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THE Σ_1-PROVABILITY LOGIC OF HA*

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Abstract. For the Heyting Arithmetic HA, HA* is defined [14, 15] as the theory $\{A \mid \mathsf{HA} \vdash A^{\Box}\}$, where $A^{\Box}$ is called the box translation of A (Definition 2.4). We characterize the Σ_1-provability logic of HA* as a modal theory iH^*_σ (Definition 3.17).

§1. Introduction. This article is a sequel of our previous article [2], in which we characterized the Σ_1-provability logic of HA as a decidable modal theory iH_σ (see Definition 3.17). Most of the materials of this article are from the article mentioned above. Our techniques and proofs are very similar to those used there. We use a crucial fact (Theorem 4.1 in this article) proved in [2]. For the sake of self-containedness as much as possible, we bring here some definitions from that article.

For an arithmetical theory T extending HA, the following axiom schema is called *the Completeness Principle*, CP_T:

$$A \to \Box_T A.$$

Recall that by the work of Gödel in [5], for each arithmetical formula A and recursively axiomatizable theory T (like *Peano Arithmetic* PA), we can formalize the statement "there exists a proof in T for A" by a sentence of the language of arithmetic, i.e., $\mathsf{Prov}_T(\ulcorner A \urcorner) := \exists x\, \mathsf{Proof}_T(x, \ulcorner A \urcorner)$, where $\ulcorner A \urcorner$ is the code of A. Now, by *interpreting* $\Box$ by $\mathsf{Prov}_T(\ulcorner A \urcorner)$, the completeness principle for theory T is read as follows:

$$A \to \mathsf{Prov}_T(\ulcorner A \urcorner).$$

Albert Visser in [14, 15] introduced an extension of HA,

$$\mathsf{HA}^* := \mathsf{HA} + \mathsf{CP}_{\mathsf{HA}^*}.$$

He called HA* as a *self-completion* of HA. Moreover, he showed that HA* may be defined as the theory $\{A \mid \mathsf{HA} \vdash A^{\Box}\}$, where $A^{\Box}$ is called the *box translation* of A (Definition 2.4).

The notion of *provability logic* goes back essentially to K. Gödel [6] in 1933. He intended to provide a semantics for Heyting's formalization of *intuitionistic logic* IPC. He defined a *translation*, or *interpretation* τ from the propositional language to the modal language such that

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$$\mathsf{IPC} \vdash A \qquad \Longleftrightarrow \qquad \mathsf{S4} \vdash \tau(A).$$

Now the question is whether we can find some modal propositional theory such that the $\Box$ operator captures the notion of *provability* in Peano Arithmetic PA. Hence the question is to find some propositional modal theory $T_{\Box}$ such that

$$T_{\Box} \vdash A \qquad \Longleftrightarrow \qquad \forall \sigma\ \mathsf{PA} \vdash \sigma(A),$$

in which σ is a mapping from the modal language to the first-order language of arithmetic, such that

- for any atomic variable p, $\sigma(p)$ is an arithmetical first-order sentence, and $\sigma(\bot) = \bot$,
- $\sigma(A \circ B) = \sigma(A) \circ \sigma(B)$, for $\circ \in \{\vee, \wedge, \rightarrow\}$,
- $\sigma(\Box A) := \exists x\, \mathsf{Proof}_{\mathsf{PA}}(x, \ulcorner \sigma(A) \urcorner)$.

It turned out that S4 is *not* a right candidate for interpreting the notion of *provability*, since $\neg \Box \bot$ is a theorem of S4, contradicting Gödel's second incompleteness theorem (Peano Arthmetic PA, does not prove its own consistency).

Martin Löb in 1955 showed [10] that the Löb's rule $(\Box A \rightarrow A / A)$ is valid. Then in 1976, Robert Solovay [12] proved that the right modal logic, in which the $\Box$ operator interprets the notion of *provability in* PA, is GL. This modal logic is well known as the Gödel–Löb logic, and has the following axioms and rules:

- all tautologies of classical propositional logic,
- $\Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B)$,
- $\Box A \rightarrow \Box\Box A$,
- Löb's axiom (L): $\Box(\Box A \rightarrow A) \rightarrow \Box A$,
- Necessitation Rule: $A / \Box A$,
- Modus ponens: $(A, A \rightarrow B) / B$.

THEOREM 1.1 (Solovay-Löb). *For any sentence A in the language of modal logic,* $\mathsf{GL} \vdash A$ *if and only if for all interpretations* σ, $\mathsf{PA} \vdash \sigma(A)$.

Now let us restrict the map σ on the atomic variables in the following sense. For any atomic variable p, $\sigma(p)$ is a Σ_1-sentence. This translation or interpretation is called a Σ_1-interpretation. On the other hand, let $\mathsf{GLV} = \mathsf{GL} + \mathsf{CP}_a$, where CP_a is the completeness principle restricted to atomic variables, i.e., $p \rightarrow \Box p$. Albert Visser [14] proved the following result:

THEOREM 1.2 (Visser). *For any sentence A in the language of modal logic,* $\mathsf{GLV} \vdash A$ *if and only if for all* Σ_1*-interpretations* σ, $\mathsf{PA} \vdash \sigma(A)$.

The question of generalizing Solovay's result from classical theories to intuitionistic ones, such as the intuitionistic counterpart of PA, well known as HA, proved to be remarkably difficult and remains a major open problem since the end of 70s [3]. For a detailed history of the origins, backgrounds and motivations of *provability logic*, we refer the readers to [3].

The following list contains crucial results about the provability logic of HA with arithmetical nature:

- John Myhill 1973 and Harvey Friedman 1975. $\mathsf{HA} \nvdash \Box_{\mathsf{HA}}(A \vee B) \rightarrow (\Box_{\mathsf{HA}} A \vee \Box_{\mathsf{HA}} B)$, [4, 11].
- Daniel Leivant 1975. $\mathsf{HA} \vdash \Box_{\mathsf{HA}}(A \vee B) \rightarrow \Box_{\mathsf{HA}}(\boxdot_{\mathsf{HA}} A \vee \boxdot_{\mathsf{HA}} B)$, in which $\boxdot_{\mathsf{HA}} A$ is a shorthand for $A \wedge \Box_{\mathsf{HA}} A$, [8].

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- Albert Visser 1981. $\mathsf{HA} \vdash \Box_{\mathsf{HA}} \neg\neg \Box_{\mathsf{HA}} A \rightarrow \Box_{\mathsf{HA}} \Box_{\mathsf{HA}} A$ and $\mathsf{HA} \vdash \Box_{\mathsf{HA}} (\neg\neg \Box_{\mathsf{HA}} A \rightarrow \Box_{\mathsf{HA}} A) \rightarrow \Box_{\mathsf{HA}} (\Box_{\mathsf{HA}} A \vee \neg \Box_{\mathsf{HA}} A)$, [14, 15].
- Rosalie Iemhoff 2001 introduced a uniform axiomatization of all known axiom schemas of the provability logic of HA in an extended language with a bimodal operator $\rhd$. In her Ph.D. dissertation [7], Iemhoff raised a conjecture that implies directly that her axiom system, iPH, restricted to the normal modal language, is equal to the provability logic of HA, [7].
- Albert Visser 2002 introduced a decision algorithm for $\mathsf{HA} \vdash A$, for all modal propositions A not containing any atomic variable, i.e., A is made up of $\top, \bot$ via the unary modal connective $\Box_{\mathsf{HA}}$ and propositional connectives $\vee, \wedge, \rightarrow$, [16].
- Mohammad Ardeshir and Mojtaba Mojtahedi 2014 characterized the Σ_1-provability logic of HA as a decidable modal theory [2], named there and here as iH_σ. Recently, Albert Visser and Jetze Zoethout [18] proved this result by an alternative method.

The authors of [1] found a *reduction* of the Solovay–Löb Theorem to the Visser Theorem *only by propositional substitutions*. It is tempting to think that the method used in [1] can be carried out in the intuitionistic case. However it seems to us that there is no obvious way of doing such a reduction for the intuitionistic case, and it should be more complicated than the classical case.

In this article, we introduce an axiomatization of a decidable modal theory iH^*_σ (see Definition 3.17) and prove that it is the Σ_1-provability logic of HA^*. This arithmetical theory is defined [14, 15] as the theory $\{A \mid \mathsf{HA} \vdash A^\Box\}$, where $A^\Box$ is called *the box translation* of A (Definition 2.4). It is worth mentioning that our proof of the Σ_1-provability logic of HA^* is in some sense, a *reduction* to the proof of the Σ_1-provability logic of HA, *only by propositional modal logic*.

