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Compressions and Probably Intersecting Families

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A family $\mathcal{A}$ of sets is said to be *intersecting* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$. It is a well-known and simple fact that an intersecting family of subsets of $[n] = \{1, 2, \ldots, n\}$ can contain at most 2^{n-1} sets. Katona, Katona and Katona ask the following question. Suppose instead $\mathcal{A} \subset \mathcal{P}[n]$ satisfies $|\mathcal{A}| = 2^{n-1} + i$ for some fixed $i > 0$. Create a new family $\mathcal{A}_p$ by choosing each member of $\mathcal{A}$ independently with some fixed probability p. How do we choose $\mathcal{A}$ to maximize the probability that $\mathcal{A}_p$ is intersecting? They conjecture that there is a nested sequence of optimal families for $i = 1, 2, \ldots, 2^{n-1}$. In this paper, we show that the families $[n]^{(\geqslant r)} = \{A \subset [n] : |A| \geqslant r\}$ are optimal for the appropriate values of i, thereby proving the conjecture for this sequence of values. Moreover, we show that for intermediate values of i there exist optimal families lying between those we have found. It turns out that the optimal families we find simultaneously maximize the number of intersecting subfamilies of each possible order.

Standard compression techniques appear inadequate to solve the problem as they do not preserve intersection properties of subfamilies. Instead, our main tool is a novel compression method, together with a way of 'compressing subfamilies', which may be of independent interest.

1. Introduction

Many problems of extremal combinatorics concern intersecting families of finite sets. A family $\mathcal{A}$ is said to be *intersecting* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$. How large an intersecting family can we find in the discrete cube $Q_n = \mathcal{P}[n] = \mathcal{P}\{1, 2, \ldots, n\}$? It is easy to achieve $|\mathcal{A}| = 2^{n-1}$, for example by taking $\mathcal{A} = \{A \subset Q_n : 1 \in A\}$. And it is easy to see that we can do no better than this: an intersecting family cannot contain both a set and its complement.

A more interesting question arises if we require our intersecting family to be *uniform.* Given a set S and a positive integer r, write $S^{(r)}$ for the collection $\{A \subset S : |A| = r\}$ of all subsets of S of size r. How large an intersecting family $\mathcal{A} \subset [n]^{(r)}$ can we find?

As in the non-uniform case, it seems natural to try taking $\mathcal{A} = \{A \in [n]^{(r)} : 1 \in A\}$, here achieving $|\mathcal{A}| = \binom{n-1}{r-1}$. And indeed, in their significant paper of 1964, Erdős, Ko and Rado [4] show that if $r \leqslant n/2$ we can do no better than this. (We remark in passing that the problem is of no interest if $r > n/2$, as then the entirety of $[n]^{(r)}$ is itself intersecting.)

In this paper we shall be concerned with two related probabilistic questions posed by Katona, Katona and Katona [8]. We begin with the non-uniform case.

Recall from above that if $\mathcal{A} \subset \mathcal{P}[n]$ is intersecting then $|\mathcal{A}| \leqslant 2^{n-1}$. Suppose that we are instead required to choose a somewhat larger family $\mathcal{A}$ and then randomly discard some of the sets in $\mathcal{A}$ to form a subfamily $\mathcal{B}$. How can we maximize the probability that $\mathcal{B}$ is intersecting? A precise statement of the problem is as follows.

Problem 1 ([8]). Let n and i be positive integers with $i \leqslant 2^{n-1}$ and let $p \in (0,1)$. Given $\mathcal{A} \subset \mathcal{P}[n]$, write $\mathcal{A}_p$ for the (random) subfamily of $\mathcal{A}$ obtained by choosing each set in $\mathcal{A}$ independently with probability p. How should we choose $\mathcal{A}$ with $|\mathcal{A}| = 2^{n-1} + i$ to maximize $\mathbb{P}(\mathcal{A}_p \text{ is intersecting})$?

Katona, Katona and Katona [8] solve the first cases of this problem, that is, for $i \leqslant \binom{n-1}{\lceil (n-3)/2 \rceil}$. They construct their optimal families by taking 'large' sets in the cube. More precisely, for n odd take all sets of size at least $(n+1)/2$ together with any i sets of size $(n-1)/2$ that contain the element 1. Similarly, for n even take all sets of size $n/2+1$, all sets of size $n/2$ that contain the element 1, and any other i sets of size $n/2$. They conjecture that a continuation of this construction gives an optimal family $\mathcal{A}$ for each i, leading to a nested sequence $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \mathcal{A}_{2^{n-1}}$ of optimal families for $i = 1, 2, \ldots, 2^{n-1}$.

In this paper, we show that the families $[n]^{(\geqslant r)} = \{A \subset \mathcal{P}[n] : |A| \geqslant r\}$ are optimal for the appropriate values of i, thereby proving the conjecture for this sequence of values. Moreover, we show that for intermediate values of i there exist optimal families lying between those we have found. Our main result is as follows.

Theorem 1.1. *Let n be a positive integer and let $p \in (0,1)$. Let r be a positive integer with $r \leqslant n/2$. Then, over all $\mathcal{A} \subset \mathcal{P}[n]$ with $|\mathcal{A}| = \sum_{j=r}^{n} \binom{n}{j}$, the probability $\mathbb{P}(\mathcal{A}_p \text{ is intersecting})$ is maximized by $\mathcal{A} = [n]^{(\geqslant r)}$.*

Moreover, suppose i is any positive integer with $i \leqslant 2^{n-1}$ and let r be such that $\sum_{j=r+1}^{n} \binom{n}{j} \leqslant 2^{n-1} + i \leqslant \sum_{j=r}^{n} \binom{n}{j}$. Then, over all $\mathcal{A} \subset \mathcal{P}[n]$ with $|\mathcal{A}| = 2^{n-1} + i$, the probability $\mathbb{P}(\mathcal{A}_p \text{ is intersecting})$ is maximized by some $\mathcal{A}$ with $[n]^{(\geqslant r+1)} \subset \mathcal{A} \subset [n]^{(\geqslant r)}$.

We remark that the result of Theorem 1.1 is independent of the value of p. In fact, for $2^{n-1} + i = \sum_{j=r}^{n} \binom{n}{j}$, the family $[n]^{(\geqslant r)}$ simultaneously maximizes the number of intersecting subfamilies of each possible order. This result may be of independent interest.

We also consider the uniform version of the problem.

Problem 2 ([8]). Let n, r and i be positive integers with $r \leqslant n/2$ and $i \leqslant \binom{n-1}{r}$, and let $p \in (0,1)$. How should we choose $\mathcal{A} \subset [n]^{(r)}$ with $|\mathcal{A}| = \binom{n-1}{r-1} + i$ to maximize $\mathbb{P}(\mathcal{A}_p \text{ is intersecting})$?

