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Symmetric solutions for a Neumann problem involving critical exponent

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In this paper, we construct three types of symmetric peaked solutions for a Neumann problem involving critical Sobolev exponent: the interior peaked solution, the boundary peaked solution and the interior-boundary peaked solution.

1. Introduction

This paper is a continuation of our study on multipeak solutions for the semilinear Neumann problem,

$$\left.\begin{aligned} -\Delta u + \lambda u = u^{2^*-1} + a u^{q-1}, \quad u > 0, \quad \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where $\Omega \subset R^N$ is a bounded domain with smooth boundary having certain symmetries (to be specified below), $N \geqslant 5$, $\lambda > 0$ is a large constant, $a > 0$ is a fixed constants, $\nu = \nu(x)$ is the unit outer normal to $\partial\Omega$ at x, $2^* = 2N/(N-2)$ is the critical Sobolev exponent and $2 < q < 2^*$.

Let $k \geqslant 1$ be an integer. A solution u_λ of (1.1) is said to be k-peaked if u_λ attains its maximum over $\bar{\Omega}$ only at k points in $\bar{\Omega}$. The construction of k-peak solutions has been studied extensively in the last few years for the problem,

$$\left.\begin{aligned} -\Delta u + \lambda u = u^{p-1}, \quad u > 0, \quad \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} = 0, \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.2}$$

where $2 < p \leqslant 2^*$. To state the results, we define the energy functional associated with (1.2):

$$E_\lambda(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2) - \frac{1}{p}\int_\Omega |u|^p.$$

A solution of (1.2) is called a least energy solution if it minimizes the functional $E_\lambda(u)$ in $\{u \in H^1(\Omega) : \langle E'_\lambda(u), u\rangle = 0, u \neq 0\}$. Ni and Takagi [21, 22] and Ni *et al*. [23] proved that for $\lambda > 0$ large, the least energy solution for (1.2) is one-peaked and its maximum point lies on the boundary and tends, as $\lambda \to +\infty$, to a point attaining the global maximum of the mean curvature of the boundary. If $2 < p < 2^*$, k-peak solutions with all peaks on $\partial\Omega$ (the boundary peak solution), or with all peaks lying in Ω (the interior peak solution), or with some peaks on $\partial\Omega$ and others in Ω (the interior-boundary peak solution), have been constructed in [6, 9, 11, 13, 15–17, 19, 32–34]. If $p = 2^*$, there are several results on the existence and multiplicity of boundary k-peak solutions. See [1,2,7,14,20,25–31,35]. However, very little is known on the existence of interior peak solution or interior boundary peak solution.

In [8], the authors have established that if $p = 2^*$, (1.2) has no interior peak solution, while under the symmetric assumption that Ω is invariant under some group action of $\Gamma \subset O(N)$ (the orthogonal group in R^N), satisfying that the orbit of each point except the origin is at least three, (1.1) has an interior single peak solution.

In this paper, we introduce a class of symmetric domains for which (1.1) with $a > 0$ has all of the three types of peaked solution: interior peaked, boundary peaked and interior-boundary peaked.

To state our results, let us first introduce some notation.

For any $x = (x_1, \dots, x_N) \in R^N$, define

$$p_i x = (x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_N).$$

Throughout this paper, we assume that $a > 0$ and Ω satisfies the following symmetry condition:

$$p_i x \in \Omega, \quad \text{if } x \in \Omega, \quad i = 1, \dots, N. \tag{1.3}$$

Let

$$I_\lambda(u) = \int_\Omega \tfrac{1}{2}(|\nabla u|^2 + \lambda u^2) - \frac{1}{2^*}\int_\Omega |u|^{2^*} - \frac{a}{q}\int_\Omega |u|^q, \qquad u \in H^1(\Omega),$$
$$H^1_s(\Omega) = \{u : u \in H^1(\Omega),\ u(p_i x) = u(x),\ \forall x \in \Omega,\ i = 1, \dots, N\}.$$

Define

$$\langle u, v\rangle_\lambda = \int_\Omega (\nabla u \nabla v + \lambda uv)$$

and $\|\cdot\|_\lambda$ is the corresponding norm.

For any $x \in R^N$, let

$$U(y) = \frac{[N(N-2)]^{(N-2)/4}}{(1 + |y|^2)^{(N-2)/2}}$$

and for $\mu > 0$, $y \in R^N$, let

$$U_{x,\mu}(y) = \mu^{(N-2)/2} U(\mu(y-x)). \tag{1.4}$$

For any $x \in \Omega$, let

$$E_{x,\mu} = \Big\{ v : v \in H^1(\Omega),\ \langle U_{\mu,x}, v\rangle_\lambda = \Big\langle \frac{\partial U_{\mu,x}}{\partial \mu}, v \Big\rangle_\lambda = 0, \quad \Big\langle \frac{\partial U_{\mu,x}}{\partial x_j}, v \Big\rangle_\lambda = 0,\ j = 1, \dots, N \Big\}.$$

For any $x_i \in \partial\Omega$, $i =, 1, \dots, l$, let

$$F_{x,\mu,l} = \Big\{ v : v \in H^1(\Omega),\ \langle U_{\mu,x_i}, v\rangle_\lambda = \Big\langle \frac{\partial U_{\mu,x_i}}{\partial \mu}, v \Big\rangle_\lambda = 0, \quad \Big\langle \frac{\partial U_{\mu,x_i}}{\partial \tau_i}, v \Big\rangle_\lambda = 0,\ i = 1, \dots, l \Big\},$$

where τ_i is any tangent unit vector of $\partial\Omega$ at x_i.

THEOREM 1.1. *Suppose that Ω satisfies (1.3) and $N \geqslant 5$. Then there is a $\lambda_0 > 0$ such that for each $\lambda > \lambda_0$, (1.1) has a solution of the form,*

$$u_{\lambda,I} = \alpha_{\lambda,0,I} U_{0,\mu_{\lambda,0}} + v_{\lambda,I}, \tag{1.5}$$

where $\alpha_{\lambda,I}$ and $\mu_{\lambda,0}$ are constants, $v_{\lambda,I} \in E_{0,\mu_{\lambda,0}} \cap H_s^1(\Omega)$, and as $\lambda \to +\infty$, $\alpha_{\lambda,0,I} \to 1$, $\|v_{\lambda,I}\| \to 0$ and $\mu_{\lambda,0}^{(N-2)(2-q)/2} \lambda \to c^* > 0$.

For the existence of boundary peak solution, let $z_i = \partial\Omega \cap \{x_i > 0,\ x_j = 0,\ j \neq i\}$, $\hat{z}_i = p_i z_i$. We have the following theorem.

THEOREM 1.2. *Suppose that Ω satisfies (1.3) and $N \geqslant 5$. Let $(z_{i_m}, \hat{z}_{i_m})$, $m = 1, \dots, k$, be k pairs of points. If one of the following conditions holds:*

(i)
$$q > \frac{2(N-1)}{N-2};$$

(ii)
$$q < \frac{2(N-1)}{N-2} \quad \textit{and} \quad H(z_{i_m}) > 0, \qquad m = 1, \dots, k,$$

where $H(y)$ is the mean curvature of $\partial\Omega$ at y;

(iii)
$$q = \frac{2(N-1)}{N-2} \quad \textit{and} \quad G_2 + H(z_{i_m}) G_3 > 0,$$

where

$$G_2 = \frac{a(2-(N-2)(q/2-1))}{q} \int_{R^N} U_{0,1}^q$$

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and

$$G_3 = \frac{1}{2}\int_{R^{N-1}} |\nabla U_{0,1}(y',0)|^2\,\mathrm{d}y' - \frac{1}{2^*}\int_{R^{N-1}} U_{0,1}^{2^*}(y',0)\,\mathrm{d}y' > 0;$$

then there is a $\lambda_0 > 0$ *such that for each* $\lambda > \lambda_0$, *(1.1) has a solution of the form,*

$$u_{\lambda,B} = \sum_{m=1}^{k} (\alpha_{\lambda,m,B} U_{z_{i_m},\mu_{\lambda,m}} + \alpha_{\lambda,m,B} U_{\hat{z}_{i_m},\mu_{\lambda,m}}) + v_{\lambda,B}, \tag{1.6}$$

where $\alpha_{\lambda,m,B}$ *and* $\mu_{\lambda,m}$ *are constants,* $v_{\lambda,B} \in F_{z,\mu_\lambda,2k} \cap H_s^1(\Omega)$, *and as* $\lambda \to +\infty$, $\alpha_{\lambda,m,B} \to 1$, $m = 1\dots,k$, $\|v_{\lambda,B}\| \to 0$ *and*

$$\mu_{\lambda,m}^{(N-2)(2-q)/2}\lambda \to c^* > 0 \quad \text{if } q > \frac{2(N-1)}{N-2};$$
$$\mu_{\lambda,m}^{-1}\lambda \to c_m^* > 0 \quad \text{if } q \leqslant \frac{2(N-1)}{N-2}.$$

The next result is about the existence of interior-boundary peak solution.

THEOREM 1.3. *Assume the hypotheses of theorem 1.2 hold and* $N \geqslant 6$. *Then there is a* $\lambda_0 > 0$ *such that for each* $\lambda > \lambda_0$, *(1.1) has a solution of the form,*

$$u_\lambda = \alpha_{\lambda,0} U_{0,\mu_{\lambda,0}} + \sum_{m=1}^{k} (\alpha_{\lambda,m} U_{z_{i_m},\mu_{\lambda,m}} + \alpha_{\lambda,m} U_{\hat{z}_{i_m},\mu_{\lambda,m}}) + v_\lambda, \tag{1.7}$$

where $\alpha_{\lambda,m}$ *and* $\mu_{\lambda,m}$ *are constants,* $v_{\lambda,B} \in E_{0,\mu_{\lambda,0}} \cap F_{z,\mu_\lambda,2k} \cap H_s^1(\Omega)$, *and as* $\lambda \to +\infty$, $\alpha_{\lambda,m} \to 1$, $m = 0,1\dots,k$, $\|v_\lambda\| \to 0$,

$$\mu_{\lambda,0}^{(N-2)(2-q)/2}\lambda \to c^* > 0,$$

and

$$\mu_{\lambda,m}^{(N-2)(2-q)/2}\lambda \to c^* > 0 \quad \text{if } q > \frac{2(N-1)}{N-2};$$
$$\mu_{\lambda,m}^{-1}\lambda \to c_m^* > 0 \quad \text{if } q \leqslant \frac{2(N-1)}{N-2}.$$

We remark that when $q < 2(N-1)/(N-2)$, the interior peak solution constructed in theorem 1.1 concentrates at the origin faster than the boundary peak solutions in theorem 1.2 do. This makes it harder to construct interior-boundary peak solutions because one needs to glue together solutions with different concentration rates. To achieve this goal, as in [10] we shall carefully estimate the interaction between different peaks to obtain a nice control of the term $\|v_\lambda - v_{\lambda,I} - v_{\lambda,B}\|$.

In this paper, the symmetry of the domain will play a crucial role to locate the peaks of the solutions. For a general domain, it is still an open problem whether (1.1) has interior peak solution or interior boundary peak solution.

