

FROM HANKEL OPERATORS TO CARLESON MEASURES IN A QUATERNIONIC VARIABLE

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Abstract We introduce and study Hankel operators defined on the Hardy space of regular functions of a quaternionic variable. Theorems analogous to those of Nehari and Fefferman are proved.

Keywords: Hardy space on the quaternionic ball; functions of a quaternionic variable; Hankel operators

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Notation

The symbol \mathbb{H} denotes the set of the quaternions $q = x_0 + x_1i + x_2j + x_3k = \operatorname{Re}(q) + \operatorname{Im}(q)$, with $\operatorname{Re}(q) = x_0$ and $\operatorname{Im}(q) = x_1i + x_2j + x_3k$, where the x_j s are real numbers and the imaginary units i, j, k are subject to the rules $ij = k, jk = i, ki = j$ and $i^2 = j^2 = k^2 = -1$. We identify the quaternions q whose imaginary part vanishes, $\operatorname{Im}(q) = 0$, with real numbers, $\operatorname{Re}(q) \in \mathbb{R}$, and, similarly, we let $\mathbb{I} = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ be the set of the imaginary quaternions. The norm $|q| \geq 0$ of q is

$$|q| = \left(\sum_{l=0}^3 x_l^2 \right)^{1/2} = (q\bar{q})^{1/2},$$

where $\bar{q} = x_0 - x_1i - x_2j - x_3k$ is the conjugate of q . The open unit ball \mathbb{B} in \mathbb{H} contains the quaternions q such that $|q| < 1$. The boundary of \mathbb{B} in \mathbb{H} is denoted by $\partial\mathbb{B}$. By the symbol \mathbb{S} we denote the unit sphere of the imaginary quaternions: $q \in \mathbb{I}$ belongs to \mathbb{S} if $|q| = 1$. For I in \mathbb{S} , the slice $L_I = L_{-I}$ in \mathbb{H} contains all quaternions having the form $q = x + yI$, with x, y in \mathbb{R} .

1. Introduction

Let \mathbb{H} be the skew-field of the quaternions. The quaternionic Hardy space $H^2(\mathbb{B})$ consists of the formal power series of the quaternionic variable q , $|q| < 1$,

$$f(q) = \sum_{n=0}^{\infty} q^n a_n, \tag{1.1}$$

such that the sequence of quaternions $\{a_n\}$ satisfies

$$\|f\|_{H^2(\mathbb{B})} := \|\{a_n\}\|_{\ell^2(\mathbb{N}, \mathbb{H})} = \left(\sum_{n=0}^{\infty} |a_n|^2 \right)^{1/2} < \infty. \quad (1.2)$$

Such functions are *regular* in \mathbb{B} in the sense of Gentili and Struppa [10]. For functions in $H^2(\mathbb{B})$, the quaternion-valued inner product is

$$\left\langle \sum q^n a_n, \sum q^n b_n \right\rangle_{H^2(\mathbb{B})} := \sum_{n=0}^{\infty} \overline{b_n} a_n.$$

We might think of $a = \{a_n\}$ as a discrete, positive time, quaternionic-valued signal, with finite energy, and, using standard notation, of $f(q) = \hat{a}(q) = \sum_{n=0}^{\infty} q^n a_n$ as its generating function. Suppose that T is a linear, stable, time-invariant, realizable filter acting on such signals. By this we mean a linear, bounded operator T on $\ell^2(\mathbb{N}, \mathbb{H})$, such that: (*time invariance*) T commutes with the shift operator, and (*realizability*) the output Ta at time n depends on a_m with $m \leq n$ only. The shift operator S is defined as

$$Sa(n) = a(n-1).$$

We have used the functional notation $a(n) = a_n$. We adopt here the convention that in vector spaces V on the quaternionic skew-field, in multiplication, scalar factors are on the right: if $v \in V$ and $\alpha \in \mathbb{H}$, then $v\alpha \in V$. Exactly as in the complex-valued case, T is time invariant and well defined on Kronecker functions δ_k ($k \in \mathbb{Z}$) if and only if $T = T_K$ is a convolution operator:

$$T_K a(n) = K * a(n) = \sum_{k \in \mathbb{Z}} K(n-k)a(k), \quad (1.3)$$

with $K = \{K(n)\}$ the sequence defined by $K = T\delta_0$. On the other hand, realizability is equivalent to the condition that $K(n) = 0$ if $n < 0$. Relation (1.3) justifies the introduction of the \star -product between functions of the form (1.1). If $g(q) = \sum_{n \geq 0} q^n b_n$, then

$$(f \star g)(q) = \sum_{n \geq 0} q^n \sum_{k=0}^n a_{n-k} b_k.$$

The \star -product is not a pointwise product (see § 2). If $\phi(q) = \hat{K}(q) = \sum_{n \geq 0} q^n K(n)$, then (1.3) becomes

$$\widehat{Ta} = \phi \star \hat{a} := M_\phi a; \quad (1.4)$$

M_ϕ is a \star -multiplication operator, where $\hat{a}(q) = \sum_{n \geq 0} q^n a(n)$.

Let $\mathcal{B}(\mathcal{H})$ be the non-commutative algebra of the bounded operators on the quaternionic Hilbert space \mathcal{H} , normed with the sup-norm. See [12] for the basic facts of functional analysis of linear spaces over the quaternions. The sharp result on boundedness for \star -multiplication operators is in [1].

Theorem A. Let $\phi: \mathbb{B} \rightarrow \mathbb{H}$ be a function that can be written as $\phi(q) = \sum_{n \geq 0} q^n \check{\phi}(n)$, where the sum converges absolutely in \mathbb{B} . Then

$$\|M_\phi\|_{\mathcal{B}(H^2(\mathbb{B}))} = \|\phi\|_{H^\infty(\mathbb{B})} := \sup_{q \in \mathbb{B}} |\phi(q)|.$$

As a consequence, if $\check{\phi}$ denotes the sequence of coefficients of ϕ , $\check{\phi} = \{\check{\phi}(n)\}_n$, then $\|T_{\check{\phi}}\|_{\mathcal{B}(\ell^2(\mathbb{N}, \mathbb{H}))} = \|\phi\|_{H^\infty(\mathbb{B})}$.

Equation (1.3), with $n, k \geq 0$, might be interpreted as multiplication times an infinite matrix $A_\alpha = [\alpha(n, k)]_{n, k \geq 0} = [K(n - k)]_{n, k \geq 0}$ with constant diagonals. In applications, it is important to consider the case of an infinite matrix with constant anti-diagonals, that is, of Hankel operators. See [14] for a detailed excursion into the theory of such operators.

Let $\alpha: \mathbb{N} \rightarrow \mathbb{H}$ be a quaternion-valued sequence. Form the infinite matrix $A_\alpha = [\alpha(j + k)]_{j, k=0}^{+\infty}$, and let Γ_α act on \mathbb{H} -valued sequences $v = (v_j)_{j=0}^{+\infty}$ by matrix multiplication:

$$(\Gamma_\alpha v)(j) = \sum_{k=0}^{\infty} \alpha(j + k)v(k).$$

Nehari’s problem. Under which conditions on α is the operator Γ_α bounded on $\ell^2(\mathbb{N}, \mathbb{H})$?

Exactly as in the complex-valued case, the problem can be reformulated in terms of regular functions; one has only to be careful in keeping account of the non-commutativity of the product in \mathbb{H} . For two positive real numbers r, s we will write $r \approx s$ if $r/C \leq s \leq Cr$ for a positive constant C and we will write $r \lesssim s$ if there exists $C > 0$ such that $r \leq Cs$. Our first main result is the following.

Theorem 1.1. Let the sequence α and the regular function $b(q) = \sum_{n=0}^{\infty} q^n \check{b}(n)$ be related via

$$\alpha(n) = \overline{\check{b}(n)} \quad \text{for } n \geq 0. \tag{1.5}$$

Set $A_b(h) := \langle h, b \rangle_{H^2(\mathbb{B})}$ when h is regular in \mathbb{B} .

The following conditions are equivalent.

- (a) $\|\Gamma_\alpha\|_{\mathcal{B}(\ell^2(\mathbb{N}, \mathbb{H}))} < \infty$.
- (b) $\sup_{f, g \neq 0} \frac{|\langle f \star g, b \rangle_{H^2(\mathbb{B})}|}{\|f\|_{H^2(\mathbb{B})} \cdot \|g\|_{H^2(\mathbb{B})}} < \infty$.
- (c) $\|b\|_{\text{BMO}(\mathbb{B})} := \sup_{I \in \mathbb{S}} \|b\|_{\text{BMO}(\mathbb{B}_I)} < \infty$, where $\|b\|_{\text{BMO}(\mathbb{B}_I)}$ is the BMO-norm of the restriction of b to the slice \mathbb{B}_I .
- (d) The measure $d\mu_b(q) = |\partial_c b(q)|^2 (1 - |q|^2) d\text{Vol}_{\mathbb{B}}(q)$ on \mathbb{B} is a Carleson measure for $H^2(\mathbb{B})$: the inequality $\int_{\mathbb{B}} |f|^2 d\mu_b \leq c(\mu_b) \|f\|_{H^2(\mathbb{B})}^2$ holds whenever f is in $H^2(\mathbb{B})$.
- (e) A_b belongs to $[H^2(\mathbb{B}) \otimes H^2(\mathbb{B})]^*$, the space dual to the weak \star -product of two copies of $H^2(\mathbb{B})$, with respect to the inner product in $H^2(\mathbb{B})$.