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Results on this problem seem rather harder to come by: Katona, Katona and Katona [8] solve only the first case $i = 1$. Using methods similar to those used to prove Theorem 1.1, we show that, for each i, there is an optimal family that is left-compressed (as explained below). Unfortunately, our methods are not sufficient to determine which amongst the left-compressed families of given order is best.

Theorem 1.2. *Let n, r and i be positive integers with $r \leqslant n/2$ and $i \leqslant \binom{n-1}{r}$, and let $p \in (0,1)$. Then there exists a left-compressed family $\mathcal{A} \subset [n]^{(r)}$ with $|\mathcal{A}| = \binom{n-1}{r-1} + i$ that maximizes* $\mathbb{P}(\mathcal{A}_p$ is intersecting) *over all subfamilies of $[n]^{(r)}$ of order $\binom{n-1}{r-1} + i$.*

Many fruitful approaches to intersection problems involve the use of compression techniques, first introduced by Erdős Ko and Rado [4] in the proof of their uniform intersection theorem mentioned above. The idea behind such techniques is that, starting from an intersecting family $\mathcal{A}$, one 'moves' certain sets in $\mathcal{A}$ to make $\mathcal{A}$ 'nicer' in some way whilst $\mathcal{A}$ retains the property of being intersecting. The proof of the Erdős–Ko–Rado theorem applies ij-compressions, defined as follows.

Let $i, j \in [n]$ with $i < j$. If $A \in [n]^{(r)}$ then the *ij-compression of A* is

$$C_{ij}A = \begin{cases} (A \cup \{i\}) - \{j\} & \text{if } j \in A, i \notin A, \\ A & \text{otherwise.} \end{cases}$$

If $\mathcal{A} \subset [n]^{(r)}$, the *$ij$-compression of $\mathcal{A}$* is

$$\mathcal{C}_{ij}\mathcal{A} = \{C_{ij}A : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : C_{ij}A \in \mathcal{A}\}.$$

Informally, we replace j by i whenever we can. We may be prevented from replacing $j \in A$ by i EITHER because i is already in A OR because $C_{ij}A$ is already in $\mathcal{A}$. When we replace $j \in A$ by i, we say that A *moves*; that is, A moves if $j \in A$, $i \notin A$ and $C_{ij}A \notin \mathcal{A}$. We say that A is *blocked* from moving by $C_{ij}A$ if $A \neq C_{ij}A$ and $C_{ij}A \in \mathcal{A}$. A family $\mathcal{A}$ is *ij-compressed* if $\mathcal{A} = \mathcal{C}_{ij}\mathcal{A}$. It is *left-compressed* if it is ij-compressed whenever $i < j$.

Erdős, Ko and Rado show that if $\mathcal{A} \subset [n]^{(r)}$ is intersecting then so is $\mathcal{C}_{ij}\mathcal{A}$. They also check that any $\mathcal{A} \subset [n]^{(r)}$ can be transformed to a left-compressed family by repeated ij-compressions. It hence suffices for them to consider only left-compressed families in their proof.

It seems at first that a similar approach to Problems 1 and 2 of Katona, Katona and Katona cannot possibly succeed. We know from [4] that compressing an intersecting family yields an intersecting family. Unfortunately, if we compress a non-intersecting family $\mathcal{A}$ then there may exist an intersecting subfamily of $\mathcal{A}$ which moves to a non-intersecting subfamily of $\mathcal{C}_{ij}\mathcal{A}$.

Here is a simple example which illustrates the main obstacle. Consider applying a 12-compression to the family $\mathcal{A} = \{13, 23, 24\}$. Only 24 moves, giving $\mathcal{C}_{12}\mathcal{A} = \{13, 23, 14\}$. But now $\mathcal{B} = \{23, 24\} \subset \mathcal{A}$ is intersecting and moves to $\{23, 14\}$ which is not. (What has gone wrong? The set 23 was blocked from moving by the set 13 which is in $\mathcal{A}$ but not in $\mathcal{B}$.)

Nevertheless, we are able to show that the family $\mathcal{C}_{ij}\mathcal{A}$ has at least as many intersecting subfamilies of each given order as does the family $\mathcal{A}$. In fact, there is a fairly natural

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injection ϕ from the collection $\mathfrak{A}$ of intersecting subfamilies of $\mathcal{A}$ to the collection $\mathfrak{C}$ of intersecting subfamilies of $\mathcal{C}_{ij}\mathcal{A}$. Starting from an intersecting family $\mathcal{B} \in \mathfrak{A}$, we form the family $\phi(\mathcal{B})$ by replacing appropriately chosen $B \in \mathcal{B}$ by $C_{ij}B$. We must obviously choose to replace those $B \in \mathcal{B}$ that move when $\mathcal{A}$ is compressed to $\mathcal{C}_{ij}\mathcal{A}$, as in this case $B \notin \mathcal{C}_{ij}\mathcal{A}$. But we also choose to replace certain $B \in \mathcal{B}$ that were blocked from moving by $C_{ij}B \in \mathcal{A}$ but for which $C_{ij}B \notin \mathcal{B}$. The choice of which such B to replace depends both on the family $\mathcal{A}$ and the subfamily $\mathcal{B}$. In Section 2 we give the details of our construction and prove that the resulting families $\phi(\mathcal{B})$ are indeed intersecting as required. This will establish Theorem 1.2.

Our launching-pad for Theorem 1.2 was the use of ij-compressions to prove the Erdős–Ko–Rado theorem. Can we find something to play a similar role for Theorem 1.1? The right place to start turns out to be from a more general compression operator first introduced by Daykin [3] in his beautiful proof of the Kruskal–Katona theorem [9, 7]. These 'UV-compressions' were independently discovered by Frankl and Füredi [5] in their proof of Harper's theorem. They also turn out to be a special case of a compression operator later developed by Bollobás and Leader [2], who use them to prove intersection theorems such as the Erdős–Ko–Rado theorem and Katona's t-intersecting theorem [10]. This proof of the t-intersecting theorem was also found independently by Ahlswede and Khachatrian [1].

When attempting to apply these methods to Problem 1, the same obstacle arises as in the proof of Theorem 1.2 and is overcome in the same way. However, further difficulties arise in this case. To preserve intersection properties in the proof of the t-intersecting theorem, it is necessary to carry out the UV-compressions in a carefully chosen order. But even when this is done, we are unable to show that the number of intersecting subfamilies of each order increases whenever an individual UV-compression is applied. Instead, it appears that we must apply a sequence of several UV-compressions together, after which there are at least as many intersecting subfamilies of each order as before. We shall explain this further in Section 3, where we prove Theorem 1.1.

Finally, in Section 4, we make some concluding remarks and mention some open problems.