In [8], the interior single peak solution was obtained by proving the minimizer of the corresponding minimization problem in some symmetric space has exactly one

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local maximum point at the origin. Now consider following problem:

$$c_\lambda = \inf\left\{ I_\lambda(u) : u \in H_s^1(\Omega),\ u \neq 0,\ \int_\Omega (|\nabla u|^2 + \lambda u^2) = \int_\Omega (|u|^{2^*} + a|u|^q) \right\}. \quad (1.8)$$

It is easy to check that $c_\lambda < S^{N/2}/N$, where S is the best constant for the Sobolev embedding $H^1(R^N) \to L^{2^*}(R^N)$. So (1.8) has a minimizer u_λ. Under the condition (1.3) here, we can only conclude that a minimizer of (1.8) has either one interior peak at the origin, or two boundary peaks at z_j, $\hat{z}_j$ for some j. In some domains satisfying (1.3), we can actually show that the minimizer of (1.8) has two boundary peaks. See the discussion in § 3. So unlike the case in [8], we can not obtain an interior peak solution as a minimizer of (1.8).

Problem (1.1) may be viewed as a prototype of pattern formation in biology and is related to the steady state problem for chemotactic aggregation model by Keller and Segel [18].

This paper is organized as follows. In § 2, we establish the existence of a symmetric interior peak solution. In § 3, we construct boundary peak solutions and in § 4, we construct interior-boundary peak solutions.

2. Existence of symmetric single peak solution

First we define

$$M_{\delta,I} = \{(\alpha, x, \mu, v) \in R_+ \times \Omega \times R_+ \times E_{x,\mu} : |\alpha - 1| \leqslant \delta,\ x \in B_\delta(0),\ \mu \geqslant \delta^{-1},$$
$$\sqrt{\lambda}/\mu^2 \leqslant \delta,\ \|v\|_\lambda \leqslant \delta\},$$

where $\delta > 0$ is a small constant. Let

$$J_I(\alpha, x, \mu, v) = I(\alpha U_{x,\mu} + v), \qquad (\alpha, x, \mu, v) \in M_{\delta,I}.$$

It is now well-known (see Bahri [3] and Bahri and Coron [4]) that if $\delta > 0$ is small enough, then for (α, x, μ, v), $u = \alpha U_{x,\mu} + v$ is a critical point of $I_\lambda(u)$ in $H^1(\Omega)$ if and only if (α, x, μ, v) is a critical point of $J_I(\alpha, , x, \mu, v)$ in $M_{\delta,I}$, which is equivalent to the existence of $a_0 \in R$, $b_0 \in R$, $c_\ell \in R$ for $\ell = 1, \ldots, N$, such that the following equations hold:

$$\frac{\partial J_I(\alpha, x, \mu, v)}{\partial \alpha} = 0, \quad (2.1)$$

$$\frac{\partial J_I(\alpha, x, \mu, v)}{\partial v} = a_0 U_{x,\mu} + b_0 \frac{\partial U_{x,\mu}}{\partial \mu} + \sum_{\ell=1}^N c_\ell \frac{\partial U_{x,\mu}}{\partial x_\ell}, \quad (2.2)$$

$$\frac{\partial J_I(\alpha, x, \mu, v)}{\partial x_j} = b_0 \left\langle \frac{\partial^2 U_{x,\mu}}{\partial \mu \partial x_j}, v \right\rangle_\lambda + \sum_{\ell=1}^N c_\ell \left\langle \frac{\partial^2 U_{x,\mu}}{\partial x_\ell \partial x_j}, v \right\rangle_\lambda, \quad (2.3)$$

$$\frac{\partial J_I(\alpha, x, \mu, v)}{\partial \mu} = b_0 \left\langle \frac{\partial^2 U_{x,\mu}}{\partial \mu^2}, v \right\rangle_\lambda + \sum_{\ell=1}^N c_\ell \left\langle \frac{\partial^2 U_{x,\mu}}{\partial \mu \partial x_\ell}, v \right\rangle_\lambda. \quad (2.4)$$

As usual, we first solve (2.1) and (2.2) so that we can reduce the problem to a finite-dimensional problem.

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Proposition 2.1. *For each fixed $x \in \Omega$ and $\mu > 0$ large, there are unique $\alpha_{\lambda,I}(x,\mu)$ and $v_{\lambda,I}(x,\mu) \in E_{x,\mu}$, such that (2.1) and (2.2) hold. Moreover,*

$$|\alpha_{\lambda,I}(x,\mu) - 1| + \|v_{\lambda,I}(x,\mu)\|_\lambda = O\biggl(\biggl(\frac{\sqrt{\lambda}}{\mu}\biggr)^{1+\theta} + \frac{1}{\mu^{(2-\sigma)/2+\theta}}\biggr),$$

for some $\theta > 0$, where $\sigma = (N-2)(q/2-1) > 0$.

Proof. As in [3] and [24], we expand $J_I(\alpha, x, \mu, v)$ at $\alpha = 1$ and $v = 0$:

$$J_I(\alpha, x, \mu, v) = J_I(1, x, \mu, 0) + \langle f_{x,\mu}, \omega\rangle + \tfrac{1}{2}\langle Q_{x,\mu}\omega, \omega\rangle + R_{x,\mu}(\omega),$$

where $\omega = (\alpha - 1, v)$,

$$\begin{aligned}\langle f_{x,\mu}, \omega\rangle &= -\int_\Omega U_{x,\mu}^{2^*-1} v - a\int_\Omega U_{x,\mu}^{q-1} v \\ &\quad + \biggl(\int_\Omega |\nabla U_{x,\mu}|^2 + \lambda\int_\Omega U_{x,\mu}^2 - \int_\Omega U_{x,\mu}^{2^*} - a\int_\Omega U_{x,\mu}^q\biggr)(\alpha - 1) \\ &=: \langle \tilde{f}_{x,\mu}, v\rangle + g_{x,\mu}(\alpha - 1),\end{aligned} \tag{2.5}$$

$$\begin{aligned}&\langle Q_{x,\mu}\omega, \omega\rangle \\ &\quad = \|v\|_\lambda^2 - (2^*-1)\int_\Omega U_{x,\mu}^{2^*-2} v^2 - a(q-1)\int_\Omega U_{x,\mu}^{q-2} v^2 \\ &\qquad + \biggl(\int_\Omega |\nabla U_{x,\mu}|^2 + \lambda\int_\Omega U_{x,\mu}^2 - (2^*-1)\int_\Omega U_{x,\mu}^{2^*} - a(q-1)\int_\Omega U_{x,\mu}^q\biggr)(\alpha - 1)^2 \\ &\qquad - 2(\alpha - 1)\biggl((2^*-1)\int_\Omega U_{x,\mu}^{2^*-1} v + a(q-1)\int_\Omega U_{x,\mu}^{q-1} v\biggr),\end{aligned} \tag{2.6}$$

$$R_{x,\mu}^{(i)}(\omega) = O(\|\omega\|^{\min(3,2^*)-i}), \qquad i = 0, 1, 2, \tag{2.7}$$

where

$$R_{x,\mu}^{(0)}(\omega) = R_{x,\mu}(\omega), \qquad R_{x,\mu}^{(1)}(\omega) = R'_{x,\mu}(\omega), \qquad R_{x,\mu}^{(2)}(\omega) = R''_{x,\mu}(\omega).$$

It is easy to check that $Q_{x,\mu}$ is invertible. So it is standard to prove that there are unique $\alpha_{x,\mu}$ and $v_{x,\mu}$ satisfying (2.1), (2.2) and

$$|\alpha_{\lambda,I}(x,\mu) - 1| + \|v_{\lambda,I}(x,\mu)\|_\lambda = \|\omega\| = O(\|f_{x,\mu}\|).$$

Now we estimate $\|f_{x,\mu}\|$. First, we have

$$\begin{aligned} g_{x,\mu} &= \int_\Omega |\nabla U_{x,\mu}|^2 + \lambda\int_\Omega U_{x,\mu}^2 - \int_\Omega U_{x,\mu}^{2^*} - a\int_\Omega U_{x,\mu}^q \\ &= \int_{\partial\Omega} \frac{\partial U_{x,\mu}}{\partial\nu} U_{x,\mu} + \lambda\int_\Omega U_{x,\mu}^2 - a\int_\Omega U_{x,\mu}^q \\ &= O\biggl(\frac{1}{\mu^{N-2}} + \frac{\lambda}{\mu^2} + \frac{1}{\mu^{2-\sigma}}\biggr) \\ &= O\biggl(\frac{\lambda}{\mu^2} + \frac{1}{\mu^{2-\sigma}}\biggr).\end{aligned} \tag{2.8}$$

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Since $q > 2 \geqslant (4N-4)/3(N-2)$, we can choose

$$p \in \left(\frac{2N}{N+2}, \frac{4N}{(N-2)q+4}\right) \quad \text{and } p \text{ close to } \frac{4N}{(N-2)q+4},$$

such that $p(N-2)(q-1) > N$. We have

$$\begin{aligned} \left|\int_\Omega U_{x,\mu}^{q-1} v\right| &\leqslant C\biggl(\int_\Omega U_{x,\mu}^{(q-1)p}\biggr)^{1/p} \|v\|_\lambda \\ &\leqslant C\frac{1}{\mu^{N/p-(N-2)(q-1)/2}}\|v\|_\lambda \\ &= C\frac{1}{\mu^{(2-\sigma)/2+\tau}}\|v\|_\lambda, \end{aligned} \tag{2.9}$$

where $\tau = N/p - \frac{1}{2}(N-2)(q-1) - \frac{1}{2}(2-\sigma) > 0$.

Now we estimate $\lambda \int_\Omega U_{x,\mu} v$. Choose

$$t \in \left(\frac{N}{N-2}, 2\right) \cap \left[\frac{2N}{N+2}, 2\right).$$

Let

$$t' = \frac{t}{t-1} \quad \text{and} \quad \eta = \frac{Nt - 2N + 2t}{2t}.$$

Then

$$\begin{aligned} \lambda\left|\int_\Omega U_{x,\mu} v\right| &\leqslant \lambda\biggl(\int_\Omega U_{x,\mu}^t\biggr)^{1/t} |v|_{t'} \\ &\leqslant C\lambda\frac{1}{\mu^{N/t-(N-2)/2}}|v|_{t'} \leqslant C\lambda\frac{1}{\mu^{N/t-(N-2)/2}}|v|_2^\eta |v|_{2^*}^{1-\eta} \\ &= C\lambda^{1-\eta/2}\frac{1}{\mu^{N/t-(N-2)/2}}(\sqrt{\lambda}|v|_2)^\eta |v|_{2^*}^{1-\eta} \\ &\leqslant C_1\lambda^{1-\eta/2}\frac{1}{\mu^{N/t-(N-2)/2}}\|v\|_\lambda \\ &= C_1\biggl(\frac{\sqrt{\lambda}}{\mu}\biggr)^{2-\eta}\|v\|_\lambda \\ &= C_1\biggl(\frac{\sqrt{\lambda}}{\mu}\biggr)^{1+\tau_1}\|v\|_\lambda, \end{aligned} \tag{2.10}$$

where $\tau_1 = 1 - \eta > 0$.

Using (2.10), we see

$$\begin{aligned} \int_\Omega U_{x,\mu}^{2^*-1} v &= -\int_\Omega \Delta U_{x,\mu} v \\ &= -\lambda\int_\Omega U_{x,\mu} v - \int_{\partial\Omega}\frac{\partial U_{x,\mu}}{\partial\nu} v = O\biggl(\frac{1}{\mu^{(N-2)/2}} + \biggl(\frac{\sqrt{\lambda}}{\mu}\biggr)^{1+\tau_1}\biggr)\|v\|_\lambda. \end{aligned} \tag{2.11}$$

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Combining (2.5), (2.8), (2.9) and (2.11), we obtain

$$|\langle \tilde{f}_{x,\mu}, v\rangle| = O\biggl(\biggl(\frac{\sqrt{\lambda}}{\mu}\biggr)^{1+\tau_1} + \frac{1}{\mu^{(2-\sigma)/2+\tau}}\biggr)\|v\|_\lambda.$$

As a result,

$$\|f_{x,\mu}\| = O\biggl(\biggl(\frac{\sqrt{\lambda}}{\mu}\biggr)^{1+\tau_1} + \frac{1}{\mu^{(2-\sigma)/2+\tau}}\biggr).$$

So we have completed the proof of this proposition. □

In the rest of this section, we fix $x = 0$. For simplicity, we denote $\alpha_{0,I} = \alpha_{\lambda,I}(0,\mu)$ and $v_I = v_{\lambda,I}(0,\mu)$.

