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On the Cauchy problems for certain Boussinesq-α equations

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We study the Cauchy problem of certain Boussinesq-α equations in n dimensions with $n = 2$ or 3. We establish regularity for the solution under $\nabla u \in L^1(0, T; \dot{B}^0_{\infty,\infty}(\mathbb{R}^n))$. As a corollary, the smooth solution of the Leray-α–Boussinesq system exists globally, when $n = 2$. For the Lagrangian averaged Boussinesq equations, a regularity criterion $\nabla\theta \in L^1(0, T; L^\infty(\mathbb{R}^2))$ is established. Other Boussinesq systems with partial viscosity are also discussed in the paper.

1. Introduction

The interactive motion of a passive scalar (e.g. temperature) and the atmosphere is modelled by the following Boussinesq equations:

$$\left.\begin{aligned} v_t + (v\cdot\nabla)v + \nabla\pi &= \theta \boldsymbol{e}_n, \\ \theta_t + v\cdot\nabla\theta &= 0, \\ \operatorname{div} v &= 0, \qquad (x,t)\in\mathbb{R}^n\times\mathbb{R}_+, \\ (v,\theta)|_{t=0} &= (v_0,\theta_0), \qquad x\in\mathbb{R}^n. \end{aligned}\right\} \tag{B}$$

Here $n = 2$ or 3, $\boldsymbol{e}_2 := (0, 1)^{\mathrm{T}}$ and $\boldsymbol{e}_3 := (0, 0, 1)^{\mathrm{T}}$. System (B) for $n = 2$ has been the subject of numerous studies [1, 2, 14]. In [1, 2], the following regularity criterion was proved:

$$\nabla\theta \in L^1(0, T; L^\infty(\mathbb{R}^2)). \tag{1.1}$$

For the three-dimensional case, we refer the reader to [5] and references therein.

When $\theta = 0$, (B) reduces to the well-known Euler equations

$$\left.\begin{aligned} v_t + (v\cdot\nabla)v + \nabla\pi &= 0, \\ \operatorname{div} v &= 0, \qquad (x,t)\in\mathbb{R}^n\times\mathbb{R}_+, \\ v|_{t=0} &= v_0, \qquad x\in\mathbb{R}^n. \end{aligned}\right\} \tag{E}$$

The global existence for the three-dimensional Euler equations (E) is a very challenging open question. The main difficulty is understanding the effect of vortex stretching, which is absent from the two-dimensional Euler equations. As part of the effort to understand the vortex-stretching effect, various simplified model equations have been proposed in the literature. An interesting recent development is the Lagrangian averaged Euler equations [7, 8]:

$$\left.\begin{aligned} v_t + (u\cdot\nabla)v + \sum_j v_j\nabla u_j + \nabla\pi = 0, \\ u - \alpha^2\Delta u = v, \\ \operatorname{div} u = \operatorname{div} v = 0, \\ v|_{t=0} = v_0, \qquad x\in\mathbb{R}^n. \end{aligned}\right\} \tag{α-E}$$

The averaged Euler models have been used to study the average behaviour of the three-dimensional Euler and Navier–Stokes equations and used as a turbulent closure model [3]. The global existence of the three-dimensional Lagrangian averaged Euler equations is still an open question, although the Lagrangian averaged-Navier–Stokes equations have been shown to have global existence [6, 12]. In [4], Fan and Ozawa proved the following regularity criterion for (α-E):

$$\nabla u \in L^1(0,T;\dot B^0_{\infty,\infty}(\mathbb{R}^n)). \tag{1.2}$$

Motivated by the above interpretation of the averaged Euler–Lagrange equations, we can clearly apply the same averaging principle to the Boussinesq equations (B). We obtain the following Lagrangian averaged Boussinesq equations:

$$v_t + u\cdot\nabla v + \sum_j v_j\nabla u_j + \nabla\pi = \theta \boldsymbol{e}_n, \tag{1.3}$$

$$\theta_t + u\cdot\nabla\theta = 0, \tag{1.4}$$

$$u - \alpha^2\Delta u = v, \tag{1.5}$$

$$\operatorname{div} u = \operatorname{div} v = 0, \tag{1.6}$$

$$(v,\theta)|_{t=0} = (v_0,\theta_0), \quad x\in\mathbb{R}^n,\ n=2,3. \tag{1.7}$$