§2. Definitions, conventions and basic facts. The propositional nonmodal language contains atomic variables, $\vee, \wedge, \rightarrow, \bot$ and the propositional modal language is the propositional non-modal language plus $\Box$. We use $\boxdot A$ as a shorthand for $A \wedge \Box A$. The notation $A[p_1|B_1, \ldots, p_n|B_n]$ stands for the simultaneous substitution of $B_1, \ldots, B_n$ for the atomic variables $p_1, \ldots, p_n$ in B, respectively. For simplicity, in this article, we write the propositional language instead of propositional *modal* language. IPC [13] is the intuitionistic propositional nonmodal logic over usual propositional nonmodal language. $\mathsf{IPC}_\Box$ is the same theory IPC in the extended language of propositional modal language, i.e., its language is propositional modal language and its axioms and rules are the same as the one in IPC. Since we have no axioms for $\Box$ in $\mathsf{IPC}_\Box$, it is obvious that for each A, $\Box A$ behaves exactly like an atomic variable inside $\mathsf{IPC}_\Box$. The first-order intuitionistic theory is denoted with IQC and CQC is its classical closure, i.e., IQC plus the principle of excluded middle. We have the usual first-order language of arithmetic which has a primitive recursive function symbol for each primitive recursive function. We use the same notations and definitions for Heyting's arithmetic HA as in [13], and Peano Arithmetic PA is HA plus the principle of excluded middle. For a set of sentences and rules $\Gamma \cup \{A\}$ in propositional non-modal, propositional modal or first-order language, $\Gamma \vdash A$ means that A is derivable from Γ in the system $\mathsf{IPC}, \mathsf{IPC}_\Box, \mathsf{IQC}$, respectively.

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DEFINITION 2.1. Suppose T is an r.e arithmetical theory and σ is a function from atomic variables to arithmetical sentences. We extend σ to all modal propositions A, inductively:

- $\sigma_T(A) := \sigma(A)$ for atomic A,
- σ_T distributes over $\wedge, \vee, \rightarrow$,
- $\sigma_T(\Box A) := \mathsf{Prov}_T(\ulcorner \sigma_T(A) \urcorner)$, in which $\mathsf{Prov}_T(x)$ is the Σ_1-predicate that formalizes provability of a sentence with Gödel number x, in the theory T.

We call σ a Σ_1-substitution, if for every atomic A, $\sigma(A)$ is a Σ_1-formula.

DEFINITION 2.2. The provability logic of a sufficiently strong theory T is defined to be a modal propositional theory $\mathcal{PL}(T)$ such that $\mathcal{PL}(T) \vdash A$ iff for arbitrary arithmetical substitutions σ_T, $T \vdash \sigma_T(A)$. If we restrict the substitutions to Σ_1-substitutions, then the new modal theory is $\mathcal{PL}_\Sigma(T)$.

LEMMA 2.3. *Let A be a nonmodal proposition and $p_i \neq p_j$, for all $0 < i < j \leq n$, are atomic variables. Then for every modal propositions $B_1, \ldots, B_n$, we have*:

$$\mathsf{IPC} \vdash A \text{ iff } \mathsf{IPC}_\Box \vdash A[p_1|\Box B_1, \ldots, p_n|\Box B_n].$$

PROOF. By simple inductions on the complexity of proofs in IPC and $\mathsf{IPC}_\Box$. ⊣

The following definition, the Beeson–Visser box-translation, is essentially from [15, Definition 4.1]. This translation is needed to define the theory HA*.

DEFINITION 2.4. For every proposition A in the modal propositional language, we associate a proposition $A^\Box$, called the box-translation of A, defined inductively as follows:

- $A^\Box := A \wedge \Box A$, for atomic A, and $\bot^\Box = \bot$,
- $(A \circ B)^\Box := A^\Box \circ B^\Box$, for $\circ \in \{\vee, \wedge\}$,
- $(A \rightarrow B)^\Box := (A^\Box \rightarrow B^\Box) \wedge \Box(A^\Box \rightarrow B^\Box)$,
- $(\Box A)^\Box := \Box(A^\Box)$.

For a first-order theory T and a first-order arithmetical formula A, the Beeson–Visser translation A^T is defined as follows:

- $A^T := A$ for atomic A,
- $(.)^T$ commutes with $\wedge, \vee$ and $\exists$,
- $(A \rightarrow B)^T := (A^T \rightarrow B^{\Box_T}) \wedge \Box_T(A^T \rightarrow B^T)$
- $(\forall x A)^T := \Box_T(\forall x A^T) \wedge \forall x A^T$.

Define NOI (No Outside Implication) as set of modal propositions A, that any occurrence of $\rightarrow$ is in the scope of some $\Box$. To be able to state an extension of Leivant's Principle (that is adequate to axiomatize the Σ_1-provability logic of HA) we need a translation on the modal language which we name Leivant's translation. We define it recursively as follows:

- $A^l := A$ for atomic A, boxed A or $A = \bot$,
- $(A \wedge B)^l := A^l \wedge B^l$,
- $(A \vee B)^l := \boxdot A^l \vee \boxdot B^l$,
- $(A \rightarrow B)^l$ is defined by cases: If $A \in \mathsf{NOI}$, define $(A \rightarrow B)^l := A \rightarrow B^l$, else define $(A \rightarrow B)^l := A \rightarrow B$.

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2.1. Definition of the modal theories. Minimal provability logic iGL, is the same as Gödel-Löb provability logic GL, without the principle of excluded middle, i.e., it has the following axioms and rules:

- all theorems of $\mathsf{IPC}_{\Box}$,
- $\Box(A \to B) \to (\Box A \to \Box B)$,
- $\Box A \to \Box\Box A$,
- Löb's axiom (L): $\Box(\Box A \to A) \to \Box A$,
- Necessitation Rule: $A/\Box A$,
- Modus ponens: $(A, A \to B)/B$.

iK4 is iGL without Löb's axiom. Note that we can get rid of the necessitation rule by adding $\Box A$ to the axioms, for each axiom A in the above list. We will use this fact later in this article. We list the following axiom schemae:

- The Completeness Principle: $\mathsf{CP} := A \to \Box A$.
- Restricted Completeness Principle to atomic formulae: $\mathsf{CP}_a := p \to \Box p$, for atomic p.
- Leivant's Principle: $\mathsf{Le} := \Box(B \vee C) \to \Box(\boxdot B \vee \boxdot C)$. [9]
- Extended Leivant's Principle: $\mathsf{Le}^+ := \Box A \to \Box A^l$.
- Trace Principle: $\mathsf{TP} := \Box(A \to B) \to (A \vee (A \to B))$. [15]

We define theories $\mathsf{iGLC} := \mathsf{iGL} + \mathsf{CP}$ and $\mathsf{H} := \mathsf{iGLC} + \mathsf{TP}$, $\mathsf{LLe} := \mathsf{iGL} + \mathsf{Le}$. Note that in the presence of CP and modus ponens, the necessitation rule is superfluous.

LEMMA 2.5. *For any modal proposition* A, $\mathsf{iK4} + \mathsf{CP} \vdash A \leftrightarrow A^{\Box}$.

PROOF. Use induction on the complexity of A. ⊣

2.2. HA^* **and** PA^***.** HA^* and PA^* were first introduced in [15]. These theories are defined as

$$\mathsf{HA}^* := \{A \mid \mathsf{HA} \vdash A^{\mathsf{HA}}\} \quad \text{and} \quad \mathsf{PA}^* := \{A \mid \mathsf{PA} \vdash A^{\mathsf{PA}}\}.$$

Visser in [15] showed that the provability logic of PA^* is H, i.e., $\mathsf{H} \vdash A$ iff for all arithmetical substitution σ, $\mathsf{PA}^* \vdash \sigma_{\mathsf{PA}^*}(A)$. That means that

$$\mathcal{PL}(\mathsf{PA}^*) = \mathcal{PL}_{\Sigma}(\mathsf{PA}^*) = \mathsf{H}.$$

LEMMA 2.6. 1. *For any arithmetical* Σ_1*-formula* A, $\mathsf{HA} \vdash A \leftrightarrow A^{\mathsf{HA}}$.
2. HA *is closed under the Beeson–Visser translation, i.e., for any arithmetical formula* A, $\mathsf{HA} \vdash A$ *implies* $\mathsf{HA} \vdash A^{\mathsf{HA}}$, *so* $\mathsf{HA} \subseteq \mathsf{HA}^*$.

PROOF. 1. See [15](4.6.iii).
2. See [15](4.14.i). ⊣

LEMMA 2.7. *For any* Σ_1*-substitution* σ *and each propositional modal sentence* A, *we have* $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(A^{\Box}) \leftrightarrow (\sigma_{\mathsf{HA}^*}(A))^{\mathsf{HA}}$ *and hence*

$$\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(A^{\Box}) \quad \textit{iff} \quad \mathsf{HA}^* \vdash \sigma_{\mathsf{HA}^*}(A)$$

PROOF. Use induction on the complexity of A. All the steps are straightforward. For the atomic case and the boxed case, use Lemma 2.6.1. Moreover, when $A = \Box B$, we can use the verifiability of the definition of HA^* in HA. ⊣

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REMARK 2.8. This lemma can be combined with the characterization of the Σ_1-provability logic of HA to derive directly a characterization of the Σ_1-provability logic of HA*:

A belongs to the Σ_1*-provability logic of* HA* *iff*
$A^{\Box}$ *belongs to the* Σ_1*-provability logic of* HA.