Lemma 2.2. *$v_\lambda(y)$ is even in each y_j, $j = 1,\dots,N$.*

Proof. For fixed j, let $\bar{v}_I(y) = v_I(p_j y)$. Then it is easy to check $\bar{v}_I \in E_{0,\mu}$ and for any $\phi \in E_{0,\mu}$, we have

$$I(\alpha_{0,I}U_{0,\mu} + \bar{v}_I + t\phi) = I(\alpha_{0,I}U_{0,\mu} + v_I + t\bar{\phi}),$$

where $\bar{\phi}(y) = \phi(p_j y)$. So $\bar{\phi} \in E_{0,\mu}$. As a result,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I(\alpha_{0,I}U_{0,\mu} + \bar{v}_I + t\phi)|_{t=0} &= \frac{\mathrm{d}}{\mathrm{d}t}I(\alpha_{0,I}U_{0,\mu} + v_I + t\bar{\phi})|_{t=0} \\ &= \biggl\langle \frac{\partial J_I(\alpha_{0,I},0,\mu,v_I)}{\partial v}, \bar{\phi}\biggr\rangle = 0.\end{aligned}$$

Similarly, we have

$$\frac{\partial J_I(\alpha_{0,I},0,\mu,\bar{v}_I)}{\partial \alpha} = \frac{\partial J_I(\alpha_{0,I},0,\mu,v_I)}{\partial \alpha} = 0.$$

So $(\alpha_{0,I},\bar{v}_I)$ also satisfies (2.1) and (2.2). By the uniqueness, we have $(\alpha_{0,I},\bar{v}_I) = (\alpha_{0,I},v_I)$. Thus $\bar{v}_I = v_I$. □

Lemma 2.3. *$c_j = 0$, $j = 1,\dots,N$.*

Proof. By lemma 2.2, v_I is even in each y_j. On the other hand, $\partial U_{0,\mu}/\partial x_i$ is odd in each y_j. So it is easy to check

$$\biggl\langle \frac{\partial J_I(\alpha_{0,I},0,\mu,v_I)}{\partial v}, \frac{\partial U_{0,\mu}}{\partial x_i}\biggr\rangle = 0, \qquad i = 1,\dots,N. \tag{2.12}$$

Also noting that

$$\biggl\langle U_{0,\mu}, \frac{\partial U_{0,\mu}}{\partial x_i}\biggr\rangle_\lambda = \biggl\langle \frac{\partial U_{0,\mu}}{\partial \mu}, \frac{\partial U_{0,\mu}}{\partial x_i}\biggr\rangle_\lambda = 0,$$

we obtain from (2.2) and (2.12) that

$$\sum_{j=1}^N c_j\biggl\langle \frac{\partial U_{0,\mu}}{\partial x_j}, \frac{\partial U_{0,\mu}}{\partial x_i}\biggr\rangle_\lambda = 0, \qquad i = 1,\dots,N.$$

So $c_j = 0$, $j = 1,\dots,N$. □

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LEMMA 2.4. *For any $\mu > 0$ large, $(\alpha_{0,I}, 0, \mu, v_I)$ satisfies (2.3).*

Proof. By lemma 2.3, (2.3) becomes

$$\frac{\partial J_I(\alpha_{0,I}, 0, \mu, v_I)}{\partial x_i} = b_0 \left\langle \frac{\partial^2 U_{0,\mu}}{\partial \mu \partial x_i}, v_I \right\rangle_\lambda.$$

But by lemma 2.2, it is easy to check that both the left-hand side and the right-hand side of the above equation are identically equals to zero. So the claim follows. □

By proposition 2.1 and lemma 2.4, it remains to find $\mu > 0$ large such that $(\alpha_{0,I}, 0, \mu, v_I)$ satisfies (2.4).

Proof of theorem 1.1. By the above discussion, we just need to solve (2.4). We have

$$\begin{aligned}
\frac{\partial J_I(\alpha_{0,I}, 0, \mu, v_I)}{\partial \mu} &= \left\langle \frac{\partial J_I(\alpha_{0,I}, 0, \mu, v_I)}{\partial v}, \frac{\partial U_{0,\mu}}{\partial \mu} \right\rangle \\
&= \alpha_{0,I} \left(\int_\Omega \nabla U_{0,\mu} \nabla \frac{\partial U_{0,\mu}}{\partial \mu} + \lambda \int_\Omega U_{0,\mu} \frac{\partial U_{0,\mu}}{\partial \mu} \right) \\
&\quad - \alpha_{0,I}^{2^*-1} \int_\Omega U_{0,\mu}^{2^*-1} \frac{\partial U_{0,\mu}}{\partial \mu} - a\alpha_{0,I}^{q-1} \int_\Omega U_{0,\mu}^{q-1} \frac{\partial U_{0,\mu}}{\partial \mu} \\
&\quad - (2^*-1)\alpha_{0,I}^{2^*-2} \int_\Omega U_{0,\mu}^{2^*-2} \frac{\partial U_{0,\mu}}{\partial \mu} v_I \\
&\quad - a(q-1)\alpha_{0,I}^{q-2} \int_\Omega U_{0,\mu}^{q-2} \frac{\partial U_{0,\mu}}{\partial \mu} v_I + O\left(\frac{1}{\mu} \|v_I\|^2 \right) \\
&= -\lambda \alpha_{0,I} \int_{R^N} U_{0,1}^2 \frac{2}{\mu^3} \\
&\quad + \frac{a\alpha_{0,I}^{q-1}}{q} \int_{R^N} U_{0,1}^q \frac{2-\sigma}{\mu^{3-\sigma}} + O\left(\left(\frac{\lambda}{\mu^2} \right)^{1+\tau} \frac{1}{\mu} + \frac{1}{\mu^{3-\sigma+\tau}} \right) \\
&= -\frac{G_1 \lambda}{\mu^3} + \frac{G_2}{\mu^{3-\sigma}} + O\left(\left(\frac{\lambda}{\mu^2} \right)^{1+\tau} \frac{1}{\mu} + \frac{1}{\mu^{3-\sigma+\tau}} \right),
\end{aligned} \tag{2.13}$$

where

$$G_1 = 2 \int_{R^N} U_{0,1}^2 > 0, \qquad G_2 = \frac{a(2-\sigma)}{q} \int_{R^N} U_{0,1}^q > 0.$$

Now we estimate b_0 in (2.2). From (2.2), we have

$$\begin{aligned}
a_0 \left\langle U_{0,\mu}, \frac{\partial U_{0,\mu}}{\partial \mu} \right\rangle_\lambda + b_0 \left\| \frac{\partial U_{0,\mu}}{\partial \mu} \right\|_\lambda^2 &= \frac{\partial J_I(\alpha_{0,I}, 0, \mu, v_I)}{\partial \mu} \\
&= O\left(\frac{\lambda}{\mu^3} + \frac{1}{\mu^{3-\sigma}} \right),
\end{aligned} \tag{2.14}$$

$$a_0 \|U_{0,\mu}\|_\lambda^2 + b_0 \left\langle U_{0,\mu}, \frac{\partial U_{0,\mu}}{\partial \mu} \right\rangle_\lambda = \frac{\partial J_I(\alpha_{0,I}, 0, \mu, v_I)}{\partial \alpha} = 0. \tag{2.15}$$

Solving (2.14) and (2.15), we obtain

$$b_0 = \mu O\left(\frac{\lambda}{\mu^2} + \frac{1}{\mu^{2-\sigma}} \right). \tag{2.16}$$

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From lemma 2.3, (2.13) and (2.16), we see that (2.4) is equivalent to

$$-\frac{G_1\lambda}{\mu^3}+\frac{G_2}{\mu^{3-\sigma}}=O\biggl(\biggl(\frac{\lambda}{\mu^2}\biggr)^{1+\tau}\frac{1}{\mu}+\frac{1}{\mu^{3-\sigma+\tau}}\biggr), \tag{2.17}$$

where $\tau>0$ is a small constant.

It is easy to prove that (2.17) has a solution μ_λ satisfying

$$\frac{\mu_\lambda}{\lambda^{1/\sigma}}\to\biggl(\frac{G_1}{G_2}\biggr)^{1/\sigma}\quad\text{as } \lambda\to+\infty.$$

□

3. Existence of symmetric multiple boundary peak solutions

For $\delta>0$ small, define

$$M_{\delta,B}=\biggl\{(\alpha,x,\mu,v)\in R_+^l\times(\partial\Omega)^l\times R_+^l\times F_{x,\mu,l}:\sum_{i=1}^l|\alpha_i-1|\leqslant\delta,$$
$$|x_i-x_j|\geqslant c_0>0,\ i\neq j,\ \mu_i\geqslant\delta^{-1},\ \frac{\sqrt{\lambda}}{\mu_i}\leqslant\delta,\ i=1,\dots,l,\ \|v\|_\lambda\leqslant\delta\biggr\}.$$

Let

$$J_B(\alpha,x,\mu,v)=I\biggl(\sum_{j=1}^l\alpha_jU_{x_j,\mu_j}+v\biggr),\qquad\forall(\alpha,x,\mu,v)\in M_{\delta,B}.$$

Let $\{\tau_{j1},\dots,\tau_{j(N-1)}\}$ form an orthogonal basis of the tangent space of $\partial\Omega$ at x_j. In order to prove that $u=\sum_{j=1}^l\alpha_jU_{x_j,\mu_j}+v$ is a critical point of $I(u)$, we just need to prove that $(\alpha,x,\mu,v)\in M_{\delta,B}$ satisfies

$$\frac{\partial J_B(\alpha,x,\mu,v)}{\partial\alpha_j}=0, \tag{3.1}$$

$$\frac{\partial J_B(\alpha,x,\mu,v)}{\partial v}=\sum_{j=1}^l\biggl(a_jU_{x_j,\mu_j}+b_j\frac{\partial U_{x_j,\mu_j}}{\partial\mu_j}\biggr)+\sum_{j=1}^k\sum_{\ell=1}^{N-1}c_{j\ell}\frac{\partial U_{x_j,\mu_j}}{\partial\tau_{j\ell}}, \tag{3.2}$$

$$\frac{\partial J_B(\alpha,x,\mu,v)}{\partial\tau_{ji}}=b_j\biggl\langle\frac{\partial^2U_{x_j,\mu_j}}{\partial\mu_j\partial\tau_{ji}},v\biggr\rangle_\lambda+\sum_{\ell=1}^{N-1}c_{j\ell}\biggl\langle\frac{\partial^2U_{x_j,\mu_j}}{\partial\tau_{j\ell}\partial\tau_{ji}},v\biggr\rangle_\lambda, \tag{3.3}$$

$$\frac{\partial J_B(\alpha,x,\mu,v)}{\partial\mu_j}=b_j\biggl\langle\frac{\partial^2U_{x_j,\mu_j}}{\partial\mu_j^2},v\biggr\rangle_\lambda+\sum_{\ell=1}^{N-1}c_{j\ell}\biggl\langle\frac{\partial^2U_{x_j,\mu_j}}{\partial\mu_j\partial\tau_{j\ell}},v\biggr\rangle_\lambda. \tag{3.4}$$

for some a_j, b_j, $c_{j\ell}\in R^1$, $j=1,\dots,l$, $i=1,\dots,N-1$.