We also consider the following version of the Leray-α–Boussinesq model:

$$\left.\begin{aligned} v_t + u\cdot\nabla v + \nabla\pi = \theta\boldsymbol{e}_n, \\ \theta_t + u\cdot\nabla\theta = 0, \\ u - \alpha^2\Delta u = v, \\ \operatorname{div} u = \operatorname{div} v = 0, \\ (v,\theta)|_{t=0} = (v_0,\theta_0), \quad x\in\mathbb{R}^n,\ n=2,3, \end{aligned}\right\} \tag{1.8}$$

and the following version of the modified Leray-α–Boussinesq model:

$$\left.\begin{aligned} v_t + v\cdot\nabla u + \nabla\pi = \theta\boldsymbol{e}_n, \\ \theta_t + u\cdot\nabla\theta = 0, \end{aligned}\right\} \tag{1.9}$$

$$\left.\begin{aligned} u-\alpha^2\Delta u &= v,\\ \operatorname{div} u=\operatorname{div} v &= 0,\\ (v,\theta)|_{t=0} &= (v_0,\theta_0), \quad x\in\mathbb{R}^n,\ n=2,3. \end{aligned}\right\} \qquad (1.9\ \textit{cont.})$$

By standard energy estimates, it is easy to prove the well-posedness of local strong solutions to problem (1.3)–(1.7), (1.8) and (1.9), and hence we omit the details here.

THEOREM 1.1. *Let* $(v_0,\theta_0)\in H^3$ *and* $\operatorname{div} v_0=0$ *in* $\mathbb{R}^n(n=2,3)$. *There exists a positive time* T *such that (1.3)–(1.7), (1.8) or (1.9) has a unique solution* (v,θ) *in* $(0,T)$ *such that*

$$(v,\theta)\in L^\infty(0,T;H^3).$$

The aim of this paper is to study the regularity conditions of the problem (1.3)–(1.7), (1.8) and (1.9). We will prove these below.

THEOREM 1.2. *Let* $(v_0,\theta_0)\in H^3$ *and* $\operatorname{div} v_0=0$ *in* $\mathbb{R}^n$. *Let* (v,θ) *be a smooth solution to (1.3)–(1.7), (1.8) or (1.9). If* u *satisfies (1.2), then the solution* (v,θ) *can be extended beyond* T.

When $n=2$, testing $(1.8)_1$ for v and using

$$\|\theta\|_{L^2}\leqslant\|\theta_0\|_{L^2},$$

we easily get

$$u\in L^\infty(0,T;H^2),$$

whence we obtain

$$\nabla u\in L^\infty(0,T;H^1)\subset L^\infty(0,T;\mathrm{BMO})\subset L^\infty(0,T;\dot B^0_{\infty,\infty}).$$

By theorem 1.2, this proves the following theorem.

THEOREM 1.3. *Let* $(v_0,\theta_0)\in H^3$ *and* $\operatorname{div} v_0=0$ *in* $\mathbb{R}^2$. *Then the two-dimensional problem (1.8) has a global-in-time smooth solution* (v,θ) *such that*

$$(v,\theta)\in L^\infty(0,T;H^3(\mathbb{R}^2))$$

for any $T>0$.

REMARK 1.4. We are unable to prove theorem 1.3 for the two-dimensional problem (1.3)–(1.7) or for (1.9).

THEOREM 1.5. *Let* $(v_0,\theta_0)\in H^3$ *with* $\operatorname{div} v_0=0$ *in* $\mathbb{R}^2$. *Let* (v,θ) *be a smooth solution to the two-dimensional problem (1.3)–(1.7). If* θ *satisfies (1.1), then the solution* (v,θ) *can be extended beyond* T.

REMARK 1.6. We cannot prove theorem 1.5 for the two-dimensional problem (1.9).