Our notation is mostly standard. We draw the reader's attention to certain points. We write $[n]$ for the set $\{1,2,\ldots,n\}$ and $[m,n]$ for the set $\{m,m+1,\ldots,n\}$. For any set S, we write $S^{(r)}$ for the set $\{A \subset S : |A| = r\}$ of all subsets of S of order r, and $S^{(\geqslant r)}$ for the set $\{A \subset S : |A| \geqslant r\}$ of all subsets of S of order at least r. If X and Y are sets we write $X - Y$ for the set $\{x \in X : x \notin Y\}$. For ease of reading, we often omit set brackets and union symbols. Thus, for example, 123 denotes the set $\{1,2,3\}$, $12XY$ denotes the set $\{1,2\} \cup X \cup Y$, and $1X \cap Y$ denotes the set $(\{1\} \cup X) \cap Y$. If $\mathcal{A}$ is a family of sets, we write $\mathfrak{I}(\mathcal{A})$ for the collection of all intersecting subfamilies of $\mathcal{A}$; that is, $\mathfrak{I}(\mathcal{A}) = \{\mathcal{B} \subset \mathcal{A} : \mathcal{B} \text{ is intersecting}\}$.

2. Left-compression

Our aim in this section is to prove Theorem 1.2.

Let $i, j \in [n]$ with $i < j$. Recall from Section 1 the definition of the *ij-compression*. If $A \in [n]^{(r)}$ then the *ij-compression of A* is

$$C_{ij}A = \begin{cases} (A \cup \{i\}) - \{j\} & \text{if } j \in A, i \notin A, \\ A & \text{otherwise.} \end{cases}$$

If $\mathcal{A} \subset [n]^{(r)}$, the *ij-compression of* $\mathcal{A}$ is

$$\mathcal{C}_{ij}\mathcal{A} = \{C_{ij}A : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : C_{ij}A \in \mathcal{A}\}.$$

It is easy to see that for any $\mathcal{A} \subset [n]^{(r)}$ we may obtain a left-compressed family by applying an appropriate sequence of ij-compressions. (For example, the quantity $\sum_{A \in \mathcal{A}} \sum_{a \in A} a$ decreases whenever we apply a non-trivial ij-compression.) So it suffices to prove that if $\mathcal{C} = \mathcal{C}_{ij}\mathcal{A}$ then $\mathbb{P}(\mathcal{C}_p$ is intersecting$) \geqslant \mathbb{P}(\mathcal{A}_p$ is intersecting$)$. This will follow immediately from the following lemma which is the heart of the proof.

Lemma 2.1. *Let $\mathcal{A} \subset [n]^{(r)}$, let $i, j \in [n]$ and let $\mathcal{C} = \mathcal{C}_{ij}\mathcal{A}$. Then there exists an injection $\phi : \mathfrak{I}(\mathcal{A}) \to \mathfrak{I}(\mathcal{C})$ such that $|\phi(\mathcal{B})| = |\mathcal{B}|$ for all $\mathcal{B} \in \mathfrak{I}(\mathcal{A})$.*

Proof. Assume without loss of generality that $i = 1$ and $j = 2$. Write $\mathfrak{A} = \mathfrak{I}(\mathcal{A})$ and $\mathfrak{C} = \mathfrak{I}(\mathcal{C})$. Let

$$\begin{aligned} \mathcal{A}_1 &= \{X \subset [3, n] : 1X \in \mathcal{A}, 2X \notin \mathcal{A}\}, \\ \mathcal{A}_2 &= \{X \subset [3, n] : 1X \notin \mathcal{A}, 2X \in \mathcal{A}\}, \\ \mathcal{A}_{12} &= \{X \subset [3, n] : 1X, 2X \in \mathcal{A}\}, \\ \mathcal{A}_0 &= \{X \in \mathcal{A} : 1, 2 \in X \text{ or } 1, 2 \notin X\}. \end{aligned}$$

Observe that $\mathcal{A}$ may be written as the disjoint union

$$\mathcal{A} = \{1X : X \in \mathcal{A}_1 \cup \mathcal{A}_{12}\} \cup \{2X : X \in \mathcal{A}_2 \cup \mathcal{A}_{12}\} \cup \mathcal{A}_0.$$

We make similar definitions and a similar observation for the family $\mathcal{C}$. We have $\mathcal{C}_1 = \mathcal{A}_1 \cup \mathcal{A}_2$, $\mathcal{C}_2 = \emptyset$, $\mathcal{C}_{12} = \mathcal{A}_{12}$ and $\mathcal{C}_0 = \mathcal{A}_0$.

For each $\mathcal{B} \in \mathfrak{A}$, we shall form $\phi(\mathcal{B}) \in \mathfrak{C}$ by replacing certain $B \in \mathcal{B}$ by $C_{12}B$. If we replace $B \in \mathcal{B}$ by $C_{12}B \neq B$ then we shall say that we *move* $B \in \mathcal{B}$.

It is clear that we cannot move B of the form $1X$ ($X \in \mathcal{A}_1 \cup \mathcal{A}_{12}$) or X ($X \in \mathcal{A}_0$) as then $C_{12}B = B$. It is equally clear that we must move B of the form $2X$ ($X \in \mathcal{A}_2$) as then $B \notin \mathcal{C}$. Thus we need only consider B of the form $2X$ ($X \in \mathcal{A}_{12}$). Moreover, if $B = 2X \in \mathcal{B}$ ($X \in \mathcal{A}_{12}$) then $C_{12}B = 1X$ so it is only possible to move B if $1X \notin \mathcal{B}$. Thus to completely define ϕ, it suffices to find a suitable answer to the following question: Given $X \in \mathcal{A}_{12}$ and $\mathcal{B} \in \mathfrak{A}$ with $2X \in \mathcal{B}$ but $1X \notin \mathcal{B}$, do we move $2X \in \mathcal{B}$? Recall that it is necessary both that the resulting family $\phi(\mathcal{B})$ be intersecting and that different families in $\mathfrak{A}$ yield different families in $\mathfrak{C}$. It turns out that this is indeed possible. We shall choose to move $2X \in \mathcal{B}$ for certain such $\mathcal{B}$ and not for others, as we now explain.

We begin by partitioning $\mathfrak{A}$ and $\mathfrak{C}$ into carefully chosen 'strata' $\mathfrak{A}_{\mathcal{X}}$ and $\mathfrak{C}_{\mathcal{X}}$ ($\mathcal{X} \in \mathfrak{X}$ for some index set $\mathfrak{X}$) in such a way that for each stratum $\mathfrak{A}_{\mathcal{X}}$ of $\mathfrak{A}$, the corresponding stratum $\mathfrak{C}_{\mathcal{X}}$ of $\mathfrak{C}$ contains all possible choices of $\phi(\mathcal{B})$ for each $\mathcal{B} \in \mathfrak{A}_{\mathcal{X}}$. For each $X \in \mathcal{A}_{12}$, the decision of whether or not to move $2X \in \mathcal{B}$ will depend only on the stratum of $\mathfrak{A}$ in which $\mathcal{B}$ lies.

So how do we define these strata? The key point is that all $\mathcal{B}$ within a single stratum of $\mathfrak{A}$ agree on every set in $\mathcal{A}$ *except* perhaps those of the form $1X$ or $2X$ ($X \in \mathcal{A}_{12}$); while for $X \in \mathcal{A}_{12}$, all $\mathcal{B}$ in the stratum contain the same *number* of the sets $1X$ and $2X$ (*i.e.*, both, precisely one, or neither).