First, we reduce the problem of finding a critical point of the form

$$\sum_{j=1}^l\alpha_jU_{x_j,\mu_j}+v$$

to a finite-dimensional problem.

PROPOSITION 3.1. *For $x_j \in \partial\Omega$, $j = 1, \dots, l$, $|x_i - x_j| \geqslant \delta > 0$, $i \neq j$, and $\mu_j > 0$ large, there are unique $\alpha_{\lambda,B}(x,\mu)$ and $v_{\lambda,B}(x,\mu) \in F_{x,\mu,l}$, such that (3.1) and (3.2) hold. Moreover,*

$$\sum_{j=1}^{l} |\alpha_{j,\lambda,B}(x,\mu) - 1| + \|v_{\lambda,B}(x,\mu)\|_\lambda = O\biggl(\sum_{j=1}^{l}\biggl(\biggl(\frac{\sqrt{\lambda}}{\mu_j}\biggr)^{1+\theta} + \frac{1}{\mu_j^{(2-\sigma)/2+\theta}} + \frac{1}{\mu_j^{1-\theta}}\biggr)\biggr),$$

for some small $\theta > 0$, where $\sigma = (N-2)(q/2-1) > 0$.

Proof. The proof of proposition 3.1 is similar to that of proposition 2.1 Thus we omit it. □

For the rest of this section, we take $l = 2k$, $k \geqslant 1$, $x_{2j-1} = z_{i_j}$, $x_{2j} = \hat{z}_{i_j}$, $\mu_{2j-1} = \mu_{2j}$, $j = 1, \dots, k$. Without loss of generality, we assume $i_j = j$. For simplicity, we let $\alpha_B = \alpha_{\lambda,B}(z,\mu)$, $v_B = v_{\lambda,B}(z,\mu)$.

LEMMA 3.2. *v_B is even in each y_j and $\alpha_{2j-1,B} = \alpha_{2j,B}$, $j = 1, \dots, k$.*

Proof. Let $\bar{v}_B(y) = v_B(p_j y)$,

$$\sigma_j\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{2(j-1)-1}, \alpha_{2(j-1)}, \alpha_{2j}, \\ \alpha_{2j-1}, \alpha_{2(j+1)-1}, \alpha_{2(j+1)}, \dots, \alpha_{2k-1}, \alpha_{2k}).$$

We have

$$J_B(\sigma_j\alpha, z, \mu, \bar{v}) = J_B(\alpha, z, \mu, v).$$

So it is easy to check that $(\sigma_j\alpha_B, \bar{v}_B)$ also satisfies (3.1) and (3.2). By uniqueness, we have $(\sigma_j\alpha_B, \bar{v}_B) = (\alpha_B, v_B)$ and the claim follows. □

LEMMA 3.3. *We have $a_{2j-1} = a_{2j}$, $b_{2j-1} = b_{2j}$ and $c_{j\ell} = 0$, $j = 1, \dots, k$, $\ell = 1, \dots, N-1$.*

Proof. First we know that all the constants a_j, b_j and c_{ij} are uniquely determined by the systems obtained by taking inner product of (3.2) with

$$U_{z_j,\mu_j}, \qquad U_{\hat{z}_j,\mu_j}, \qquad \frac{\partial U_{z_j,\mu_j}}{\partial \mu_j}, \qquad \frac{\partial U_{\hat{z}_j,\mu_j}}{\partial \mu_j}, \qquad \frac{\partial U_{z_j,\mu_j}}{\partial \tau_{2j-1,\ell}}, \qquad \frac{\partial U_{\hat{z}_j,\mu_j}}{\partial \tau_{2j,\ell}}.$$

So if we can choose A_j and B_j suitably, such that $a_{2j-1} = a_{2j} = A_j$, $b_{2j-1} = b_{2j} = B_j$, $c_{j\ell} = 0$ satisfy the same system, then our claim follows.

Let A_j and B_j be determined by the following systems:

$$\biggl\langle \frac{\partial J_B(\alpha_B, z, \mu, v_B)}{\partial v}, U_{\hat{z}_i,\mu_i} \biggr\rangle = \sum_{j=1}^{k} A_j \langle U_{z_j,\mu_j} + U_{\hat{z}_j,\mu_j}, U_{\hat{z}_i,\mu_i} \rangle_\lambda \\ + \sum_{j=1}^{k} B_j \biggl\langle \frac{\partial U_{z_j,\mu_j}}{\partial \mu_j} + \frac{\partial U_{\hat{z}_j,\mu_j}}{\partial \mu_j}, U_{\hat{z}_i,\mu_i} \biggr\rangle_\lambda, \tag{3.5}$$

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$$\left\langle \frac{\partial J_B(\alpha_B, z, \mu, v_B)}{\partial v}, \frac{\partial U_{\hat{z}_i,\mu_i}}{\partial \mu_i} \right\rangle = \sum_{j=1}^{k} A_j \left\langle U_{z_j,\mu_j} + U_{\hat{z}_j,\mu_j}, \frac{\partial U_{\hat{z}_i,\mu_i}}{\partial \mu_i} \right\rangle_\lambda + \sum_{j=1}^{k} B_j \left\langle \frac{\partial U_{z_j,\mu_j}}{\partial \mu_j} + \frac{\partial U_{\hat{z}_j,\mu_j}}{\partial \mu_j}, \frac{\partial U_{\hat{z}_i,\mu_i}}{\partial \mu_i} \right\rangle_\lambda . \tag{3.6}$$

It is easy to check that A_j and B_j are uniquely determined by (3.5) and (3.6). For such A_j and B_j, we also have

$$\begin{aligned}\left\langle \frac{\partial J_B(\alpha_B, z, \mu, v_B)}{\partial v}, U_{z_i,\mu_i} \right\rangle &= \left\langle \frac{\partial J_B(\alpha_B, z, \mu, v)}{\partial v}, U_{\hat{z}_i,\mu_i} \right\rangle \\ &= \sum_{j=1}^{k} A_j \langle U_{z_j,\mu_j} + U_{\hat{z}_j,\mu_j}, U_{\hat{z}_i,\mu_i} \rangle_\lambda + \sum_{j=1}^{k} B_j \left\langle \frac{\partial U_{z_j,\mu_j}}{\partial \mu_j} + \frac{\partial U_{\hat{z}_j,\mu_j}}{\partial \mu_j}, U_{\hat{z}_i,\mu_i} \right\rangle_\lambda \\ &= \sum_{j=1}^{k} A_j \langle U_{z_j,\mu_j} + U_{\hat{z}_j,\mu_j}, U_{z_i,\mu_i} \rangle_\lambda + \sum_{j=1}^{k} B_j \left\langle \frac{\partial U_{z_j,\mu_j}}{\partial \mu_j} + \frac{\partial U_{\hat{z}_j,\mu_j}}{\partial \mu_j}, U_{z_i,\mu_i} \right\rangle_\lambda .\end{aligned} \tag{3.7}$$

Similarly, we have

$$\left\langle \frac{\partial J_B(\alpha_B, z, \mu, v_B)}{\partial v}, \frac{\partial U_{z_j,\mu_i}}{\partial \mu_i} \right\rangle = \sum_{j=1}^{k} A_j \left\langle U_{z_j,\mu_j} + U_{\hat{z}_j,\mu_j}, \frac{\partial U_{z_i,\mu_i}}{\partial \mu_i} \right\rangle_\lambda + \sum_{j=1}^{k} B_j \left\langle \frac{\partial U_{z_j,\mu_j}}{\partial \mu_j} + \frac{\partial U_{\hat{z}_j,\mu_j}}{\partial \mu_j}, \frac{\partial U_{z_i,\mu_i}}{\partial \mu_i} \right\rangle_\lambda . \tag{3.8}$$

Moreover,

$$\left\langle \frac{\partial J_B(\alpha_B, z, \mu, v_B)}{\partial v}, \frac{\partial U_{z_i,\mu_i}}{\partial \tau_{i\ell}} \right\rangle = 0 = \sum_{j=1}^{k} A_j \left\langle U_{z_j,\mu_j} + U_{\hat{z}_j,\mu_j}, \frac{\partial U_{z_i,\mu_i}}{\partial \tau_{i\ell}} \right\rangle_\lambda + \sum_{j=1}^{k} B_j \left\langle \frac{\partial U_{z_j,\mu_j}}{\partial \mu_j} + \frac{\partial U_{\hat{z}_j,\mu_j}}{\partial \mu_j}, \frac{\partial U_{z_i,\mu_i}}{\partial \tau_{i\ell}} \right\rangle_\lambda , \tag{3.9}$$

$$\left\langle \frac{\partial J_B(\alpha_B, z, \mu, v_B)}{\partial v}, \frac{\partial U_{\hat{z}_i,\mu_i}}{\partial \tau_{i\ell}} \right\rangle = 0 = \sum_{j=1}^{k} A_j \left\langle U_{z_j,\mu_j} + U_{\hat{z}_j,\mu_j}, \frac{\partial U_{\hat{z}_i,\mu_i}}{\partial \tau_{i\ell}} \right\rangle_\lambda + \sum_{j=1}^{k} B_j \left\langle \frac{\partial U_{z_j,\mu_j}}{\partial \mu_j} + \frac{\partial U_{\hat{z}_j,\mu_j}}{\partial \mu_j}, \frac{\partial U_{\hat{z}_i,\mu_i}}{\partial \tau_{i\ell}} \right\rangle_\lambda . \tag{3.10}$$

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So we see $a_{2j-1} = a_{2j} = A_j$, $b_{2j-1} = b_{2j} = B_j$ and $c_{j\ell} = 0$ is the solution of the systems. □

LEMMA 3.4. *For any $\mu_j > 0$ large, (α_B, z, μ, v_B) satisfies (3.3).*

Proof. By lemma 3.2, $\sum_{j=1}^{k}(\alpha_{2j,B}U_{z_j,\mu_j} + \alpha_{2j,B}U_{\hat{z}_j,\mu_j}) + v_B$ is an even function of y_j, $j = 1, \dots, N$. Thus

$$\frac{\partial J_B(\alpha_B, z, \mu, v_B)}{\partial \tau_{2j,\ell}} = 0.$$

On the other hand, it is easy to check the right-hand side of (3.3) is zero. So our claim follows. □

LEMMA 3.5. *If μ_j $(j = 1, \dots, k)$ satisfies*

$$\frac{\partial J_B(\alpha_B, z, \mu, v_B)}{\partial \mu_{2j-1}}\bigg|_{\mu_{2j-1}=\mu_{2j}=\mu_j} = b_j \left\langle \frac{\partial^2 U_{z_j,\mu_j}}{\partial \mu_j^2}, v_B \right\rangle_\lambda, \tag{3.11}$$

then μ_j $(j = 1, \dots, k)$ also satisfies

$$\frac{\partial J_B(\alpha_B, z, \mu, v_B)}{\partial \mu_{2j}}\bigg|_{\mu_{2j-1}=\mu_{2j}=\mu_j} = b_j \left\langle \frac{\partial^2 U_{\hat{z}_j,\mu_j}}{\partial \mu_j^2}, v_B \right\rangle_\lambda.$$

Proof. We have

$$\begin{aligned}\frac{\partial J_B(\alpha_B, z, \mu, v_B)}{\partial \mu_{2j}}\bigg|_{\mu_{2j-1}=\mu_{2j}=\mu_j} &= \left\langle \frac{\partial J_B(\alpha_B, z, \mu, v_B)}{\partial v}, \frac{\partial U_{\hat{z}_i,\mu_i}}{\partial \mu_j} \right\rangle_\lambda \\ &= \left\langle \frac{\partial J_B(\alpha_B, z, \mu, v_B)}{\partial v}, \frac{\partial U_{z_i,\mu_i}}{\partial \mu_j} \right\rangle_\lambda\end{aligned}$$