DEFINITION 1.7 (Triebel [15]). Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be the Littlewood–Paley dyadic decomposition of unity that satisfies

$$\hat\phi\in C_0^\infty(B_2\setminus B_{1/2}),\quad \hat\phi_j(\xi)=\hat\phi(2^{-j}\xi)$$

and

$$\sum_{j\in\mathbb{Z}} \hat{\phi}_j(\xi) = 1$$

for any $\xi \neq 0$. The homogeneous Besov space $\dot{B}^s_{p,q} := \{f \in \mathcal{S}' : \|f\|_{\dot{B}^s_{p,q}} < \infty\}$ is introduced by the norm

$$\|f\|_{\dot{B}^s_{p,q}} := \left(\sum_{j\in\mathbb{Z}} \|2^{js}\phi_j * f\|^q_{L^p}\right)^{1/q}$$

for $s \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$.

In the proofs below, we will use the following bilinear commutator and product estimates due to Kato and Ponce [9] and Kenig *et al.* [10]:

$$\|\Lambda^r(fg) - f\Lambda^r g\|_{L^p} \leqslant C(\|\nabla f\|_{L^{p_1}}\|\Lambda^{r-1}g\|_{L^{q_1}} + \|\Lambda^r f\|_{L^{p_2}}\|g\|_{L^{q_2}}), \tag{1.10}$$

$$\|\Lambda^r(fg)\|_{L^p} \leqslant C(\|f\|_{L^{p_1}}\|\Lambda^r g\|_{L^{q_1}} + \|\Lambda^r f\|_{L^{p_2}}\|g\|_{L^{q_2}}), \tag{1.11}$$

with

$$r > 0, \quad \Lambda := (-\Delta)^{1/2} \quad \text{and} \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}.$$

We will also use the following Gagliardo–Nirenberg inequality (see [13] for a generalized form):

$$\|\nabla\Delta u\|^2_{L^4} \leqslant C\|\nabla u\|_{L^\infty}\|\nabla\Delta^2 u\|_{L^2}. \tag{1.12}$$

2. Proof of theorem 1.2

This section is devoted to the proof of theorem 1.2. Since we deal with the regularity conditions for smooth solutions, we only need to establish the *a priori* estimates for smooth solutions.

Since (1.8) and (1.9) are easier, we study only the problem (1.3)–(1.7).

First, from (1.4) and (1.6), it follows easily that

$$\|\theta(t)\|_{L^2\cap L^n} \leqslant \|\theta_0\|_{L^2\cap L^n}. \tag{2.1}$$

Testing (1.3) by u, using (1.5), (1.6) and (2.1), we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int u^2 + \alpha^2|\nabla u|^2 \,\mathrm{d}x = \int \theta \boldsymbol{e}_n \cdot u \,\mathrm{d}x \leqslant \|\theta_0\|_{L^2}\|u\|_{L^2},$$

whence we obtain

$$\|u\|_{L^\infty(0,T;H^1)} \leqslant C. \tag{2.2}$$

Testing (1.3) by v, using (1.6) and (2.1), we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int v^2 \,\mathrm{d}x \leqslant C\|\nabla u\|_{L^\infty}\|v\|^2_{L^2} + \|\theta_0\|_{L^2}\|v\|_{L^2}. \tag{2.3}$$

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Applying Λ^3 to (1.3), testing by $\Lambda^3 v$, using (1.6), (1.10) and (1.11), we infer that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\Lambda^3 v|^2\,\mathrm{d}x &\leqslant \left|\int (\Lambda^3(u\cdot\nabla v) - u\cdot\nabla\Lambda^3 v)\cdot\Lambda^3 v\,\mathrm{d}x\right| \\ &\quad + \sum_j \left|\int \Lambda^3(v_j\nabla u_j)\cdot\Lambda^3 v\,\mathrm{d}x\right| + \left|\int \Lambda^3\theta\boldsymbol{e}_n\cdot\Lambda^3 v\,\mathrm{d}x\right| \\ &\leqslant C(\|\nabla u\|_{L^\infty}\|\Lambda^3 v\|_{L^2} + \|\Lambda^3 u\|_{L^4}\|\nabla v\|_{L^4} \\ &\quad + \|v\|_{L^4}\|\Delta^2 u\|_{L^4} + \|\Lambda^3\theta\|_{L^2})\|\Lambda^3 v\|_{L^2}. \qquad (2.4)\end{aligned}$$