This means that we have a decision algorithm for the Σ_1-provability logic of HA*. The rest of this article is devoted to axiomatize the Σ_1-provability logic of HA*.

§3. Propositional modal logics.

3.1. NNIL formulae and the related topics. The class of *No Nested Implications in the Left*, NNIL formulae in a propositional language was introduced in [17], and more explored in [16]. The crucial result of [16] is providing an algorithm that as input, gets a non-modal proposition A and returns its best NNIL approximation A^* from below, i.e., $\mathsf{IPC} \vdash A^* \to A$ and for all NNIL formula B such that $\mathsf{IPC} \vdash B \to A$, we have $\mathsf{IPC} \vdash B \to A^*$. In the following, we explain this algorithm and explain how to extend it to the modal propositions.

Intuitively, the class of NNIL formulae contains those formulae which no $\to$ occurs in the left hand side of any $\to$. For a formal definition of the class of NNIL propositions, let us first define a complexity measure ρ on nonmodal propositions as follows:

- $\rho p = \rho\bot = \rho\top = 0$, where p is an atomic proposition,
- $\rho(A \wedge B) = \rho(A \vee B) = max(\rho A, \rho B)$,
- $\rho(A \to B) = max(\rho A + 1, \rho B)$,

Then $\mathsf{NNIL} = \{A \mid \rho A \leq 1\}$.

In the following definition, we define some complexity measure $\mathfrak{o}$ over modal propositions. We need this measure only for showing that the TNNIL-algorithm terminates.

DEFINITION 3.1. We define a measure complexity for modal propositions D as follows:

- $\mathfrak{C}_{\to}(D) := \{E \in Sub(D) \mid E$ is an implication that is not in the scope of a $\Box\}$. In other words, $\mathfrak{C}_{\to}(A)$ is the set of outer occurrences of $\to$,
- $\mathfrak{c}_{\to}(D) := max\{|\mathfrak{C}(E)| \mid E \in \mathfrak{C}(D)\}$, where $|X|$ is the number of elements of X. In other words, $\mathfrak{c}_{\to}(A)$ is the maximum number of outer occurrences of $\to$ that appear in some outer implication,
- $\mathfrak{c}_0 D :=$ the number of occurrences of logical connectives which is not in the scope of a $\Box$,
- $\mathfrak{c}_{\Box} D :=$ the maximum number of nested boxes. To be more precise,
  - $\mathfrak{c}_{\Box} D := 0$ for atomic D,
  - $\mathfrak{c}_{\Box} D := max\{\mathfrak{c}_{\Box} D_1, \mathfrak{c}_{\Box} D_2\}$, where $D = D_1 \circ D_2$ and $\circ \in \{\wedge, \vee, \to\}$,
  - $\mathfrak{c}_{\Box}\Box D := 1 + \mathfrak{c}_{\Box} D$,
- $\mathfrak{c} D := (\mathfrak{c}_{\Box} D, \mathfrak{c}_{\to} D, \mathfrak{c}_0 D)$.

The measure $\mathfrak{c}D$ is ordered lexicographically, i.e., $(d, i, c) < (d', i', c')$ iff $d < d'$ or $d = d', i < i'$ or $d = d', i = i', c < c'$.

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DEFINITION 3.2. For any two nonmodal propositions A and B, we define $[A]B$ and $[A]'B$, by induction on the complexity of B:

- $[A]p = [A]'p = p$, for atomic p, $\top$ and $\bot$,
- $[A](B_1 \circ B_2) = [A](B_1) \circ [A](B_2)$, $[A]'(B_1 \circ B_2) = [A]'(B_1) \circ [A]'(B_2)$ for $\circ \in \{\vee, \wedge\}$,
- $[A](B_1 \to B_2) = A \to (B_1 \to B_2)$, $[A]'(B_1 \to B_2) = (A' \wedge B_1) \to B_2$, in which A' is the result of replacing every occurrence of $B_1 \to B_2$ in A by B_2.

3.1.1. NNIL-*algorithm.* For each proposition A, A^* is produced, by induction on the complexity measure $\mathfrak{c}A$, as follows. For details see [16].

1. A is atomic, take $A^* := A$,
2. $A = B \wedge C$, take $A^* := B^* \wedge C^*$,
3. $A = B \vee C$, take $A^* := B^* \vee C^*$,
4. $A = B \to C$, we have several sub-cases. In the following, an occurrence of E in D is called an *outer occurrence*, if E is not in the scope of an implication.
   - 4.a. C contains an outer occurrence of a conjunction. Let C_1 and C_2 be the result of replacing that particular occurrence of conjunction by its left and right conjunct in C, respectively. Then define $A_1 := B \to C_1$ and $A_2 := B \to C_2$ and let $A^* := A_1^* \wedge A_2^*$.
   - 4.b. B contains an outer occurrence of a disjunction. Let B_1 and B_2 be the result of replacing that particular occurrence of disjunction by its left and right disjunct in B, respectively. Then define $A_1 := B_1 \to C$ and $A_2 := B_2 \to C$ and let $A^* := A_1^* \wedge A_2^*$.
   - 4.c. $B = \bigwedge X$ and $C = \bigvee Y$ and X, Y are sets of implications or atoms. We have several subcases:
     - 4.c.i. X contains atomic p. Set $D := \bigwedge(X \setminus \{p\})$ and take $A^* := p \to (D \to C)^*$.
     - 4.c.ii. X contains $\top$. Define $D := \bigwedge(X \setminus \{\top\})$ and take $A^* := (D \to C)^*$.
     - 4.c.iii. X contains $\bot$. Take $A^* := \top$.
     - 4.c.iv. X contains only implications. For any $D = E \to F \in X$, let

$$B \downarrow D := \bigwedge((X \setminus \{D\}) \cup \{F\}).$$

Let $Z := \{E \mid E \to F \in X\} \cup \{C\}$ and $A_0 := [B]Z := \bigvee\{[B]E \mid E \in Z\}$. Now if $\mathfrak{c}A_0 < \mathfrak{c}A$, we take

$$A^* := \bigwedge\{((B \downarrow D) \to C)^* | D \in X\} \wedge A_0^*,$$

otherwise, first set $A_1 := [B]'Z$ and then take

$$A^* := \bigwedge\{((B \downarrow D) \to C)^* | D \in X\} \wedge A_1^*.$$

Note that the clause involving A_1 is only needed to ensure that the algorithm terminates. It is also worth mentioning that A_0 and A_1 are IPC-equivalent.

We can extend ρ to all modal language with $\rho(\Box A) := 0$. The class of NNIL propositions may be defined for propositional modal language as well, i.e., we call a modal proposition A to be $\mathsf{NNIL}_\Box$, if $\rho(A) \leq 1$ (for extended ρ). We also define two other classes of propositions:

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DEFINITION 3.3. TNNIL (Thoroughly NNIL) is the smallest class of propositions such that

- TNNIL contains all atomic propositions,
- if $A, B \in \mathsf{TNNIL}$, then $A \vee B, A \wedge B, \Box A \in \mathsf{TNNIL}$,
- if all $\to$ occurring in A are contained in the scope of a $\Box$ (or equivalently $A \in \mathsf{NOI}$) and $A, B \in \mathsf{TNNIL}$, then $A \to B \in \mathsf{TNNIL}$.

Finally we define $\mathsf{TNNIL}^{\Box}$ as the set of all the propositions like $A(\Box B_1, \ldots, \Box B_n)$, such that $A(p_1, \ldots, p_n)$ is an arbitrary non-modal proposition and $B_1, \ldots, B_n \in \mathsf{TNNIL}$.

We can use the same algorithm with slight modifications treating propositions inside $\Box$ as well. First we extend Definition 3.2 to capture the modal language.

DEFINITION 3.4. For any two modal propositions A, B, we define $[A]B$ and $[A]'B$ by induction on the complexity of B. We extend Definition 3.2 by the following items:

- $[A]\Box B_1 = [A]'\Box B_1 := \Box B_1$.

Moreover, we adapt the definition for $\to$ as follows:

- $[A](B_1 \to B_2) = A \to (B_1 \to B_2)$, $[A]'(B_1 \to B_2) = (A' \wedge B_1) \to B_2$, in which A' is the result of replacing every outer occurrence of $B_1 \to B_2$ (i.e., those which are not in the scope of $\Box$) in A by B_2.

For a set Γ of propositions, we define

$$[A]\Gamma := \bigvee_{B \in \Gamma} [A](B) \qquad \text{and} \qquad [A]'\Gamma := \bigvee_{B \in \Gamma} [A]'(B).$$

It is clear that we are treating a boxed formula as an atomic variable.

3.1.2. $\mathsf{NNIL}_{\Box}$-algorithm. We use NNIL-algorithm with the following changes to produce a similar NNIL-algorithm for a modal language.