(since $\alpha_{2j-1,B} = \alpha_{2j,B}, v_B(p_j y) = v_B(y)$)

$$\begin{aligned}&= b_j \left\langle \frac{\partial^2 U_{z_j,\mu_j}}{\partial \mu_j^2}, v_B \right\rangle_\lambda = b_j \left\langle \frac{\partial^2 U_{\hat{z}_j,\mu_j}}{\partial \mu_j^2}, v_B(p_j y) \right\rangle_\lambda \\ &= b_j \left\langle \frac{\partial^2 U_{\hat{z}_j,\mu_j}}{\partial \mu_j^2}, v_B \right\rangle_\lambda.\end{aligned}$$

□

Proof of theorem 1.2. Similar to (2.13) in the proof of theorem 1.1, we can check that

$$\begin{aligned}&\frac{\partial J_B(\alpha_B, z, \mu, v_B)}{\partial \mu_{2j-1}}\bigg|_{\mu_{2j-1}=\mu_{2j}=\mu_j} \\ &\quad = \frac{G_3 H(z_j)}{\mu_j^2} - \frac{G_1 \lambda}{\mu_j^3} + \frac{G_2}{\mu_j^{3-\sigma}} + O\left(\sum_{j=1}^{k}\left(\left(\frac{\lambda}{\mu_j^2}\right)^{1+\tau}\frac{1}{\mu_j} + \frac{1}{\mu_j^{2+\tau}} + \frac{1}{\mu_j^{3-\sigma+\tau}}\right)\right).\end{aligned} \tag{3.12}$$

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Besides, similar to (2.14) and (2.15), using (3.12), we can check

$$b_j = \mu_j O\biggl(\sum_{i=1}^{k}\biggl(\frac{\lambda}{\mu_i^2} + \frac{1}{\mu_i} + \frac{1}{\mu_j^{2-\sigma}}\biggr)\biggr). \tag{3.13}$$

Combining (3.12) and (3.13), we see that (3.11) is equivalent to

$$\frac{G_3 H(z_j)}{\mu_j^2} - \frac{G_1\lambda}{\mu_j^3} + \frac{G_2}{\mu_j^{3-\sigma}} = O\biggl(\sum_{j=1}^{k}\biggl(\biggl(\frac{\lambda}{\mu_j^2}\biggr)^{1+\tau}\frac{1}{\mu_j} + \frac{1}{\mu_j^{2+\tau}} + \frac{1}{\mu_j^{3-\sigma+\tau}}\biggr)\biggr), \tag{3.14}$$

where

$$G_3 = \frac{1}{2}\int_{R^{N-1}} |\nabla U_{0,1}(y',0)|^2 \,\mathrm{d}y' - \frac{1}{2^*}\int_{R^{N-1}} U_{0,1}^{2^*}(y',0)\,\mathrm{d}y' > 0,$$

$y' = (y_1,\dots,y_{N-1})$.

(i) If $3-\sigma < 2$, that is, $q > 2(N-1)/(N-2)$, then (3.14) has a solution $(\mu_{1,\lambda},\dots,\mu_{k,\lambda})$ satisfying

$$\mu_{j,\lambda}^{(N-2)(1-q/2)}\lambda \to \frac{G_2}{G_1} > 0.$$

(ii) If $3-\sigma > 2$ and $H(z_j) > 0$, $j = 1,\dots,k$, that is, $q < 2(N-1)/(N-2)$ and $H(z_j) > 0$, $j = 1,\dots,k$, then (3.14) has a solution $(\mu_{1,\lambda},\dots,\mu_{k,\lambda})$ satisfying

$$\mu_{j,\lambda}\lambda^{-1} \to \frac{G_1}{G_3 H(z_j)} > 0.$$

(iii) If $3-\sigma = 2$ and $G_2 + G_3 H(z_j) > 0$, $j = 1,\dots,k$, that is, $q = 2(N-1)/(N-2)$ and $G_2 + G_3 H(z_j) > 0$, $j = 1,\dots,k$, then (3.14) has a solution $(\mu_{1,\lambda}\dots,\mu_{k,\lambda})$ satisfying

$$\mu_{j,\lambda}\lambda^{-1} \to \frac{G_1}{G_2 + G_3 H(z_j)} > 0.$$

□

Before we close this section, we discuss briefly the least energy solution in the symmetric space.

PROPOSITION 3.6. *Let $u_\lambda \in H_s^1(\Omega)$ be the minimizer of (1.8). Suppose that $2 < q < 2(N-1)/(N-2)$. We have the following.*

(i) *If $H(z_j) < 0$, $j = 1,\dots,N$, then u_λ is an interior single peak solution with its peak in the origin.*

(ii) *If $H(z_j) > 0$ for some j, then u_λ is a boundary two peak solution with its peaks in $z_{j_0}, \hat{z}_{j_0}$, where $H(z_{j_0}) = \max_{0\leqslant j\leqslant N} H(z_j)$.*

Proof. By the symmetry of the domain, we know that u_λ has either one interior peak in the origin, or has two peaks in $z_i, \hat{z}_i$ for some i.

(i) If $H(z_j) < 0$, $j = 1,\dots,N$, from (3.14), we see that (1.1) has no boundary two solution with its peak in $z_i, \hat{z}_i$. So in this case, u_λ is an interior single peak solution.

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(ii) Suppose that $H(z_j) > 0$ for some j. Then for any symmetric boundary two peak solution $u_{\lambda,j}$ with its peak in $z_j, \hat{z}_j$, $H(z_j) > 0$, we have the following energy expansion:

$$I_\lambda(u_{\lambda,j}) = \frac{S^{N/2}}{N} - \frac{c_j}{\lambda} + O\biggl(\frac{1}{\lambda^{2+\tau}}\biggr)$$

for some $c_j > 0$ depending on $H(z_j)$. But for an interior single peak solution $u_{\lambda,0}$ with its peak at the origin, we have

$$I_\lambda(u_{\lambda,0}) = \frac{S^{N/2}}{N} - \frac{c_0}{\lambda^{2/\sigma-1}} + O\biggl(\frac{1}{\lambda^{2/\sigma-1+\tau/\sigma}}\biggr),$$

for some $c_0 > 0$. So we see

$$I_\lambda(u_{\lambda,j}) < I_\lambda(u_{\lambda,0}),$$

since $2/\sigma - 1 > 1$. □

4. Existence of symmetric interior-boundary peak solutions

Let $\delta > 0$ small. Define

$$M_\delta = \biggl\{(\alpha, x, \mu, v) \in R_+^{l+1} \times (\Omega \times (\partial\Omega)^l) \times R_+^{l+1} \times (E_{x_0,\mu_0} \cap F_{x,\mu,l}) : \sum_{i=0}^{l} |\alpha_i - 1| \leqslant \delta,\ x_0 \in B_\delta(0),\ |x_i - x_j| \geqslant c_0 > 0,\ 1 \leqslant i < j \leqslant l, \mu_i \geqslant \delta^{-1},\ \frac{\sqrt{\lambda}}{\mu_i^2} \leqslant \delta,\ i = 0, \dots, l,\ \|v\|_\lambda \leqslant \delta\biggr\}.$$

Let

$$J(\alpha, x, \mu, v) = I\biggl(\sum_{j=0}^{l} \alpha_j U_{x_j,\mu_j} + v\biggr), \qquad \forall (\alpha, x, \mu, v) \in M_\delta.$$

In this section, we will construct an interior-boundary peak solution of the form $u = \sum_{j=0}^{l} \alpha_j U_{x_j,\mu_j} + v$. As in the previous sections, this is equivalent to find $(\alpha, x, \mu, v) \in R_+^{l+1} \times (\Omega \times (\partial\Omega)^l) \times R_+^{l+1} \times (E_{x_0,\mu_0} \cap F_{x,\mu,l})$ satisfies

$$\frac{\partial J(\alpha, x, \mu, v)}{\partial \alpha_j} = 0, \tag{4.1}$$

$$\frac{\partial J(\alpha, x, \mu, v)}{\partial v} = a_0 U_{x,\mu_0} + b_0 \frac{\partial U_{x_0,\mu}}{\partial \mu_0} + \sum_{\ell=1}^{N} c_{0\ell} \frac{\partial U_{x_0,\mu_0}}{\partial x_{0\ell}} + \sum_{j=1}^{l} \biggl(a_j U_{x_j,\mu_j} + b_j \frac{\partial U_{x_j,\mu_j}}{\partial \mu_j}\biggr) + \sum_{j=1}^{l} \sum_{\ell=1}^{N-1} c_{j\ell} \frac{\partial U_{x_j,\mu_j}}{\partial \tau_{j\ell}}, \tag{4.2}$$

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$$\frac{\partial J(\alpha,x,\mu,v)}{\partial x_{0i}}=b_0\left\langle\frac{\partial^2 U_{x_0,\mu_0}}{\partial\mu_0\partial x_{0i}},v\right\rangle_\lambda+\sum_{\ell=1}^{N}c_{0\ell}\left\langle\frac{\partial^2 U_{x_0,\mu_0}}{\partial x_{0\ell}\partial x_{0i}},v\right\rangle_\lambda, \tag{4.3}$$

$$\frac{\partial J(\alpha,x,\mu,v)}{\partial \tau_{ji}}=b_j\left\langle\frac{\partial^2 U_{x_j,\mu_j}}{\partial\mu_j\partial \tau_{ji}},v\right\rangle_\lambda+\sum_{\ell=1}^{N-1}c_{j\ell}\left\langle\frac{\partial^2 U_{x_j,\mu_j}}{\partial \tau_{j\ell}\partial \tau_{ji}},v\right\rangle_\lambda, \tag{4.4}$$

$$\frac{\partial J(\alpha,x,\mu,v)}{\partial \mu_0}=b_0\left\langle\frac{\partial^2 U_{x_0,\mu_0}}{\partial\mu_0^2},v\right\rangle_\lambda+\sum_{\ell=1}^{N}c_{0\ell}\left\langle\frac{\partial^2 U_{x_0,\mu_0}}{\partial \mu_0\partial x_{0\ell}},v\right\rangle_\lambda, \tag{4.5}$$

$$\frac{\partial J(\alpha,x,\mu,v)}{\partial \mu_j}=b_j\left\langle\frac{\partial^2 U_{x_j,\mu_j}}{\partial\mu_j^2},v\right\rangle_\lambda+\sum_{\ell=1}^{N-1}c_{j\ell}\left\langle\frac{\partial^2 U_{x_j,\mu_j}}{\partial \mu_j\partial \tau_{j\ell}},v\right\rangle_\lambda, \tag{4.6}$$

for some a_j, b_j, $c_{j\ell}\in R^1$.

First, we reduce the problem of finding a critical point of the form

$$\sum_{j=0}^{l}\alpha_j U_{x_j,\mu_j}+v$$

to a finite-dimensional problem.