We use (1.12), (1.5) and (2.2) to bound

$$\begin{aligned}\|\Lambda^3 u\|_{L^4}\|\nabla v\|_{L^4} &\leqslant C\|\Lambda^3 u\|_{L^4}(\|\nabla u\|_{L^4} + \|\Lambda^3 u\|_{L^4}) \\ &\leqslant C\|\Lambda^3 u\|_{L^4}(\|\nabla u\|_{L^2}^{8/(8+n)}\|\Lambda^3 u\|_{L^4}^{n/(8+n)} + \|\Lambda^3 u\|_{L^4}) \\ &\leqslant C\|\Lambda^3 u\|_{L^4}(1 + \|\Lambda^3 u\|_{L^4}) \\ &\leqslant C + C\|\Lambda^3 u\|_{L^4}^2 \\ &\leqslant C + C\|\nabla u\|_{L^\infty}\|\nabla\Delta^2 u\|_{L^2} \\ &\leqslant C + C\|\nabla u\|_{L^\infty}\|\Lambda^3 v\|_{L^2}, \qquad (2.5)\end{aligned}$$

where (in the second inequality) we used

$$\|\nabla u\|_{L^4} \leqslant C\|\nabla u\|_{L^2}^{8/(8+n)}\|\Lambda^3 u\|_{L^4}^{n/(8+n)}.$$

In what follows, we will use the following two Gagliardo–Nirenberg inequalities (also found in [13]):

$$\|\Lambda^4 u\|_{L^4} \leqslant C\|\nabla u\|_{L^\infty}^{\beta}\|\Lambda^5 u\|_{L^2}^{1-\beta}, \qquad (2.6)$$

$$\|\Delta u\|_{L^4} \leqslant C\|\nabla u\|_{L^\infty}^{1-\beta}\|\Lambda^5 u\|_{L^2}^{\beta}, \qquad (2.7)$$

with $\beta = (4-n)/(16-2n)$.

Using (2.6), (2.7), (1.5) and (2.2), we bound

$$\begin{aligned}\|v\|_{L^4}\|\Delta^2 u\|_{L^4} &\leqslant C(\|u\|_{L^4} + \|\Delta u\|_{L^4})\|\Delta^2 u\|_{L^4} \\ &\leqslant C(1 + \|\Delta u\|_{L^4})\|\Delta^2 u\|_{L^4} \\ &\leqslant C\|\Delta^2 u\|_{L^4} + C\|\Delta u\|_{L^4}\|\Delta^2 u\|_{L^4} \\ &\leqslant C\|\nabla u\|_{L^2}^{1/4-n/16}\|\Lambda^5 u\|_{L^2}^{3/4+n/16} + C\|\nabla u\|_{L^\infty}\|\Lambda^5 u\|_{L^2} \\ &\leqslant C + C\|\Lambda^3 v\|_{L^2} + C\|\nabla u\|_{L^\infty}\|\Lambda^3 v\|_{L^2}. \qquad (2.8)\end{aligned}$$

Inserting (2.7) and (2.8) into (2.4), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\Lambda^3 v|^2\,\mathrm{d}x \leqslant C\|\nabla u\|_{L^\infty}\|\Lambda^3 v\|_{L^2}^2 + C\|\Lambda^3 v\|_{L^2}^2 + C\|\Lambda^3\theta\|_{L^2}^2 + C. \qquad (2.9)$$

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Taking Λ^3 to (1.4), testing by $\Lambda^3\theta$, using (1.10), (2.1), we deduce that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\Lambda^3\theta|^2\,\mathrm{d}x &\\ &\leqslant\left|\int(\Lambda^3(u\cdot\nabla\theta)-u\cdot\nabla\Lambda^3\theta)\Lambda^3\theta\,\mathrm{d}x\right|\\ &\leqslant C\|\nabla u\|_{L^\infty}\|\Lambda^3\theta\|_{L^2}^2+C\|\nabla\theta\|_{L^4}\|\Lambda^3u\|_{L^4}\|\Lambda^3\theta\|_{L^2}\\ &\leqslant C\|\nabla u\|_{L^\infty}\|\Lambda^3\theta\|_{L^2}^2+C\|\theta\|_{L^n}^{1/2}\|\Lambda^3\theta\|_{L^2}^{1/2}\cdot\|\nabla u\|_{L^\infty}^{1/2}\|\Lambda^5u\|_{L^2}^{1/2}\cdot\|\Lambda^3\theta\|_{L^2}\\ &\leqslant C\|\nabla u\|_{L^\infty}\|\Lambda^3\theta\|_{L^2}^2+C\|\Lambda^3\theta\|_{L^2}^{3/2}\cdot\|\nabla u\|_{L^\infty}^{1/4}\cdot\|\nabla u\|_{L^\infty}^{1/4}\|\Lambda^5u\|_{L^2}^{1/2}\\ &\leqslant C\|\nabla u\|_{L^\infty}\|\Lambda^3\theta\|_{L^2}^2+C\|\nabla u\|_{L^\infty}^{1/3}\|\Lambda^3\theta\|_{L^2}^2+C\|\nabla u\|_{L^\infty}\|\Lambda^3v\|_{L^2}^2. \quad (2.10)\end{aligned}$$