We now proceed to the details of the construction. Suppose

$$\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_{12,(0)}, \mathcal{X}_{12,(1)}, \mathcal{X}_{12,(2)}, \mathcal{X}_0),$$

where $\mathcal{X}_1 \subset \mathcal{A}_1$, $\mathcal{X}_2 \subset \mathcal{A}_2$, $\mathcal{X}_0 \subset \mathcal{A}_0$ and $\mathcal{X}_{12,(0)}$, $\mathcal{X}_{12,(1)}$, $\mathcal{X}_{12,(2)}$ form a disjoint partition of $\mathcal{A}_{12}$. Let $\mathfrak{A}_{\mathcal{X}} \subset \mathfrak{A}$ be the collection of intersecting families $\mathcal{B} \subset \mathcal{A}$ satisfying the following conditions.

(i) For $X \in \mathcal{A}_1$, $1X \in \mathcal{B} \iff X \in \mathcal{X}_1$.
(ii) For $X \in \mathcal{A}_2$, $2X \in \mathcal{B} \iff X \in \mathcal{X}_2$.
(iii) For $X \in \mathcal{A}_0$, $X \in \mathcal{B} \iff X \in \mathcal{X}_0$.
(iv) For $X \in \mathcal{A}_{12}$:

- if $X \in \mathcal{X}_{12,(0)}$ then $1X, 2X \notin \mathcal{B}$,
- if $X \in \mathcal{X}_{12,(1)}$ then $1X \in \mathcal{B}$ or $2X \in \mathcal{B}$ but not both,
- if $X \in \mathcal{X}_{12,(2)}$ then $1X, 2X \in \mathcal{B}$.

Let $\mathfrak{C}_{\mathcal{X}} \subset \mathfrak{C}$ be the collection of intersecting families $\mathcal{B} \subset \mathcal{C}$ satisfying conditions (i), (iii) and (iv), and the additional condition:

(ii)′ For $X \in \mathcal{A}_2$, $1X \in \mathcal{B} \iff X \in \mathcal{X}_2$.

Observe that $\mathfrak{A}$ and $\mathfrak{C}$ can be written as disjoint unions $\mathfrak{A} = \bigcup_{\mathcal{X}} \mathfrak{A}_{\mathcal{X}}$ and $\mathfrak{C} = \bigcup_{\mathcal{X}} \mathfrak{C}_{\mathcal{X}}$, the union in each case ranging over all permissible values of $\mathcal{X}$. Moreover, for each $\mathcal{X}$ there is a positive integer m such that $|\mathcal{B}| = m$ for every $\mathcal{B} \in \mathfrak{A}_{\mathcal{X}} \cup \mathfrak{C}_{\mathcal{X}}$. Hence it suffices to construct, for each $\mathcal{X}$, an injection $\phi_{\mathcal{X}} : \mathfrak{A}_{\mathcal{X}} \to \mathfrak{C}_{\mathcal{X}}$.

So fix $\mathcal{X}$. Let

$$\mathcal{Y} = \{X \in \mathcal{X}_{12,(1)} : 2X \in \mathcal{B} \text{ for all } \mathcal{B} \in \mathfrak{A}_{\mathcal{X}}\}.$$

Define $\phi_{\mathcal{X}} : \mathfrak{A}_{\mathcal{X}} \to \mathfrak{C}_{\mathcal{X}}$ by

$$\phi_{\mathcal{X}}(\mathcal{B}) = \big(\mathcal{B} \cup \{1X : X \in \mathcal{X}_2 \cup \mathcal{Y}\}\big) - \{2X : X \in \mathcal{X}_2 \cup \mathcal{Y}\}.$$

In order to check that $\phi_{\mathcal{X}}$ is well-defined, we must check that $\phi_{\mathcal{X}}(\mathcal{B})$ is intersecting for each $\mathcal{B} \in \mathfrak{A}_{\mathcal{X}}$. It will then be clear that $\phi_{\mathcal{X}}$ is an injection from $\mathfrak{A}_{\mathcal{X}}$ to $\mathfrak{C}_{\mathcal{X}}$.

Assume for a contradiction that $\mathcal{B} \in \mathfrak{A}_{\mathcal{X}}$ but that $\mathcal{D} = \phi_{\mathcal{X}}(\mathcal{B})$ is not intersecting. So there are sets $A, B \in \mathcal{D}$ with $A \cap B = \emptyset$. As $\mathcal{B}$ is intersecting, we cannot have both $A, B \in \mathcal{B}$, so assume without loss of generality that $A \notin \mathcal{B}$. Then $A = 1X$ for some $X \in \mathcal{X}_2 \cup \mathcal{Y}$ and $2X \in \mathcal{B}$. Now we must have $B \in \mathcal{B}$ (as otherwise we would have $1 \in B$). So $B \cap 2X \neq \emptyset$ but $B \cap 1X = \emptyset$. Hence $B = 2Y$ for some $Y \subset [3, n]$ with $X \cap Y = \emptyset$. We cannot have $1Y \in \mathcal{B}$ (as $1Y \cap 2X = \emptyset$), so $Y \in \mathcal{X}_2 \cup \mathcal{X}_{12,(1)}$. But, as $2Y \in \mathcal{D}$, we have $Y \notin \mathcal{X}_2 \cup \mathcal{Y}$. So $Y \in \mathcal{X}_{12,(1)} - \mathcal{Y}$; that is, there is some $\mathcal{E} \in \mathfrak{A}_{\mathcal{X}}$ with $1Y \in \mathcal{E}$. But $X \in \mathcal{X}_2 \cup \mathcal{Y}$ and so $2X \in \mathcal{E}$. But $1Y \cap 2X = \emptyset$, a contradiction (as $\mathcal{E}$ is intersecting).

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Thus $\phi_{\mathcal{X}}$ is an injection from $\mathfrak{A}_{\mathcal{X}}$ to $\mathfrak{C}_{\mathcal{X}}$ for each $\mathcal{X}$. Putting together all of the $\phi_{\mathcal{X}}$, we obtain the required injection $\phi : \mathfrak{I}(\mathcal{A}) \to \mathfrak{I}(\mathcal{C})$. □

We immediately obtain the main result of this section.