Moreover,

$$\begin{aligned} \|\Gamma_\alpha\|_{\mathcal{B}(\ell^2(\mathbb{N},\mathbb{H}))} &= \sup_{f,g \neq 0} \frac{|\langle f \star g, b \rangle_{H^2(\mathbb{B})}|}{\|f\|_{H^2(\mathbb{B})} \cdot \|g\|_{H^2(\mathbb{B})}} \approx \|A_b\|_{(H^2(\mathbb{B}) \otimes H^2(\mathbb{B}))^*} \\ &\approx \|b\|_{\text{BMO}(\mathbb{B})} \\ &\approx C(\mu_b)^{1/2}. \end{aligned}$$

In (d), $\partial_c b(q) = \sum_{n=1}^\infty n\check{b}(n)q^{n-1}$ is the Cullen derivative of b . See §2 for basic terminology and facts regarding regular functions.

The statement summarizes a circle of ideas which, in the complex-valued case, are due to Nehari [13], Fefferman [9], and Coifman *et al.* [6]. The weak \star -product $H^2(\mathbb{B}) \otimes H^2(\mathbb{B})$ is defined as the space of the functions φ that are regular in \mathbb{B} , and such that

$$\|\varphi\|_{H^2(\mathbb{B}) \otimes H^2(\mathbb{B})} := \inf \left\{ \sum_{j=1}^\infty \|f_j\|_{H^2(\mathbb{B})} \cdot \|g_j\|_{H^2(\mathbb{B})} : \varphi = \sum_{j=1}^\infty f_j \star g_j \right\} < \infty. \tag{1.6}$$

The proof of Theorem 1.1 partly makes direct use of the corresponding results in one complex variable, and partly adapts their proofs to the quaternionic setting, but some new ideas are required. On the one hand there is no available inner–outer factorization for H^2 functions in the quaternionic sense. On the other hand, the geometric properties of H^2 's reproducing kernel are not as clear as in the one complex variable case.

In §5 we discuss several equivalent definitions of BMOA in the quaternionic context. There we show in a rather indirect way the curious fact that in (1.6) the infimum might be taken over decompositions with two summands only, $\varphi = \sum_{j=1}^2 f_j \star g_j$. Having just one summand would imply a good factorization for regular functions in $H^1(\mathbb{B})$, but we do not have a result of this strength.

In order to make more concrete the definition of Carleson measures, we also provide a geometric characterization of them, which might have some independent interest. For $q = re^{J\theta} \in \mathbb{B}$, with $r \geq 0$, $\theta \in \mathbb{R}$, and $J \in \mathbb{S}$, let

$$S(q) = \{\varrho e^{I\alpha} \in \mathbb{B} : |\alpha - \theta| \leq 1 - r, 0 < 1 - \varrho \leq 2(1 - r), I \in \mathbb{S}\}$$

be the symmetric box in \mathbb{B} , indexed by q , which is independent of the particular J .

Theorem 1.2. *A measure μ on \mathbb{B} is a Carleson measure for the Hardy space $H^2(\mathbb{B})$ if and only if for any $q \in \mathbb{B}$ the measure of the symmetric box $S(q)$ satisfies*

$$\mu(S(q)) \lesssim 1 - |q|.$$

The theorem corresponds to Carleson’s characterization of the corresponding measures on the complex unit disc; see [5]. The ‘only if’ part of the proof requires some care. The fact that it suffices to test the measure over symmetric boxes only reflects the fact that regular functions are affine with respect to the imaginary unit, or, more geometrically, the fact that the ‘invariant metric’ for functions regular on \mathbb{B} is bounded on copies of \mathbb{S} inside \mathbb{B} (see [3]).

The paper is structured as follows. In §2 we provide some background material on regular functions of a quaternionic variable. Section 3, with the proof of Theorem 1.2, and §4, with the characterization of the symbols of bounded Hankel operators, are essentially independent of each other. In §5 we discuss some properties of the quaternionic version of analytic BMO. Some questions we could not answer are summarized in §6.

2. Preliminaries

In this section we recall the definition of slice regular functions over the quaternions \mathbb{H} (in what follows, simply ‘regular’ functions), together with some basic properties. We will restrict our attention to functions defined on the quaternionic unit ball $\mathbb{B} = \{q \in \mathbb{H} : |q| < 1\}$. We refer the reader to [11] for all details and proofs.

Let \mathbb{S} denote the two-dimensional sphere of imaginary units of \mathbb{H} , $\mathbb{S} = \{q \in \mathbb{H} \mid q^2 = -1\}$. One can ‘slice’ the space \mathbb{H} in copies of the complex plane that intersect along the real axis,

$$\mathbb{H} = \bigcup_{I \in \mathbb{S}} (\mathbb{R} + \mathbb{R}I), \quad \mathbb{R} = \bigcap_{I \in \mathbb{S}} (\mathbb{R} + \mathbb{R}I),$$

where $L_I := \mathbb{R} + \mathbb{R}I \cong \mathbb{C}$, for any $I \in \mathbb{S}$. Each element $q \in \mathbb{H}$ can be expressed as $q = x + yI_q$, where x, y are real (if $q \in \mathbb{R}$, then $y = 0$) and I_q is an imaginary unit. To have a unique decomposition (outside the real axis) we choose $y \geq 0$.

A function $f : \mathbb{B} \rightarrow \mathbb{H}$ is called (*slice*) *regular* if for any $I \in \mathbb{S}$ the restriction f_I of f to $\mathbb{B}_I = \mathbb{B} \cap L_I$ is *holomorphic*, i.e. it has continuous partial derivatives and it is such that

$$\bar{\partial}_I f_I(x + yI) = \frac{1}{2} \left(\frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + yI) = 0$$

for all $x + yI \in \mathbb{B}_I$.

A wide class of examples of regular functions is given by power series with quaternionic coefficients of the form $\sum_{n=0}^\infty q^n a_n$ that converge in open balls centred at the origin. In fact, a function f is regular on \mathbb{B} if and only if f has a power series expansion $f(q) = \sum_{n=0}^\infty q^n a_n$ converging in \mathbb{B} .

The *slice (or Cullen) derivative* of a function f that is regular on \mathbb{B} is the regular function defined as

$$\partial_c f(x + yI) = \frac{1}{2} \left(\frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) f_I(x + yI).$$

Slice regular functions defined on \mathbb{B} present a peculiar property.

Theorem 2.1 (representation and extension formula). *Let $f : \mathbb{B} \rightarrow \mathbb{H}$ be a regular function and let $x + y\mathbb{S} \subset \mathbb{B}$. Then, for any $I, J \in \mathbb{S}$,*

$$f(x + yJ) = \frac{1}{2}[f(x + yI) + f(x - yI)] + \frac{1}{2}JI[f(x - yI) - f(x + yI)].$$

Moreover, the previous formula allows one to uniquely extend any holomorphic function $g_I : \mathbb{B}_I \rightarrow \mathbb{H}$ to a slice regular function on \mathbb{B} , denoted by $\text{ext}(g_I)$.

A basic result that establishes a relation between regular functions and holomorphic functions of one complex variable is the following.

Lemma 2.2 (splitting lemma). *Let f be a regular function on \mathbb{B} . Then for any $I \in \mathbb{S}$ and for any $J \in \mathbb{S}$, $J \perp I$, there exist two holomorphic functions $F, G: \mathbb{B}_I = \mathbb{B} \cap L_I \rightarrow L_I$ such that*

$$f(x + yI) = F(x + yI) + G(x + yI)J$$

for any $x + yI \in \mathbb{B}_I$.

Since the pointwise product of functions does not preserve slice regularity, a new multiplication operation for regular functions is defined. In the special case of power series, the *regular product* (or \star -product) of $f(q) = \sum_{n=0}^\infty q^n a_n$ and $g(q) = \sum_{n=0}^\infty q^n b_n$ is

$$f \star g(q) = \sum_{n \geq 0} q^n \sum_{k=0}^n a_k b_{n-k}.$$

The \star -product is related to the standard pointwise product by the following formula.

Proposition 2.3. *Let f, g be regular functions on \mathbb{B} . Then*

$$f \star g(q) = \begin{cases} 0 & \text{if } f(q) = 0, \\ f(q)g(f(q)^{-1}qf(q)) & \text{if } f(q) \neq 0. \end{cases}$$

The reproducing kernel for the quaternionic Hardy space $H^2(\mathbb{B})$ is expressed as

$$k_w(q) = \sum_{n=0}^\infty q^n \bar{w}^n = (1 - q\bar{w})^{-\star},$$

where the power $-\star$ denotes the reciprocal in the slice regular sense [11]. See [1] for this and other basic facts concerning $H^2(\mathbb{B})$.

3. Carleson measures on \mathbb{B}

A non-negative Borel measure μ on \mathbb{B} is called a *Carleson measure* for the Hardy space $H^2(\mathbb{B})$ if it satisfies the embedding inequality

$$\int_{\mathbb{B}} |f(q)|^2 d\mu(q) \leq c(\mu) \|f\|_{H^2(\mathbb{B})}^2$$

for any $f \in H^2(\mathbb{B})$, with a constant $c(\mu)$ depending on μ alone.