Here we have used the Gagliardo–Nirenberg inequalities:

$$\|\nabla\theta\|_{L^4}^2\leqslant C\|\theta\|_{L^n}\|\Lambda^3\theta\|_{L^2}, \quad (2.11)$$

$$\|\Lambda^3u\|_{L^4}^2\leqslant\|\nabla u\|_{L^\infty}\|\Lambda^5u\|_{L^2}. \quad (2.12)$$

Combining (2.9), (2.10), (2.3) and using the logarithmic Sobolev inequality [11]

$$\|\nabla u\|_{L^\infty}\leqslant C(1+\|\nabla u\|_{\dot B^0_{\infty,\infty}}\log(e+\|\Lambda^5u\|_{L^2})), \quad (2.13)$$

we conclude that

$$\|(v,\theta)\|_{L^\infty(0,T;H^3)}\leqslant C. \quad (2.14)$$

This completes the proof.

Remark 2.1. Consider the following three-dimensional Boussinesq-α system with partial viscosity:

$$\left.\begin{aligned}v_t+u\cdot\nabla v+\sum_j v_j\nabla u_j+\nabla\pi-\Delta v=\theta\boldsymbol{e}_3,\\ \theta_t+u\cdot\nabla\theta=0,\\ u-\alpha^2\Delta u=v,\\ \operatorname{div}u=\operatorname{div}v=0,\\ (v,\theta)|_{t=0}=(v_0,\theta_0).\end{aligned}\right\} \quad (2.15)$$

It is easy to prove that theorem 1.2 holds for (2.15). Specifically, testing $(2.15)_1$ by u and using (2.1), we have

$$u\in L^2(0,T;H^2). \quad (2.16)$$

Testing $(2.15)_1$ by v, using (2.16), we easily get

$$u\in L^2(0,T;H^3),$$

whence we obtain

$$\nabla u\in L^2(0,T;L^\infty)\subset L^2(0,T;\mathrm{BMO})\subset L^2(0,T;\dot B^0_{\infty,\infty}).$$

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THEOREM 2.2. *Let* $(v_0, \theta_0) \in H^3$ *and* $\operatorname{div} v_0 = 0$ *in* $\mathbb{R}^3$. *Then (2.15) has a unique smooth solution* (v, θ) *such that*

$$(v, \theta) \in L^\infty(0, T; H^3), \quad v \in L^2(0, T; H^4),$$

for any $T > 0$.

REMARK 2.3. Theorem 2.2 also holds for the Leray-α–Boussinesq system with partial viscosity:

$$\left.\begin{aligned} v_t + u \cdot \nabla v + \nabla \pi - \Delta v &= \theta \boldsymbol{e}_3, \\ \theta_t + u \cdot \nabla \theta &= 0, \\ u - \alpha^2 \Delta u &= v, \\ \operatorname{div} u = \operatorname{div} v &= 0, \\ (v, \theta)|_{t=0} &= (v_0, \theta_0). \end{aligned}\right\} \tag{2.17}$$

REMARK 2.4. Theorem 2.2 also holds for the following modified Leray-α–Boussinesq system with partial viscosity:

$$\left.\begin{aligned} v_t + v \cdot \nabla u + \nabla \pi - \Delta v &= \theta \boldsymbol{e}_3, \\ \theta_t + u \cdot \nabla \theta &= 0, \\ u - \alpha^2 \Delta u &= v, \\ \operatorname{div} u = \operatorname{div} v &= 0, \\ (v, \theta)|_{t=0} &= (v_0, \theta_0). \end{aligned}\right\} \tag{2.18}$$

3. Proof of theorem 1.5

First, we still have to prove (2.1) and (2.2).