Proposition 4.1. *For $x_0\in\Omega$ and $x_j\in\partial\Omega$, $j=1,\dots,l$, $|x_i-x_j|\geqslant\delta>0$, $i\neq j$, and $\mu_j>0$ large, there are unique $\alpha_\lambda(x,\mu)$ and $v_\lambda(x,\mu)\in E_{x_0,\mu_0}\cap F_{x,\mu,l}$, such that (4.1) and (4.2) hold. Moreover,*

$$\sum_{j=0}^{l}|\alpha_{\lambda,j}(x,\mu)-1|+\|v_\lambda(x,\mu)\|_\lambda=O\biggl(\sum_{j=0}^{l}\biggl(\biggl(\frac{\sqrt{\lambda}}{\mu_j}\biggr)^{1+\theta}+\frac{1}{\mu_j^{(2-\sigma)/2+\theta}}+\frac{1}{\mu_j^{1-\theta}}\biggr)\biggr),$$

for some small $\theta>0$, where $\sigma=(N-2)(q/2-1)>0$.

Proof. The proof of proposition 4.1 is similar to that of proposition 2.1 Thus we omit it. □

For the rest of this section, we take $l=2k$, $k\geqslant 1$, $x_0=0$, $x_{2j-1}=z_{i_j}$, $x_{2j}=\hat z_{i_j}$, $\mu_{2j-1}=\mu_{2j}$, $j=1,\dots,k$. Without loss of generality, we assume $i_j=j$. For simplicity, we let $\alpha=\alpha_\lambda((0,z),\mu)$, $v=v_\lambda((0,z),\mu)$.

Arguing as in § 2 and § 3, we know that v is even in y_j, $a_{2j-1}=a_{2j}$, $b_{2j-1}=b_{2j}$ and $c_{ij}=0$. Similar to the proof of theorems 1.1 and 1.2, using the estimate in proposition 4.1, we see that to get a solution of the form

$$\alpha U_{0,\mu_0}+\sum_{j=1}^{k}(\alpha_{2j}U_{z_j,\mu_j}+\alpha_{2j}U_{\hat z_j,\mu_j})+v$$

is equivalent to solve the following system:

$$-\frac{G_1\lambda}{\mu_0^3}+\frac{G_2}{\mu_0^{3-\sigma}}=O\biggl(\sum_{j=0}^{k}\biggl(\biggl(\frac{\lambda}{\mu_j^2}\biggr)^{1+\tau}\frac{1}{\mu_j}+\frac{1}{\mu_j^{2+\tau}}+\frac{1}{\mu_j^{3-\sigma+\tau}}\biggr)\biggr), \tag{4.7}$$

$$\frac{G_3H(z_j)}{\mu_j^2}-\frac{G_1\lambda}{\mu_j^3}+\frac{G_2}{\mu_j^{3-\sigma}}=O\biggl(\sum_{j=0}^{k}\biggl(\biggl(\frac{\lambda}{\mu_j^2}\biggr)^{1+\tau}\frac{1}{\mu_j}+\frac{1}{\mu_j^{2+\tau}}+\frac{1}{\mu_j^{3-\sigma+\tau}}\biggr)\biggr). \tag{4.8}$$

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If $3-\sigma \leqslant 2$, then $1/\mu^2$ is a higher-order term than $1/\mu^{3-\sigma}$ for $3-\sigma<2$ and is the same order of $1/\mu^{3-\sigma}$ for $3-\sigma=2$. So we can solve (4.7) and (4.8) to get a solution $\mu_{j,\lambda}$ satisfying $\mu^{\sigma}_{j,\lambda}\lambda^{-1}\to c>0$. Thus we only need to consider the case $3-\sigma>2$. From (4.8), we expect $\mu_{j,\lambda}\lambda^{-1}\to c>0$. So the right-hand side of (4.7) is of order $1/\lambda^{2+\tau}$, which is not a higher-order term of the left-hand side of (4.7). Thus we are not able to determine μ_0 from (4.7).

In order to glue together the interior peak solution and the boundary peak solution, which have different concentration rates, more work is needed.

The main work of this section is to improve the estimate of (4.7). We shall prove that (4.5) is equivalent to

$$-\frac{G_1\lambda}{\mu_0^3}+\frac{G_2}{\mu_0^{3-\sigma}}=O\biggl(\biggl(\frac{\lambda}{\mu_0^2}\biggr)^{1+\tau}\frac{1}{\mu_0}+\frac{1}{\mu_0^{3-\sigma+\tau}}\biggr). \tag{4.9}$$

To get (4.9), we need to estimate $\|v-v_I-v_B\|_\lambda$. Let

$$f(u)=|u|^{2^*-2}u+a|u|^{q-2}u.$$

First, we have the following pointwise estimate for v_I and v_B.

Lemma 4.2. *Let v_I be the function obtained in proposition 2.1. Then for fixed small $\varepsilon>0$,*

$$v_I=O\biggl(\biggl(\frac{\sqrt{\lambda}}{\mu_0}\biggr)^{1+\tau}+\frac{1}{\mu_0^{(2-\sigma)/2+\tau}}\biggr)\qquad \forall y\in\bar{\Omega}\setminus B_\varepsilon(0).$$

Proof. By (2.2), we know that v_I satisfies

$$\biggl\langle\frac{\partial J_I}{\partial v},\varphi\biggr\rangle=a_0\langle U_{0,\mu_0},\varphi\rangle_\lambda+b_0\biggl\langle\frac{\partial U_{0,\mu_0}}{\partial\mu_0},\varphi\biggr\rangle_\lambda.$$

Thus, v_I satisfies

$$\begin{aligned}-\Delta v_I+\lambda v_I&=f(\alpha_0U_{0,\mu_0}+v_I)-\alpha_0U_{0,\mu_0}^{2^*-1}-\lambda\alpha_0U_{0,\mu_0}\\&\quad+a_0(U_{0,\mu_0}^{2^*-1}+\lambda U_{0,\mu_0})+b_0\biggl((2^*-1)U_{0,\mu_0}^{2^*-2}\frac{\partial U_{0,\mu_0}}{\partial\mu_0}+\lambda\frac{\partial U_{0,\mu_0}}{\partial\mu_0}\biggr)\\&=:F(y),\qquad y\in\Omega;\\ \frac{\partial v_I}{\partial\nu}&=-a_0U_{0,\mu_0}-b_0\frac{\partial U_{0,\mu_0}}{\partial\mu_0},\qquad y\in\partial\Omega.\end{aligned}$$

Moreover, we have the following estimate,

$$\|v_I\|_\lambda=O\biggl(\biggl(\frac{\sqrt{\lambda}}{\mu_0}\biggr)^{1+\tau}+\frac{1}{\mu_0^{(2-\sigma)/2+\theta}}\biggr),$$

which is small if $\delta>0$ is small. We also have

$$|F(y)|\leqslant\frac{C\lambda}{\mu_0^{(N-2)/2}}+C|v_I|^{2^*-1}+C|v_I|^{q-1},\qquad |y|\geqslant\varepsilon/2>0.$$

Using [5] and [12, theorem 8.17], we have that

$$|v_I(y)|_{L^\infty(B_{\varepsilon/2}(y))} \leqslant C|v_I|_{L^{2N/(N-2)}(B_\varepsilon(y))} + \frac{C\lambda}{\mu_0^{(N-2)/2}} = O\biggl(\biggl(\frac{\sqrt{\lambda}}{\mu_0}\biggr)^{1+\tau} + \frac{1}{\mu_0^{(2-\sigma)/2+\tau}}\biggr) \qquad \forall |y| \geqslant \varepsilon > 0.$$

□

Similarly, we have the following lemma.

LEMMA 4.3. *Let v_B be the function obtained in proposition 3.1. Then*

$$v_B = O\biggl(\sum_{j=1}^{k}\biggl(\biggl(\frac{\sqrt{\lambda}}{\mu_j}\biggr)^{1+\tau} + \frac{1}{\mu_j^{(2-\sigma)/2+\tau}}\biggr)\biggr) \quad \forall y \in \bar{\Omega} \setminus \cup_{j=1}^{k}(B_\varepsilon(z_j) \cup B_\varepsilon(\hat{z}_j)).$$

Now we are ready to estimate $\|v - v_I - v_B\|_\lambda$.

PROPOSITION 4.4. *We have*

$$\|v - v_I - v_B\|_\lambda = O\biggl(\biggl(\frac{\sqrt{\lambda}}{\mu_0}\biggr)^{1+\tau} + \frac{1}{\mu_0^{(2-\sigma)/2+\tau}}\biggr).$$

Proof. By (2.1) and (4.1), we have

$$\biggl\langle \alpha_0 U_{0,\mu_0} + \sum_{j=1}^{k} \alpha_{2j}(U_{z_j,\mu_j} + U_{\hat{z}_j,\mu_j}), U_{0,\mu_0}\biggr\rangle_\lambda - \int_\Omega f\biggl(\alpha_0 U_{0,\mu_0} + \sum_{j=1}^{k}\alpha_{2j}(U_{z_j,\mu_j} + U_{\hat{z}_j,\mu_j}) + v\biggr)U_{0,\mu_0} = 0 \quad (4.10)$$

and

$$\langle \alpha_{0,I}U_{0,\mu_0}, U_{0,\mu_0}\rangle_\lambda - \int_\Omega f(\alpha_{0,I}U_{0,\mu_0} + v_I)U_{0,\mu_0} = 0. \quad (4.11)$$

Let

$$G(x) = \alpha_{0,I}U_{0,\mu_0} + v_I + v_B.$$

Since

$$\langle U_{z_j,\mu_j}, U_{0,\mu_0}\rangle_\lambda = O\biggl(\frac{1}{(\mu_0\mu_j)^{(N-2)/2}} + \frac{\lambda}{(\mu_0\mu_j)^{(N-2)/2}}\biggr)$$

and (by lemma 4.3)

$$\begin{aligned}\int_\Omega v_B^{2^*-1}U_{0,\mu_0} &= \int_{\Omega\setminus B_\varepsilon(0)} v_B^{2^*-1}U_{0,\mu_0} + \int_{\Omega\cap B_\varepsilon(0)} v_B^{2^*-1}U_{0,\mu_0} \\ &\leqslant \frac{C}{\mu_0^{(N-2)/2}}\|v_B\|^{2^*-1} + C|v_B|^{2^*-1}_{L^\infty(B_\varepsilon(0))}\int_\Omega U_{0,\mu_0} \\ &= o\biggl(\frac{1}{\mu_0^{(N-2)/2}}\biggr)\end{aligned}$$

for $j > 0$, we have

$$\begin{aligned}
&\langle(\alpha_0-\alpha_{0,I})U_{0,\mu_0},U_{0,\mu_0}\rangle_\lambda+o\biggl(\frac{1}{\mu_0^{(N-2)/2}}\biggr)\\
&\qquad=\int_\Omega f\biggl(\alpha_{0,\mu_0}U_{0,\mu_0}+\sum_{j=1}^k\alpha_{2j,\lambda}(U_{z_j,\mu_j}+U_{\hat z_j,\mu_j})+v_\lambda\biggr)U_{0,\mu_0}\\
&\qquad\quad-\int_\Omega f(\alpha_{0,\mu_0,I}U_{0,\mu_0}+v_I)U_{0,\mu_0}\\
&\qquad=\int_\Omega f\biggl(\alpha_{0,\mu_0}U_{0,\mu_0}+\sum_{j=1}^k\alpha_{2j,\lambda}(U_{z_j,\mu_j}+U_{\hat z_j,\mu_j})+v_\lambda\biggr)U_{0,\mu_0}\\
&\qquad\quad-\int_\Omega f(G(x))U_{0,\mu_0}+o\biggl(\frac{1}{\mu_0^{(N-2)/2}}\biggr)\\
&\qquad=\int_\Omega f'(G(x))(\alpha_{0,\mu_0}-\alpha_{0,\mu_0,I})U^2_{0,\mu_0}+\int_\Omega f'(G(x))(v-v_I-v_B)U_{0,\mu_0}\\
&\qquad\quad+O(|\alpha_{0,\mu_0}-\alpha_{0,\mu_0,I}|^{1+\tau})\\
&\qquad\quad+O(\|v-v_I-v_B\|^{1+\tau})+o\biggl(\frac{1}{\mu_0^{(N-2)/2}}\biggr),
\end{aligned}\tag{4.12}$$

for some $\tau > 0$.