We can always decompose a measure μ on the unit ball as $\mu = \mu_1 + \mu_2$, where $\mu_1(\mathbb{B} \cap \mathbb{R}) = 0$ and $\text{supp}(\mu_2) \subseteq \mathbb{B} \cap \mathbb{R}$. Moreover, μ is a Carleson measure if and only if the measures μ_1 and μ_2 are Carleson as well.

Thanks to the disintegration theorem (see [2, Theorem 2.28]), any finite measure μ on \mathbb{B} , such that $\mu(\mathbb{B} \cap \mathbb{R}) = 0$, can be uniquely decomposed as

$$d\mu(x + yI) = d\mu_I^\dagger(x + yI) d\nu(I), \tag{3.1}$$

where ν is the measure on the sphere \mathbb{S} defined by

$$\nu(E) = \mu(\{x + yI \in \mathbb{B} \mid y > 0 \text{ and } I \in E\})$$

and μ_I^+ is a (probability) measure on $\mathbb{B}_I^+ = \{x + yI \in \mathbb{B}_I : y \geq 0\}$. Hence we can write

$$\int_{\mathbb{B}} \varphi(x + yI) \, d\mu(x + yI) = \int_{\mathbb{S}} \int_{\mathbb{B}_I^+} \varphi(x + yI) \, d\mu_I^+(x + yI) \, d\nu(I)$$

for any $\varphi: \mathbb{B} \rightarrow \mathbb{H}$. If in general μ is a finite measure on \mathbb{B} , we can decompose $\mu = \mu_{\mathbb{R}} + \tilde{\mu}$, where $\tilde{\mu}(\mathbb{B} \cap \mathbb{R}) = 0$ and $\text{supp } \mu_{\mathbb{R}} \subseteq \mathbb{B} \cap \mathbb{R}$, so that

$$\int_{\mathbb{B}} \varphi(x + yI) \, d\mu(x + yI) = \int_{\mathbb{B} \cap \mathbb{R}} \varphi(x) \, d\mu_{\mathbb{R}}(x) + \int_{\mathbb{S}} \int_{\mathbb{B}_I^+} \varphi(x + yI) \, d\tilde{\mu}_I^+(x + yI) \, d\nu(I), \tag{3.2}$$

where $\tilde{\mu}_I^+$ is obtained from $\tilde{\mu}$ using the disintegration theorem.

A stronger property for a measure defined on the unit ball \mathbb{B} is to be a Carleson measure on each slice \mathbb{B}_I . For any $I \in \mathbb{S}$ let $\mu_I = \mu_{\mathbb{R}} + \tilde{\mu}_I^+ + \tilde{\mu}_{-I}^+$ be the ‘restriction’ of μ to the slice \mathbb{B}_I .

A finite Borel measure μ is called *slice Carleson* for $H^2(\mathbb{B})$ if there exists a constant $c(\mu)$ such that

$$\begin{aligned} \int_{\mathbb{B}_I} |f(x + yI)|^2 \, d\mu_I(x + yI) &= \int_{\mathbb{B} \cap \mathbb{R}} |f(x)|^2 \, d\mu_{\mathbb{R}}(x) + \int_{\mathbb{B}_I^+} |f(x + yI)|^2 \, d\tilde{\mu}_I^+(x + yI) \\ &\quad + \int_{\mathbb{B}_{-I}^+} |f(x + y(-I))|^2 \, d\tilde{\mu}_{-I}^+(x + y(-I)) \\ &\leq c(\mu) \|f\|_{H^2(\mathbb{B})}^2 \end{aligned} \tag{3.3}$$

for any $f \in H^2(\mathbb{B})$ and I in \mathbb{S} .

For any $I \in \mathbb{S}$, the norm $\|f\|_{H^2(\mathbb{B}_I)}^2$ can be interpreted as the complex H^2 -norm of the restriction $f_I(x + yI) = \sum_{n=0}^{\infty} (x + yI)^n a_n$ of f to \mathbb{B}_I :

$$\|f\|_{H^2(\mathbb{B}_I)}^2 = \sum_{n=0}^{\infty} |a_n|^2 = \|f\|_{H^2(\mathbb{B})}^2.$$

Proposition 3.1. *Let μ be a finite Borel measure. If μ is slice Carleson for the Hardy space $H^2(\mathbb{B})$, then it is also Carleson.*

Proof. Using the same notation as in (3.2), since $\mu(\mathbb{B}) = \nu(\mathbb{S}) < +\infty$ we can apply the disintegration theorem to write

$$\begin{aligned} \int_{\mathbb{B}} |f(q)|^2 \, d\mu(q) &= \int_{\mathbb{B} \cap \mathbb{R}} |f(x)|^2 \, d\mu_{\mathbb{R}}(x) + \int_{\mathbb{S}} d\nu(I) \int_{\mathbb{B}_I^+} |f(z)|^2 \, d\tilde{\mu}_I^+(z) \\ &\lesssim \|f\|_{H^2(\mathbb{B})}^2 + \nu(\mathbb{S}) \|f\|_{H^2(\mathbb{B})}^2 \\ &\lesssim \|f\|_{H^2(\mathbb{B})}^2 \end{aligned}$$

for any $f \in H^2(\mathbb{B})$. □

It is not difficult to find examples of infinite measures μ on \mathbb{B} such that μ_I is a Carleson measure on each slice \mathbb{B}_I .

As in the classical complex setting (see, for example, [8]), it is possible to give a characterization of Carleson measures in terms of ‘boxes’. For $q = re^{j\theta} \in \mathbb{B}$, denote by $A_I(q)$ the arc of $\partial\mathbb{B}_I$ defined as

$$A_I(q) = \{e^{I\alpha} \in \partial\mathbb{B}_I : |\alpha - \theta| \leq 1 - r\},$$

and let $S_I(q)$ be the ‘box’ in \mathbb{B}_I defined by

$$S_I(q) = \{\varrho e^{I\alpha} \in \mathbb{B}_I : e^{I\alpha} \in A_I(q), 0 < 1 - \varrho \leq 2(1 - r)\}.$$

Definition 3.2. Let $q \in \mathbb{B}$. The symmetric subset of \mathbb{B} obtained as

$$S(q) = \bigcup_{I \in \mathbb{S}} S_I(q)$$

is called a *symmetric box*.

For slice Carleson measures the characterization follows quite easily from the classical case.

Proposition 3.3. *A finite Borel measure μ on \mathbb{B} is slice Carleson for $H^2(\mathbb{B})$ if and only if, for any $I \in \mathbb{S}$ and $z \in \mathbb{B}_I$, $\mu_I(S_I(z)) \lesssim |A_I(z)|$, where $|A_I(z)|$ denotes the length of the arc $A_I(z)$.*

The main ingredients of the proof are the classical Carleson theorem (see, for example, [8, Theorem 9.3]) and the splitting lemma (Lemma 2.2).

The characterization theorem for general Carleson measures requires the following technical result.

Lemma 3.4. *Let $w \in \mathbb{D}$ and let $s(w) = \{z \in \mathbb{D} : 1 - |z| \leq 2(1 - |w|), |\arg(z) - \arg(w)| \leq 1 - |w|\}$. Then there exists a constant $c > 0$ such that*

$$\frac{1}{c} \leq \left| \frac{1 - z(w + \bar{w})/2}{1 - zw} \right| \leq c$$

for any $z \in s(w)$.

Proof. We can suppose that $\operatorname{Re} w > 0$, since the statement is invariant by multiplication by -1 , and $\operatorname{Im} w \geq 0$, since $\overline{s(w)} = s(\bar{w})$. We will write $a + ib \approx \alpha + i\beta$ if $|\alpha|/c \leq |a| \leq c|\alpha|$ and $|\beta|/c \leq |b| \leq c|\beta|$ for some positive constant c . Hence $a + ib \approx \alpha + i\beta$ implies $|a + ib| \approx |\alpha + i\beta|$.