Taking Λ to (1.4), testing by $\Lambda\theta$, using (1.6), (1.10) and (2.2), we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int |\Lambda\theta|^2 \,\mathrm{d}x \leqslant \|\nabla u\|_{L^2} \|\nabla\theta\|_{L^\infty} \|\Lambda\theta\|_{L^2} \leqslant C \|\nabla\theta\|_{L^\infty} \|\Lambda\theta\|_{L^2},$$

whence we obtain

$$\|\theta\|_{L^\infty(0,T;H^1)} \leqslant C, \tag{3.1}$$

due to Gronwall's inequality and (1.1).

Taking curl to (1.3), defining $\omega := \operatorname{curl} v$, we have

$$\omega_t + u \cdot \nabla\omega = \operatorname{curl}(\theta \boldsymbol{e}_2) = \partial_1 \theta. \tag{3.2}$$

Testing (3.2) by ω, using (1.6) and (3.1), we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \omega^2 \,\mathrm{d}x \leqslant \int \partial_1\theta\omega \,\mathrm{d}x \leqslant \|\partial_1\theta\|_{L^2} \|\omega\|_{L^2},$$

whence we obtain

$$\|\omega\|_{L^\infty(0,T;L^2)} \leqslant C. \tag{3.3}$$

This proves

$$\|u\|_{L^\infty(0,T;H^3)} \leqslant C,$$

whence we obtain

$$\|\nabla u\|_{L^\infty(0,T;\dot{B}^0_{\infty,\infty})} \leqslant C. \tag{3.4}$$

This completes the proof thanks to theorem 1.2.

REMARK 3.1. Consider the following two-dimensional Boussinesq-α system with partial viscosity:

$$v_t + u\cdot\nabla v + \sum_j v_j \nabla u_j + \nabla\pi = \theta \boldsymbol{e}_2, \tag{3.5}$$

$$\theta_t + u\cdot\nabla\theta = \partial_1^2\theta, \tag{3.6}$$

$$u - \alpha^2\Delta u = v, \tag{3.7}$$

$$\operatorname{div} u = \operatorname{div} v = 0, \tag{3.8}$$

$$(v,\theta)|_{t=0} = (v_0,\theta_0). \tag{3.9}$$

Testing (3.6) by θ, using (3.8), we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \theta^2\,\mathrm{d}x + \int |\partial_1\theta|^2\,\mathrm{d}x = 0,$$

whence we obtain

$$\|\partial_1\theta\|_{L^2(0,T;L^2)} \leqslant C. \tag{3.10}$$

Taking curl to (3.5), we have (3.2). Thus, we arrive at (3.4). By theorem 1.5, this proves the following theorem.

THEOREM 3.2. *Let $(v_0,\theta_0) \in H^3$ and $\operatorname{div} v_0 = 0$ in $\mathbb{R}^2$. Then the problem (3.5)–(3.9) has a unique smooth solution (v,θ) such that*

$$(v,\theta) \in L^\infty(0,T;H^3), \quad \partial_1\theta \in L^2(0,T;H^3),$$

for any $T > 0$.

REMARK 3.3. Consider the following two-dimensional Boussinesq-α system with partial viscosity:

$$v_t + u\cdot\nabla v + \sum_j v_j\nabla u_j + \nabla\pi - \partial_1^2 v = \theta\boldsymbol{e}_2, \tag{3.11}$$

$$\theta_t + u\cdot\nabla\theta = 0, \tag{3.12}$$

$$u - \alpha^2\Delta u = v, \tag{3.13}$$

$$\operatorname{div} u = \operatorname{div} v = 0, \tag{3.14}$$

$$(v,\theta)|_{t=0} = (v_0,\theta_0). \tag{3.15}$$

Taking curl to (3.11), denoting $\omega := \operatorname{curl} v$, we have

$$\omega_t + u\cdot\nabla\omega - \partial_1^2\omega = \partial_1\theta. \tag{3.16}$$

Testing (3.16) by ω, using (3.14) and (2.1), we see that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\omega^2\,\mathrm{d}x + \int|\partial_1\omega|^2\,\mathrm{d}x = \int\partial_1\theta\cdot\omega\,\mathrm{d}x = -\int\theta\cdot\partial_1\omega\,\mathrm{d}x \leqslant \|\theta_0\|_{L^2}\|\partial_1\omega\|_{L^2},$$

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whence we obtain

$$\omega \in L^\infty(0,T;L^2).$$

Thus, we arrive at (3.4). By theorem 1.5, this proves the following theorem.