1. A is atomic or boxed, take $A^* = A$.
4. An occurrence of E in D is called an *outer occurrence*, if E is neither in the scope of an implication nor in the scope of a boxed formula.

4. c(i). X contains atomic or boxed formula p. We set $D := \bigwedge(X \setminus \{p\})$ and take $A^* := p^* \to (D \to C)^*$.

REMARK 3.5. In fact, we have two ways to find out $\mathsf{NNIL}_{\Box}$ approximation of a modal proposition.

First: simply apply $\mathsf{NNIL}_{\Box}$-algorithm to a modal proposition A and compute A^*.

Second: let $B_1, \ldots, B_n$ be all boxed sub-formulae of A which are not in the scope of any other boxes. Let $A'(p_1, \ldots, p_n)$ be unique nonmodal proposition such that $\{p_i\}_{1 \le i \le n}$ are fresh atomic variables not occurred in A and $A = A'[p_1|B_1, \ldots, p_n|B_n]$. Let $\gamma(A) := (A')^*[p_1|B_1, \ldots, p_n|B_n]$. Then it is easy to observe that $\mathsf{IPC}_{\Box} \vdash \gamma(A) \leftrightarrow A^*$.

The above-defined algorithm is not deterministic, however by the following Theorem, we know that A^* is unique up to $\mathsf{IPC}_{\Box}$ equivalence.

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THEOREM 3.6. *For each modal proposition A, the* $\mathsf{NNIL}_\Box$ *algorithm with input A terminates and the output formula A^* is an* $\mathsf{NNIL}_\Box$ *proposition such that* $\mathsf{IPC}_\Box \vdash A^* \to A$.

PROOF. See [2, Theorem 4.5]. ⊣

3.1.3. TNNIL*-algorithm.* Here we define A^+ as TNNIL-formula approximating A. Informally speaking, to find A^+, we first compute A^* and then replace all outer boxed formula $\Box B$ in A by $\Box B^+$. To be more accurate, we define A^+ by induction on $\mathfrak{c}_\Box A$. Suppose that for all B with $\mathfrak{c}_\Box B < \mathfrak{c}_\Box A$, we have defined B^+. Suppose that $A'(p_1, \ldots, p_n)$ and $\Box B_1, \ldots, \Box B_n$ such that $A = A'[p_1|\Box B_1, \ldots, p_n|\Box B_n]$, where A' is a nonmodal proposition and $p_1, \ldots, p_n$ are fresh atomic variables (not occurred in A). It is clear that $\mathfrak{c}_\Box B_i < \mathfrak{c}_\Box A$ and then we can define $A^+ := (A')^*[p_1|\Box B_1^+, \ldots, p_n|\Box B_n^+]$.

LEMMA 3.7. *For any modal proposition A,*

1. *for all Σ_1-substitution σ we have* $\mathsf{HA} \vdash \Box\sigma_{\mathsf{HA}}(A) \leftrightarrow \Box\sigma_{\mathsf{HA}}(A^+)$ *and hence* $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(A)$ *iff* $\mathsf{HA} \vdash \sigma_{\mathsf{HA}}(A^+)$.
2. $\mathsf{iGL} \vdash A_1 \to A_2$ *implies* $\mathsf{iGL} \vdash A_1^+ \to A_2^+$, *and* $i\mathsf{K4} \vdash A_1 \to A_2$ *implies* $\mathsf{iK4} \vdash A_1^+ \to A_2^+$.

PROOF. See [2, Corollary 4.8]. ⊣

3.1.4. $\mathsf{TNNIL}^\Box$*-algorithm.*

COROLLARY 3.8. *There exists an algorithm, which we call* $\mathsf{TNNIL}^\Box$*-algorithm, such that for any modal proposition A, it halts and produces a proposition $A^- \in \mathsf{TNNIL}^\Box$ such that* $\mathsf{IPC}_\Box \vdash A^+ \to A^-$.

PROOF. Let $A := B(\Box C_1, \ldots, \Box C_n)$ where $B(p_1, \ldots, p_n)$ is nonmodal. Clearly, such B exists. Then define $A^- := B(\Box C_1^+, \ldots, \Box C_n^+)$. Now definition of A^+ implies $A^+ = (A^-)^*$ and hence Theorem 3.6 implies that A^- has desired property. ⊣

LEMMA 3.9. *For each modal proposition A and Σ_1-substitution σ,* $\mathsf{HA} \vdash \sigma_{\mathsf{HA}} A \leftrightarrow \sigma_{\mathsf{HA}} A^-$.

PROOF. Use definition of $(.)^-$ and Lemma 3.7.1. ⊣

REMARK 3.10. Note that $\mathsf{iGLC} \vdash A \leftrightarrow B$ does not imply $\mathsf{iGLC} \vdash A^+ \leftrightarrow B^+$. A counter-example is $A := \neg\neg p$ and $B := \neg \boxdot (\neg p)$. We have $A^+ = A^* = p$ and $\mathsf{iGLC} \vdash B^+ \leftrightarrow (\Box\neg p \to p)$. Now one can use Kripke models [2, Section 4.5] to show $\mathsf{iGLC} \nvdash \neg\neg p \to (\Box\neg p \to p)$.

REMARK 3.11. In the $\mathsf{NNIL}_\Box$-algorithm, if we replace the operation $(\cdot)^*$ by $(\cdot)^\dagger$, and change the step 1 to
1. $A^\dagger := A$, if A is atomic, and $(\Box B)^\dagger := \Box B^\dagger$,
then the new algorithm also halts, and for any modal proposition A, we have $\mathsf{iK4} \vdash A^\dagger \leftrightarrow A^+$.

3.2. The box translation and propositional theories.

DEFINITION 3.12. A modal theory T is called to be *closed under the box-translation* if for every proposition A, $T \vdash A$ implies $T \vdash A^\Box$.

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PROPOSITION 3.13. *For an arbitrary subset X of $\{\mathsf{CP}, \mathsf{CP}_a, \mathsf{L}\}$, $\mathsf{iK4} + X$ is closed under the box-translation.*

PROOF. Let $\mathsf{iK4} + X \vdash A$. We show that $\mathsf{iK4} + X \vdash A^\Box$. The proof can be carried out by induction on the complexity of the derivation of A from $\mathsf{iK4} + X$.

1. For any instance of an axiom A of IPC, we clearly have $\mathsf{iK4} \vdash A^\Box$.
2. For the modal axioms of $\mathsf{iK4}$, we have
$$(\Box A \to \Box\Box A)^\Box = \boxdot(\Box A^\Box \to \Box\Box A^\Box)$$
and also
$$\mathsf{iK4} \vdash [(\Box(A \to B) \wedge \Box A) \to \Box B]^\Box \leftrightarrow \boxdot[(\Box(A^\Box \to B^\Box) \wedge \Box A^\Box) \to \Box B^\Box].$$
3. For any axiom $A \in X$, we observe that $\mathsf{iK4} + X \vdash A^\Box$.
4. Now assume that the last step of the derivation $\mathsf{iK4} + X \vdash A$ uses modus ponens. Then $\mathsf{iK4} + X \vdash B \to A$ and $\mathsf{iK4} + X \vdash B$, with lower complexity and hence induction hypothesis implies that $\mathsf{iK4} + X \vdash B^\Box$ and $\mathsf{iK4} + X \vdash B^\Box \to A^\Box$. Then $\mathsf{iK4} + X \vdash A^\Box$.
5. Assume that the last step of the derivation $\mathsf{iK4} + X \vdash A$ uses necessitation. Then $A = \Box B$ and $\mathsf{iK4} + X \vdash B$, with lower complexity and hence induction hypothesis implies that $\mathsf{iK4} + X \vdash B^\Box$. Then $\mathsf{iK4} + X \vdash A^\Box$. ⊣

The following two lemmas will be used in the proof of Theorem 3.19.

LEMMA 3.14. *For any modal propositions A, A' and B, and any propositional modal theory T containing the axioms and the inference rule of $\mathsf{IPC}_\Box$,*

1. $\mathsf{iK4} + \Box A^\Box \vdash ([A]B)^\Box \leftrightarrow ([A^\Box]B^\Box)$.
2. *$T \vdash A \leftrightarrow A'$ implies $T \vdash [A]B \leftrightarrow [A']B$.*

PROOF. Both parts can be proved by induction on the complexity of B. We give the argument for the first item and leave the second one to the reader.

