns out that the tail region is free of atoms altogether.

Claim 1. *Suppose that Assumption 3 holds. Then* $\mathbb{P}(\zeta = x) = 0$ *for every* $x \in (-\infty, R_-) \cup (R_+, \infty)$.

Proof. Assume to the contrary that there exists $x \in (-\infty, R_-) \cup (R_+, \infty)$ such that $\mathbb{P}(\zeta = x) > 0$. Since $\mathbb{P}(\zeta = x) = \mathbb{P}(\bigcap_{n=1}^{\infty}\{\zeta \in (x - 1/n, x]\})$, by the continuity of probability, it follows that there exists N such that $\mathbb{P}(\zeta \in (x - 1/N, x]) \leq (1/2C + 1)\mathbb{P}(\zeta = x)$, which implies that $\mathbb{P}(\zeta \in (x - 1/N, x)) \leq 1/2C\mathbb{P}(\zeta = x)$. Therefore, we have

$$\begin{aligned}\mathbb{P}\bigg(\zeta \in \bigg(x - \frac{1}{2N}, x - \frac{1}{N}\bigg]\bigg) &\leq \mathbb{P}\bigg(\zeta \in \bigg(x - \frac{1}{N}, x\bigg)\bigg) \\ &\leq \frac{\mathbb{P}(\zeta = x)}{2C} \\ &\leq \frac{1}{2C}\mathbb{P}\bigg(\zeta \in \bigg(x - \frac{1}{2N}, x\bigg]\bigg),\end{aligned}$$

which contradicts Assumption 3. □

Theorem 3. *Suppose that* $K = 1$ *and* ζ *satisfies Assumption 3 for some* R_+ *and* R_-. *Then* $\mathcal{X}' \not\to \infty$ *a.s. and, consequently, by Theorem 1 we have* $F(t) \to 0$ *a.s. Additionally,*

$$\Big\{\liminf_{t\to\infty} x_{(1)}(t) > R_+\Big\} \cup \Big\{\limsup_{t\to\infty} x_{(N-1)}(t) < R_-\Big\} \subseteq \{\mathcal{X}'(t) \to \phi \text{ for some } \phi\}$$

except perhaps a set of measure 0. *Finally, if* $R_- > R_+$ *then* $\mathbb{P}(\mathcal{X}'(t) \to \phi \text{ for some } \phi) = 1$.

Remark 6. The last part of Theorem 3 applies to many distributions for which $\operatorname{supp}\zeta = \mathbb{R}$, for example, to normal, Laplace, or Cauchy distributions (one can take $R_+ = -1$ and $R_- = +1$).

Proof. We begin with the first statement of the theorem. Given some $L \geq 1$, from now on assume that $A_L = \{\sqrt{2F(0)} < L/2, |\zeta_{0;k}| < L, k = 1, \ldots, N\}$ occurs, this will imply that $D(t) \leq L/2$ for all t. Since the distance between any two points in the core at time t is bounded by $D(t)$, it follows that if one core point diverges to $+\infty$ so must all the other points. Similarly, if one of the points diverges to $-\infty$ so must all of the rest. Therefore, it is enough to show that $\mathbb{P}(\{\mathcal{X}'(t) \to +\infty\} \cup \{\mathcal{X}'(t) \to -\infty\}) = 0$. We will now prove that $\mathcal{X}'(t) \not\to +\infty$ a.s.; the proof that $\mathcal{X}'(t) \not\to -\infty$ a.s. is completely analogous.

Let $\pi_a = \inf\{t : \sqrt{2F(t)} < a/2\}$, $\eta_{1,a} = \tau_{1,a} = \pi_a$, and, for $k > 1$, let

$$\tau_{k,a} = \inf\{t > \eta_{k-1,a} : x_{(1)}(t) > R_+ + a\},$$
$$\eta_{k,a} = \inf\{t > \tau_{k,a} : x_{(1)}(t) < R_+ + a\}, \qquad \gamma_{k,t,a} = \min(\eta_{k,a}, \tau_{k,a} + t),$$

where $x_{(1)}(t)$ denotes the left-most point of the core at time t. If $\tau_{k,a} = \infty$ for some k then we set $\eta_{m,a} = \tau_{m,a} = \infty$ for all $m \geq k$. It is obvious that on A_L, $\pi_L = 0$. Furthermore,

$$\{\tau_{k,L} = \infty\} \cap \{\eta_{k-1,L} < \infty\} \subseteq \Big\{\limsup_{t\to\infty} x_{(1)}(t) \leq R_+ + a\Big\} \subseteq \{\mathcal{X}'(t) \not\to +\infty\}.$$

Let $C_k = \{\eta_{k,L} < \infty\}$ and note that

$$\bigg(\bigcap_{k=2}^{\infty} C_k\bigg) \subseteq \{\mathcal{X}'(t) \subseteq B_{R_+ + 2L}(0) \text{ i.o.}\} \subseteq \{\mathcal{X}'(t) \not\to +\infty\}.$$

Since $(\bigcap_{k=1}^{\infty} C_k) \subseteq \{\mathcal{X}'(t) \not\to +\infty\}$, if we could also show that

$$\mathbb{P}\bigg(\bigg(\bigcap_{k=2}^{\infty} C_k\bigg)^c \cap \{\mathcal{X}'(t) \to +\infty\}\bigg) = \mathbb{P}\bigg(\bigg(\bigcup_{k=2}^{\infty}\{\eta_{k,L} = \infty\}\bigg) \cap \{\mathcal{X}'(t) \to +\infty\}\bigg) = 0, \quad (11)$$

then it would follow that $\mathbb{P}(A_L \cap \{\mathcal{X}'(t) \not\to +\infty\}) = \mathbb{P}(A_L)$ and since $\mathbb{P}(\bigcup_{L=1}^{\infty} A_L) = 1$, it would then follow from the continuity of probability that $\mathbb{P}(\mathcal{X}'(t) \to +\infty) = 0$.

Now we show that $\mathbb{P}(\{\eta_{k,L} = \infty\} \cap \{\mathcal{X}'(t) \to +\infty\}) = 0$ for every $k > 1$, which will establish (11). For this purpose (and for the purpose of showing the other statements of the theorem), we will need the following lemma.

Lemma 9. *For some fixed $k > 1$ and $a > 0$, let*

$$h_c(s) = (\sqrt{F(s)} + c[\mu'(s) + \max(0, -R_+)])\, \mathbf{1}_{A_L}.$$

Then there exists $c > 0$ such that $\lim_{t\to\infty} h_c(\gamma_{k,t,a})$ exists a.s. on $\tau_{k,a} < \infty$.

Proof. We will show that $h_c(\gamma_{k,t,a})$ is a nonnegative supermartingale with respect to $\{\mathcal{F}_{\gamma_{k,t,a}}\}_{t\geq 0}$, and then the result will follow from the supermartingale convergence theorem. In order to simplify the notation, from now on we set $\gamma_t = \gamma_{k,t,a}$ throughout the proof of this lemma. First, observe that the positivity of $h_c(\gamma_t)$ is ensured by the term $c\max(0, -R_+)$, and by the definition of γ_t and π_a. Therefore, from now on we can assume that $R_+ \geq 0$ w.l.o.g.