Noting that

$$\|U_{0,\mu_0}\|^2-(2^*-1)\int_\Omega U_{0,\mu_0}^{2^*}-a(q-1)\int_\Omega U_{0,\mu_0}^q=-(2^*-2)(A+o(1))<0,$$

$\langle U_{0,\mu_0},U_{z_{2j},\mu_j}\rangle_\varepsilon=o(1)$ and $\int_\Omega f'(G(x))U_{0,\mu_0}U_{z_{2j},\mu_j}=o(1)$ for $j>0$, where $o(1)\to 0$ as $\mu_0,\mu_j\to+\infty$, we can solve (4.12) to get

$$|\alpha_0-\alpha_{0,I}|=o\biggl(\frac{1}{\mu_0^{(N-2)/2}}\biggr)+O(\|v-v_I-v_B\|).\tag{4.13}$$

Similarly, we have

$$|\alpha_{2j}-\alpha_{2j,B}|=o\biggl(\frac{1}{\mu_0^{(N-2)/2}}\biggr)+O(\|v-v_I-v_B\|).\tag{4.14}$$

On the other hand, by (2.2), (3.2) and (4.2), we have

$$\begin{aligned}
&\biggl\langle\alpha_0U_{0,\mu_0}+\sum_{j=1}^k\alpha_{2j}(U_{z_j,\mu_j}+U_{\hat z_j,\mu_j})+v,\phi\biggr\rangle_\lambda\\
&\quad-\int_\Omega f\biggl(\alpha_0U_{0,\mu_0}+\sum_{j=1}^k\alpha_{2j}(U_{z_j,\mu_j}+U_{\hat z_j,\mu_j})+v\biggr)\phi=0,\qquad\phi\in E_{0,\mu_0}\cap F_{z,\mu,2k},
\end{aligned}\tag{4.15}$$

$$\langle\alpha_{0,I}U_{0,\mu_0}+v_I,\phi\rangle_\lambda-\int_\Omega f(\alpha_{0,I}U_{0,\mu_0}+v_I)\phi=0,\qquad\phi\in E_{0,\mu_0},\tag{4.16}$$

and

$$\Big\langle \sum_{j=1}^{k}\alpha_{2j,B}(U_{z_j,\mu_j}+U_{\hat z_j,\mu_j})+v_B,\phi\Big\rangle_\lambda - \int_\Omega f\Big(\sum_{j=1}^{k}\alpha_{2j,B}(U_{z_j,\mu_j}+U_{\hat z_j,\mu_j})+v_B\Big)\phi=0, \qquad \phi\in F_{z,\mu,2k}. \tag{4.17}$$

Thus,

$$\begin{aligned}\langle v-v_I-v_B,\phi\rangle_\lambda &= \int_\Omega f\Big(\alpha_0U_{0,\mu_0}+\sum_{j=1}^{k}\alpha_{2j}(U_{z_j,\mu_j}+U_{\hat z_j,\mu_j})+v\Big)\phi\\ &\quad-\int_\Omega f(\alpha_{0,I}U_{0,\mu_0}+v_I)\phi\\ &\quad-\int_\Omega f\Big(\sum_{j=1}^{k}\alpha_{2j,B}(U_{z_j,\mu_j}+U_{\hat z_j,\mu_j})+v_B\Big)\phi,\\ &\qquad\qquad \forall\phi\in E_{0,\mu_0}\cap F_{z,\mu,2k}.\end{aligned} \tag{4.18}$$

Let

$$G_1(x)=\alpha_{0,I}U_{0,\mu_0}+v_I+\sum_{j=1}^{k}\alpha_{2j,B}(U_{z_j,\mu_j}+U_{\hat z_j,\mu_j})+v_B. \tag{4.19}$$

By lemma 4.2, for $p=2^*$ or $p=q$, we have

$$\begin{aligned}&\int_\Omega (U_{0,\mu_0}+|v_I|)(U_{z_j,\mu_j}+|v_B|)^{p-2}|\phi|\\ &\quad\leqslant o(1)\int_{\Omega\cap B_\varepsilon(0)}(U_{0,\mu_0}+|v_I|)|\phi|\\ &\qquad + C\bigg(\frac{1}{\mu_0^{(N-2)/2}}+\bigg(\frac{\sqrt{\lambda}}{\mu_0^2}\bigg)^{1+\tau}\bigg)\int_{\Omega\setminus B_\varepsilon(0)}(U_{z_j,\mu_j}+|v_B|)^{p-2}|\phi|\\ &\quad= o\bigg(\frac{1}{\mu_0^{(N-2)/2}}+\|v_I\|+\frac{1}{\mu_0^{(N-2)/2}}+\bigg(\frac{\sqrt{\lambda}}{\mu_0^2}\bigg)^{1+\tau}\bigg)\|\phi\|\\ &\quad= o\bigg(\bigg(\frac{\sqrt{\lambda}}{\mu_0^2}\bigg)^{1+\tau}+\frac{1}{\mu_0^{(N-2)/2}}\bigg)\|\phi\|.\end{aligned} \tag{4.20}$$

Using the following inequality:

$$\begin{aligned}||a+b|^p-|a|^p-|b|^p| &\leqslant \begin{cases} C|a|^{p-1}|b|, & \text{if } |a|\geqslant|b|,\\ C|a||b|^{p-1}, & \text{if } |a|<|b|,\end{cases}\\ &\leqslant C|a||b|^{p-1},\end{aligned}$$

if $1<p\leqslant 2$,

$$||a+b|^p-|a|^p-|b|^p|\leqslant C(|a|^{p-1}|b|+|a||b|^{p-1}),$$

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if $p > 2$, and noting (4.20), we see

$$\begin{aligned}\int_\Omega f(G_1(x))\phi &= \int_\Omega f(\alpha_{0,I}U_{0,\mu_0}+v_I)\phi + \int_\Omega f\biggl(\sum_{j=1}^k \alpha_{2j,B}(U_{z_j,\mu_j}+U_{\hat z_j,\mu_j})+v_B\biggr)\phi \\ &\quad + O\biggl(\int_\Omega (U_{0,\mu_0}+|v_I|)\biggl(\sum_{j=1}^k (U_{z_j,\mu_j}+U_{\hat z_j,\mu_j})+|v_B|\biggr)^{2^*-2}|\phi|\biggr) \\ &\quad + O\biggl(\int_\Omega (U_{0,\mu_0}+|v_I|)\biggl(\sum_{j=1}^k (U_{z_j,\mu_j}+U_{\hat z_j,\mu_j})+|v_B|\biggr)^{q-2}|\phi|\biggr) \\ &= \int_\Omega f(\alpha_{0,I}U_{0,\mu_0}+v_I)\phi + \int_\Omega f\biggl(\sum_{j=1}^k \alpha_{2j,B}(U_{z_j,\mu_j}+U_{\hat z_j,\mu_j})+v_B\biggr)\phi \\ &\quad + o\biggl(\biggl(\frac{\sqrt{\lambda}}{\mu_0^2}\biggr)^{1+\tau}+\frac{1}{\mu_0^{(N-2)/2}}\biggr)\|\phi\|. \end{aligned} \tag{4.21}$$

Combining (4.18) and (4.21), we obtain

$$\begin{aligned}\langle v-v_I-v_B,\phi\rangle_\lambda &= \int_\Omega f\biggl(\alpha_0 U_{0,\mu_0}+\sum_{j=1}^k \alpha_{2j}(U_{z_j,\mu_j}+U_{\hat z_j,\mu_j})+v-f(G_1(x))\biggr)\phi \\ &\qquad + o\biggl(\biggl(\frac{\sqrt{\lambda}}{\mu_0^2}\biggr)^{1+\tau}+\frac{1}{\mu_0^{(N-2)/2}}\biggr)\|\phi\| \\ &= \int_\Omega f'(G_1(x))(v-v_I-v_B)\phi \\ &\qquad + \int_\Omega f'(G_1(x))(\alpha_0-\alpha_{0,I})U_{0,\mu_0}\phi \\ &\qquad + \int_\Omega f'(G_1(x))\sum_{j=1}^k(\alpha_{2j}-\alpha_{2j,B})(U_{z_j,\mu_j}+U_{\hat z_j,\mu_j})\phi \\ &\qquad + O\biggl(|\alpha_0-\alpha_{0,I}|^{1+\sigma}+\sum_{j=1}^k|\alpha_{2j}-\alpha_{2j,B}|^{1+\sigma}\biggr)\|\phi\| \\ &\qquad + O(\|v-v_I-v_B\|_\lambda^{1+\sigma})\|\phi\|_\lambda \\ &\qquad + o\biggl(\biggl(\frac{\sqrt{\lambda}}{\mu_0^2}\biggr)^{1+\tau}+\frac{1}{\mu_0^{(N-2)/2}}\biggr)\|\phi\|. \end{aligned} \tag{4.22}$$

Since for any $\phi \in E_{0,\mu_0}\cap F_{z,\mu,2k}$, we have

$$\begin{aligned}\int_\Omega f'(G_1(x))U_{z_j,\mu_j}\phi &= (2^*-1)\int_\Omega (G_1^{2^*-2}(x)-U_{z_j,\mu_j}^{2^*-2})U_{z_j,\mu_j}\phi + o(1)\|\phi\| \\ &= o(1)\|\phi\|. \end{aligned} \tag{4.23}$$

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Consequently, from (4.22), (4.13) and (4.23), we obtain

$$
\begin{aligned}
\langle v - v_I - v_B, \phi\rangle_\lambda = (2^* - 1)\int_\Omega G_1^{2^*-2}(x)(v - v_I - v_B)\phi \\
+ o(\|v - v_I - v_B\|_\lambda \|\phi\|_\lambda) \\
+ o\biggl(\biggl(\frac{\sqrt{\lambda}}{\mu_0^2}\biggr)^{1+\tau} + \frac{1}{\mu_0^{(N-2)/2}}\biggr)\|\phi\|_\lambda, \quad \forall \phi \in E_{0,\mu_0} \cap F_{z,\mu,2k}.
\end{aligned}
\tag{4.24}
$$

Choose β_j and γ_{ij} such that

$$
\begin{aligned}
\phi = v - v_I - v_B + \beta_0 U_{0,\mu_0} + \sum_{j=1}^{k}(\beta_{2j-1}U_{z_j,\mu_j} + \beta_{2j}U_{\hat{z}_j,\mu_j}) \\
+ \eta_0 \frac{\partial U_{0,\mu_0}}{\partial \mu_0} + \sum_{j=1}^{k}\biggl(\eta_{2j-1}\frac{\partial U_{z_j,\mu_j}}{\partial \mu_j} + \eta_{2j}\frac{\partial U_{\hat{z}_j,\mu_j}}{\partial \mu_j}\biggr) \\
+ \sum_{i=1}^{N}\gamma_{0i}\frac{\partial U_{0,\mu_0}}{\partial x_{0i}} + \sum_{j=1}^{k}\sum_{i=1}^{N-1}\biggl(\gamma_{2j-1,i}\frac{\partial U_{z_j,\mu_j}}{\partial \tau_{2j-1,i}} + \gamma_{2j}\frac{\partial U_{\hat{z}_j,\mu_j}}{\partial \tau_{2j,i}}\biggr) \\
\in E_{0,\mu_0} \cap F_{z,\mu,2k}.
\end{aligned}
$$

Noting that

$$
\langle v_I, U_{z_j,\mu_j}\rangle_\lambda = O(\|v_I\|_\lambda), \qquad \langle v_B, U_{0,\mu_0}\rangle_\lambda = O\biggl(\frac{1}{\mu_0^{(N-2)/2}}\biggr),
$$

etc., we obtain

$$
\beta_j, \eta_j, \gamma_{ij} = O\biggl(\biggl(\frac{\sqrt{\lambda}}{\mu_0^2}\biggr)^{1+\tau} + \frac{1}{\mu_0^{(2-\sigma)/2+\tau}}\biggr).
$$

So we find from (4.24) that

$$
\|\phi\|_\lambda^2 \leqslant O\biggl(\biggl(\frac{\sqrt{\lambda}}{\mu_0^2}\biggr)^{1+\tau} + \frac{1}{\mu_0^{(2-\sigma)/2+\tau}}\biggr)\|\phi\|_\lambda
$$

and the result follows. □

As a direct consequence of proposition 4.4, we have the following expansion.