The only nontrivial case is when B is an implication. Let $B := C \to D$. By Definitions 3.4 and 2.4,
$$([A](C \to D))^\Box = \boxdot(A^\Box \to ((C^\Box \to D^\Box) \wedge \Box(C^\Box \to D^\Box)))$$
and also
$$[A^\Box](C \to D)^\Box = (A^\Box \to (C^\Box \to D^\Box)) \wedge \Box(C^\Box \to D^\Box).$$
Now it is easy to observe that
$$\mathsf{iK4} + \Box A^\Box \vdash ([A](C \to D))^\Box \leftrightarrow ([A^\Box](C \to D)^\Box).$$ ⊣

NOTATION 3.15. *In the rest of the article, we use $A \equiv B$ as a shorthand for $\mathsf{iK4} \vdash A \leftrightarrow B$.*

LEMMA 3.16. *Let $A = B \to C$ be a modal proposition such that $B = \bigwedge X$ and $C = \bigvee Y$, where X is a set of implications and Y is a set of atomic, boxed or implicative propositions. Then*
$$(A^\Box)^+ \equiv \boxdot\left(\bigwedge_{E \to F \in X} \Box\left((E \to F)^\Box\right)^+ \to \left(\bigwedge\left\{\left((B \downarrow D \to C)^\Box\right)^+ \mid D \in X\right\} \wedge \left(([B]Z)^\Box\right)^+\right)\right)$$
where $Z = \{E \mid E \to F \in X\} \cup \{C\}$.

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PROOF. To simplify notations, Let us indicate

- the sets of all atomic and boxed propositions by At and Bo, respectively,
- $X' := \{E^\Box \to F^\Box \mid E \to F \in X\}$,
- $Z' := Z^\Box = \{E^\Box \mid E \to F \in X\} \cup \{C^\Box\}$,
- $B' := \bigwedge X'$,
- for any $I \subseteq Y$, $C^I := \bigvee_{E\to F\in I} \Box(E^\Box \to F^\Box) \vee \bigvee_{E\in I\cap \mathsf{At}} \Box E \vee \bigvee_{E\to F\in Y\setminus I}(E^\Box \to F^\Box) \vee \bigvee_{E\in (Y\setminus I)\cap \mathsf{At}} E \vee \bigvee_{E\in \mathsf{Bo}\cap Y} E^\Box$,
- and $Z^I := \{E^\Box \mid E \to F \in X\} \cup \{C^I\}$.

By repeated application of distributivity of conjunction over disjunction, which is valid in IPC , we have

$$C^\Box \equiv \bigwedge_{I\subseteq Y} C^I \quad \text{and} \quad \Big(\bigvee Z\Big)^\Box \equiv \bigwedge_{I\subseteq Y}\Big(\bigvee Z^I\Big). \tag{1}$$

Note that $A^\Box = (B^\Box \to C^\Box) \wedge \Box(B^\Box \to C^\Box)$, and then by definition of $(\cdot)^+$,

$$(A^\Box)^+ = (B^\Box \to C^\Box)^+ \wedge \Box(B^\Box \to C^\Box)^+.$$

Now we compute the left conjunct:

$$\Big(B^\Box \to C^\Box\Big)^+ = \bigwedge_{I\subseteq Y}\Big(B^\Box \to C^I\Big)^+ \tag{2}$$

$$\equiv \bigwedge_{I\subseteq Y}\Bigg(\bigwedge_{E\to F\in X} \Box\Big(E^\Box \to F^\Box\Big)^+ \to \Bigg(\Bigg(\bigwedge_{E\to F\in X}\Big(E^\Box \to F^\Box\Big)\Bigg) \to C^I\Bigg)^+\Bigg) \tag{3}$$

$$\equiv \bigwedge_{E\to F\in X} \Box\Big((E\to F)^\Box\Big)^+ \to \bigwedge_{I\subseteq Y}\Big(B' \to C^I\Big)^+ \tag{4}$$

$$\equiv \bigwedge_{E\to F\in X} \Box\Big((E\to F)^\Box\Big)^+ \to \bigwedge_{I\subseteq Y}\Bigg(\bigwedge\Big\{\Big(B'\downarrow D' \to C^I\Big)^+ \mid D'\in X'\Big\} \wedge \Big([B']Z^I\Big)^+\Bigg) \tag{5}$$

$$\equiv \bigwedge_{E\to F\in X} \Box\Big((E\to F)^\Box\Big)^+ \to \Bigg(\bigwedge\Big\{\Big(B'\downarrow D' \to C^\Box\Big)^+ \mid D'\in X'\Big\} \wedge ([B']Z')^+\Bigg) \tag{6}$$

and hence

$$\Big(A^\Box\Big)^+ \equiv \boxdot\Bigg(\bigwedge_{E\to F\in X} \Box\Big((E\to F)^\Box\Big)^+ \to \Bigg(\bigwedge\Big\{\Big((B\downarrow D\to C)^\Box\Big)^+ \mid D\in X\Big\} \wedge ([B']Z')^+\Bigg)\Bigg). \tag{7}$$

Note that 2 and 3 hold by $\mathsf{NNIL}_\Box$-algorithm, 4 holds by properties of iK4, 5 holds by TNNIL-algorithm, 6 holds by TNNIL-algorithm and equation 1, and finally equation 7 is derived from 6 by deduction in iK4 and TNNIL-algorithm. Now it is enough to show that the last formula is equivalent to the following one in iK4:

$$\boxdot\Bigg(\bigwedge_{E\to F\in X} \Box\Big((E\to F)^\Box\Big)^+ \to \Bigg(\bigwedge\Big\{\Big((B\downarrow D\to C)^\Box\Big)^+ \mid D\in X\Big\} \wedge \Big(([B]Z)^\Box\Big)^+\Bigg)\Bigg). \tag{8}$$

To show this, it is enough to show

$$\mathsf{iK4} \vdash \bigwedge_{E\to F\in X} \Box\Big((E\to F)^\Box\Big)^+ \to \Bigg(\Big(([B]Z)^\Box\Big)^+ \leftrightarrow ([B']Z')^+\Bigg).$$

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Then by Lemma 3.7.2, it is enough to show $\mathsf{iK4} \vdash \bigwedge_{E\to F\in X} \Box(E \to F)^{\Box} \to ((|B|Z)^{\Box} \leftrightarrow [B']Z')$. Since $\bigwedge_{E\to F\in X} \Box(E \to F)^{\Box} \equiv \Box B^{\Box}$, then it is enough to show $\mathsf{iK4} + \Box B^{\Box} \vdash ([B]Z)^{\Box} \leftrightarrow [B']Z'$. Now, by Lemma 3.14.1, we have $\mathsf{iK4} + \Box B^{\Box} \vdash ([B]Z)^{\Box} \leftrightarrow [B^{\Box}]Z^{\Box}$. Hence we should show $\mathsf{iK4} + \Box B^{\Box} \vdash [B^{\Box}]Z^{\Box} \leftrightarrow [B']Z'$. We have $Z' = Z^{\Box}$ and $\mathsf{iK4} + \Box B^{\Box} \vdash B^{\Box} \leftrightarrow B'$. Then by Lemma 3.14.2, $\mathsf{iK4} + \Box B^{\Box} \vdash [B^{\Box}]Z^{\Box} \leftrightarrow [B']Z'$. ⊣

3.3. Axiomatizing TNNIL-algorithm. In this section, we introduce the axiom set X such that $\mathsf{iK4} + X \vdash (A^{\Box})^{-} \leftrightarrow A^{\Box}$. Note that we may simply choose $X := \{(A^{\Box})^{-} \leftrightarrow A^{\Box} \mid A \text{ is arbitrary proposition}\}$. However, we want to reduce X to some smaller efficient set of formulae.

We use some modal variant of Visser's $\blacktriangleright_\sigma$ in [16]. It is exactly the same as the relation $\blacktriangleright$ in [2] (Section 4.3) except for item B2, which is a little bit different:

- B2′. Let X be a set of implications, $B := \bigwedge X$ and $A := B \to C$. Also assume that $Z := \{E|E \to F \in X\} \cup \{C\}$. Then $A \blacktriangleright [B]Z$,

The relation $\blacktriangleright^*$ is defined to be the smallest relation on modal propositional sentences satisfying:

- A1. If $\mathsf{iK4} \vdash A \to B$, then $A \blacktriangleright^* B$.
- A2. If $A \blacktriangleright^* B$ and $B \blacktriangleright^* C$, then $A \blacktriangleright^* C$.
- A3. If $C \blacktriangleright^* A$ and $C \blacktriangleright^* B$, then $C \blacktriangleright^* A \wedge B$.
- A4. If $A \blacktriangleright^* B$, then $\Box A \blacktriangleright^* \Box B$.
- B1. If $A \blacktriangleright^* C$ and $B \blacktriangleright^* C$, then $A \vee B \blacktriangleright^* C$.
- B2. Let X be a set of implications, $B := \bigwedge X$ and $A := B \to C$. Also assume that $Z := \{E|E \to F \in X\} \cup \{C\}$. Then $A \wedge \Box B \blacktriangleright^* [B]Z$.
- B3. If $A \blacktriangleright^* B$, then $p \to A \blacktriangleright^* p \to B$, in which p is atomic or boxed.

$A \blacktriangleright\!\blacktriangleleft^* B$ means $A \blacktriangleright^* B$ and $B \blacktriangleright^* A$.

