

A NOTE ON THE INCREASING DIRECTIONALLY CONCAVE MONOTONICITY IN QUEUES

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In this article, we study comparison theorems for stochastic functionals like $V(\infty; C) = \sup_{0 \leq t} \{M(t) - C(t)\}$ or $V(T; C) = \sup_{0 \leq t \leq T} \{M(t) - C(t)\}$, where $\{M(t)\}$ and $\{C(t)\}$ are two independent nondecreasing processes with stationary increments. We will tacitly assume that the considered stochastic functionals are proper random variables. We prove that $V(T; C') \leq_{\text{icx}} V(T; C) \leq_{\text{icx}} V(T; C'')$, where $C'(dt) = \mathbb{E}[C(1) - C(0)] dt$ and $C''(dt) = c(0) dt$, provided $dC(t)$ is absolute continuous with density $c(t)$. Similarly, we show that $V(\infty; C') \leq_{\text{icx}} V(\infty; C) \leq_{\text{icx}} V(\infty; C'')$. For proofs, we develop the theory of the \leq_{idcv} ordering defined by increasing directionally concave functions. We apply the theory to $M/G/1$ priority queues and $M/G/1$ queues with positive and negative customers.

1. INTRODUCTION

In this note, we study comparison theorems for stochastic functionals like $V(\infty; C) = \sup_{0 \leq t} \{M(t) - C(t)\}$ or $V(T; C) = \sup_{0 \leq t \leq T} \{M(t) - C(t)\}$, where $\{M(t)\}$ and $\{C(t)\}$ are two independent nondecreasing processes with stationary increments. We will tacitly assume that the considered stochastic functionals are proper random variables. Typically, $M(t)$ is a cumulative service time up to t . For example, in [1,4,6–11,13], $M(t) = \sum_{j=1}^{N(t)} S_j$, where $\{N(t)\}$ is a counting Cox process and $\{S_j\}$ is a sequence of independent random variables, independent of the process $\{N(t)\}$, $C(t) = t$, and the comparisons are made with respect to different Cox processes. The seminal problem was a question by Ross [13] about the optimality of the con-

stant arrival rate in the class of arrival rates with the same asymptotic rate, which was solved in [10]. The worst case among all of the stationary arrival rates with the same asymptotic intensity was detected in [12]. In this article the use of the so-called increasing directionally convex (idcx) functions was proposed to derive the result. In his article, Ross [13] posed a problem on the monotonicity of $\mathbb{E}V(M_a)$ with respect to a class $\{M_a\}$ parameterized by a . This problem has been considered in many articles, and the direct question of Ross for two-state background continuous time Markov chains (CTMCs) was solved in [4]. The case of more states of the background CTMC was addressed in [1]. Recently, quite a general result in terms of a concept based on \leq_{idcx} -ordering was given in [8]. Bonald, Borst, and Proutière [2] addressed how to find insensitive bounds in some wireless data network by the use of \leq_{idcx} -ordering.

The study of \leq_{idcv} -ordering was motivated by two problems from the theory of queues, posed to the author by Borst and Boxma. For these problems, we keep $\{M(t)\}$ fixed and study comparison theorems with respect to $\{C(t)\}$. Whereas in the former studies the orderings \leq_{sm} and \leq_{idcx} were of importance, here we use the dual ordering \leq_{idcv} to the ordering \leq_{idcx} defined by the class of increasing directionally concave functions. These are multivariate extensions of icx and icv orderings of univariate random variables. We also define the notion \leq_{idcv} -regularity, a dual one to \leq_{idcx} -regularity studied in [8]. We apply the theory to $M/G/1$ with varying instantaneous service speed—in particular, priority queues and $M/G/1$ queues with positive and negative customers.

2. ORDERINGS

We give definitions of needed integral orderings and classes of functions defining these orderings. It is standard to denote by cx and cv respectively the class of convex and concave functions $f: \mathbb{R} \rightarrow \mathbb{R}$. For multivariable functions, we have the following fundamental notion.

DEFINITION 2.1: A function $f: \mathbb{R}^k \rightarrow \mathbb{R}$ is said to be supermodular (sm) if for any x and $y \in \mathbb{R}^k$,

$$f(x) + f(y) \leq f(x \wedge y) + f(x \vee y),$$

where the operators \wedge and \vee denote respectively coordinatewise minimum and maximum.

A suitable extensions of cx and cv classes for the multivariate case are as follows.