PROPOSITION 4.5. *Let (α, v) be the map obtained in proposition 4.4. Suppose that $\sigma < 1$. Then we have*

$$
\frac{\partial J}{\partial \mu_0} = -\frac{G_1\lambda}{\mu_0^3} + \frac{G_2}{\mu_0^{3-\sigma}} + O\biggl(\biggl(\frac{\lambda}{\mu_0^2}\biggr)^{1+\tau}\frac{1}{\mu_0} + \frac{1}{\mu_0^{3-\sigma+\tau}}\biggr), \tag{4.25}
$$

$$
\biggl\langle \frac{\partial J}{\partial v}, U_{0,\mu_0}\biggr\rangle = O\biggl(\frac{\lambda}{\mu_0^2} + \frac{1}{\mu_0^{2-\sigma}}\biggr). \tag{4.26}
$$

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Proof. Let

$$H(y)=\alpha_0U_{0,\mu_0}+\sum_{j=1}^{k}\alpha_{2j}(U_{z_j,\mu_j}+U_{\hat z_j,\mu_j}).$$

Then

$$\begin{aligned}\frac{\partial J}{\partial\mu_0}&=\left\langle H,\frac{\partial U_{0,\mu_0}}{\partial\mu_0}\right\rangle_\lambda-\int_\Omega f(H(y)+v)\frac{\partial U_{0,\mu_0}}{\partial\mu_0}\\&=-\int_\Omega[f(H(y)+v)-f(H(y))-f'(H(y))v]\frac{\partial U_{0,\mu_0}}{\partial\mu_0}\\&\quad-\int_\Omega(f'(H(y))-(2^*-1)(\alpha_0U_{0,\mu_0})^{2^*-2})v\frac{\partial U_{0,\mu_0}}{\partial\mu_0}\\&\quad-(2^*-1)\int_\Omega(\alpha_0U_{0,\mu_0})^{2^*-2}\frac{\partial U_{0,\mu_0}}{\partial\mu_0}v\\&\quad+\left\langle H,\frac{\partial U_{0,\mu_0}}{\partial\mu_0}\right\rangle_\lambda-\int_\Omega f(H(y))\frac{\partial U_{0,\mu_0}}{\partial\mu_0}\\&=:I_1+I_2+I_3+I_4,\end{aligned}\tag{4.27}$$

where I_i, $i=1,2,3,4$, is the natural splitting of the last formula.

We have

$$\begin{aligned}|I_1|&\leqslant\frac{C}{\mu_0}\int_{\Omega\cap B_\varepsilon(0)}U_{0,\mu_0}^{2^*-2}|v|^2+O\left(\frac{1}{\mu_0^3}\right)\\&\leqslant\frac{C}{\mu_0}\int_{\Omega\cap B_\varepsilon(0)}U_{0,\mu_0}^{2^*-2}|v-v_I-v_B|^2+\frac{C}{\mu_0}\int_{\Omega\cap B_\varepsilon(0)}U_{0,\mu_0}^{2^*-2}|v_B|^2\\&\quad+\frac{C}{\mu_0}\int_{\Omega\cap B_\varepsilon(0)}U_{0,\mu_0}^{2^*-2}|v_I|^2+O\left(\frac{1}{\mu_0^3}\right)\\&\leqslant\frac{C}{\mu_0}\|v-v_I-v_B\|^2+\frac{C}{\mu_0^3}+\frac{C}{\mu_0}\|v_I\|^2\\&=O\left(\left(\frac{\lambda}{\mu_0^2}\right)^{1+\tau}\frac{1}{\mu_0}+\frac{1}{\mu_0^{3-\sigma+\tau}}\right).\end{aligned}\tag{4.28}$$

Now we estimate I_2:

$$\begin{aligned}|I_2|&\leqslant\frac{C}{\mu_0}\int_{\Omega\cap B_\varepsilon(0)}(U_{0,\mu_0}^{2^*-2}+U_{0,\mu_0}^{q-1})|v|+\frac{C}{\mu_0^{1+(N-2)/2}}\\&\leqslant\frac{C}{\mu_0}\left(\int_\Omega U_{0,\mu_0}^{(2^*-2)2N/(N+2)}\right)^{(N+2)/(2N)}\|v\|\\&\quad+\frac{C}{\mu_0}\int_{\Omega\cap B_\varepsilon(0)}U_{0,\mu_0}^{q-1}|v-v_I-v_B|+\frac{C}{\mu_0}\int_{\Omega\cap B_\varepsilon(0)}U_{0,\mu_0}^{q-1}|v_I|\\&\quad+\frac{C}{\mu_0}\int_{\Omega\cap B_\varepsilon(0)}U_{0,\mu_0}^{q-1}|v_B|+\frac{C}{\mu_0^{1+(N-2)/2}}\\&=O\left(\frac{1}{\mu_0^3}\right).\end{aligned}\tag{4.29}$$

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As for the estimate of I_3, we have

$$\begin{aligned}|I_3| &\leqslant C\left|\int_{\partial\Omega}\frac{\partial}{\partial\nu}\left(\frac{\partial U_{0,\mu_0}}{\partial\mu_0}\right)v\right| + O\left(\frac{\lambda}{\mu_0}\int_\Omega U_{0,\mu_0}|v|\right)\\ &\leqslant O\left(\frac{\lambda}{\mu_0^{1+(N-2)/2}} + \frac{\lambda}{\mu_0}\int_\Omega U_{0,\mu_0}|v-v_I-v_B|\right.\\ &\qquad\left. + \frac{\lambda}{\mu_0}\int_\Omega U_{0,\mu_0}|v_I| + \frac{\lambda}{\mu_0}\int_\Omega U_{0,\mu_0}|v_B|\right)\\ &= O\left(\left(\frac{\lambda}{\mu_0^2}\right)^{1+\tau}\frac{1}{\mu_0} + \frac{1}{\mu_0^{3-\sigma+\tau}}\right). \end{aligned} \tag{4.30}$$

Finally, we estimate I_4.

$$\begin{aligned}I_4 &= \left\langle H, \frac{\partial U_{0,\mu_0}}{\partial\mu_0}\right\rangle - \int_\Omega U_{0,\mu_0}^{2^*-1}\frac{\partial U_{0,\mu_0}}{\partial\mu_0} - a\int_\Omega U_{0,\mu_0}^{q-1}\frac{\partial U_{0,\mu_0}}{\partial\mu_0} + O\left(\frac{1}{\mu_0^{1+(N-2)/2}}\right)\\ &= -\frac{G_1\lambda}{\mu_0^3} + \frac{G_2}{\mu_0^{3-\sigma}} + O\left(\left(\frac{\lambda}{\mu_0^2}\right)^{1+\tau}\frac{1}{\mu_0} + \frac{1}{\mu_0^{3-\sigma+\tau}}\right). \end{aligned} \tag{4.31}$$

So (4.25) follows from (4.28), (4.29), (4.30) and (4.31). We can prove (4.26) in a similar way. □

LEMMA 4.6. *Let a_j, b_j be the constants in proposition 4.1. Then*

$$a_0, \mu_0 b_0 = O\left(\frac{\lambda}{\mu_0^2} + \frac{1}{\mu_0^{2-\sigma}}\right).$$

Proof. We have

$$a_0\langle U_{0,\mu_0}, U_{0,\mu_0}\rangle_\lambda + b_0\left\langle \frac{\partial U_{0,\mu_0}}{\partial\mu_0}, U_{0,\mu_0}\right\rangle_\lambda = \left\langle \frac{\partial J}{\partial v}, U_{0,\mu_0}\right\rangle + O\left(\frac{1}{\mu_0^{(N-2)/2}}\right), \tag{4.32}$$

$$a_0\left\langle U_{0,\mu_0}, \frac{\partial U_{0,\mu_0}}{\partial\mu_0}\right\rangle_\lambda + b_0\left\langle \frac{\partial U_{0,\mu_0}}{\partial\mu_0}, \frac{\partial U_{0,\mu_0}}{\partial\mu_0}\right\rangle_\lambda = \frac{\partial J}{\partial\mu_0} + O\left(\frac{1}{\mu_0^{(N-2)/2}}\right). \tag{4.33}$$

Using proposition 4.5, we can solve the above system to obtain the desired estimates. □

As a direct consequence of proposition 4.5 and lemma 4.6, we have the following proposition.

PROPOSITION 4.7. *Let (α, v) be the map obtained in proposition 4.4. Then (4.5) is equivalent to*

$$-\frac{G_1\lambda}{\mu_0^3} + \frac{G_2}{\mu_0^{3-\sigma}} = O\left(\left(\frac{\lambda}{\mu_0^2}\right)^{1+\tau}\frac{1}{\mu_0} + \frac{1}{\mu_0^{3-\sigma+\tau}}\right).$$

Proof of theorem 1.3. From the discussion in the beginning of this section, we only need to deal with the case $2-\sigma > 1$. By proposition 4.7 and (4.8), we see that (4.5)

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and (4.6) are equivalent to

$$-\frac{G_1\lambda}{\mu_0^3}+\frac{G_2}{\mu_0^{3-\sigma}}=O\biggl(\biggl(\frac{\lambda}{\mu_0^2}\biggr)^{1+\tau}\frac{1}{\mu_0}+\frac{1}{\mu_0^{3-\sigma+\tau}}\biggr), \tag{4.34}$$

$$-\frac{G_1\lambda}{\mu_j^3}+\frac{G_3H(z_j)}{\mu_j^2}=O\biggl(\sum_{j=0}^{k}\biggl(\biggl(\frac{\lambda}{\mu_j^2}\biggr)^{1+\tau}\frac{1}{\mu_j}+\frac{1}{\mu_j^{2+\tau}}\biggr)\biggr). \tag{4.35}$$

Let $\mu_0=t_0\lambda^{1/\sigma}$, $\mu_j=t_j\lambda$. Then (4.34) and (4.35) become

$$-\frac{G_1}{t_0^3}+\frac{G_2}{t_0^{3-\sigma}}=o(1), \tag{4.36}$$

$$-\frac{G_1}{t_j^3}+\frac{G_3H(z_j)}{t_j^2}=o(1), \tag{4.37}$$

where $o(1)\to 0$ as $\lambda\to+\infty$. It is easy to see that (4.36) and (4.37) has a solution $(t_{0\lambda},\dots,t_{k\lambda})$ satisfying

$$t_{0\lambda}\to\biggl(\frac{G_1}{G_2}\biggr)^{1/\sigma},\qquad t_{k\lambda}\to\frac{G_1}{G_3H(z_j)}\quad\text{as }\lambda\to+\infty.$$

□

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