DEFINITION 3.17. We define

$$\mathsf{iH}^*_\sigma := \mathsf{iGL} + \mathsf{CP} + \{\Box A \to \Box B | A \blacktriangleright^* B\}.$$

Note that the Σ_1-provability logic of HA is proved in [2] to be

$$\mathsf{iH}_\sigma := \mathsf{iGL} + \mathsf{CP}_a + \mathsf{Le}^+ + \{\Box A \to \Box B | A \blacktriangleright B\},$$

in which CP_a is the Completeness Principle restricted to atomic propositions.

LEMMA 3.18. *For any propositional modal sentences A, B, $A \blacktriangleright^* B$ implies $A^{\Box} \blacktriangleright^* B^{\Box}$.*

PROOF. It is clear that $A \blacktriangleright^* B$ iff there exists a Hilbert-type sequence of relations $\{A_i \blacktriangleright^* B_i\}_{0\le i\le n}$ such that $A_n = A, B_n = B$ and for each $i \le n$, $A_i \blacktriangleright^* B_i$ is an instance of axioms A1 or B2, or it is derived by making use of some previous members of sequence and some of the rules A2–A4 or B1 or B3. Hence we are authorized to use induction on the length of such sequence for $A \blacktriangleright^* B$ to show $A^{\Box} \blacktriangleright^* B^{\Box}$. The only nontrivial steps are axioms A1, B2 and the rule B3.

- Suppose that $A \blacktriangleright^* B$ is an instance of A1, i.e., $\mathsf{iK4} \vdash A \to B$. Then by Proposition 3.13, we have $\mathsf{iK4} \vdash A^{\Box} \to B^{\Box}$ and hence again by A1, $A^{\Box} \blacktriangleright^* B^{\Box}$, as desired.
- For treating B2, suppose that $A := B \to C$, $B = \bigwedge X$, X is a set of implications and $Z := \{E|E \to F \in X\} \cup \{C\}$. We must show $(A \wedge \Box B)^{\Box} \blacktriangleright^* ([B]Z)^{\Box}$.

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Define $X' := \{E^{\Box} \to F^{\Box} | E \to F \in X\}$, $B' := \bigwedge X'$. Hence by B2, $B' \to C^{\Box} \wedge \Box B' \blacktriangleright^* [B']Z^{\Box}$. Note that we have $\Box B' \equiv \Box B^{\Box}$ and also $\mathsf{iK4} + \Box B' \vdash B' \leftrightarrow B^{\Box}$.

Now by using properties of $\blacktriangleright^*$ (A1–A3) and Lemma 3.14(2), we can deduce $(B^{\Box} \to C^{\Box}) \wedge \Box B^{\Box} \blacktriangleright^* [B^{\Box}]Z^{\Box}$. Then Lemma 3.14(1) implies $(B^{\Box} \to C^{\Box}) \wedge \Box B^{\Box} \blacktriangleright^* ([B]Z)^{\Box}$. Then by A1 and A2, we can deduce $((B \to C) \wedge \Box B)^{\Box} \blacktriangleright^* ([B]Z)^{\Box}$, as desired.

- For the rule B3, assume that $A = p \to A'$, $B = p \to B'$, $A' \blacktriangleright^* B'$ and p is atomic or boxed. By induction hypothesis, we get $A'^{\Box} \blacktriangleright^* B'^{\Box}$. We have two subcases.
  - Let p be atomic. By B3, we have $p \to A'^{\Box} \blacktriangleright^* p \to B'^{\Box}$ and hence $\Box p \to (p \to A'^{\Box}) \blacktriangleright^* \Box p \to (p \to B'^{\Box})$. Then by A1 and A2, we have $(p \wedge \Box p) \to A'^{\Box} \blacktriangleright^* (p \wedge \Box p) \to B'^{\Box}$. Then A4 implies $\Box[(p \wedge \Box p) \to A'^{\Box}] \blacktriangleright^* \Box[(p \wedge \Box p) \to B'^{\Box}]$. Finally A1, A2 and A3 implies that $(p \to A')^{\Box} \blacktriangleright^* (p \to B')^{\Box}$, as desired.
  - Let $p = \Box C$. From $A'^{\Box} \blacktriangleright^* B'^{\Box}$ and B3, we have $p^{\Box} \to A'^{\Box} \blacktriangleright^* p^{\Box} \to B'^{\Box}$. Then A4 implies $\Box[p^{\Box} \to A'^{\Box}] \blacktriangleright^* \Box[p^{\Box} \to B'^{\Box}]$. Finally A1, A2 and A3 implies that $(p \to A')^{\Box} \blacktriangleright^* (p \to B')^{\Box}$, as desired. ⊣

The following theorem is analogous to the Theorem 4.18 in [2]:

THEOREM 3.19. *For any modal proposition A, $A^{\Box} \blacktriangleright\blacktriangleleft^* (A^{\Box})^{+}$.*

Before proving this theorem, we state a corollary.

COROLLARY 3.20. $\mathsf{iH}^*_{\sigma} \vdash A^{\Box} \leftrightarrow (A^{\Box})^{-}$.

PROOF. Let $A^{\Box} = B(\Box C_1, \Box C_2, \ldots, \Box C_n)$ where $B(p_1, \ldots, p_n)$ is a nonmodal proposition and for each $1 \leq j \leq n$, C_j is of the form $D_j^{\Box}$. Hence for each $1 \leq j \leq n$, $\mathsf{iK4} \vdash \Box C_j \leftrightarrow \Box C_j^{\Box}$. By definition of $(A^{\Box})^{-}$, we have $(A^{\Box})^{-} = B(\Box C_1^{+}, \ldots, \Box C_n^{+})$. Now by Lemma 3.7, we can deduce that $\mathsf{iK4} \vdash B(\Box C_1^{+}, \ldots, \Box C_n^{+}) \leftrightarrow B(\Box (C_1^{\Box})^{+}, \ldots, \Box (C_n^{\Box})^{+})$. Then Theorem 3.19 implies that $\mathsf{iH}^*_{\sigma} \vdash \Box (C_i^{\Box})^{+} \leftrightarrow \Box C_i^{\Box}$. Hence $\mathsf{iH}^*_{\sigma} \vdash (A^{\Box})^{-} \leftrightarrow A^{\Box}$. ⊣

PROOF (Theorem 3.19). We prove by induction on $\mathfrak{c}(A^{\Box})$. Suppose that we have the desired result for each proposition B with $\mathfrak{c}(B^{\Box}) < \mathfrak{c}(A^{\Box})$. We treat A by the following cases.

1. (A1) A is atomic. Then $(A^{\Box})^{+} = A^{\Box}$, by definition, and result holds trivially.

2,3. (A1–A4, B1) $A = \Box B, A = B \wedge C, A = B \vee C$. All these cases hold by induction hypothesis. In boxed case, we use of induction hypothesis and A4. In conjunction case, we use of A1–A3 and in disjunction case, we use A1,A2 and B1.

4. $A = B \to C$. There are several sub-cases. Similar to the definition of NNIL-algorithm, an occurrence of a sub-formula B of A is said to be an *outer occurrence* in A, if it is neither in the scope of a $\Box$ nor in the scope of $\to$.

   4.a.(A1–A3) C contains an outer occurrence of a conjunction. We can treat this case using the induction hypothesis and TNNIL-algorithm.

   4.b.(A1–A3) B contains an outer occurrence of a disjunction. We can treat this case by induction hypothesis and TNNIL-algorithm.

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4.c. $B = \bigwedge X$ and $C = \bigvee Y$, where X, Y are sets of implications, atoms and boxed formulae. We have several subcases.

4.c.i.(A1–A4, B3) X contains atomic variables. Let p be an atomic variable in X. Set $D := \bigwedge(X \setminus \{p\})$. Then

$$\begin{aligned}(A^{\Box})^{+} &\equiv \boxdot[(p \wedge \Box p) \to (D^{\Box} \to C^{\Box})^{+}] \\ &\equiv \boxdot[(p \wedge \Box p) \to ((D \to C)^{\Box})^{+}].\end{aligned}$$

On the other hand, we have by induction hypothesis and A1,A2, and B3, that

$$[(p \wedge \Box p) \to ((D \to C)^{\Box})^{+}] \bowtie^{*} (p \wedge \Box p) \to ((D \to C)^{\Box})$$

which by use of A4 implies:

$$\Box[(p \wedge \Box p) \to ((D \to C)^{\Box})^{+}] \bowtie^{*} \Box[(p \wedge \Box p) \to ((D \to C)^{\Box})].$$

And by use of A1–A3 we have

$$\boxdot[(p \wedge \Box p) \to ((D \to C)^{\Box})^{+}] \bowtie^{*} \boxdot[(p \wedge \Box p) \to ((D \to C)^{\Box})].$$

Finally by A1 and A2 we have: $(A^{\Box})^{+} \bowtie^{*} A^{\Box}$.

4.c.i′.(A1–A4, B3) X contains boxed formula. Similar to the previous case.

4.c.ii,iii.(A1, A2) X contains $\top$ or $\bot$. Trivial.