DEFINITION 2.2: A function $f: \mathbb{R}^k \rightarrow \mathbb{R}$ is said to be directionally convex (dcx) (directionally concave or [dvcv]) if for any x_1, x_2 , and $y \in \mathbb{R}^k$ such that for $x_1 \leq x_2$ and $y \geq 0$,

$$f(x_1 + y) - f(x_1) \leq (\geq) f(x_2 + y) - f(x_2).$$

We write “i” before “sm,” “dcx,” or “dcv” for restrictions to increasing functions. Note also that the directional convexity is not the same as ordinary convexity. Important properties of dcx and dcv functions are as follows. If f is dcx (dcv) and twice differentiable, then $\partial^2 f(x)/\partial x_i \partial x_j \geq (\leq) 0$ for all i and j . Note that for sm, we have only $\partial^2 f(x)/\partial x_i \partial x_j \geq (\leq) 0$ for all $i \neq j$. More on sm and dcx functions can be found in Müller and Stoyan [9]. Functions dcv and idcv appeared in the doctoral thesis of Meester [6]. It was pointed out there that the composition $f \circ d(y)$ of an icv function f with idcv function d on \mathbb{R}^k is also an idcv function of y [6, Lemma 2.2.6]. Note that $f(x)$ is idcx if and only if $-f(-x)$ is idcv. We will use the fact that idcx functions form a cone (see, e.g., [9]).

To each class of the above-considered functions we relate the stochastic ordering, which in the terminology of Müller and Stoyan [9] are integral orderings. Let \mathcal{F} be a class of functions like cx, icx, cv, icv, sm, ism, dcx, idcx, dcv, and idcv. We say that two random vectors (in the case of cx, icx, cv, and icv random variables) X and Y are $<_{\mathcal{F}}$ -ordered if $\mathbf{E}[f(X)] \leq \mathbf{E}[f(Y)]$ for any function $f \in \mathcal{F}$, such that the expectations are finite. In this way, we define orderings $\leq_{\text{icx}}, \leq_{\text{ism}}, \leq_{\text{idcx}}, \leq_{\text{icv}}$, and \leq_{idcv} used in this note. For ordering of stochastic processes, we have the following definition.

DEFINITION 2.3:

1. For two \mathbb{R} -valued stochastic processes $\{X(t)\}_{t \in \mathbb{R}}$ and $\{Y(t)\}_{t \in \mathbb{R}}$, we say that $\{X(t)\}_{t \in \mathbb{R}}$ is smaller than $\{Y(t)\}_{t \in \mathbb{R}}$ in the \mathcal{F} -ordering and write $\{X(t)\}_{t \in \mathbb{R}} \leq_{\mathcal{F}} \{Y(t)\}_{t \in \mathbb{R}}$, if for any positive integer k and any $-\infty < t_1 \leq \dots \leq t_k < +\infty$, $(X(t_1), \dots, X(t_k)) \leq_{\mathcal{F}} (Y(t_1), \dots, Y(t_k))$.
2. Let $\{C(t)\}_{t \in \mathbb{R}_+}$ and $\{C'(t)\}$ be two increasing stochastic processes defining random measures $dC(t)$ and $dC'(t)$, respectively. It is said that $C <_{\mathcal{F}} C'$, or $dC(t) <_{\mathcal{F}} dC'(t)$, if for any k and $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2 \leq \dots \leq s_k \leq t_k$,

$$(C(s_1, t_1], \dots, C(s_k, t_k]) <_{\mathcal{F}} (C'(s_1, t_1], \dots, C'(s_k, t_k]),$$

where we denote $C(a, b] = C(b) - C(a)$.

3. We say that a stationary stochastic process $\{Y_t\}_{t \in T}$, where $T = \mathbb{Z}$ or \mathbb{R} is \leq_{idcv} -regular (\leq_{idcx} -regular) if for each k and all idcv functions $f: T^{k+1} \rightarrow \mathbb{R}$, function ϕ , defined by

$$\phi(t_1, \dots, t_k) = \mathbf{E}f(Y_0, Y_{t_1}, Y_{t_1+t_2}, \dots, Y_{t_1+\dots+t_k}),$$

is increasing (decreasing) in $t_1, t_2, \dots, t_k \in T$.

We first recall that from the Lorenz inequality (see, e.g., [9]), we have the following lemma.

LEMMA 2.4: If $Z_0 =_{\text{st}} Z_1 =_{\text{st}} \dots =_{\text{st}} Z_n$, then $(Z_0, \dots, Z_k) \leq_{\text{sm}} (Z_0, \dots, Z_0)$.

The notion of \leq_{idcx} -regularity, also called monotonicity in lag, was studied in Miyoshi and Rolski [8]. For later use, we need the following quite obvious lemma.