First consider the case in which w is near the real axis. Let $w = (1 - \varepsilon)e^{i\delta}$ and $z = (1 - \varepsilon')e^{i(\delta' + \delta)}$, where $0 < \varepsilon \leq \varepsilon_0$ for a fixed $\varepsilon_0 > 0$, $0 \leq \delta \leq \delta_0$ and $|\delta'|, \varepsilon' \leq \varepsilon$. For ε_0 and δ_0 small enough we have $\varepsilon + \varepsilon' + \varepsilon\varepsilon' \approx \varepsilon + \varepsilon'$ and $(2\delta + \delta')^2 + |2\delta + \delta'| \approx |2\delta + \delta'|$

so that

$$\begin{aligned} 1 - zw &= 1 - (1 - \varepsilon)e^{i\delta}(1 - \varepsilon')e^{i(\delta+\delta')} \\ &= 1 - (1 - \varepsilon)(1 - \varepsilon') + (1 - \varepsilon)(1 - \varepsilon')(1 - e^{i(2\delta+\delta')}) \\ &\approx \varepsilon + \varepsilon' + 1 - \cos(2\delta + \delta') - i \sin(2\delta + \delta') \\ &\approx \varepsilon + \varepsilon' + (2\delta + \delta')^2 - i(2\delta + \delta'). \end{aligned}$$

With regard to the norm, we have

$$|1 - zw| \approx \varepsilon + \varepsilon' + (2\delta + \delta')^2 + |2\delta + \delta'| \approx \varepsilon + |2\delta + \delta'|.$$

We also have

$$\begin{aligned} 1 - z \frac{w + \bar{w}}{2} &= 1 - (1 - \varepsilon)(1 - \varepsilon')e^{i(\delta+\delta')} \cos \delta \\ &= 1 - (1 - \varepsilon)(1 - \varepsilon') + (1 - \varepsilon)(1 - \varepsilon')(1 - \cos \delta) \\ &\quad + (1 - \varepsilon)(1 - \varepsilon') \cos \delta (1 - e^{i(\delta+\delta')}) \\ &\approx \varepsilon + \varepsilon' + \delta^2 + 1 - \cos(\delta + \delta') - i \sin(\delta + \delta'), \end{aligned}$$

which, in terms of norm, implies that

$$\left| 1 - z \frac{w + \bar{w}}{2} \right| \approx \varepsilon + \varepsilon' + \delta^2 + (\delta + \delta')^2 + |\delta + \delta'| \approx \varepsilon + \delta^2 + |\delta + \delta'|.$$

Moreover, if $|2\delta + \delta'| \geq 5\varepsilon$, since $|\delta'| \leq \varepsilon$, we necessarily get $\delta \geq 2\varepsilon$. Therefore,

$$|\delta + \delta'| \approx |2\delta + \delta'| \approx \delta \geq \delta^2,$$

i.e.

$$|1 - zw| \approx \left| 1 - z \frac{w + \bar{w}}{2} \right| \approx \varepsilon + \delta.$$

If $|2\delta + \delta'| \leq 5\varepsilon$, then $|\delta| \leq 5\varepsilon/2$ and

$$|1 - zw| \approx \left| 1 - z \frac{w + \bar{w}}{2} \right| \approx \varepsilon \approx \varepsilon + \delta.$$

Consider now the case in which w is far away from the real axis. Let $\varphi: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be the fractional linear transformation defined by

$$z \mapsto \frac{1 - z(w + \bar{w})/2}{1 - zw}.$$

The real axis is mapped by φ to the circle \mathcal{C} passing through

$$\varphi(0) = 1, \quad \varphi(1) = \frac{1 - (w + \bar{w})/2}{1 - w} \quad \text{and} \quad \varphi\left(\frac{2}{w + \bar{w}}\right) = 0;$$

the upper half-space is mapped inside the disc bounded by \mathcal{C} (since, for instance, $\varphi(1/\bar{w}) = 1/2$). This directly implies that for any $z \in s(w)$, if $w = (1 - \varepsilon)e^{i\delta}$ with $0 \leq \varepsilon \leq \varepsilon_0$ and $\delta \geq \varepsilon_0$, we have $|\varphi(z)| \leq c$ for some constant c .

Hence, we are left to show that when $z \in s(w)$, $|\varphi(z)|$ is bounded away from 0. Consider

$$\left| \varphi(z) - \frac{1}{2} \right| = \left| \frac{1 - z(w + \bar{w})/2}{1 - zw} - \frac{1}{2} \right| = \frac{1}{2} \left| \frac{1 - z\bar{w}}{1 - zw} \right|.$$

Let ε_0 be sufficiently small and let $\pi/2 \geq \delta \geq \delta_0$. For any $w = (1 - \varepsilon)e^{i\delta}$ and for any $z \in s(w)$ we have that $|z - 1/\bar{w}| \leq \frac{1}{2}|z - 1/w|$, which implies $|\varphi(z) - \frac{1}{2}| \leq \frac{1}{4}$, and hence $|\varphi(z)| \geq 1/4$.

The last case to consider is when $|w| \leq 1 - \varepsilon_0$. We have the estimates

$$\frac{1 - |w|}{1 + |w|} \leq \left| \frac{1 - z(w + \bar{w})/2}{1 - zw} \right| \leq \frac{1 + |w|}{1 - |w|},$$

which, setting $r_0 = 1 - \varepsilon_0$, lead to

$$\frac{1 - r_0}{1 + r_0} \leq |\varphi(z)| \leq \frac{1 + r_0}{1 - r_0},$$

thus concluding the proof. □

The announced characterization result can then be stated as follows.

Theorem 3.5. *A measure μ on \mathbb{B} is a Carleson measure for the Hardy space $H^2(\mathbb{B})$ if and only if for any $q = re^{j\theta} \in \mathbb{B}$ the measure of the symmetric box $S(q)$ satisfies*

$$\mu(S(q)) \lesssim |A_I(q)|,$$

where I is any imaginary unit and $|A_I(q)|$ denotes the length of the arc $A_I(q)$.

Proof. Consider first the case in which $\text{supp } \mu \subseteq (\mathbb{B} \cap \mathbb{R})$. In this case μ is a Carleson measure if and only if it is slice Carleson: since $\mathbb{B} \cap \mathbb{R}$ is contained in each slice \mathbb{B}_I we have

$$\int_{\mathbb{B}_I} |f(z)|^2 d\mu(z) = \int_{\mathbb{B}} |f(q)|^2 d\mu(q) \lesssim \|f\|_{H^2(\mathbb{B})}^2$$

for any $I \in \mathbb{S}$. Therefore, the statement follows by Proposition 3.3.

Now let $\mu(\mathbb{B} \cap \mathbb{R}) = 0$. Suppose first that μ is Carleson for $H^2(\mathbb{B})$ and let $w = u + vI_w \in \mathbb{B}$. Consider the function

$$K(q) := \frac{1}{A(\mathbb{S})} \int_{\mathbb{S}} k_{u+vI}(q) dA_{\mathbb{S}}(I),$$

where $k_{u+vI_w}(q) = k_w(q) = (1 - q\bar{w})^{-*}$ is the reproducing kernel of $H^2(\mathbb{B})$ and $dA_{\mathbb{S}}$ denotes the usual area element on the sphere \mathbb{S} . Then, using the fact that $k_w(q) = \overline{k_q(w)}$

for any $w, q \in \mathbb{B}$ and the representation formula, we can write

$$\begin{aligned}
 K(q) &= \frac{1}{A(\mathbb{S})} \int_{\mathbb{S}} k_{u+vI}(q) \, dA_{\mathbb{S}}(I) = \frac{1}{A(\mathbb{S})} \int_{\mathbb{S}} \overline{k_q(u+vI)} \, dA_{\mathbb{S}}(I) \\
 &= \frac{1}{A(\mathbb{S})} \int_{\mathbb{S}} \left(\frac{1-IJ}{2} k_q(u+vJ) + \frac{1+IJ}{2} k_q(u-vJ) \right) \, dA_{\mathbb{S}}(I) \\
 &= \frac{1}{A(\mathbb{S})} \left(\int_{\mathbb{S}} \overline{k_q(u+vJ)} \frac{1-JI}{2} \, dA_{\mathbb{S}}(I) + \int_{\mathbb{S}} \overline{k_q(u-vJ)} \frac{1+JI}{2} \, dA_{\mathbb{S}}(I) \right) \\
 &= \frac{1}{2} (k_{u+vJ}(q) + k_{u-vJ}(q)), \tag{3.4}
 \end{aligned}$$

where J is any imaginary unit. The function $K(q) \in H^2(\mathbb{B})$ since it is the superposition of functions in H^2 and, for any $J \in \mathbb{S}$,

$$\|K\|_{H^2(\mathbb{B})} \leq \frac{1}{2} (\|k_{u+vJ}\|_{H^2(\mathbb{B})} + \|k_{u-vJ}\|_{H^2(\mathbb{B})}) = \|k_{u+vJ}\|_{H^2(\mathbb{B})}.$$

Then, since μ is a Carleson measure, we can write

$$\begin{aligned}
 \int_{\mathbb{B}} |K(q)|^2 \, d\mu(q) &\lesssim \|K\|_{H^2(\mathbb{B})}^2 \leq \|k_{u+vJ}\|_{H^2(\mathbb{B})}^2 = \frac{1}{1-(u^2+v^2)} \\
 &= \frac{1}{1-|w|^2} \leq \frac{1}{1-|w|}. \tag{3.5}
 \end{aligned}$$

Let us study the modulus $|K(q)|$, when q belongs to the symmetric box $S(w)$. Recalling (3.4) together with the fact that we can choose the imaginary unit J appearing in (3.4) to be the same as the one of q , we are left to estimate the modulus of K slicewise, namely, we need to estimate the quantity

$$|K(z)| = \frac{1}{2} |k_w(z) + k_{\bar{w}}(z)| = \frac{1}{2} \left| \frac{1}{1-z\bar{w}} + \frac{1}{1-zw} \right| = \left| \frac{1-z(w+\bar{w})/2}{1-zw} \right| \left| \frac{1}{1-z\bar{w}} \right|$$

with $z \in S_{I_w}(w)$. Thanks to Lemma 3.4 and to classical estimates of the reproducing kernel of the complex Hardy space (see, for example, [8, proof of Theorem 9.3]) we obtain that for any $J \in \mathbb{S}$

$$|K(x+yJ)| \gtrsim \frac{1}{1-|w|^2}.$$

With the notation of (3.2), we get

$$\begin{aligned}
 \int_{\mathbb{B}} |K(q)|^2 \, d\mu(q) &= \int_{\mathbb{S}} \int_{\mathbb{B}_I^+} |K(x+yI)|^2 \, d\mu_I^+(x+yI) \, d\nu(I) \\
 &\geq \int_{\mathbb{S}} \int_{S_I(w)} |K(x+yI)|^2 \, d\mu_I^+(x+yI) \, d\nu(I) \\
 &\gtrsim \int_{\mathbb{S}} \int_{S_I(w)} \frac{1}{(1-|w|^2)^2} \, d\mu_I^+(x+yI) \, d\nu(I) = \frac{\mu(S(w))}{(1-|w|^2)^2}. \tag{3.6}
 \end{aligned}$$