4.c.iv.(A1–A4, B2, B3) X contains only implications. This case needs the axiom B2 and it seems to be an interesting case.

By Lemma 3.16,

$$(A^{\Box})^{+} \equiv \boxdot\left(\bigwedge_{E \to F \in X} \Box\left((E \to F)^{\Box}\right)^{+} \to \left(\bigwedge\left\{\left((B \downarrow D \to C)^{\Box}\right)^{+} \mid D \in X\right\} \wedge \left(([B]Z)^{\Box}\right)^{+}\right)\right).$$

Then by induction hypothesis, A1–A4 and B3 we have:

$$\begin{aligned}(A^{\Box})^{+} &\bowtie^{*} \boxdot\left(\bigwedge_{E \to F \in X} \Box(E \to F)^{\Box} \to \left(\bigwedge\left\{(B \downarrow D \to C)^{\Box} \mid D \in X\right\} \wedge ([B]Z)^{\Box}\right)\right) \\ &\bowtie^{*} \left(\Box B \to \left(\bigwedge\{B \downarrow D \to C \mid D \in X\} \wedge [B]Z\right)\right)^{\Box}.\end{aligned}$$

We show that for each $E \in Z$,

$$(*) \qquad \mathsf{iK4} \vdash \left(\bigwedge\{(B \downarrow D) \to C \mid D \in X\} \wedge [B]E\right) \to A.$$

If $E = C$, we are done by $\mathsf{IPC}_{\Box} \vdash [B]C \to (B \to C)$. So suppose some $E \to F \in X$. We reason in iK4. Assume $\bigwedge\{(B \downarrow D \to C \mid D \in X\}$, $[B]E$ and B. We want to derive C. We have $\left(\bigwedge(X \setminus \{E \to F\}) \wedge F\right) \to C$, $[B]E$ and B. From B and $[B]E$, we derive E. Also from B, we derive $E \to F$, and so F. Hence we have $\bigwedge(X \setminus \{E \to F\}) \wedge F$, which implies C, as desired.

Now (*) implies

$$\mathsf{iK4} \vdash \overbrace{\left(\bigwedge\{(B \downarrow D \to C \mid D \in X\} \wedge [B]Z\right)}^{G} \to A.$$

Then by Proposition 3.13, we have $\mathsf{iK4} \vdash (G^{\Box} \wedge B^{\Box}) \to C^{\Box}$. This implies $\mathsf{iK4} \vdash (B^{\Box} \to (G^{\Box} \wedge B^{\Box})) \to (B^{\Box} \to C^{\Box})$, and hence $\mathsf{iK4} \vdash (B^{\Box} \to G^{\Box}) \to$

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$(B^\Box \to C^\Box)$. Then because $B^\Box \to \Box B^\Box$, we have $\mathsf{iK4} \vdash (\Box(B^\Box) \to G^\Box) \to (B^\Box \to C^\Box)$. Hence by necessitation, we derive $\mathsf{iK4} \vdash (\Box B \to (\bigwedge\{(B \downarrow D \to C \mid D \in X\} \wedge [B]Z))^\Box \to A^\Box$. Hence $(A^\Box)^+ \blacktriangleright^* A^\Box$.

To show the other way around, i.e., $A^\Box \blacktriangleright^* (A^\Box)^+$, by Lemma 3.18, it is enough to show

$$A \blacktriangleright^* \left(\Box B \to \left(\bigwedge\{B \downarrow D \to C \mid D \in X\} \wedge [B]Z\right)\right)$$

or equivalently

$$A \wedge \Box B \blacktriangleright^* \left(\bigwedge\{B \downarrow D \to C \mid D \in X\} \wedge [B]Z\right).$$

We have $\mathsf{IPC}_\Box \vdash A \to \bigwedge\{B \downarrow D \to C \mid D \in X\}$, and hence by A1, $A \wedge \Box B \blacktriangleright^* \bigwedge\{B \downarrow D \to C \mid D \in X\}$. On the other hand, $A \wedge \Box B \blacktriangleright^* [B]Z$, which by A3, implies

$$A \wedge \Box B \blacktriangleright^* \left(\bigwedge\{B \downarrow D \to C \mid D \in X\} \wedge [B]Z\right). \quad \dashv$$

§4. The Σ_1-Provability logic of HA^*. In this section, we will show that iH^*_σ is the provability logic of HA^* for Σ_1-substitutions.

Before we continue with the soundness and completeness theorems, let us state the main Theorem from [2] that plays a crucial role in the rest of this article.

THEOREM 4.1. *Let $A \in \mathsf{TNNIL}^\Box$ be a modal proposition such that $\mathsf{iGLC} \nvdash A$. Then there exists some arithmetical Σ_1-substitution σ such that $\mathsf{HA} \nvdash \sigma_{\mathsf{HA}}(A)$.*

PROOF. For the rather long proof of this fact, see [2], Theorems 4.26 and 5.1. $\dashv$

4.1. The soundness theorem. Let us define some notions from [16]. Let T be a first-order arithmetical theory. We say that a first-order sentence A, Σ_1-preserves B ($A \rhd_{T,\Sigma_1} B$), if for each Σ_1-sentence C, if $T \vdash C \to A$, then $T \vdash C \to B$. For modal propositions A and B, we define $A \rhd_{T,\Sigma_1,\Sigma_1} B$ iff for each arithmetical Σ_1-substitution σ_T, we have $\sigma_T(A) \rhd_{T,\Sigma_1} \sigma_T(B)$. For arbitrary modal sentences A, B, the notation $A \mathrel{|\!\sim}_{T,\Sigma_1} B$ means that $T \vdash \sigma_T(A)$ implies $T \vdash \sigma_T(B)$, for arbitrary Σ_1-substitution σ_T. All the above relations with a superscript of HA, means "an arithmetical formalization of that relation in HA", for example, $A \rhd^{\mathsf{HA}}_{\mathsf{HA}^*,\Sigma_1} B$ means $\mathsf{HA} \vdash$ "$A \rhd_{\mathsf{HA}^*,\Sigma_1} B$".

LEMMA 4.2. 1. *For each first-order sentences A, B, $A \rhd^{\mathsf{HA}}_{\mathsf{HA}^*,\Sigma_1} B$ iff $A^{\mathsf{HA}} \rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1} B^{\mathsf{HA}}$,*

2. *For each propositional modal A, B, $A \rhd^{\mathsf{HA}}_{\mathsf{HA}^*,\Sigma_1,\Sigma_1} B$ iff $A^\Box \rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1,\Sigma_1} B^\Box$.*

PROOF. To prove part 1, use Lemma 2.6.1 and definitions of $\rhd^{\mathsf{HA}}_{\mathsf{HA}^*,\Sigma_1}$ and $\rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1}$.

To prove part 2, note that $A \rhd^{\mathsf{HA}}_{\mathsf{HA}^*,\Sigma_1,\Sigma_1} B$ iff for all Σ-substitution σ, $\sigma_{\mathsf{HA}^*}(A) \rhd^{\mathsf{HA}}_{\mathsf{HA}^*,\Sigma_1} \sigma_{\mathsf{HA}^*}(B)$ iff for all Σ-substitution σ, $\sigma_{\mathsf{HA}^*}(A)^{\mathsf{HA}} \rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1} \sigma_{\mathsf{HA}^*}(B)^{\mathsf{HA}}$ (by previous part) iff for all Σ-substitution σ, $\sigma_{\mathsf{HA}}(A^\Box) \rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1} \sigma_{\mathsf{HA}}(B^\Box)$ iff $A^\Box \rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1,\Sigma_1} B^\Box$. $\dashv$

LEMMA 4.3. *$\rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1}$ is closed under $B1$.*

PROOF. See [16], 9.1. $\dashv$

COROLLARY 4.4. *$\rhd^{\mathsf{HA}}_{\mathsf{HA}^*,\Sigma_1}$ is closed under $B1$.*

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PROOF. Immediate corollary of Lemmas 4.2 and 4.3. ⊣

LEMMA 4.5. $\rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1,\Sigma_1}$ *satisfies* $A1$–$A4$, $B1$, $B2'$, *and* $B3$.

PROOF. Proof of closure under A1–A4 and B3 is straightforward. Closure under B1 is by Lemma 4.3. For a proof of case B2′, see [16].9.2. Note that B_2' is named as B_2 in [16]. ⊣

COROLLARY 4.6. $\rhd^{\mathsf{HA}}_{\mathsf{HA}^*,\Sigma_1,\Sigma_1}$ *satisfies* $B2$.