LEMMA 2.5: *If a stationary process $\xi(t)$ is \leq_{idcv} -regular, then for a monotone function f , the process $f(\xi(t))$ is \leq_{idcv} -regular.*

3. GENERAL FRAMEWORK

Let $M(t)$ and $\{C(t)\}$ be two increasing processes with stationary increments such that $M(0) = 0$ and $C(0) = 0$ and let $\mathbb{E}C(0,1) = \gamma$. Our aim is to study bounds and comparison theorems for

$$V = V(\infty; C) = \sup_{0 \leq t} \{M(t) - C(t)\} \tag{3.1}$$

or

$$V_T = V(T; C) = \sup_{0 \leq t \leq T} \{M(t) - C(t)\}. \tag{3.2}$$

We have the following representations for V :

$$V = \lim_{i \rightarrow \infty} \sup \left\{ \sum_{k=1}^j (M(I_{i,k}) - C(I_{i,k})), j = 0, \dots, i2^i \right\}, \tag{3.3}$$

where $I_{i,k} = ((k - 1)/2^i, k/2^i]$ for $k = 1, \dots, i2^i$ and $\sum_{k=1}^0 \cdot = 0$ conventionally. We have

$$V_T = \lim_{i \rightarrow \infty} (X + M(T) - C(T)) \vee \max \left\{ \sum_{k=j}^{2^i} (M(I_{i,k}^T) - C(I_{i,k}^T)), j = 1, \dots, 2^i \right\}, \tag{3.4}$$

where $I_{i,k}^T = ((k - 1)(T/2^i), k(T/2^i)]$ for $k = 1, \dots, 2^i$, and X, M , and C are independent.

Let $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ be icx, which is also assumed to be continuous on the whole \mathbb{R}_+ . Similarly to Lemma 1 of [12], we have from the monotone convergence theorem and the continuity of f ,

$$\begin{aligned} & \mathbb{E}[f(V(\infty; C))] \\ &= \lim_{i \rightarrow \infty} \mathbb{E} \left[f \left(\max \left\{ \sum_{k=1}^j M(I_{i,k}) - \sum_{k=1}^j C(I_{i,k}), j = 0, \dots, i2^i \right\} \right) \right] \end{aligned} \tag{3.5}$$

$$= \lim_{i \rightarrow \infty} \mathbb{E}[f \circ g_{i2^i}(M(I_{i,1}) - C(I_{i,1}), \dots, M(I_{i,i2^i}) - C(I_{i,i2^i}))], \tag{3.6}$$

where $g_k(x) = \max\{0, x_1, \dots, x_1 + \dots + x_k\}$ for $x = (x_1, \dots, x_k) \in \mathbb{R}^k$. Similarly, we can represent $f(V_T)$ by the use of functions $x \rightarrow \max\{0, y + x_1 + \dots + x_k, x_2 + \dots + x_k, \dots, x_1\}$. The above functions are idcx.

As in Lemma 2 of [12], $g_k(x)$ is idcx (and, moreover, it is a convex function too). Furthermore, $d_x(y) = -f \circ g_k(x - y)$ is an idcv function of y . This can be

justified as follows. The composition $f \circ g_k(x + y)$ of an icx function f with an idcx function g_k on \mathbb{R}^k is also an idcx function of $y \in \mathbb{R}^k$ [6, Lemma 2.2.6]. Therefore, $-f \circ g_k(x - y)$ is idcv. Hence, we have the following lemma.

LEMMA 3.1: *Let f be icx. Then*

$$\psi_k(y_1, \dots, y_k) = -\mathbf{E}[f \circ g_k(M(I_1) - y_1, \dots, M(I_k) - y_k)] \tag{3.7}$$

is idcv and, hence, $-\psi_k(y_1, \dots, y_k)$ is a decreasing sm function.

Further, to a given process $\{C(t)\}$, we now associate two increasing processes $\{C'(t)\}$ and $\{C''(t)\}$ defined as follows:

- $C'(dt) = \gamma dt$.
- $C''(dt) = c(0) dt$, provided $dC(t)$ is absolute continuous and $c(t)$ is a stationary nonnegative process, called an intensity process. We always assume that intensity processes have Riemann integrable sample paths.

For further reference, we state the following lemma without proof.

LEMMA 3.2: *If ϕ is sm of k variables, then the function $\hat{\phi}: \mathbb{R}^{I_1} \times \dots \times \mathbb{R}^{I_k} \rightarrow \mathbb{R}$ of $I_1 \times \dots \times I_k$ variables*

$$\hat{\phi}(x_{11}, \dots, x_{1, I_1}, \dots, x_{k1}, \dots, x_{k, I_k}) = \phi \left(\sum_{j=1}^{I_1} c_{1j} x_{1j}, \dots, \sum_{j=1}^{I_k} c_{kj} x_{kj} \right)$$

is sm, provided all c_{ij} are positive numbers.