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We have

$$\begin{aligned}
\mathbb{E}|h_c(s)| &\le \mathbb{E}[(\sqrt{F(s)} + c|\mu'(s)|)\,\mathbf{1}_{A_L}] \\
&\le \mathbb{E}\bigg[\bigg(\sqrt{F(0)} + c\bigg(|\mu'(0)| + \sum_{l=1}^{s} |\mu'(l) - \mu'(l-1)|\bigg)\bigg)\,\mathbf{1}_{A_L}\bigg] \\
&\le \mathbb{E}\bigg[\bigg(\frac{L}{2\sqrt{2}} + c\bigg(|\mu'(0)| + \sum_{l=1}^{s} D(l)\bigg)\bigg)\,\mathbf{1}_{A_L}\bigg] \\
&\le \mathbb{E}\bigg[\bigg(\frac{L}{2\sqrt{2}} + c\bigg(L + \sum_{l=1}^{s} \sqrt{2F(l)}\bigg)\bigg)\,\mathbf{1}_{A_L}\bigg] \\
&\le \mathbb{E}[(L + c(L + s\sqrt{2F(0)}))\,\mathbf{1}_{A_L}] \\
&\le L\bigg(1 + c\bigg(1 + \frac{s}{2}\bigg)\bigg) \\
&< \infty,
\end{aligned}$$

where we used Lemma 3, the fact that $|\mu'(0)| \le \max_{x \in \mathcal{X}'(0)} |x| \le L$, $|\mu'(s+1) - \mu'(s)| \le D(s+1)$, $s \ge 0$, and that F is nonincreasing. Hence, $\mathbb{E}|h_c(s)| < \infty$. We use $\mathbf{1}_A$ to denote the indicator function on the event A.

Since $\{\gamma_t < \eta_{k,a}\} \in \mathcal{F}_{\gamma_t}$, we have

$$\begin{aligned}
\mathbb{E}[h_c(\gamma_{t+1}) - h_c(\gamma_t) \mid \mathcal{F}_{\gamma_t}] &= \mathbb{E}[(h_c(\gamma_{t+1}) - h_c(\gamma_t))(\mathbf{1}_{\{\gamma_t = \eta_{k,a}\}} + \mathbf{1}_{\{\gamma_t < \eta_{k,a}\}}) \mid \mathcal{F}_{\gamma_t}] \\
&= \mathbb{E}[(h_c(\gamma_t + 1) - h_c(\gamma_t))\,\mathbf{1}_{\{\gamma_t < \eta_{k,a}\}} \mid \mathcal{F}_{\gamma_t}] \\
&= \mathbb{E}[h_c(\gamma_t + 1) - h_c(\gamma_t) \mid \mathcal{F}_{\gamma_t}]\,\mathbf{1}_{\{\gamma_t < \eta_{k,a}\}} \\
&\le \max(0, \mathbb{E}[(h_c(\gamma_t + 1) - h_c(\gamma_t)) \mid \mathcal{F}_{\gamma_t}])\,\mathbf{1}_{\{\gamma_t < \eta_{k,a}\}} \\
&\le \max(0, \mathbb{E}[(h_c(\gamma_t + 1) - h_c(\gamma_t)) \mid \mathcal{F}_{\gamma_t}]).
\end{aligned}$$

It suffices now to show that $\mathbb{E}(h(\gamma_t + 1) - h(\gamma_t) \mid \mathcal{F}_{\gamma_t}) \le 0$ a.s. Since $\gamma_t \le \eta_{k,a}$, we can deduce that

$$x_{(1)}(\gamma_t) \ge x_{(1)}(\eta_{k,a}) \ge x_{(1)}(\eta_{k,a} - 1) - D(\eta_{k,a} - 1) > R_+ + a - \sqrt{2F(\pi_a)} > R_+ + \tfrac{1}{2}a. \quad (12)$$

From the above inequalities we see that all the core points lie to the right of R_+ at time γ_t, since this region is free of atoms, we can conclude that $D(\gamma_t) > 0$ a.s. Recall that the points of the core at time γ_t are ordered as $x_{(1)}(\gamma_t) \le \cdots \le x_{(N-1)}(\gamma_t)$, and let $\zeta = \zeta_{\gamma_t + 1}$.

We now introduce some new variables, where we drop the time indices for the sake of brevity:

$$\begin{gathered}
D = D(\gamma_t), \qquad \mathcal{F} = \mathcal{F}_{\gamma_t}, \qquad y_k = \frac{x_{(k)}(\gamma_t) - x_{(1)}(\gamma_t)}{D}, \qquad \zeta' = \frac{\zeta - x_{(1)}(\gamma_t)}{D}, \\
F_o = \sqrt{F(\{y_1, \ldots, y_{N-1}\})}, \qquad F_n = \sqrt{F(\{y_1, \ldots, y_{N-1}, \zeta'\}')}, \\
\mu'_o = \mu(\{y_1, \ldots, y_{N-1}\}), \qquad \mu'_n = \mu(\{y_1, \ldots, y_{N-1}, \zeta'\}').
\end{gathered}$$

At time γ_t the transformed core consists of the new points $(y_1, \ldots, y_k)$ such that $0 = y_1 \le \cdots \le y_{N-1} = 1$. Note that we will always reject ζ' if $\zeta' < -1$ but this is equivalent to $\zeta < x_{(1)}(\gamma_t) - D$, which is bounded below by $x_{(1)}(\gamma_t) - a/2$. By (12) this is strictly larger than R_+ so we can conclude that ζ is accepted into the core only if it lies to the right of R_+.

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Furthermore, if $a > -1$ then, since ζ is independent of $\mathcal{F}$, it follows that

$$\begin{aligned}\mathbb{P}(\zeta' \in (a+u, a+2u]) &= \mathbb{P}(\zeta \in ((a+u)D + x_{(1)}(\gamma_t), (a+2u)D + x_{(1)}(\gamma_t)]) \\ &\le C\mathbb{P}(\zeta \in (aD + x_{(1)}(\gamma_t), (a+u)D + x_{(1)}(\gamma_t)]) \\ &= C\mathbb{P}(\zeta' \in (a, a+u]), \end{aligned} \tag{13}$$

hence, Assumption 3 translates to ζ'. If we combine (13) with the same type of argument as in (10), we see that if $-1 < a < b < c$ then

$$\mathbb{P}(\zeta' \in (b, c]) \le \sum_{k=1}^{\lceil (c-a)/(b-a) \rceil} C^{k-1}\mathbb{P}(\zeta' \in (a, b]). \tag{14}$$

Due to the translation invariance of $\sqrt{F}$ and μ, we have

$$\mu'(\gamma_t + 1) - \mu'(\gamma_t) = D(\mu_n' - \mu_o'), \qquad F(\gamma_t + 1) - F(\gamma_t) = D(\sqrt{F_n} - \sqrt{F_o}),$$

implying

$$\frac{1}{D}(h(\gamma_t + 1) - h(\gamma_t)) = \sqrt{F_n} - \sqrt{F_o} + c(\mu_n' - \mu_o').$$

Denote $\Delta h = \sqrt{F_n} - \sqrt{F_o} + c(\mu_n' - \mu_o')$; since $D > 0$ a.s., it follows that

$$\mathbb{E}[(h(\gamma_{t+1}) - h(\gamma_t)) \mid \mathcal{F}] \le 0$$

is equivalent to $\mathbb{E}[\Delta h \mid \mathcal{F}] \le 0$.