PROOF. Let A, B, C, X, Z be as stated in defining B2. We must prove $A \wedge \Box B \rhd^{\mathsf{HA}}_{\mathsf{HA}^*,\Sigma_1,\Sigma_1} [B]Z$. Hence by Lemma 4.2, it is enough to show $(A \wedge \Box B)^{\Box} \rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1,\Sigma_1} ([B]Z)^{\Box}$. Let $X' := \{E^{\Box} \to F^{\Box} | E \to F \in X\}$, $B' := \bigwedge X'$, $C' := C^{\Box}$, $Z' := \{E^{\Box} | E \to F \in X\} \cup \{C'\}$. Now Because $\rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1,\Sigma_1}$ satisfies B2′ (Lemma 4.5), we have $(B' \to C') \rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1,\Sigma_1} [B']Z'$. Note that $Z^{\Box} = Z'$ and $\mathsf{IPC}_{\Box} \vdash (B' \wedge \Box B') \leftrightarrow B^{\Box}$. Hence by Lemma 3.14.2, $\mathsf{iK4} + \Box B' \vdash [B']Z' \leftrightarrow [B^{\Box}]Z^{\Box}$. Also by Lemma 3.14.1, $\mathsf{iK4} + \Box B' \vdash [B^{\Box}]Z^{\Box} \leftrightarrow ([B]Z)^{\Box}$. So $\mathsf{iK4} + \Box B' \vdash [B']Z' \leftrightarrow ([B]Z)^{\Box}$. Now because $\rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1,\Sigma_1}$ satisfies A1, we have $\Box B' \rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1,\Sigma_1} [B']Z' \leftrightarrow ([B]Z)^{\Box}$. Now one can easily observe that because $\rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1,\Sigma_1}$ is closed under A1–A3, we can deduce $(B' \to C') \wedge \Box B' \rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1,\Sigma_1} ([B]Z)^{\Box}$. This by using A1–A3 implies $((B \to C) \wedge \Box B)^{\Box} \rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1,\Sigma_1} ([B]Z)^{\Box}$. Hence by Lemma 4.2.2, $(B \to C) \wedge \Box B \rhd^{\mathsf{HA}}_{\mathsf{HA}^*,\Sigma_1,\Sigma_1} [B]Z$, as desired. ⊣

COROLLARY 4.7. $\rhd^{\mathsf{HA}}_{\mathsf{HA}^*,\Sigma_1,\Sigma_1}$ *is closed under* $B3$.

PROOF. Let p be atomic or boxed and assume some A, B such that $A \rhd^{\mathsf{HA}}_{\mathsf{HA}^*,\Sigma_1,\Sigma_1} B$. Then by Lemma 4.2.2, $A^{\Box} \rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1,\Sigma_1} B^{\Box}$. Because $\rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1,\Sigma_1}$ satisfies A1–A3 and B3, we get $p^{\Box} \to A^{\Box} \rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1,\Sigma_1} p^{\Box} \to B^{\Box}$. Now by A4, $\Box[p^{\Box} \to A^{\Box}] \rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1,\Sigma_1} \Box[p^{\Box} \to B^{\Box}]$, which implies $(p \to A)^{\Box} \rhd^{\mathsf{HA}}_{\mathsf{HA},\Sigma_1,\Sigma_1} (p \to B)^{\Box}$. Now by Lemma 4.2.2, $p \to A \rhd^{\mathsf{HA}}_{\mathsf{HA}^*,\Sigma_1,\Sigma_1} p \to B$, as desired. ⊣

LEMMA 4.8. *We have the following inclusions*:

$$\blacktriangleright^* \subseteq \rhd^{\mathsf{HA}}_{\mathsf{HA}^*,\Sigma_1,\Sigma_1} \subseteq \mathrel{|\!\sim}^{\mathsf{HA}}_{\mathsf{HA}^*,\Sigma_1}.$$

PROOF. The second inclusion is a trivial. We only prove the first inclusion. We show that $\rhd^{\mathsf{HA}}_{\mathsf{HA}^*,\Sigma_1,\Sigma_1}$ is closed under A1–A4 and B1–B3. One can observe that $\rhd^{\mathsf{HA}}_{\mathsf{HA}^*,\Sigma_1,\Sigma_1}$ is closed under A1–A4 and we leave this to the reader. Closure under B1, B2, and B3 is by Corollaries 4.4,4.6 and 4.7, respectively. ⊣

THEOREM 4.9 (Soundness). iH^*_σ *is sound for* Σ_1*-arithmetical interpretations in* HA^*, *i.e.*, $\mathsf{iH}^*_\sigma \subseteq \mathcal{PL}_\Sigma(\mathsf{HA}^*)$.

PROOF. We show that for arbitrary Σ-substitution, σ_{HA^*}, and for any A, if $\mathsf{iH}^*_\sigma \vdash A$, then $\mathsf{HA}^* \vdash \sigma_{\mathsf{HA}^*}(A)$. This can be done by induction on the complexity of $\mathsf{iH}^*_\sigma \vdash A$. All inductive steps clearly holds, except for the axioms $\Box A \to \Box B$ with $A \blacktriangleright^* B$. This case is a direct consequence of Lemma 4.8. ⊣

4.2. The completeness theorem.

THEOREM 4.10. Σ_1*-arithmetical interpretations in* HA^* *are complete for* iH^*_σ, *i.e.*,

$$\mathcal{PL}_\Sigma(\mathsf{HA}^*) \subseteq \mathsf{iH}^*_\sigma.$$

PROOF. We prove the Completeness Theorem contra-positively. Let $\mathsf{iH}^*_\sigma \nvdash A(p_1, \ldots, p_n)$. Lemma 2.5 implies $\mathsf{iH}^*_\sigma \nvdash A^{\Box}$ and hence by Corollary 3.20, $\mathsf{iH}^*_\sigma \nvdash$

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$(A^{\Box})^{-}$. This, by Theorem 3.6, implies $\mathsf{iH}^*_\sigma \nvdash ((A^{\Box})^{-})^*$ and hence $\mathsf{iH}^*_\sigma \nvdash (A^{\Box})^{+}$, and a fortiori, $\mathsf{iGLC} \nvdash (A^{\Box})^{+}$. Hence by Theorem 4.1, there exists some Σ_1-substitution σ, such that $\mathsf{HA} \nvdash \sigma_{\mathsf{HA}}((A^{\Box})^{+})$. Hence by Lemma 3.7.1, $\mathsf{HA} \nvdash \sigma_{\mathsf{HA}}(A^{\Box})$ and by Lemma 2.7, $\mathsf{HA}^* \nvdash \sigma_{\mathsf{HA}^*}(A)$. ⊣

COROLLARY 4.11. *For any modal proposition* A, $\mathsf{iH}^*_\sigma \vdash A$ *iff* $\mathsf{iH}_\sigma \vdash A^{\Box}$.

PROOF. By Theorems 4.9 and 4.10 and Lemma 2.7. ⊣

COROLLARY 4.12. iH^*_σ *is decidable.*

PROOF. A proof can be given either with inspections in the proof of the Completeness Theorem (4.10) or by using the decidability of iH_σ [2] and Corollary 4.11. ⊣

4.3. Open problems.

1. The statement of Corollary 4.11 is *purely propositional*. However, our proof of this corollary is based on Theorem 4.10, that has *arithmetical* theme. A tempting problem is to find a *direct propositional proof* for this corollary. Then we can derive Theorem 4.10.
2. We conjecture that the full provability logic of HA^* is the logic iH^*, axiomatized as follows
$$\mathsf{iH}^* := \mathsf{iGL} + \mathsf{CP} + \{\Box A \to \Box B : A \blacktriangleright^* B\},$$
in which the relation $\blacktriangleright^*$ is defined as the smallest relation satisfying:
   - A1. If $\mathsf{iK4} \vdash A \to B$, then $A \blacktriangleright^* B$,
   - A2. If $A \blacktriangleright^* B$ and $B \blacktriangleright^* C$, then $A \blacktriangleright^* C$,
   - A3. If $C \blacktriangleright^* A$ and $C \blacktriangleright^* B$, then $C \blacktriangleright^* A \wedge B$,
   - A4. If $A \blacktriangleright^* B$, then $\Box A \blacktriangleright^* \Box B$,
   - B1. If $A \blacktriangleright^* C$ and $B \blacktriangleright^* C$, then $A \vee B \blacktriangleright^* C$,
   - B2. Let X be a set of implications, $B := \bigwedge X$ and $A := B \to C$. Also assume that $Z := \{E | E \to F \in X\} \cup \{C\}$. Then $A \wedge \Box B \blacktriangleright^* \{B\}Z$,
   - B3. If $A \blacktriangleright^* B$, then $\Box C \to A \blacktriangleright^* \Box C \to B$.

   The notation $\{A\}(B)$, for modal propositions A and B, is defined inductively:
   - $\{A\}(\Box B) = \Box B$ and $\{A\}(\bot) = \bot$.
   - $\{A\}(B_1 \circ B_2) = \{A\}(B_1) \circ \{A\}(B_2)$, for $\circ \in \{\vee, \wedge\}$,
   - $\{A\}(B) = A \to B$ for all of the other cases, i.e., when B is atomic variable or implication.

   And $\{A\}\Gamma$, for a set Γ of modal propositions, is defined as $\bigvee_{B \in \Gamma}\{A\}(B)$.

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