Theorem 2.2. *Let n, r and i be positive integers with $r \leqslant n/2$ and $i \leqslant \binom{n-1}{r}$, and let $p \in (0,1)$. Then there exists a left-compressed family $\mathcal{A} \subset [n]^{(r)}$ with $|\mathcal{A}| = \binom{n-1}{r-1} + i$ that maximizes $\mathbb{P}(\mathcal{A}_p$ is intersecting$)$ over all subfamilies of $[n]^{(r)}$ of order $\binom{n-1}{r-1} + i$.*

Proof. Let $\mathcal{A} \subset [n]^{(r)}$ be a family of order k maximizing $\mathbb{P}(\mathcal{A}_p$ is intersecting$)$ over all families of order k. Carry out a sequence of ij-compressions $\mathcal{C}_{i_1 j_1}, \mathcal{C}_{i_2 j_2}, \ldots, \mathcal{C}_{i_m j_m}$ to obtain families $\mathcal{A}_1 = \mathcal{C}_{i_1 j_1}\mathcal{A}$, $\mathcal{A}_2 = \mathcal{C}_{i_2 j_2}\mathcal{A}_1, \ldots, \mathcal{A}_m = \mathcal{C}_{i_m j_m}\mathcal{A}_{m-1}$, with $\mathcal{A}_m$ left-compressed.

It follows from Lemma 2.1 that, for any family $\mathcal{B}$ and any i, j, we have

$$\mathbb{P}((\mathcal{C}_{ij}\mathcal{B})_p \text{ is intersecting}) \geqslant \mathbb{P}(\mathcal{B}_p \text{ is intersecting}).$$

Hence, by induction,

$$\mathbb{P}((\mathcal{A}_m)_p \text{ is intersecting}) \geqslant \mathbb{P}(\mathcal{A}_p \text{ is intersecting}).$$

But the family $\mathcal{A}$ was chosen to maximize this probability, so in fact we have that

$$\mathbb{P}((\mathcal{A}_m)_p \text{ is intersecting}) = \mathbb{P}(\mathcal{A}_p \text{ is intersecting})$$

and $\mathcal{A}_m$ is our required left-compressed family. □

3. Main result

We now turn to the proof of our main result, Theorem 1.1. As we remarked in Section 1, we begin from the proof of Katona's t-intersecting theorem using UV-compressions. In Section 3.1 we define UV-compressions, briefly outline this proof of the t-intersecting theorem, and explain where the difficulties lie in translating these methods to solve Problem 1. In Section 3.2 we define our new compression operators. Finally, in Section 3.3 we prove Theorem 1.1.

3.1. Background

Let n be a positive integer and let U, $V \subset [n]$ be disjoint. If $A \subset [n]$ then the *UV-compression of A* is

$$C_{UV}A = \begin{cases} (A \cup U) - V & \text{if } V \subset A, U \cap A = \emptyset, \\ A & \text{otherwise.} \end{cases}$$

If $\mathcal{A} \subset \mathcal{P}[n]$, the *$UV$-compression of $\mathcal{A}$* is

$$\mathcal{C}_{UV}\mathcal{A} = \{C_{UV}A : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : C_{UV}A \in \mathcal{A}\}.$$

As with ij-compressions, it is generally helpful to think of the compression 'moving' certain sets by replacing V with U where possible. Indeed, ij-compressions are simply the special case of UV-compressions where U and V are both singleton sets.

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Again as with ij-compressions, a typical application aims to compress an initial family to make it 'nicer' in some way whilst preserving some property of the family. However, one must often take great care over the order in which the compressions are applied.

A well-known example is the t-intersecting theorem. A family $\mathcal{A} \subset \mathcal{P}[n]$ is said to be *t-intersecting* if $|A \cap B| \geqslant t$ for all $A, B \in \mathcal{A}$. How large can such a family be?

Assume for simplicity that $n+t$ is even. One obvious example is to take $\mathcal{A} = [n]^{(\geqslant \frac{n+t}{2})}$. Katona [6] showed that this was best possible. We sketch a later proof based on UV-compressions.

The proof begins with a t-intersecting family $\mathcal{A}$ and aims to transform it into a family $\mathcal{B}$ with $[n]^{(\geqslant r+1)} \subset \mathcal{B} \subset [n]^{(\geqslant r)}$. This can be done by a sequence of UV-compressions with $|V| < |U|$ in each case. (In fact, we need only use UV compressions with $|U| = |V| + 1$.) If the resulting family $\mathcal{B}$ is t-intersecting then the theorem is proved.

Unfortunately, this need not always be the case: the family $\mathcal{A} = \{45, 46\}$ is 1-intersecting but $\mathcal{C}_{123,45}\mathcal{A} = \{123, 46\}$ is not. However, this problem can be resolved by carrying out the simplest available compression at each stage: here $\mathcal{A}$ is not $(12,4)$-compressed, and $\mathcal{C}_{12,4}\mathcal{A} = \{125, 126\}$ *is* 1-intersecting. (To be precise, it is now easy to check that if $\mathcal{A}$ is t-intersecting and $U'V'$-compressed for all $U' \subset U$ and $V' \subset V$ with $|U'| > |V'|$ and $(U', V') \neq (U, V)$, then $\mathcal{C}_{UV}\mathcal{A}$ is t-intersecting.) This suffices to prove the t-intersecting theorem.

We now consider how this can be applied to Problem 1. We begin with a family $\mathcal{A} \subset \mathcal{P}[n]$ which we aim to compress to a family $\mathcal{A}'$ with $[n]^{(\geqslant r+1)} \subset \mathcal{A}' \subset [n]^{(\geqslant r)}$ by means of UV-compressions with $|U| = |V| + 1$. Our initial hope might be that if these compressions are applied in an appropriate order then, as with ij-compressions in Section 2, the number of intersecting subfamilies of each possible order increases after each compression.

We may clearly apply a UV-compression with $|V| = 0$; each intersecting subfamily of $\mathcal{A}$ moves to an intersecting subfamily of $\mathcal{C}_{UV}\mathcal{A}$.

If $|V| = 1$ then, as with ij-compressions, it is possible for an intersecting subfamily of $\mathcal{A}$ to move to a non-intersecting subfamily of $\mathcal{C}_{UV}\mathcal{A}$. But this problem can be resolved precisely as it was for ij-compressions in the proof of Lemma 2.1.

The real problem first arises when $|V| = 2$. Now we are unable to show that $\mathcal{C}_{UV}\mathcal{A}$ contains more intersecting subfamilies of each order than does $\mathcal{A}$, even if we assume that we have already performed all simpler compressions (although we do not have a counterexample).

Why does the proof of Lemma 2.1 not carry over? Suppose, say, we perform the compression $\mathcal{C}_{123,45}$ on $\mathcal{A}$, and $\mathcal{B} \subset \mathcal{A}$ is intersecting. Perhaps when forming $\phi(\mathcal{B})$ we replace $45 \in \mathcal{B}$ with 123. Now, if also $4 \in \mathcal{B}$ then 4 does not move but $4 \cap 123 = \emptyset$. However, we know that $\mathcal{A}$ is $(12,4)$-compressed so maybe we can replace 4 with 12. But what if, say, $34 \in \mathcal{B}$? Now 34 does not intersect 12 . . .

At some point in the proof, it appears that we need to perform an illegal replacement, say $34 \to 125$. And we cannot assume that $\mathcal{A}$ is $(125,34)$-compressed as then we do not obtain a well-founded order in which to carry out our compressions.