If the new point ζ is sampled then either 0, 1, or ζ' is eliminated in the next step. There are three different cases, either $\zeta' < 0$, $\zeta' \in (0, 1)$, or $\zeta' > 1$ (recall that ζ has no atoms under Assumption 3). The new centre of mass for the whole configuration is thus

$$\mu_n = \frac{\zeta' + M\mu_o'}{M+1},$$

where $M = N - 1$. If point 0 is eliminated then the centre of mass of the new core is $\mu_n' = \zeta'/M + u_o'$. If point 1 is eliminated then $\mu_n' = (\zeta' - 1)/M + \mu_o'$. Note that, by Claim 1, our probability measure is nonatomic to the right of R_+ and, therefore, the probability of a tie between which point should be eliminated is 0; consequently, we can disregard these events.

In the $\zeta' < 0$ case, only ζ' or 1 can be eliminated. Point 1 is eliminated if and only if $\mu_n - \zeta' < 1 - \mu_n$. This is equivalent to $\zeta' > (M(2\mu_o' - 1) - 1)/(M - 1)$. So in this case point 1 is eliminated if and only if $\zeta' \in ((M(2\mu_o' - 1) - 1)/(M - 1), 0)$. Denote this event by

$$L_1 = \left\{\min\left(\frac{M(2\mu_o' - 1) - 1}{M - 1}, 0\right) < \zeta' < 0\right\}.$$

In the $\zeta' \in (0, 1)$ case, ζ' is never eliminated, but one of points 0 or 1 must be. Point 0 is eliminated if and only if $\mu_n > 1 - \mu_n$, which is equivalent to $\zeta' > (M+1)/2 - M\mu_o'$, hence, $\zeta' \in (\min((M+1)/2 - M\mu_o', 1), 1)$. Let

$$B_0 = \left\{\min\left(\frac{M+1}{2} - M\mu_o', 1\right) < \zeta' < 1\right\}.$$

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Point 1 is eliminated if $\zeta' \in (0, 1) \setminus [\min((M+1)/2 - M\mu_o', 1), 1]$. Let

$$B_1 = \left\{0 < \zeta' < \min\left(\frac{M+1}{2} - M\mu_o', 1\right)\right\}.$$

Finally, in the $\zeta' > 1$ case, only ζ' or 0 can be eliminated. Point 0 will be eliminated if $\zeta' - \mu_n$ and if and only if $\zeta' < 2M\mu_o'/(M-1)$, that is, if $\zeta' \in (1, \max(2M\mu_o'/(M-1), 1))$. Let

$$R_0 = \left\{1 < \zeta' < \max\left(\frac{2M\mu_o'}{M-1}, 1\right)\right\}.$$

We begin with the $M = 2$ case. We have $\mu_o' = \frac{1}{2}$, $F_o = \frac{1}{2}$, $L_1 = \{-1 < \zeta' < 0\}$, $B_1 = \{0 < \zeta' < \frac{1}{2}]\}$, $B_0 = \{\frac{1}{2} < \zeta' < 1\}$, and $R_0 = \{1 < \zeta' < 2\}$. If point 1 is eliminated, then $F_n = \zeta'^2/2$, moreover, note that in this case, $\mu_o' - \mu_n'$ is nonpositive. If point 0 is eliminated then $\mu_n' = (1 + \zeta')/2$. We have

$$\begin{aligned}
\mathbb{E}(\Delta h \mid \mathcal{F}) &= \mathbb{E}[(\mu_n' - \mu_o') + c(F_n - F_o) \mid \mathcal{F}] \\
&\le c\mathbb{E}[(F_n - F_o)\,\mathbf{1}_{\{L_1 \cup B_1\}} \mid \mathcal{F}] + \mathbb{E}[(\mu_n' - \mu_o')\,\mathbf{1}_{\{R_0 \cup B_0\}} \mid \mathcal{F}] \\
&\le \tfrac{1}{2}c\mathbb{E}[(\zeta'^2 - 1)\,\mathbf{1}_{B_1} \mid \mathcal{F}] + \tfrac{1}{2}\mathbb{E}[\zeta'\,\mathbf{1}_{\{R_0 \cup B_0\}} \mid \mathcal{F}] \\
&\le \tfrac{1}{2}c\Big(\tfrac{1}{4} - 1\Big)\mathbb{P}\Big(0 < \zeta' < \tfrac{1}{2}\Big) + \tfrac{2}{2}\mathbb{P}\Big(\tfrac{1}{2} < \zeta' < 2\Big) \\
&\le -\tfrac{3}{8}c\mathbb{P}\Big(0 < \zeta' < \tfrac{1}{2}\Big) + (1 + C + C^2 + C^3)\mathbb{P}\Big(0 < \zeta' < \tfrac{1}{2}\Big),
\end{aligned}$$

where we used (14) in the last inequality. It is obvious that the last expression can be made negative for large enough $c > 0$, as required.

Let us now consider the $M \ge 3$ case. First, we note that $\mu_o' \in (1/M, (M-1)/M)$ a.s., where the lower bound is approached as $y_2, \dots, y_{M-1}$ all go to 0, while the upper bound is approached as $y_2, \dots, y_{M-1}$ all go to 1. If we now denote by K_0 the event that 0 is eliminated, and K_1 the event that 1 is eliminated, then we have $K_0 = R_0 \cup B_0$ and $K_1 = L_1 \cup B_1$. Furthermore,

$$\mu_n' - \mu_o' = \frac{\zeta'}{M}\,\mathbf{1}_{K_0} + \frac{\zeta' - 1}{M}\,\mathbf{1}_{K_1}.$$

We also have

$$\begin{aligned}
F_n = &\left(F_o + \frac{M-1}{M}\zeta'^2 - 2\mu_o'\zeta'\right)\mathbf{1}_{K_0} \\
&+ \left(F_o + \frac{M-1}{M}\zeta'^2 - \frac{2(M\mu_o' - 1)}{M}\zeta' + \frac{2(M\mu_o' - 1)}{M} - \frac{M-1}{M}\right)\mathbf{1}_{K_1}.
\end{aligned}$$