Comparing (3.5) and (3.6) we conclude that

$$\mu(S(w)) \lesssim \frac{(1 - |w|^2)^2}{1 - |w|} \leq 1 - |w| \approx |A_I(w)|.$$

Let us now suppose that μ is a measure on \mathbb{B} such that for any symmetric box $S(q)$, $\mu(S(q)) \lesssim |A_I(q)|$. With the notation of (3.2), the hypothesis on μ yields that, for any $w \in \mathbb{B}$,

$$\begin{aligned} |A_I(w)| \gtrsim \mu(S(w)) &= \int_{S(w)} d\mu(q) \\ &= \int_{\mathbb{S}} \left(\int_{S_I(w)} d\mu_I^+(z) \right) d\nu(I) \\ &= \int_{\mathbb{S}} \left(\int_{S_{J_0}(u+vJ_0)} d\mu_I^{+\text{proj}}(x + yJ_0) \right) d\nu(I), \end{aligned}$$

where J_0 is any (fixed) imaginary unit, $S_{J_0}(w)$ is the projection of the symmetric box $S(w)$ on the fixed semi-disc $\mathbb{B}_{J_0}^+$ and $\mu_I^{+\text{proj}}$ is the projection of the measure μ_I^+ on the same slice,

$$\mu_I^{+\text{proj}}(E) = \mu_I^+(\{x + yI : y > 0 \text{ and } x + yJ_0 \in E\}), \quad d\mu_I^{+\text{proj}}(x + yJ_0) = d\mu_I^+(x + yI)$$

for any $E \subseteq \mathbb{B}_{J_0}^+$. Then the measure

$$\int_{\mathbb{S}} d\mu_I^{+\text{proj}}(x + yJ_0) d\nu(I)$$

is Carleson for $H^2(\mathbb{B}_{J_0})$. Let $f \in H^2(\mathbb{B})$. Using the representation formula, if I denotes the imaginary unit of q , and J_0 is any imaginary unit, $J_0 \neq \pm I$, we have

$$\begin{aligned} \int_{\mathbb{B}} |f(q)|^2 d\mu(q) &\lesssim \int_{\mathbb{S}} \int_{\mathbb{B}_{J_0}^+} (|f(x + yJ_0)|^2 + |f(x - yJ_0)|^2) d\mu_I^+(x + yI) d\nu(I) \\ &= \int_{\mathbb{B}_{J_0}^+} |f(x + yJ_0)|^2 \int_{\mathbb{S}} d\mu_I^{+\text{proj}}(x + yJ_0) d\nu(I) \\ &\quad + \int_{\mathbb{B}_{J_0}^+} |f^c(x + yJ_0)|^2 \int_{\mathbb{S}} d\mu_I^{+\text{proj}}(x + yJ_0) d\nu(I), \end{aligned}$$

where we used the fact that $|f(\bar{q})| = |\overline{f(q)}| = |f^c(q)|$. Since the regular conjugate $f^c(q) = \sum_{n \geq 0} q^n \overline{a_n}$ of a function $f(q) = \sum_{n \geq 0} q^n a_n \in H^2(\mathbb{B})$ belongs to $H^2(\mathbb{B})$ as well (see [7]) and has same H^2 -norm, thanks to Proposition 3.3 and to the fact that the H^2 -norm is the same on each slice, we thus conclude that

$$\int_{\mathbb{B}} |f(q)|^2 d\mu(q) \lesssim \|f\|_{H^2(\mathbb{B}_{J_0})}^2 + \|f^c\|_{H^2(\mathbb{B}_{J_0})}^2 = 2\|f\|_{H^2(\mathbb{B})}^2. \quad \square$$

4. Hankel bilinear forms

Let $b: \mathbb{B} \rightarrow \mathbb{H}$ be a regular function. The *Hankel operator* associated with the symbol b is the bilinear operator $T_b: H^2(\mathbb{B}) \times H^2(\mathbb{B}) \rightarrow \mathbb{H}$ defined by

$$T_b(f, g) = \langle f \star g, b \rangle_{H^2(\mathbb{B})}$$

for any $f, g \in H^2(\mathbb{B})$.

As stated in the introduction, a natural problem in this setting is to find necessity and sufficiency conditions on the symbol b so that the corresponding T_b is a bounded operator, i.e. so that there exists a constant $c(b)$ depending only on b such that

$$|T_b(f, g)| \leq c(b) \|f\|_{H^2(\mathbb{B})} \|g\|_{H^2(\mathbb{B})}$$

for any $f, g \in H^2(\mathbb{B})$.

First, let us show that the boundedness of T_b is equivalent to the boundedness of the operator Γ_α with $\alpha = \{\tilde{b}_n\}$, thus proving the first equivalence of Theorem 1.1.

Proposition 4.1. *Let $b(q) = \sum_{n \geq 0} q^n \hat{b}_n$ be a regular function on \mathbb{B} and let $\alpha = \{\alpha_n\}_n$ be a quaternionic sequence such that $\overline{\alpha_n} = \hat{b}_n$ for any n . Then the operator $T_b: (f, g) \mapsto \langle f \star g, b \rangle_{H^2(\mathbb{B})}$ is bounded on $H^2(\mathbb{B}) \times H^2(\mathbb{B})$ if and only if the operator*

$$\Gamma_\alpha: \{a_n\}_n \mapsto \left\{ \sum_{k \geq 0} \alpha_{n+k} a_k \right\}_n$$

is bounded on $\ell^2(\mathbb{N}, \mathbb{H})$. Moreover,

$$\|\Gamma_\alpha\|_{\mathcal{B}(\ell^2(\mathbb{N}, \mathbb{H}))} = \sup_{f, g \neq 0} \frac{|\langle f \star g, b \rangle_{H^2(\mathbb{B})}|}{\|f\|_{H^2(\mathbb{B})} \cdot \|g\|_{H^2(\mathbb{B})}}.$$

Proof. Let G_α be the bilinear operator associated with Γ_α , defined on $(c = \{c_n\}_{n \in \mathbb{N}}, d = \{d_n\}_{n \in \mathbb{N}})$ in $\ell^2(\mathbb{N}, \mathbb{H}) \times \ell^2(\mathbb{N}, \mathbb{H})$ by

$$G_\alpha(c, d) := \langle d, \overline{\Gamma_\alpha c} \rangle_{\ell^2(\mathbb{N}, \mathbb{H})} = \sum_{n \geq 0} \sum_{k \geq 0} \alpha_{n+k} c_k d_n.$$

It is not difficult to see that G_α is bounded if and only if Γ_α is, and

$$\sup_{c, d \neq 0} \frac{|G_\alpha(c, d)|}{\|c\|_{\ell^2(\mathbb{N}, \mathbb{H})} \cdot \|d\|_{\ell^2(\mathbb{N}, \mathbb{H})}} = \|\Gamma_\alpha\|_{\mathcal{B}(\ell^2(\mathbb{N}, \mathbb{H}))}.$$

Let $f(q) := \check{c}(q) = \sum_{n \geq 0} q^n c_n$ and $g(q) := \check{d}(q) = \sum_{n \geq 0} q^n d_n$ be functions in $H^2(\mathbb{B})$. Then

$$G_\alpha(c, d) = \sum_{n \geq 0} \sum_{j \geq 0} \alpha_j c_{j-n} d_n = \sum_{j \geq 0} \alpha_j \sum_{n=0}^j c_{j-n} d_n = \langle f \star g, b \rangle_{H^2(\mathbb{B})} \tag{4.1}$$

and

$$\sup_{c, d \neq 0} \frac{|G_\alpha(c, d)|}{\|c\|_{\ell^2(\mathbb{N}, \mathbb{H})} \cdot \|d\|_{\ell^2(\mathbb{N}, \mathbb{H})}} = \sup_{f, g \neq 0} \frac{|\langle f \star g, b \rangle_{H^2(\mathbb{B})}|}{\|f\|_{H^2(\mathbb{B})} \cdot \|g\|_{H^2(\mathbb{B})}},$$

which concludes the proof. □

When a viable factorization theory is not available, and this is one such case, a more general tool was developed in the important article [6]. Consider the weak \star -product space $H^2(\mathbb{B}) \otimes H^2(\mathbb{B})$, namely, the space of all linear combinations of \star -products of pairs of functions in $H^2(\mathbb{B})$,

$$H^2(\mathbb{B}) \otimes H^2(\mathbb{B}) = \left\{ \Phi: \mathbb{B} \rightarrow \mathbb{H}: \Phi(q) = \sum_j f_j \star g_j(q), \text{ with } f_j, g_j \in H^2(\mathbb{B}) \right\},$$

endowed with the norm

$$\|\Phi\|_{H^2(\mathbb{B}) \otimes H^2(\mathbb{B})} = \inf \left\{ \sum_j \|f_j\|_{H^2(\mathbb{B})} \|g_j\|_{H^2(\mathbb{B})} : \Phi = \sum_j f_j \star g_j, \text{ with } f_j, g_j \in H^2(\mathbb{B}) \right\}.$$

The first step in studying the boundedness of the operator T_b is the following theorem.