THEOREM 3.4. *Let* $(v_0,\theta_0) \in H^3$ *and* $\operatorname{div} v_0 = 0$ *in* $\mathbb{R}^2$. *Then the problem (3.11)–(3.15) has a unique smooth solution* (v,θ) *such that*

$$(v,\theta) \in L^\infty(0,T;H^3), \qquad \partial_1 v \in L^2(0,T;H^3)$$

for any $T > 0$.

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References

1 D. Chae and H. S. Nam. Local existence and blow-up criterion for the Boussinesq equations. *Proc. R. Soc. Edinb.* A **127** (1997), 935–946.
2 D. Chae, S. K. Kim and H. S. Nam. Local existence and blow-up criterion of Hölder continuous solutions of the Boussinesq equations. *Nagoya Math. J.* **155** (1999), 55–80.
3 S. Y. Chen, C. Foias, D. D. Holm, E. J. Olson, E. S. Titi and S. Wynne. The Camassa–Holm equations as a closure model for turbulent channel and pipe flows. *Phys. Fluids* **11** (1999), 2343–2353.
4 J. Fan and T. Ozawa. On the regularity criteria for the generalized Navier–Stokes equations and Lagrangian averaged Euler equations. *Diff. Integ. Eqns* **21** (2008), 443–457.
5 J. Fan and Y. Zhou. A note on regularity criterion for the 3D Boussinesq system with partial viscosity. *Appl. Math. Lett.* **22** (2009), 802–805.
6 C. Foias, D. D. Holm and E. S. Titi. The three-dimensional viscous Camassa–Holm equations, and their relation to the Navier–Stokes equations and turbulence theory. *J. Dynam. Diff. Eqns* **14** (2002), 1–35.
7 D. D. Holm, J. E. Marsden and T. S. Ratiu. Euler–Poincaré models of ideal fluids with nonlinear dispersion. *Phys. Rev. Lett.* **349** (1998), 4173–4177.
8 D. D. Holm, J. E. Marsden and T. S. Ratiu. Euler–Poincaré equations and semidirect products with applications to continuum theories. *Adv. Math.* **137** (1998), 1–81.
9 T. Kato and G. Ponce. Commutator estimates and the Euler and Navier–Stokes equations. *Commun. Pure Appl. Math.* **41** (1988), 891–907.
10 C. Kenig, G. Ponce and L. Vega. Well-posedness of the initial value problem for the Korteweg–de Vries equations. *J. Am. Math. Soc.* **4** (1991), 323–347.
11 H. Kozono, T. Ogawa and Y. Taniuchi. The critical Sobolev inequalities in Besov spaces and regularity criterion to some semilinear equations. *Math. Z.* **242** (2002), 251–278.
12 J. Marsden and S. Shkoller. Global well-posedness for the Lagrangian averaged Navier–Stokes (LANS-α) equations on bounded domains. *Phil. Trans. R. Soc. Lond.* A **359** (2001), 1449–1468.
13 Y. Meyer. Oscillating patterns in some nonlinear evolution equations. In *Mathematical foundation of turbulent viscous flows*, Lecture Notes in Mathematics, vol. 1871, pp. 101–187 (Springer, 2006).
14 Y. Taniuchi. A note on the blow-up criterion for the inviscid 2D Boussinesq equations. In *The Navier–Stokes equations: theory and numerical methods* (ed. R. Salvi), Lecture Notes in Pure and Applied Mathematics, vol. 223, pp. 131–140 (Boca Raton, FL: CRC Press, 2002).
15 H. Triebel. *Theory of function spaces* (Birkhäuser, 1983).

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