The solution is to perform four compressions together: instead of comparing $\mathcal{A}$ with $\mathcal{C}_{123,45}\mathcal{A}$, we compare it with $\mathcal{C} = \mathcal{C}_{123,45}\mathcal{C}_{125,34}\mathcal{C}_{134,25}\mathcal{C}_{145,23}\mathcal{A}$. It is now possible to arrange that all of the necessary replacements are legal, yielding a proof that $\mathcal{C}$ contains at least as many intersecting subfamilies of each possible order as does $\mathcal{A}$.

3.2. (U, v, f)-compressions

It is convenient to define a new compression operator which carries out all of the necessary compressions simultaneously. In fact, it moves sets in such a way that we no longer need to worry about carrying out simpler compressions first.

Let X be a set. A *pairing function* on X is a function $f : X \to X$ such that $f \circ f$ is the identity and f has no fixed point. We may think of f as 'pairing' the elements of X. Note that if X is finite then it must have even order.

Let $U \subset [n]$ be of even order, let $v \in [n] - U$ and let $f : U \to U$ be a pairing function. We define the (U, v, f)-compression on $\mathcal{P}[n]$ by

$$C_{U,v,f}(A) = \begin{cases} A & \text{if } v \in A, \\ f(A \cap U) \cup \{v\} \cup (A - U) & \text{if } v \notin A \end{cases}$$

for $A \in \mathcal{P}[n]$, and

$$\mathcal{C}_{U,v,f}(\mathcal{A}) = \{C_{U,v,f}(A) : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : C_{U,v,f}(A) \in \mathcal{A}\}$$

for $\mathcal{A} \subset \mathcal{P}[n]$.

We remark that in the case where $\mathcal{A}$ is already $U'V'$-compressed for all disjoint pairs (U', V') with $V' \subset U$, $U' \subset U \cup \{v\}$, $|V'| < |U|/2$, $|U'| \leqslant |U|/2 + 1$ and $|U'| > |V'|$, then $\mathcal{C}_{U,v,f}$ can be written as a composition of UV compressions. Indeed, in this case $\mathcal{C}_{U,v,f} = \mathcal{C}_{U_1V_1}\mathcal{C}_{U_2V_2}\cdots\mathcal{C}_{U_kV_k}$, where $V_1, V_2, \ldots, V_k$ are the subsets of U of order $|U|/2$ and $U_i = (U - V_i) \cup \{v\}$.

As an illustration, we prove that (U, v, f)-compressions preserve the property of a family being intersecting.

Proposition 3.1. *Let $\mathcal{A} \subset \mathcal{P}[n]$ be intersecting, let $U \subset [n]$ be of even order, let $v \in [n] - U$ and let $f : U \to U$ be a pairing function. Then $\mathcal{C}_{U,v,f}\mathcal{A}$ is intersecting.* □

Proof. Write $\mathcal{C} = \mathcal{C}_{U,v,f}\mathcal{A}$. Suppose that $\mathcal{C}$ is not intersecting. Choose A, $B \in \mathcal{C}$ with $A \cap B = \emptyset$. As $\mathcal{A}$ is intersecting, we cannot have both A, $B \in \mathcal{A}$, so assume without loss of generality that $A \notin \mathcal{A}$. Then $A = vf(W)X$ for some $W \subset U$ and $X \subset [n] - (U \cup \{v\})$ with $WX \in \mathcal{A}$. As $A \cap B = \emptyset$, we must have $v \notin B$ and thus $B \in \mathcal{A}$ and $B = TY$ for some $T \subset U$ and $Y \subset [n] - (U \cup \{v\})$. Moreover, $T \cap f(W) = \emptyset$ and $X \cap Y = \emptyset$. Now, as $v \notin B$ and $B \in \mathcal{C}$ we must have $vf(T)Y \in \mathcal{A}$. Now consider WX, $vf(T)Y \in \mathcal{A}$. We have $W \cap f(T) = f(f(W) \cap T) = \emptyset$ and $X \cap Y = \emptyset$, and so $WX \cap vf(T)Y = \emptyset$. But this is a contradiction, as $\mathcal{A}$ is intersecting. □

Note that, unlike with standard UV-compressions, there is no restriction here on the order in which these compressions may be applied. However, we remark in passing that the order *would* be important if we wanted to retain the property of $\mathcal{A}$ being 2-intersecting; in this case we would again have to apply compressions with smaller U first.

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For example, taking $\mathcal{A} = \{23, 1236\}$, $U = \{2345\}$, $v = 1$, $f(2,3,4,5) = (4,5,2,3)$, we have $\mathcal{A}$ 2-intersecting but $C_{U,v,f}\mathcal{A} = \{145, 1236\}$ not 2-intersecting.

3.3. Proof of main result

The heart of the proof is the following lemma. The proof of the lemma mirrors the proof of Lemma 2.1, but using our (U, v, f)-compressions in place of ij-compressions.

Lemma 3.2. *Let $\mathcal{A} \subset \mathcal{P}[n]$, let U, v and f be as above and let $\mathcal{C} = \mathcal{C}_{U,v,f}(\mathcal{A})$. Then there exists an injection $\phi : \mathfrak{I}(\mathcal{A}) \to \mathfrak{I}(\mathcal{C})$ such that $|\phi(\mathcal{B})| = |\mathcal{B}|$ for all $\mathcal{B} \in \mathfrak{I}(\mathcal{A})$.*

Proof. Write $\mathfrak{A} = \mathfrak{I}(\mathcal{A})$, $\mathfrak{C} = \mathfrak{I}(\mathcal{C})$ and $S = [n] - (U \cup \{v\})$. For each $W \subset U$, let

$$\mathcal{A}_1^W = \{X \subset S : WX \notin \mathcal{A}, vf(W)X \in \mathcal{A}\},$$
$$\mathcal{A}_2^W = \{X \subset S : WX \in \mathcal{A}, vf(W)X \notin \mathcal{A}\},$$
$$\mathcal{A}_{12}^W = \{X \subset S : WX, vf(W)X \in \mathcal{A}\}.$$

Observe that $\mathcal{A}$ may be written as the disjoint union

$$\mathcal{A} = \bigcup_{W \subset U} \left(\{vf(W)X : X \in \mathcal{A}_1^W \cup \mathcal{A}_{12}^W\} \cup \{WX : X \in \mathcal{A}_2^W \cup \mathcal{A}_{12}^W\}\right).$$

We make similar definitions and a similar observation for the family $\mathcal{C}$. For each $W \subset U$ we have $\mathcal{C}_1^W = \mathcal{A}_1^W \cup \mathcal{A}_2^W$, $\mathcal{C}_2^W = \emptyset$ and $\mathcal{C}_{12}^W = \mathcal{A}_{12}^W$.