Theorem 4.2. *Let $b: \mathbb{B} \rightarrow \mathbb{H}$ be a regular function and set $\Lambda_b(h) := \langle h, b \rangle_{H^2(\mathbb{B})}$ when h is regular in \mathbb{B} . Then there exists a constant $c(b)$, depending only on b , such that*

$$|T_b(f, g)| \leq c(b) \|f\|_{H^2(\mathbb{B})} \|g\|_{H^2(\mathbb{B})}$$

for any $f, g \in H^2(\mathbb{B})$, if and only if Λ_b belongs to the dual space $(H^2(\mathbb{B}) \otimes H^2(\mathbb{B}))^*$.

On the other hand, for all Λ in $(H^2(\mathbb{B}) \otimes H^2(\mathbb{B}))^*$ there exists a unique regular $b: \mathbb{B} \rightarrow \mathbb{H}$ such that $\Lambda = \Lambda_b$. Moreover,

$$\|\Lambda_b\|_{(H^2(\mathbb{B}) \otimes H^2(\mathbb{B}))^*} \approx \sup_{f \neq 0 \neq g \in H^2(\mathbb{B})} \frac{|T_b(f, g)|}{\|f\|_{H^2(\mathbb{B})} \cdot \|g\|_{H^2(\mathbb{B})}}.$$

Proof. The argument is standard. We translate it into quaternionic language for the ease of the reader. First consider a regular b such that $\Lambda_b \in (H^2(\mathbb{B}) \otimes H^2(\mathbb{B}))^*$. Then, for any $f, g \in H^2(\mathbb{B})$,

$$\begin{aligned} |\langle f \star g, b \rangle_{H^2(\mathbb{B})}| &\leq \|f \star g\|_{H^2(\mathbb{B}) \otimes H^2(\mathbb{B})} \|\Lambda_b\|_{(H^2(\mathbb{B}) \otimes H^2(\mathbb{B}))^*} \\ &\leq \|f\|_{H^2(\mathbb{B})} \|g\|_{H^2(\mathbb{B})} \|\Lambda_b\|_{(H^2(\mathbb{B}) \otimes H^2(\mathbb{B}))^*}, \end{aligned}$$

where the last equality follows from the definition of the norm on $H^2(\mathbb{B}) \otimes H^2(\mathbb{B})$. Now let Λ be a linear functional in $(H^2(\mathbb{B}) \otimes H^2(\mathbb{B}))^*$. Since $H^2(\mathbb{B})$ embeds in $H^2(\mathbb{B}) \otimes H^2(\mathbb{B})$ continuously, as a dense subspace, Λ can be identified with a unique element of $(H^2(\mathbb{B}))^* = H^2(\mathbb{B})$. Hence, $\Lambda = \Lambda_b$ for some b in $H^2(\mathbb{B})$, and

$$\sup_{f \neq 0 \neq g \in H^2(\mathbb{B})} \frac{|T_b(f, g)|}{\|f\|_{H^2(\mathbb{B})} \cdot \|g\|_{H^2(\mathbb{B})}} \lesssim \|\Lambda_b\|_{(H^2(\mathbb{B}) \otimes H^2(\mathbb{B}))^*}.$$

In the other direction, let $\Phi \in H^2(\mathbb{B}) \otimes H^2(\mathbb{B})$. Then, for any decomposition of Φ of the form $\Phi = \sum_j f_j \star g_j$, if T_b is bounded, we have

$$|\langle \Phi, b \rangle_{H^2(\mathbb{B})}| \leq \sum_j |\langle f_j \star g_j, b \rangle_{H^2(\mathbb{B})}| \leq \sum_j \|f_j\|_{H^2(\mathbb{B})} \|g_j\|_{H^2(\mathbb{B})} c(b)$$

for some constant $c(b)$ depending only on the function b . Therefore, taking the infimum on all possible decompositions of the function Φ , we get

$$|\langle \Phi, b \rangle_{H^2(\mathbb{B})}| \leq \inf \left\{ \sum_j \|f_j\|_{H^2(\mathbb{B})} \|g_j\|_{H^2(\mathbb{B})} c(b) \mid f_j, g_j \in H^2(\mathbb{B}) \right\} = c(b) \|\Phi\|_{H^2(\mathbb{B}) \oplus H^2(\mathbb{B})},$$

thus showing that $A_b \in (H^2(\mathbb{B}) \oplus H^2(\mathbb{B}))^*$ and that

$$\sup_{f \neq 0 \neq g \in H^2(\mathbb{B})} \frac{|T_b(f, g)|}{\|f\|_{H^2(\mathbb{B})} \cdot \|g\|_{H^2(\mathbb{B})}} \gtrsim \|A_b\|_{(H^2(\mathbb{B}) \oplus H^2(\mathbb{B}))^*}. \quad \square$$

Another way to characterize the functions b for which the Hankel operator T_b is a bounded operator is in terms of Carleson measures.

First we need a preliminary result (which holds in analogy with the complex case) concerning the H^2 -norm. The calculations proving it are identical to those of the complex case.

Proposition 4.3. *Let $d\text{Vol}_{\mathbb{B}}$ denote the volume form on \mathbb{B} defined by $d\text{Vol}_{\mathbb{B}}(x + yI) = \frac{1}{4} dA_{\mathbb{S}}(I) dx dy$. Then, for any $f \in H^2(\mathbb{B})$,*

$$\begin{aligned} \|f\|_{H^2(\mathbb{B})}^2 &= |f(0)|^2 + \frac{1}{\text{Vol}(\mathbb{B})} \int_{\mathbb{B}} |\partial_c f(q)|^2 \log |q|^{-2} d\text{Vol}_{\mathbb{B}}(q) \\ &\approx |f(0)|^2 + \frac{1}{\text{Vol}(\mathbb{B})} \int_{\mathbb{B}} |\partial_c f(q)|^2 (1 - |q|^2) d\text{Vol}_{\mathbb{B}}(q). \end{aligned}$$

For a proof of the equivalence between the complex analogues of the two integral terms

$$\frac{1}{\text{Vol}(\mathbb{B})} \int_{\mathbb{B}} |\partial_c f(q)|^2 \log |q|^{-2} d\text{Vol}_{\mathbb{B}}(q) \quad \text{and} \quad \frac{1}{\text{Vol}(\mathbb{B})} \int_{\mathbb{B}} |\partial_c f(q)|^2 (1 - |q|^2) d\text{Vol}_{\mathbb{B}}(q),$$

see, for example, [15, proof of Theorem 8.1.10]. By polarization, we can use as an inner product of $H^2(\mathbb{B})$ the following:

$$\langle f, g \rangle_{H^2(\mathbb{B})} = \overline{g(0)} f(0) + \frac{1}{\text{Vol}(\mathbb{B})} \int_{\mathbb{B}} \overline{\partial_c g(q)} \partial_c f(q) (1 - |q|^2) d\text{Vol}_{\mathbb{B}}(q). \quad (4.2)$$

The following result can be interpreted as an analogue of the Nehari theorem in the quaternionic setting.

Theorem 4.4. *Let $b: \mathbb{B} \rightarrow \mathbb{H}$ be a regular function. Then the Hankel operator T_b is bounded, i.e. there exists a constant $c(b)$ depending only on b such that*

$$|\langle f \star g, b \rangle_{H^2(\mathbb{B})}|^2 \leq c(b) \|f\|_{H^2(\mathbb{B})}^2 \|g\|_{H^2(\mathbb{B})}^2 \quad (4.3)$$

for any $f, g \in H^2(\mathbb{B})$ if and only if the measure on the unit ball \mathbb{B} defined by

$$|\partial_c b(q)|^2 (1 - |q|^2) d\text{Vol}_{\mathbb{B}}(q) \quad (4.4)$$

is a Carleson measure for $H^2(\mathbb{B})$.