Observe that $\Delta h = h_0\,\mathbf{1}_{K_0} + h_1\,\mathbf{1}_{K_1}$, where

$$h_i = \sqrt{F_o + \Delta_i(\zeta', \mu_o')} + c\frac{\zeta'}{M} - \sqrt{F_o}, \qquad i = 0, 1,$$

$$\Delta_i(x, y) = \frac{1}{M}\begin{cases}(M-1)x^2 - 2Mxy, & i = 0,\\ (M-1)(x^2 - 1) + 2(1-x)(My - 1), & i = 1.\end{cases}$$

Using this notation, we obtain

$$\begin{aligned}
\mathbb{E}[\Delta h \mid \mathcal{F}] &= \mathbb{E}[h_0\,\mathbf{1}_{K_0} \mid \mathcal{F}] + \mathbb{E}[h_1\,\mathbf{1}_{K_1} \mid \mathcal{F}] \\
&= \mathbb{E}[h_0\,\mathbf{1}_{R_0} \mid \mathcal{F}] + \mathbb{E}[h_0\,\mathbf{1}_{B_0} \mid \mathcal{F}] + \mathbb{E}[h_1\,\mathbf{1}_{L_1} \mid \mathcal{F}] + \mathbb{E}[h_1\,\mathbf{1}_{B_1} \mid \mathcal{F}] \\
&= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3,
\end{aligned}$$

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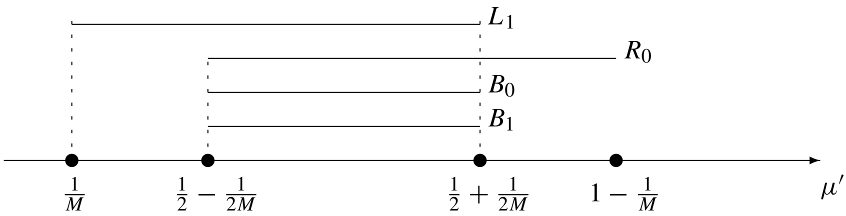

FIGURE 2: Possible locations of ζ'.

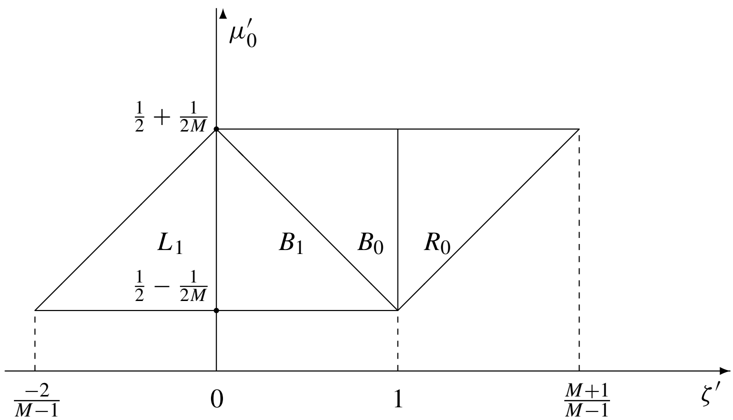

FIGURE 3: Possible combinations of ζ' and μ'_o.

where

$$\begin{aligned}
\mathrm{I}_1 &= (\mathbb{E}[h_1 \mathbf{1}_{L_1} \mid \mathcal{F}])\, \mathbf{1}_{\{\mu'_o \in (1/M,(M-1)/2M)\}}, \\
\mathrm{I}_2 &= (\mathbb{E}[h_1 \mathbf{1}_{L_1} \mid \mathcal{F}] + \mathbb{E}[h_1 \mathbf{1}_{B_1} \mid \mathcal{F}] + \mathbb{E}[h_0 \mathbf{1}_{R_0} \mid \mathcal{F}] \\
&\quad + \mathbb{E}[h_0 \mathbf{1}_{B_0} \mid \mathcal{F}])\, \mathbf{1}_{\{\mu'_o \in ((M-1)/2M,(M+1)/2M)\}}, \\
\mathrm{I}_3 &= (\mathbb{E}[h_0 \mathbf{1}_{R_0} \mid \mathcal{F}])\, \mathbf{1}_{\{\mu'_o \in ((M+1)/2M,(M-1)/M)\}} .
\end{aligned}$$

(See Figure 2 for the locations of ζ' for the events L_1, B_1, B_0 and R_0.) It suffices to show that all the three terms in the expression for $\mathbb{E}[\Delta h \mid \mathcal{F}]$ are nonpositive. The fact that $\mathrm{I}_1 \le 0$ is obvious, since if point 1 is eliminated then the core centre of mass must move leftwards, while F is always nonincreasing. The term I_2 is a little more complicated and requires more careful study. We illustrate the possible combinations of ζ' and μ'_o in Figure 3. We now present the following elementary statement.

Claim 2. *Let* $\Delta < 0$. *Then*

$$\sqrt{F_o + \Delta} - \sqrt{F_o} \le -\frac{\Delta}{2M}.$$

Proof. The inequality follows from the fact that $\sqrt{F_0} \le \sqrt{M/2} \le M$ and the trivial inequality $\sqrt{x+y} - \sqrt{x} \le y/2\sqrt{x}$ valid for all $x > 0$ and $x + y \ge 0$. □

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Next, we obtain an upper bound for $\Delta_1(x, y)$ on the rectangle

$$A_1 = \left\{(x, y)\colon \frac{M-1}{2M} \le y \le \frac{1}{2},\ 0 \le x \le \frac{1}{2}\right\}.$$

The critical point for $\Delta_1(\cdot, \cdot)$ is at $(1, 1)$ which falls outside A_1, hence, we only need to study the boundary points of A_1 to bound the maximum of Δ_1 on A_1. We have

$$\Delta_1 = \begin{cases} -\dfrac{M-1}{M} + \dfrac{2(My-1)}{M} \le -\dfrac{1}{M}, & x = 0, \\ -\dfrac{3(M-1)}{4M} + \dfrac{My-1}{M} \le -\dfrac{M+1}{4M}, & x = \frac{1}{2}, \\ \dfrac{M-1}{M}x^2 + \dfrac{3-M}{M}x - \dfrac{2}{M} \le -\dfrac{1}{M}, & y = (M-1)/2M, \\ \dfrac{M-1}{M}(x^2 - x) + \dfrac{x-1}{M} \le -\dfrac{1}{M}, & y = \frac{1}{2}. \end{cases}$$

Since $M \ge 3$,

$$-\frac{M+1}{4M} \le -\frac{1}{M}$$

and, therefore,

$$\Delta_1 \le -\frac{1}{M} \quad \text{on } A_1.$$

Combining these bounds with Claim 2, we obtain, for $(M-1)/2M \le \mu_o' \le \frac{1}{2}$ and $0 \le \zeta' \le \frac{1}{2}$ (which is a subset of $B_1 \cap \{(M-1)/2M \le \mu_o' \le \frac{1}{2}\}$),

$$\sqrt{F_o + \Delta_1(\zeta', \mu_o')} - \sqrt{F_o} \le -\frac{1}{2M^2}. \tag{15}$$