As in the proof of Lemma 2.1, we proceed by partitioning $\mathfrak{A}$ and $\mathfrak{C}$ into appropriate strata and then moving certain sets within each stratum. However, there are now many more cases of sets which might or might not move: for every subset $W \subset U$, given $X \in \mathcal{A}_{12}^W$ and $B \in \mathfrak{A}$ with $WX \in \mathcal{B}$ but $vf(W)X \notin \mathcal{B}$, we must determine whether or not to move $WX \in \mathcal{B}$. Hence the definition of the strata, while analogous to that in Lemma 2.1, appears rather more complex.

Suppose $\mathcal{X} = (\mathcal{X}_1^W, \mathcal{X}_2^W, \mathcal{X}_{12,(0)}^W, \mathcal{X}_{12,(1)}^W, \mathcal{X}_{12,(2)}^W)_{W \subset U}$ where, for each $W \subset U$, we have $\mathcal{X}_1^W \subset \mathcal{A}_1^W$, $\mathcal{X}_2^W \subset \mathcal{A}_2^W$ and $\mathcal{X}_{12,(0)}^W$, $\mathcal{X}_{12,(1)}^W$ and $\mathcal{X}_{12,(2)}^W$ forming a disjoint partition of $\mathcal{A}_{12}^W$. Let $\mathfrak{A}_\mathcal{X} \subset \mathfrak{A}$ be the collection of intersecting families $\mathcal{B} \subset \mathcal{A}$ satisfying, for each $W \subset U$, the following conditions.

(i) For $X \in \mathcal{A}_1^W$, $vf(W)X \in \mathcal{B} \iff X \in \mathcal{X}_1^W$.
(ii) For $X \in \mathcal{A}_2^W$, $WX \in \mathcal{B} \iff X \in \mathcal{X}_2^W$.
(iii) For $X \in \mathcal{A}_{12}^W$:

- if $X \in \mathcal{X}_{12,(0)}^W$ then $WX, vf(W)X \notin \mathcal{B}$,
- if $X \in \mathcal{X}_{12,(1)}^W$ then $WX \in \mathcal{B}$ or $vf(W)X \in \mathcal{B}$ but not both,
- if $X \in \mathcal{X}_{12,(2)}^W$ then $WX, vf(W)X \in \mathcal{B}$.

Let $\mathfrak{C}_\mathcal{X}$ be the collection of intersecting families $\mathcal{B} \subset \mathcal{C}$ satisfying, for each $W \subset U$, conditions (i) and (iii) and the additional condition:

(ii)′ For $X \in \mathcal{A}_2^W$, $vf(W)X \in \mathcal{B} \iff X \in \mathcal{A}_2^W$.

Observe that $\mathfrak{A}$ and $\mathfrak{C}$ can be written as disjoint unions $\mathfrak{A} = \bigcup_\mathcal{X} \mathfrak{A}_\mathcal{X}$ and $\mathfrak{C} = \bigcup_\mathcal{X} \mathfrak{C}_\mathcal{X}$, and that, for each $\mathcal{X}$, there is a positive integer m such that $|\mathcal{B}| = m$ for all $B \in \mathfrak{A}_\mathcal{X} \cup \mathfrak{C}_\mathcal{X}$. Hence, as before, it suffices to construct, for each $\mathcal{X}$, an injection $\phi_\mathcal{X} : \mathfrak{A}_\mathcal{X} \to \mathfrak{C}_\mathcal{X}$.

So fix $\mathcal{X}$. For each $W \subset U$ let

$$\mathcal{Y}^W = \{X \in \mathcal{X}^W_{12,(1)} : WX \in \mathcal{B} \text{ for all } \mathcal{B} \in \mathfrak{A}_{\mathcal{X}}\}.$$

Define $\phi_{\mathcal{X}} : \mathfrak{A}_{\mathcal{X}} \to \mathfrak{C}_{\mathcal{X}}$ by

$$\phi_{\mathcal{X}}(\mathcal{B}) = \mathcal{B} \cup \bigcup_{W \subset U} \{vf(W)X : X \in \mathcal{X}^W_2 \cup \mathcal{Y}^W\} - \bigcup_{W \subset U} \{WX : X \in \mathcal{X}^W_2 \cup \mathcal{Y}^W\}.$$

Again, all that we need to check is that $\phi_{\mathcal{X}}(\mathcal{B})$ is intersecting for each $\mathcal{B} \in \mathfrak{A}_{\mathcal{X}}$.

Assume for a contradiction that $\mathcal{B} \in \mathfrak{A}_{\mathcal{X}}$ but that $\mathcal{D} = \phi_{\mathcal{X}}(\mathcal{B})$ is not intersecting. So there are sets A, $B \in \mathcal{D}$ with $A \cap B = \emptyset$. As $\mathcal{B}$ is intersecting, we cannot have both A, $B \in \mathcal{B}$, so assume without loss of generality $A \notin \mathcal{B}$. Then $A = vf(W)X$ for some $W \subset U$, $X \in \mathcal{X}^W_2 \cup \mathcal{Y}^W$ and $WX \in \mathcal{B}$. Now, we must have $B \in \mathcal{B}$ (as otherwise we would have $v \in B$ and so $A \cap B \neq \emptyset$). So $B \cap WX \neq \emptyset$ but $B \cap vf(W)X = \emptyset$. Hence $B = TY$ for some $T \subset U$ and $Y \subset [n] - (U \cup \{v\})$ with $T \cap f(W) = \emptyset$ and $X \cap Y = \emptyset$.

It is easy to check that $vf(T) \cap W = \emptyset$. Indeed, suppose instead that there is some $a \in vf(T) \cap W$. As $v \notin W$ we must have $a \neq v$ and so $a = f(t)$ for some $t \in T$. But then $t = f(a) \in f(W)$, contradicting $T \cap f(W) = \emptyset$.

Now, we have $vf(T) \cap W = \emptyset$ and $X \cap Y = \emptyset$, so $vf(T)Y \cap WX = \emptyset$. But $WX \in \mathcal{B}$ so $vf(T)Y \notin \mathcal{B}$. Now, $vf(T)Y \notin \mathcal{B}$ but $TY \in \mathcal{B}$ so $Y \in \mathcal{X}^T_2 \cup \mathcal{X}^T_{12,(1)}$. But $TY = B \in \mathcal{D}$ so $Y \notin \mathcal{X}^T_2 \cup \mathcal{Y}^T$. Hence $Y \in \mathcal{X}^T_{12,(1)} - \mathcal{Y}^T$; that is, there is some $\mathcal{E} \in \mathfrak{A}_{\mathcal{X}}$ with $vf(T)Y \in \mathcal{E}$.

But $vf(W)X = A \in \mathcal{D}$ and $vf(W)X \notin \mathcal{B}$ so $X \in \mathcal{X}^W_2 \cup \mathcal{Y}^W$. Hence $WX \in \mathcal{E}$. But now $vf(T)Y$, $WX \in \mathcal{E}$ with $vf(T)Y \cap WX = \emptyset$, a contradiction. □

We now obtain our main result.