Proof. Suppose first that T_b is bounded. Let $I \in \mathbb{S}$ and consider any $f, g \in H^2(\mathbb{B})$ such that $f, g: \mathbb{B}_I \rightarrow L_I$, namely, such that $f, g \in H^2(\mathbb{D}) \subset H^2(\mathbb{B}_I)$; according to the splitting lemma, let b be decomposed as

$$b(z) = b_1(z) + b_2(z)J$$

with $b_1, b_2: \mathbb{B}_I \rightarrow L_I$ holomorphic and $J \in \mathbb{S}$ orthogonal to I . Then, recalling that the norm and the inner product of $H^2(\mathbb{B})$ can be computed on any slice,

$$\begin{aligned} |\langle f \star g, b \rangle_{H^2(\mathbb{B})}|^2 &= |\langle f \star g, b \rangle_{H^2(\mathbb{B}_I)}|^2 = |\langle fg, b_1 + b_2J \rangle_{H^2(\mathbb{B}_I)}|^2 \\ &= |\langle fg, b_1 \rangle_{H^2(\mathbb{D})}|^2 + |\langle fg, b_2 \rangle_{H^2(\mathbb{D})}|^2, \end{aligned}$$

which, recalling (4.3), leads to

$$|\langle fg, b_k \rangle_{H^2(\mathbb{D})}|^2 \leq c(b) \|f\|_{H^2(\mathbb{B})}^2 \|g\|_{H^2(\mathbb{B})}^2$$

for both $k = 1, 2$. Thanks to the analogous result (see, for example, [15]) in the complex case, we get that $|b'_k(z)|^2(1 - |z|^2) dx dy$ is a (complex) Carleson measure for $H^2(\mathbb{D}) \subset H^2(\mathbb{B}_I)$ for $k = 1, 2$, and hence, for any $f \in H^2(\mathbb{B}_I)$,

$$\begin{aligned} \int_{\mathbb{B}_I} |f(z)|^2 |\partial_c b(z)|^2 (1 - |z|^2) dx dy &= \int_{\mathbb{B}_I} |f(z)|^2 (|b'_1(z) + b'_2(z)J|^2) (1 - |z|^2) dx dy \\ &= \int_{\mathbb{B}_I} |f(z)|^2 |b'_1(z)|^2 (1 - |z|^2) dx dy \\ &\quad + \int_{\mathbb{B}_I} |f(z)|^2 |b'_2(z)|^2 (1 - |z|^2) dx dy \\ &\lesssim \|f\|_{H^2(\mathbb{B}_I)}^2, \end{aligned}$$

which, recalling (3.3), implies that the quaternionic measure $|\partial_c b(z)|^2(1 - |z|^2) dx dy$ is Carleson for $H^2(\mathbb{B}_I)$. The fact that I was any imaginary unit yields that the measure $|\partial_c b(q)|^2(1 - |q|^2) d\text{Vol}_{\mathbb{B}}(q)$ is slice Carleson for $H^2(\mathbb{B})$ and therefore, thanks to Proposition 3.1, Carleson.

Let us now prove the opposite implication. Using (4.2) and the Leibniz rule for the slice derivative, if $z = x + yI \in \mathbb{B}_I$, we can write

$$\begin{aligned} |\langle f \star g, b \rangle_{H^2(\mathbb{B})}| &\lesssim |\overline{b(0)}(f \star g)(0)| + \left| \int_{\mathbb{B}} \overline{\partial_c b(q)} \partial_c (f \star g)(q) (1 - |q|^2) d\text{Vol}_{\mathbb{B}}(q) \right| \\ &\leq \left| \frac{1}{4} \int_{\mathbb{S}} dA_{\mathbb{S}}(I) \int_{\mathbb{B}_I} \overline{\partial_c b(\tilde{z})} \partial_c (f \star g)(z) (1 - |z|^2) dx dy \right| \\ &\lesssim \int_{\mathbb{S}} dA_{\mathbb{S}}(I) \int_{\mathbb{B}_I} |\partial_c b(z)| |\partial_c f(z)| |g(\hat{z})| (1 - |z|^2) dx dy \\ &\quad + \int_{\mathbb{S}} dA_{\mathbb{S}}(I) \int_{\mathbb{B}_I} |\partial_c b(z)| |f(z)| |\partial_c g(\tilde{z})| (1 - |z|^2) dx dy = \mathcal{I}_1 + \mathcal{I}_2, \end{aligned}$$

where \hat{z} and \tilde{z} are points lying on the same two sphere as z , determined in view of the expression of the \star -product in terms of the pointwise product. (See Proposition 2.3. Here

we are omitting the discrete subset of \mathbb{B}_I where f or $\partial_c f$ vanish.) To estimate the integral \mathcal{I}_1 we first use the Cauchy–Schwarz inequality, and then Proposition 4.3, to get

$$\begin{aligned} \mathcal{I}_1^2 &\lesssim \int_{\mathbb{S}} dA_{\mathbb{S}}(I) \int_{\mathbb{B}_I} |\partial_c b(z)|^2 |g(\hat{z})|^2 (1 - |z|^2) dx dy \int_{\mathbb{S}} dA_{\mathbb{S}}(I) \int_{\mathbb{B}_I} |\partial_c f(z)|^2 (1 - |z|^2) dx dy \\ &\lesssim \|f\|_{H^2(\mathbb{B})}^2 \int_{\mathbb{S}} dA_{\mathbb{S}}(I) \int_{\mathbb{B}_I} |\partial_c b(z)|^2 |g(\hat{z})|^2 (1 - |z|^2) dx dy. \end{aligned} \tag{4.5}$$

The representation formula allows us to express $g(\hat{z})$ in terms of $g(z)$ and $g(\bar{z})$, so that we get

$$\begin{aligned} \int_{\mathbb{S}} dA_{\mathbb{S}}(I) \int_{\mathbb{B}_I} |\partial_c b(z)|^2 |g(\hat{z})|^2 (1 - |z|^2) dx dy \\ \lesssim \int_{\mathbb{S}} dA_{\mathbb{S}}(I) \int_{\mathbb{B}_I} |\partial_c b(z)|^2 (|g(z)|^2 + |g(\bar{z})|^2) (1 - |z|^2) dx dy. \end{aligned} \tag{4.6}$$

Now $|g(\bar{z})| = |\overline{g(z)}| = |g^c(z)|$ and $\|g^c\|_{H^2(\mathbb{B})} = \|g\|_{H^2(\mathbb{B})}$ (see, for example, [7]); hence, using (4.5), (4.6) and recalling that $|\partial_c b(q)|^2 (1 - |q|^2) d\text{Vol}_{\mathbb{B}}(q)$ is a Carleson measure for $H^2(\mathbb{B})$, we conclude that

$$\mathcal{I}_1^2 \lesssim \|g\|_{H^2(\mathbb{B})}^2 \|f\|_{H^2(\mathbb{B})}^2.$$

With analogous arguments, we get the same estimate on the second integral \mathcal{I}_2 , thus completing the proof. □

5. Quaternionic BMOA

Definition 5.1. Let $f \in H^1(\mathbb{B})$ and, for any interval $a = (\alpha, \beta)$ of \mathbb{R} such that $|a| := |\beta - \alpha| \leq 2\pi$, denote by $f_{I,a}$ the average value of (the radial limit of) f on the arc $(e^{\alpha I}, e^{\beta I}) \subseteq \partial\mathbb{B}_I$,

$$f_{I,a} = \frac{1}{|a|} \int_a f(e^{\theta I}) d\theta.$$

We say that $f \in \text{BMOA}(\mathbb{B}_I)$ if

$$\|f\|_{\text{BMOA}(\mathbb{B}_I)} := \sup_{a \subset \mathbb{R}, |a| \leq 2\pi} \left\{ \frac{1}{|a|} \int_a |f(e^{\theta I}) - f_{I,a}| d\theta \right\} < +\infty.$$

We say that $f \in \text{BMOA}(\mathbb{B})$ if

$$\|f\|_{\text{BMOA}(\mathbb{B})} := \sup_{I \in \mathbb{S}} \|f\|_{\text{BMOA}(\mathbb{B}_I)} < +\infty.$$

The fact that $f \in H^1(\mathbb{B})$ yields that almost everywhere at the boundary f satisfies the representation formula (see [7]), which leads to the following statement.

Proposition 5.2. *Let $f \in H^1(\mathbb{B})$. Then $f \in \text{BMOA}(\mathbb{B})$ if and only if $f \in \text{BMOA}(\mathbb{B}_I)$ for some $I \in \mathbb{S}$. More precisely,*

$$\|f\|_{\text{BMOA}(\mathbb{B}_I)} \leq \|f\|_{\text{BMOA}(\mathbb{B})} \leq 2\|f\|_{\text{BMOA}(\mathbb{B}_I)}.$$

Proof. The necessity condition is trivial. Suppose then that $f \in \text{BMOA}(\mathbb{B}_I)$ for $I \in \mathbb{S}$. Thanks to the representation formula, the average value of f on a different slice \mathbb{B}_J , $J \in \mathbb{S}$, $J \neq \pm I$, can be computed as

$$\begin{aligned} f_{J,a} &= \frac{1}{|a|} \int_a f(e^{\theta J}) \, d\theta = \frac{1}{|a|} \int_a \left(\frac{1 - JI}{2} f(e^{\theta I}) + \frac{1 + JI}{2} f(e^{-\theta I}) \right) \, d\theta \\ &= \frac{1 - JI}{2} f_{I,a} + \frac{1 + JI}{2} f_{I,-a}, \end{aligned}$$

where if $a = (\alpha, \beta)$, then $-a = (-\beta, -\alpha)$. The statement follows by simple calculations. □

Theorem 5.3. *Let $f \in H^1(\mathbb{B})$. Then $f \in \text{BMOA}(\mathbb{B})$ if and only if the measure $|\partial_c f(q)|^2(1 - |q|^2) \, d\text{Vol}_{\mathbb{B}}(q)$ is a Carleson measure for $H^2(\mathbb{B})$.*