On the other hand, if $\mu_o' \ge \frac{1}{2}$ and $0 \le \zeta' \le 1$ then

$$\Delta_0(\zeta', \mu_o') \le \left(\frac{M-1}{M} - 2\mu_o'\right)\zeta' \le -\frac{\zeta'}{M}$$

and, therefore, by Claim 2,

$$\sqrt{F_o + \Delta_0(\zeta', \mu_o')} - \sqrt{F_o} \le -\frac{\zeta'}{2M^2}. \tag{16}$$

Our next task is to find an upper bound for $\Delta_0(x, y)$ on the rectangle

$$A_2 = \left\{(x, y)\colon \frac{1}{2} \le y \le \frac{M+1}{2M},\ 1 \le x \le \frac{2M-1}{2M-2}\right\}.$$

The function $\Delta_0(\cdot, \cdot)$ has its only critical point at $(0, 0)$ which falls outside this rectangle, so again we only need to study the boundary values of the rectangle. If $x = 1$ then

$$\Delta_0 = \frac{M-1}{M} - 2y \le \frac{M-1}{M} - 1 = -\frac{1}{M}.$$

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If $x = (2M-1)/(2M-2)$ then

$$\Delta_0 = -\frac{(4My - 2M + 1)(2M-1)}{4M(M-1)} =: f_1(y),$$

and this function has a critical point at

$$y = \frac{M}{2M-2} > \frac{M+1}{2M}$$

which, thus, lies outside of the border of A_2. Substituting in the endpoints, we obtain

$$f_1\left(\frac{1}{2}\right) = -\frac{2M-1}{4M(M-1)} \le -\frac{1}{2M}, \qquad f_1\left(\frac{M+1}{2M}\right) = -\frac{3(2M-1)}{4M(M-1)} \le -\frac{3}{4M} \le -\frac{1}{2M}.$$

If $y = \frac{1}{2}$ then

$$\Delta_0 = \frac{M-1}{M}x^2 - x \le -\frac{1}{4M} \quad \text{for all } 1 \le x \le \frac{2M-1}{2M-2}.$$

If $y = (M+1)/(2M)$ then

$$\Delta_0 = \frac{M-1}{M}x^2 - \frac{M+1}{M}x \le -\frac{1}{M} \quad \text{for } 1 \le x \le \frac{2M-1}{2M-2}.$$

As a result, we conclude that

$$\Delta_0 \le -\frac{1}{4M} \quad \text{on } A_2.$$

Combining this with Claim 2, we find that when $\frac{1}{2} \le \mu'_o \le (M+1)/2M$ and $1 \le \zeta' \le (2M-1)/2(M-1)$ (this is a subset of $R_0 \cap \{\frac{1}{2} \le \mu'_o \le (M+1)/2M\}$),

$$\sqrt{F_o + \Delta_0(\zeta', \mu'_o)} - \sqrt{F_o} \le -\frac{1}{8M^2}. \tag{17}$$

We will also again make use of the fact that, by definition, $h_1\,\mathbf{1}_{L_1} \le 0$ and $h_1\,\mathbf{1}_{B_1} \le 0$; therefore,

$$\begin{aligned}(\mathbb{E}[h_1\,\mathbf{1}_{L_1} \mid \mathcal{F}] + \mathbb{E}[h_1\,\mathbf{1}_{B_1} \mid \mathcal{F}])\,\mathbf{1}_{\{\mu'_o \in ((M-1)/2M,(M+1)/2M)\}}\\ \le \mathbb{E}[h_1\,\mathbf{1}_{\{B_1\}} \mid \mathcal{F}]\,\mathbf{1}_{\{\mu'_o \in ((M-1)/2M,1/2)\}}.\end{aligned}$$

Now we make the following bounds:

$$\begin{aligned}
\mathrm{I}_2 &\le \mathbb{E}[h_1\,\mathbf{1}_{B_1} \mid \mathcal{F}]\,\mathbf{1}_{\{\mu'_o \in ((M-1)/2M,1/2)\}} + (\mathbb{E}[h_0\,\mathbf{1}_{R_0} \mid \mathcal{F}]\\
&\quad + \mathbb{E}[h_0\,\mathbf{1}_{B_0} \mid \mathcal{F}])\,\mathbf{1}_{\{\mu'_o \in ((M-1)/2M,(M+1)/2M)\}}\\
&\le \mathbb{E}[(\sqrt{F_o + \Delta_1(\zeta', \mu'_o)} - \sqrt{F_o})\,\mathbf{1}_{B_1} \mid \mathcal{F}]\,\mathbf{1}_{\{\mu'_o \in ((M-1)/2M,1/2)\}}\\
&\quad + \mathbb{E}[(\sqrt{F_o + \Delta_0(\zeta', \mu'_o)} - \sqrt{F_o})(\mathbf{1}_{B_0} + \mathbf{1}_{R_0}) \mid \mathcal{F}]\,\mathbf{1}_{\{\mu'_o \in (1/2,(M+1)/2M)\}}\\
&\quad + \frac{c}{M}(\mathbb{E}[\zeta'\,\mathbf{1}_{B_0} \mid \mathcal{F}] + \mathbb{E}[\zeta'\,\mathbf{1}_{R_0} \mid \mathcal{F}])\,\mathbf{1}_{\{\mu'_o \in ((M-1)/2M,(M+1)/2M)\}}\\
&\le \mathbb{E}[(\sqrt{F_o + \Delta_1(\zeta', \mu'_o)} - \sqrt{F_o})\,\mathbf{1}_{\{0 \le \zeta' \le 1/2\}} \mid \mathcal{F}]\,\mathbf{1}_{\{\mu'_o \in ((M-1)/2M,1/2)\}}\\
&\quad + \mathbb{E}[(\sqrt{F_o + \Delta_0(\zeta', \mu'_o)} - \sqrt{F_o})(\mathbf{1}_{B_0} + \mathbf{1}_{\{1 \le \zeta' \le (2M-1)/2(M-1)\}}) \mid \mathcal{F}]\\
&\qquad \times \mathbf{1}_{\{\mu'_o \in (1/2,(M+1)/2M)\}}\\
&\quad + \frac{c}{M}(\mathbb{E}[\zeta'\,\mathbf{1}_{B_0} \mid \mathcal{F}] + \mathbb{E}[\zeta'\,\mathbf{1}_{R_0} \mid \mathcal{F}])\,\mathbf{1}_{\{\mu'_o \in ((M-1)/2M,(M+1)/2M)\}},
\end{aligned} \tag{18}$$