Theorem 3.1. *Let n be a positive integer and $p \in (0,1)$. Let r be a positive integer with $r \leqslant n/2$. Then, over all $\mathcal{A} \subset \mathcal{P}[n]$ with $|\mathcal{A}| = \sum_{j=r}^{n} \binom{n}{j}$, the probability $\mathbb{P}(\mathcal{A}_p$ is intersecting) is maximized by $\mathcal{A} = [n]^{(\geqslant r)}$.*

Moreover, suppose i is any positive integer with $i \leqslant 2^{n-1}$ and let r be such that $\sum_{j=r+1}^{n} \binom{n}{j} \leqslant 2^{n-1} + i \leqslant \sum_{j=r}^{n} \binom{n}{j}$. Then, over all $\mathcal{A} \subset \mathcal{P}[n]$ with $|\mathcal{A}| = 2^{n-1} + i$, the probability $\mathbb{P}(\mathcal{A}_p$ is intersecting) is maximized by some $\mathcal{A}$ with $[n]^{(\geqslant r+1)} \subset \mathcal{A} \subset [n]^{(\geqslant r)}$.

Proof. It clearly suffices to prove the second statement as the first follows immediately. Starting from any family $\mathcal{A}$, we observe that it is possible to obtain a family $\mathcal{C}$ with $[n]^{(\geqslant r+1)} \subset \mathcal{C} \subset [n]^{(\geqslant r)}$ for some r by a sequence of (U, v, f)-compressions. Indeed, suppose that $\mathcal{A}$ is not already of the required form. Then it is easy to see that there are some disjoint sets W, $V \in [n]$ with $|W| = |V| + 1$ and $\mathcal{A}$ not WV-compressed. Take $v = \min W$, $U = (W - \{v\}) \cup V$ and $f : U \to U$ a pairing function with $f(W - \{v\}) = V$. Then $\mathcal{A}$ is not (U, v, f)-compressed, so we may apply $C_{U,v,f}$ to obtain a new family. But every time we apply a non-trivial (U, v, f)-compression the quantity $\sum_{A \in \mathcal{A}} |A|$ increases, and so this process must terminate with some $\mathcal{A}$ of the required form. Hence Theorem 1.1 follows from Lemma 3.2 precisely as Theorem 1.2 follows from Lemma 2.1. □

Examining the proof of Lemma 3.2, we note that $\mathcal{C}_{U,v,f}\mathcal{A}$ has at least as many intersecting subfamilies of each possible order as does $\mathcal{A}$. Hence our optimal families simultaneously maximize the number of intersecting subfamilies of every possible order. This may be of independent interest.

Corollary 3.2. *Let $\mathcal{A} \subset \mathcal{P}[n]$ with $|A| = \sum_{j=r}^{n} \binom{n}{r}$. Then the family $[n]^{(\geqslant r)}$ has at least as many intersecting subfamilies of every possible order as has $\mathcal{A}$.*

In fact, Theorem 1.1 also solves Problem 1 in the cases where $2^{n-1} + i = \left(\sum_{j=r}^{n} \binom{n}{r}\right) \pm 1$. Moreover, Lemma 2.1 holds for $\mathcal{A} \subset \mathcal{P}[n]$ as well as for $\mathcal{A} \subset [n]^{(r)}$ with an identical proof, allowing us to solve Problem 1 in the cases where $2^{n-1} + i = \left(\sum_{j=r}^{n} \binom{n}{r}\right) \pm 2$. For completeness, we state these results explicitly.

Corollary 3.3. *Let n and r be positive integers with $r \leqslant n/2$.*

- *Over all $\mathcal{A} \subset \mathcal{P}[n]$ with $|\mathcal{A}| = \left(\sum_{j=r}^{n} \binom{n}{r}\right) + 1$, the probability $\mathbb{P}(\mathcal{A}_p$ is intersecting) is maximized by*
$$\mathcal{A} = [n]^{(\geqslant r)} \cup \{123 \ldots r-1\}.$$
- *Over all $\mathcal{A} \subset \mathcal{P}[n]$ with $|\mathcal{A}| = \left(\sum_{j=r}^{n} \binom{n}{r}\right) - 1$. the probability $\mathbb{P}(\mathcal{A}_p$ is intersecting) is maximized by*
$$\mathcal{A} = [n]^{(\geqslant r)} - \{(n-r+1)(n-r+2)\ldots n\}.$$
- *Over all $\mathcal{A} \subset \mathcal{P}[n]$ with $|\mathcal{A}| = \left(\sum_{j=r}^{n} \binom{n}{r}\right) + 2$, the probability $\mathbb{P}(\mathcal{A}_p$ is intersecting) is maximized by*
$$\mathcal{A} = [n]^{(\geqslant r)} \cup \{123 \ldots r-1, 123 \ldots (r-2)r\}.$$
- *Over all $\mathcal{A} \subset \mathcal{P}[n]$ with $|\mathcal{A}| = \left(\sum_{j=r}^{n} \binom{n}{r}\right) - 2$, the probability $\mathbb{P}(\mathcal{A}_p$ is intersecting) is maximized by*
$$\mathcal{A} = [n]^{(\geqslant r)} - \{(n-r)(n-r+2)(n-r+3)\ldots n, (n-r+1)(n-r+2)\ldots n\}.$$

Moreover, in each case the optimal family simultaneously maximizes the number of intersecting subfamilies of every possible order.

4. Concluding remarks

Theorem 1.1 gives us substantial information about the structure of the optimal families solving Problem 1: they consist of the top layers of the cube together with some collection of sets from the next layer down. However, we know little about what happens within the layers. The analogue of Lemma 2.1 for families in $\mathcal{P}[n]$ gives that we may take our optimal family to be left-compressed, but this still leaves open many possibilities. In particular, we would be interested to know if there is indeed a nested sequence of optimal families as conjectured by Katona, Katona and Katona [8].

We observed in Corollaries 3.2 and 3.3 that in all the cases of Problem 1 that we could solve, the optimal family simultaneously maximized the number of intersecting subfamilies of every given order. We would like to know if this is always possible.

Question 1. Let $N \leqslant 2^n$. Does there exist a family $\mathcal{A} \subset \mathcal{P}[n]$ of order N which simultaneously maximizes the number of intersecting subfamilies of every possible order?

Finally, we noted in Section 3.1 that we were unable to prove that UV-compressions (applied in appropriate order) always increased the number of intersecting subfamilies of each order. However, we also have no counterexample. Hence we ask the following question.

Question 2. Let $\mathcal{A} \subset \mathcal{P}[n]$ and $U, V \subset [n]$ be disjoint with $|U| > |V|$. Suppose $\mathcal{A}$ is $U'V'$-compressed for all $U' \subset U$ and $V' \subset V$ with $(U', V') \neq (U, V)$ and $|U'| > |V'|$. Must $\mathcal{C}_{UV}\mathcal{A}$ have at least as many intersecting subfamilies of every possible order as has $\mathcal{A}$?

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