Proof. Let $I, J \in \mathbb{S}$ with $J \perp I$, and consider the splitting of f on \mathbb{B}_I with respect to J , $f = F + GJ$, where $F, G: \mathbb{B}_I \rightarrow L_I$ are holomorphic functions. Notice that for any $a \in \mathbb{R}$, $|a| \leq 2\pi$,

$$f_{I,a} = \frac{1}{|a|} \int_a f(e^{\theta I}) \, d\theta = \frac{1}{|a|} \int_a F(e^{\theta I}) + G(e^{\theta I})J \, d\theta = F_a + G_a J,$$

where F_a, G_a are the average values of F and G respectively. Moreover, thanks to the orthogonality of I and J we get

$$\frac{1}{|a|} \int_a |f(e^{\theta I}) - f_{I,a}| \, d\theta = \frac{1}{|a|} \int_a (|F(e^{\theta I}) - F_a|^2 + |G(e^{\theta I}) - G_a|^2)^{1/2} \, d\theta. \tag{5.1}$$

Then, if $f \in \text{BMOA}(\mathbb{B})$, we also have that both F and G belong to the (complex) space $\text{BMOA}(\mathbb{D})$. Thanks to classical results (see, for example, [15, Theorem 8.3.5]) we then get that both $|F'(z)|^2(1 - |z|^2) \, dx \, dy$ and $|G'(z)|^2(1 - |z|^2) \, dx \, dy$ are Carleson measures for the complex Hardy space $H^2(\mathbb{D}) \subset H^2(\mathbb{B}_I)$. Hence,

$$|\partial_c f(z)|^2(1 - |z|^2) \, dx \, dy = (|F'(z)|^2 + |G'(z)|^2)(1 - |z|^2) \, dx \, dy$$

is a Carleson measure for $H^2(\mathbb{B}_I)$. Since I was any imaginary unit, we have that $|\partial_c f(q)|^2(1 - |q|^2) \, d\text{Vol}_{\mathbb{B}}(q)$ is slice Carleson for $H^2(\mathbb{B})$, and therefore it is a Carleson measure for $H^2(\mathbb{B})$.

If, on the other hand, $|\partial_c f(q)|^2(1 - |q|^2) \, d\text{Vol}_{\mathbb{B}}(q)$ is a Carleson measure for $H^2(\mathbb{B})$, thanks to Proposition 4.4 we get that the Hankel operator T_f associated with f ,

$$T_f: H^2(\mathbb{B}) \times H^2(\mathbb{B}) \rightarrow \mathbb{H}, \quad (g_1, g_2) \mapsto \langle g_1 \star g_2, f \rangle_{H^2(\mathbb{B})},$$

is bounded. In particular, T_f is bounded also when restricted to $H^2(\mathbb{D}) \times H^2(\mathbb{D}) \subset H^2(\mathbb{B}_I) \times H^2(\mathbb{B}_I)$. Namely, for any $g_1, g_2 \in H^2(\mathbb{B})$ that map \mathbb{B}_I to L_I for some $I \in \mathbb{S}$, if the splitting of f on \mathbb{B}_I with respect to $J \perp I$ is $f = F + GJ$, then

$$\begin{aligned} +\infty > |\langle g_1 \star g_2, f \rangle_{H^2(\mathbb{B})}|^2 &= |\langle g_1 g_2, F + GJ \rangle_{H^2(\mathbb{B}_I)}|^2 \\ &= |\langle g_1 g_2, F \rangle_{H^2(\mathbb{D})}|^2 + |\langle g_1 g_2, G \rangle_{H^2(\mathbb{D})}|^2. \end{aligned}$$

Hence, thanks to classical results (see, for example, [14]) we have that both F and G belong to $\text{BMOA}(\mathbb{D})$. Recalling (5.1), we get that $f = F + GJ \in \text{BMOA}(\mathbb{B}_I)$, which, by Proposition 5.2, leads to the conclusion. \square

Combining Theorems 4.2, 4.4 and 5.3, we get the following identification.

Corollary 5.4. *The two spaces $(H^2(\mathbb{B}) \otimes H^2(\mathbb{B}))^*$ and $\text{BMOA}(\mathbb{B})$ coincide.*

A natural question is then whether the spaces $H^1(\mathbb{B})$ and $H^2(\mathbb{B}) \otimes H^2(\mathbb{B})$ coincide or not. A partial answer is given by the following result.

Proposition 5.5. $H^1(\mathbb{B}) = H^2(\mathbb{B}) \star H^2(\mathbb{B}) + H^2(\mathbb{B}) \star H^2(\mathbb{B})$.

Proof. On the one hand $H^2(\mathbb{B}) \star H^2(\mathbb{B}) + H^2(\mathbb{B}) \star H^2(\mathbb{B}) \subseteq H^1(\mathbb{B})$ since $H^2(\mathbb{B}) \star H^2(\mathbb{B}) \subseteq H^1(\mathbb{B})$, as proven in [7]. On the other hand, let $f \in H^1(\mathbb{B})$, $I, J \in \mathbb{S}$ with $I \perp J$, and let $F, G: \mathbb{B}_I \rightarrow L_I$ be such that $f_I = F + GJ$ on \mathbb{B}_I . Then, since $F, G \in H^1(\mathbb{D}) \subseteq H^1(\mathbb{B}_I)$ (see [7]), thanks to the classical factorization for the complex Hardy space $H^1(\mathbb{D})$, we can write $F(z) = F_1(z)F_2(z)$ and $G(z) = G_1(z)G_2(z)$ with $F_1, F_2, G_1, G_2 \in H^2(\mathbb{D}) \subseteq H^2(\mathbb{B}_I)$. Hence, on the slice \mathbb{B}_I we have

$$\begin{aligned} f_I(z) &= F_1(z)F_2(z) + G_1(z)G_2(z)J \\ &= F_1 \star F_2(z) + G_1 \star G_2(z)J, \end{aligned}$$

which can be (uniquely) extended on the entire \mathbb{B} by means of the extension lemma as

$$\begin{aligned} f(q) &= \text{ext}(F_1 \star F_2 + G_1 \star G_2 J)(q) \\ &= \text{ext}(F_1) \star \text{ext}(F_2)(q) + \text{ext}(G_1) \star \text{ext}(G_2)(q)J, \end{aligned}$$

where the last equality follows from the identity principle. Since $F_1, F_2, G_1, G_2 \in H^2(\mathbb{B}_I)$ we get that their regular extensions belong to $H^2(\mathbb{B})$, and therefore that $f \in H^2(\mathbb{B}) \star H^2(\mathbb{B}) + H^2(\mathbb{B}) \star H^2(\mathbb{B})$. \square

As a consequence we have that $H^1(\mathbb{B}) \subseteq H^2(\mathbb{B}) \otimes H^2(\mathbb{B})$.

Theorem 5.6. *The dual space of $H^1(\mathbb{B})$ is $\text{BMOA}(\mathbb{B})$.*

Proof. Corollary 5.4, Proposition 5.5 and a standard density argument yield that $\text{BMOA}(\mathbb{B}) \subseteq (H^1(\mathbb{B}))^*$.

To prove the opposite inclusion, let $L \in (H^1(\mathbb{B}))^*$ be a bounded linear functional on $H^1(\mathbb{B})$. Then, for any $I \in \mathbb{S}$, $L \in (H^1(\mathbb{B}_I))^*$. Since, for any $I \in \mathbb{S}$, $H^1(\mathbb{B}_I)$ is a linear subspace of $L^1(\partial\mathbb{B}_I)$ (see [7]), the Hahn–Banach theorem in the quaternionic setting (see [4]) yields that L extends to a bounded linear functional on $L^1(\partial\mathbb{B}_I)$. As in the complex case, $(L^1(\partial\mathbb{B}_I))^* = L^\infty(\partial\mathbb{B}_I)$, and hence there exists $\varphi \in L^\infty(\partial\mathbb{B}_I)$ such that

$$L(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \overline{\varphi(e^{\theta I})} f(e^{\theta I}) \, d\theta$$

for any $f \in L^1(\partial\mathbb{B}_I)$. The same duality relation holds, restricted to $f \in H^1(\mathbb{B})$, even if we replace φ by its projection $g = P\varphi$ onto the functions that are holomorphic on \mathbb{B}_I ,

with values in the quaternions (here the splitting lemma is used), and it follows from one complex-dimensional Nehari theory that g lies in $\text{BMOA}(\mathbb{B}_I)$. Thanks to Proposition 5.2 we conclude that $g \in \text{BMOA}(\mathbb{B})$. \square

Corollary 5.7. $H^1(\mathbb{B}) = H^2(\mathbb{B}) \oplus H^2(\mathbb{B})$.

Proof. Equality of duals implies norm equivalence in the spaces. \square

6. Concluding remarks

We conclude with some problems connected with the paper's subject that we could not answer.

- We know that $H^1(\mathbb{B}) = H^2(\mathbb{B}) \oplus H^2(\mathbb{B})$. It would be interesting to know if $H^2(\mathbb{B}) \star H^2(\mathbb{B}) = H^1(\mathbb{B})$; that is, if a good factorization theory exists in the quaternionic setting.
- Our testing function for the Carleson measure theorem is an average of reproducing kernels. It would be interesting to know if the 'reproducing kernel thesis' holds; that is, if the inequality

$$\int_{\mathbb{B}} |k_w|^2 d\mu \leq c(\mu) \|k_w\|_{H^2(\mathbb{B})}^2,$$

with $c(\mu)$ independent of w , implies that μ is Carleson for $H^2(\mathbb{B})$.

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