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where we used the fact that

$$\left\{0<\zeta'<\frac{1}{2}\right\}\cap\left\{\frac{M-1}{2M}<\mu_o'<\frac{1}{2}\right\}\subseteq\left\{\frac{M-1}{2M}<\mu_o'<\frac{1}{2}\right\}\cap B_1,$$
$$\left\{1\le\zeta'\le\frac{2M-1}{2(M-1)}\right\}\cap\left\{\frac{1}{2}<\mu_o'<\frac{M+1}{2M}\right\}\subseteq\left\{\frac{1}{2}<\mu_o'<\frac{M+1}{2M}\right\}\cap R_0,$$

and on B_1, we have

$$h_1\le\sqrt{F_o+\Delta_1(\zeta',\mu_o')}-\sqrt{F_o}.$$

We now study the terms in (18). Note that the term in the last line of (18) (a.s.) is equal to

$$\frac{c}{M}(\mathbb{E}[\zeta'\mathbf{1}_{B_0}\mid\mathcal{F}]+\mathbb{E}[\zeta'\mathbf{1}_{R_0}\mid\mathcal{F}])(\mathbf{1}_{\{\mu_o'\in((M-1)/2M,1/2)\}}+\mathbf{1}_{\{\mu_o'\in(1/2,(M+1)/2M)\}}),$$

while from (16) and (17), it follows that

$$\begin{aligned}&\mathbb{E}[(\sqrt{F_o+\Delta_0(\zeta',\mu_o')}-\sqrt{F_o})(\mathbf{1}_{B_0}+\mathbf{1}_{\{1\le\zeta'\le 2M-1/2(M-1)\}})\mid\mathcal{F}]\,\mathbf{1}_{\{\mu_o'\in(1/2,(M+1)/2M)\}}\\&\quad\le\left(\mathbb{E}\left[-\frac{\zeta'}{2M^2}\mathbf{1}_{B_0}\,\middle|\,\mathcal{F}\right]-\frac{1}{8M^2}\mathbb{P}\left(1\le\zeta'\le\frac{2M-1}{2(M-1)}\right)\right)\mathbf{1}_{\{\mu_o'\in(1/2,(M+1)/2M)\}}.\end{aligned}$$

From (15), it also follows that

$$\begin{aligned}&\mathbb{E}[(\sqrt{F_o+\Delta_1(\zeta',\mu_o')}-\sqrt{F_o})\,\mathbf{1}_{\{0<\zeta'<1/2\}}\mid\mathcal{F}]\,\mathbf{1}_{\{\mu_o'\in((M-1)/2M,1/2)\}}\\&\quad\le-\frac{1}{2M^2}\mathbb{P}\left(0<\zeta'<\frac{1}{2}\right)\mathbf{1}_{\{\mu_o'\in((M-1)/2M,1/2)\}}.\end{aligned}$$

Furthermore, we note that

$$\mathbb{E}[\zeta'\mathbf{1}_{B_0}\mid\mathcal{F}]\le\mathbb{P}(B_0)\quad\text{and}\quad\mathbb{E}[\zeta'\mathbf{1}_{R_0}\mid\mathcal{F}]\le\frac{M}{M-1}\mathbb{P}(R_0)\quad\text{for }\frac{M-1}{2M}<\mu_o'<\frac{1}{2},$$

while

$$\mathbb{E}[\zeta'\mathbf{1}_{R_0}\mid\mathcal{F}]\le\frac{M+1}{M-1}\mathbb{P}(R_0)\quad\text{when }\mu_o'<\frac{M+1}{2M}.$$

We can now conclude with

$$\begin{aligned}\mathrm{I}_2&\le\left[-\frac{1}{2M^2}\mathbb{P}\left(0<\zeta'<\frac{1}{2}\right)+\frac{c}{M}\left(\mathbb{P}(B_0)+\frac{M}{M-1}\mathbb{P}(R_0)\right)\right]\mathbf{1}_{\{\mu_o'\in((M-1)/2M,1/2)\}}\\&\quad+\mathbb{E}\left[\left(\frac{c\zeta'}{M}-\frac{\zeta'}{2M^2}\right)\mathbf{1}_{B_0}\,\middle|\,\mathcal{F}\right]\mathbf{1}_{\{\mu_o'\in(1/2,(M+1)/2M)\}}\\&\quad+\left(-\frac{1}{8M^2}\mathbb{P}\left(1<\zeta'<\frac{2M-1}{2(M-1)}\right)+\frac{c}{M}\frac{M+1}{M-1}\mathbb{P}(R_0)\right)\mathbf{1}_{\{\mu_o'\in(1/2,(M+1)/2M)\}}\\&\le\left[\frac{c}{M}\left(C_1+\frac{M}{M-1}C_2\right)-\frac{1}{2M^2}\right]\mathbb{P}\left(0<\zeta'<\frac{1}{2}\right)\mathbf{1}_{\{\mu_o'\in((M-1)/2M,1/2)\}}\\&\quad+\left[C_3\frac{c}{M}\frac{M+1}{M-1}-\frac{1}{8M^2}\right]\mathbb{P}\left(1<\zeta'<\frac{2M-1}{2(M-1)}\right)\mathbf{1}_{\{\mu_o'\in(1/2,(M+1)/2M)\}}\\&\quad+\mathbb{E}\left[\zeta'\left(\frac{c}{M}-\frac{1}{2M^2}\right)\mathbf{1}_{B_0}\,\middle|\,\mathcal{F}\right]\mathbf{1}_{\{\mu_o'\in(1/2,(M+1)/2M)\}},\end{aligned}$$

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where

$$C_1 = \frac{\mathbb{P}(\zeta' \in (0,1))}{\mathbb{P}(0 < \zeta' < 1/2)} \geq \frac{\mathbb{P}(B_0)}{\mathbb{P}(0 < \zeta' < 1/2)},$$
$$C_2 = \frac{\mathbb{P}(\zeta' \in (1,2))}{\mathbb{P}(0 < \zeta' < 1/2)} \geq \frac{\mathbb{P}(R_0)}{\mathbb{P}(0 < \zeta' < 1/2)},$$
$$C_3 = \frac{\mathbb{P}(\zeta' \in (1,2))}{\mathbb{P}(1 < \zeta' < (2M-1)/2(M-1))} \geq \frac{\mathbb{P}(R_0)}{\mathbb{P}(1 < \zeta' < (2M-1)/2(M-1))}.$$

From (14), it follows that these constants are all bounded above by some polynomial in C whose power depends only on M; also note that $\zeta' \geq 0$ on $B_0 \cap \{\frac{1}{2} \leq \mu_0' \leq (M+1)/2M\}$. Therefore, it is obvious that we can pick c small enough to make the first two terms in the last displayed inequality above nonpositive. The last term is trivially nonpositive due to the fact that $\zeta' \geq 0$ on B_0.

Now we show that $\mathrm{I}_3 \leq 0$. We begin by finding an upper bound for $\Delta_0(x, y)$ on the rectangle

$$A_3 = \left\{(x,y)\colon \frac{M+1}{2M} \leq y \leq \frac{M-1}{M}, 1 \leq x \leq \frac{M}{M-1}\right\}.$$

We already know it is sufficient to study the boundary of this rectangle, since no extreme points lie inside. If $x = 1$ then

$$\Delta_0 = \frac{M-1}{M} - 2y\frac{M+1}{2M} \leq -\frac{2}{M}.$$

If $x = M/(M-1)$ then

$$\Delta_0 = \frac{M}{M-1} - \frac{2M}{M-1}y \leq -\frac{1}{M-1}.$$

If $y = (M+1)/2M$ then

$$\Delta_0 = \frac{M-1}{M}x^2 - \frac{M+1}{M}x \leq -\frac{1}{M}.$$

If $y = (M-1)/M$ then

$$\Delta_0 = \frac{M-1}{M}(x^2 - 2x) \leq -\frac{2-M}{M} \leq -\frac{1}{M-1}.$$

Hence,

$$\Delta_0 \leq -\frac{1}{M} \quad \text{on } A_3,$$

and combining this with Claim 2, we find that if $(M+1)/2M \leq \mu_o' \leq (M-1)/M$ then

$$\begin{aligned}
\mathrm{I}_3 &= \mathbb{E}[h_0 \mathbf{1}_{R_0} \mid \mathcal{F}] \\
&\leq \mathbb{E}[(\sqrt{F_o + \Delta_0(\zeta', \mu_o')} - \sqrt{F_o})\, \mathbf{1}_{\{1 \leq \zeta' \leq M/(M-1)\}} \mid \mathcal{F}] + \frac{c}{M}\mathbb{E}[\zeta' \mathbf{1}_{R_0} \mid \mathcal{F}] \\
&\leq \mathbb{E}\left[\left(\sqrt{F_o - \frac{1}{M-1}} - \sqrt{F_o}\right) \mathbf{1}_{\{1 \leq \zeta' \leq M/(M-1)\}} \,\middle|\, \mathcal{F}\right] + \frac{c}{M}\mathbb{E}[2\, \mathbf{1}_{R_0}], \qquad (19)
\end{aligned}$$

where we used the fact that

$$\left\{\frac{M+1}{2M} < \mu_o' < \frac{M-1}{M}\right\} \cap \left\{1 \leq \zeta' \leq \frac{M}{M-1}\right\} \subseteq \left\{\frac{M+1}{2M} < \mu_o' < \frac{M-1}{M}\right\} \cap R_0$$

for the first term and $\zeta' < 2$ on R_0 (since $\mu_0' < (M-1)/M$) for the second term. If we apply Claim 2 to the first term in (19), and again apply the fact that $\zeta' < 2$ on R_0 for the second term, then we see that it is less than or equal to

$$\begin{aligned}&\bigg(\sqrt{F_o - \frac{1}{M-1}} - \sqrt{F_o}\bigg)\mathbb{P}\bigg(\zeta' \in \bigg(1, \frac{M}{M-1}\bigg)\bigg) + 2\frac{c}{M}\mathbb{P}(\zeta' \in (1,2)) \\ &\quad \le \bigg(-\frac{1}{2M(M-1)} + 2\frac{c}{M}C_4\bigg)\mathbb{P}\bigg(\zeta' \in \bigg(1, \frac{M}{M-1}\bigg)\bigg),\end{aligned}$$

where

$$C_4 = \frac{\mathbb{P}(1 < \zeta' < 2)}{\mathbb{P}(1 < \zeta' < M/(M-1))}$$

which is again bounded above by some polynomial in C according to (14). Therefore, it is clear that we can again pick c small enough to also make this term nonpositive which proves that $\mathbb{E}[\Delta h \mid \mathcal{F}] \le 0$ and, hence, h_k is a nonnegative supermartingale. □

Now we continue with the proof of Theorem 3. Fix k and $a = L$, and let c be defined by Lemma 9. If we denote by h_∞ the a.s. limit of $h_c(\gamma_{k,t,L})$ as $t \to \infty$ on $\{\tau_{k,L} < \infty\} \cap \{\eta_{k,L} = \infty\}$, then

$$h_\infty = \lim_{t\to\infty} (\sqrt{F(\tau_{k,L} + t)} + c\mu'(\tau_{k,L} + t))\,\mathbf{1}_{A_L} = \Big(\sqrt{F_\infty} + \lim_{t\to\infty} c\mu'(t)\Big)\,\mathbf{1}_{A_L},$$

that is, $\lim_{t\to\infty} \mu'(t) \in \mathbb{R}$ on A_L, implying $\mathcal{X}'(t) \not\to +\infty$.

We now prove the second statement of the theorem. Note that we have just proved that $F(t) \to 0$ a.s. and, hence, $\pi_{1/n} < \infty$ a.s. for all $n > 0$. First, we show that

$$\mathbb{P}\Big(\Big\{\liminf_{t\to\infty} x_{(1)}(t) > R_+\Big\} \cap \{\mathcal{X}'(t) \text{ does not converge}\}\Big) = 0. \tag{20}$$

Indeed, let $E_n = \{\liminf_{t\to\infty} x_{(1)}(t) \ge R_+ + 1/n\}$, then $\{\liminf_{t\to\infty} x_{(1)}(t) > R_+\} = \bigcup_{n=1}^\infty E_n$ and it suffices to prove that $\mathbb{P}(E_n \cap \{\mathcal{X}'(t) \text{ does not converge}\}) = 0$. Note that

$$E_n \subseteq \bigcup_{k=1}^\infty (\{\eta_{k,\,1/n} = \infty\} \cap \{\tau_{k,\,1/n} < \infty\}) \subseteq \bigcup_{k=1}^\infty \Big\{\lim_t \gamma_{k,\,t,\,1/n} = \infty\Big\}.$$

By Lemma 9, $h_c(\gamma_{k,t,1/n})$ has an a.s. limit for some $c > 0$ on $\{\eta_{k,\,1/n} = \infty\} \cap \{\tau_{k,\,1/n} < \infty\} \cap A_L$; thus,

$$\mathbb{P}\Big(A_L \cap \{\eta_{k,\,1/n} = \infty\} \cap \{\tau_{k,\,1/n} < \infty\} \cap \Big\{\lim_{t\to\infty} \mu'(t) \text{ does not exist}\Big\}\Big) = 0.$$

Using the continuity of probability again, applied to the sets A_L, $L \to \infty$, we can discard the term A_L in the expression above. Since $F(t) \to 0$ a.s., from the first part of the theorem, we have

$$\Big\{\lim_{t\to\infty} \mu'(t) \text{ exists}\Big\} = \{\mathcal{X}'(t) \to \phi \text{ for some } \phi\},$$

except perhaps a set of measure zero; therefore,

$$\begin{aligned}&\mathbb{P}(E_n \cap \{\lim \mathcal{X}'(t) \text{ does not exist}\}) \\ &\quad = \mathbb{P}(E_n \cap \{\lim \mu'(t) \text{ does not exist}\}) \\ &\quad \le \mathbb{P}\Big(\{\eta_{k,\,1/n} = \infty\} \cap \{\tau_{k,1/n} < \infty\} \cap \Big\{\lim_{t\to\infty} \mu'(t) \text{ does not exist}\Big\}\Big) \\ &\quad = 0.\end{aligned}$$

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Noting that $E_n \subseteq E_{n+1}$, (20) follows from the continuity of probability; the proof of the respective statement for lim sup is completely analogous, and they together are equivalent to the second statement of the theorem.

We now prove the last statement of the theorem. Assume that $R_+ < R_-$ in Assumption 3. Let $u = \liminf_{t\to\infty} x_{(1)}(t)$ and $v = \limsup_{t\to\infty} x_{(N-1)}(t)$. Consider the event

$$A_{a,b} = \{u < a\} \cap \{v > b\} \quad \text{for some } a < b.$$

If $b \le R_-$ or $a \ge R_+$, we have already showed that we have convergence, so suppose that $b > R_-$ and $a < R_+$. We now make the observation that the interval $[R_+, R_-]$ is regular with parameters $\delta = \frac{2}{3}$ and $r = 1/2C$ (see Definition 2), and in the event of $A_{a,b}$, we cross the interval $(a + (b-a)/2, b - (b-a)/2)$ i.o. However, since this interval also inherits the regularity property, this would contradict Proposition 2, which states that a regular interval cannot be visited i.o. a.s. and so $\mathbb{P}(A_{a,b}) = 0$. From this, we can conclude that

$$\mathbb{P}(\{\mathcal{X}'(t) \to \phi \text{ for some } \phi\}^c) = \mathbb{P}\bigg(\bigcup_{a,b\in\mathbb{Q},\, a<R_+,\, b>R_-} A_{a,b}\bigg) = 0,$$

that is, the core converges to a point a.s. □

4.1. Strengthening Theorem 2.

In the $d = 1$ case, we can obtain stronger results than for the general case $\zeta \in \mathbb{R}^d$, $d \ge 1$. For any interval $(a, b) \subset \mathbb{R}$ and any $\delta \in (0, 1)$, we define a δ-*truncation* of (a, b) as

$$(a, b)_\delta = \bigg(a + \frac{\delta}{2}(b-a), b - \frac{\delta}{2}(b-a)\bigg).$$

Definition 2. The interval (a_1, b_1) is called *regular* if there are $\delta, r \in (0, 1)$ such that, for any $(a_2, b_2) \subseteq (a_1, b_1)$, we have

$$\mathbb{P}(\zeta \in (a_2, b_2)_\delta \mid \zeta \in (a_2, b_2)) \ge r. \tag{21}$$

Remark 7. We can iterate (21) in order to establish

$$\mathbb{P}(\zeta \in (\cdots(a_2, b_2\underbrace{)_\delta)\cdots)_\delta}_{k \text{ times}} \mid \zeta \in (a_2, b_2)) \ge r^k, \qquad k \ge 2,$$

and the iteration of the δ-truncation eventually shrinks an interval to a point while r^k is still in $(0, 1)$. Hence, it is not difficult to check that if Definition 2 holds for some $\delta \in (0, 1)$ it holds for all δ in this interval.

Assumption 4. *Suppose that any interval* (a, b) *such that* $\mathbb{P}(\zeta \in (a, b)) > 0$ *contains a regular interval.*

Remark 8. The property above seems to hold for all common distributions; the authors were not able, in fact, to construct even a single counterexample, nor, unfortunately, to show that none exists.

Theorem 4. *Under Assumptions 1 and 4,* $\mathcal{X}'(t) \to \phi \in [-\infty, +\infty]$ *a.s.*

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The proof of this theorem immediately follows from the next proposition, since, if $\{\mathcal{X}'(t) \not\to \pm\infty\} = \{\mu'(t) \not\to \pm\infty\}$ occurs, then $\mu'(t)$ either converges to a finite number or crosses some interval i.o. However, every interval contains some regular interval by Assumption 4 and, by Theorem 1, $D(t) \to 0$ a.s. if $\mu'(t) \not\to \pm\infty$, so the core must converge in this case.

Proposition 2. *For any a, b such that $a < b$, with probability 1 $\mu'(t)$ cannot cross the interval (a, b) infinitely many times.*

Proof. Suppose the contrary. From Assumption 4, it follows that (a, b) contains some regular interval, say (a_1, b_1), which must also be crossed i.o. Now the rest of the proof is almost the same as that of Theorem 2 so we only highlight the differences, which lie in how Assumption 4 is used (in place of the stronger Assumption 2) when we define our 'absorbing' region G. Here we let $G = (a_1, b_1)$ and assume w.l.o.g. that $a_1 = 0$ and $b_1 = R$. Let $\zeta(t)$ and M satisfy the conditions of Theorem 2 and define $\tau_0 = \tau_0^{(M)}$ such that

$$\mathcal{X}'(\tau_0) \subseteq \left[\frac{1}{4}R, \frac{3}{4}R\right], \qquad F(\tau_0) \le \frac{R^2}{M^2 4^M}.$$

We define the events A'_m, A''_m, A_m for $m = 1, 2, \dots$ as in (9) with the only change that

$$A''_m = A''_{m,M} = \{\mathcal{X}'(\tau_{(m+M)^2}) \subseteq (2^{-(m+2)}R, [1 - 2^{-(m+2)}]R)\}.$$

Since G is regular, Lemma 7 can still be applied. The rest of the proof is identical to that of Theorem 2. □

Corollary 3. *Suppose that* $\operatorname{supp}\zeta$ *is bounded. Hence, under Assumptions 1 and 4 we have $\mathcal{X}'(t) \to \phi \in \mathbb{R}$ a.s.*

Corollary 4. *Suppose that $K = 1$ and that Assumption 4 is valid in some interval $[a, b]$, and, in addition, Assumption 3 is valid for some $R_- \ge a$ and $R_+ \le b$. Then $\mathcal{X}'(t) \to \phi \in \mathbb{R}$ a.s.*

Proof. Let $u = \liminf_{t\to\infty} x_{(1)}(t)$ and $v = \limsup_{t\to\infty} x_{(N-1)}(t)$. Consider the event

$$A_{c,d} = \{u \le c\} \cap \{v \ge d\} \quad \text{for some } c < d.$$

If $d < R_-$ or $c > R_+$, we already know from Theorem 3 that we have convergence, so suppose that both $c, d \in [a, b]$. In this case the interval $(c + (d-c)/2, d - (d-c)/2) \subset [c, d]$ is visited i.o. but, since this interval inherits the property of Assumption 4, it follows from Proposition 2 that $\mathbb{P}(A_{c,d}) = 0$. Therefore,

$$\mathbb{P}(\mathcal{X}'(t) \text{ does not converge}) = \mathbb{P}\left(\bigcup_{c,d\in\mathbb{Q},\, d<b,\, c>a} A_{c,d}\right) = 0,$$

that is, the core converges to a point a.s. □

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