LEMMA 3.3: *We have for an sm function ϕ ,*

$$\mathbf{E}[\phi(C(I_1), \dots, C(I_k))] \geq \mathbf{E}[\phi(C''(I_1), \dots, C''(I_k))],$$

where I_1, \dots, I_k are finite intervals. Hence, for an idcv function ϕ ,

$$\mathbf{E}[\phi(C(I_1), \dots, C(I_k))] \leq \mathbf{E}[\phi(C''(I_1), \dots, C''(I_k))].$$

PROOF: For the proof we have to use the following facts: \leq_{sm} -order is generated by continuous nonnegative sm functions [9, Thm. 3.9.13]; hence, for a continuous sm function ϕ , we have the convergence in distribution of

$$\phi \left(\sum_{j: j/n \in I_1} c(j/n)/n, \dots, \sum_{j: j/n \in I_k} c(j/n)/n \right) \rightarrow_d \phi \left(\int_{I_1} c(s) ds, \dots, \int_{I_k} c(s) ds \right).$$

By Lemma 3.2, $\phi(\sum_{j: j/n \in I_1} c(j/n)/n, \dots, \sum_{j: j/n \in I_k} c(j/n)/n)$ can be considered an sm function of variables $c(j/n)$. From the Lorentz inequality

$$\begin{aligned} \mathbb{E}\phi \left(\sum_{j: j/n \in I_1} c(j/n)/n, \dots, \sum_{j: j/n \in I_k} c(j/n)/n \right), \\ \leq \mathbb{E}\phi \left(\sum_{j: j/n \in I_1} c(0)/n, \dots, \sum_{j: j/n \in I_k} c(0)/n \right). \end{aligned}$$

Hence,

$$\begin{aligned} & \mathbb{E}\phi\left(\int_{I_1} c(s) ds, \dots, \int_{I_k} c(s) ds\right) \\ &= \lim_{n \rightarrow \infty} \mathbb{E}\phi\left(\sum_{j:j/n \in I_1} c(j/n)/n, \dots, \sum_{j:j/n \in I_k} c(j/n)/n\right) \\ &\leq \lim_{n \rightarrow \infty} \mathbb{E}\phi\left(\sum_{j:j/n \in I_1} c(0)/n, \dots, \sum_{j:j/n \in I_k} c(0)/n\right) \\ &= \mathbb{E}\phi(c(0)|I_1, \dots, c(0)|I_k). \quad \blacksquare \end{aligned}$$

Hence, we can say that $-f(V(\infty; C))$ and $-f(V(T; C))$ are idcv representable; that is, their expectations are limits of functions, as in Lemma 3.3.

In the next proposition, we consider three stochastic functionals: $V(T; C)$ as defined in (3.1),

$$V(\infty; C'') = \sup_{t \geq 0} \left\{ M(t) - \int_0^t c(t) dt \right\}$$

and

$$V(\infty; C') = \sup_{t \geq 0} \{M(t) - \gamma t\}.$$

Similarly, we define $V(T; C)$, $V(T; C')$, and $V(T; C'')$.

In point (iv) of Proposition 3.4, we assume that $C(dt) = c(t) dt$ admits an intensity process $c(t)$ and consider a family of random variables $V(T; C_a)$ defined by random measures $C_a(dt) = c_a(t) dt$, where $a > 0$, admitting an intensity process $c_a(t)$ respectively defined by $c_a(t) = c(at)$, where $\{c(t)\}$ is the intensity process of some random measure C . Similarly to [9], we can prove the following a monotonicity of $V(T; C_a)$ with respect to $a > 0$. In Proposition 3.4, T is finite or infinite and it is tacitly assumed that all stochastic functionals V are proper random variables.

PROPOSITION 3.4:

(i)

$$V(T; C') \leq_{\text{icx}} V(T; C).$$

(ii) If C is absolute continuous with intensity $c(t)$, then

$$V(T; C) \leq_{\text{icx}} V(T; C'').$$

(iii) If $C_1 \leq_{\text{idcv}} C_2$, then $V(T; C_1) \geq_{\text{icx}} V(T; C_2)$.

(iv) Suppose that $C_a(dt) = c_a(t) dt$ ($a > 0$) is a family of random measures. If $\{c(t)\}$ is \leq_{idcv} -regular, then $V(T; C_a)$ is \leq_{icx} -decreasing for $a > 0$.

PROOF:

(i) Assume T infinite. Use (3.6) and the conditional Jensen inequality to obtain

$$\begin{aligned} \mathbf{E}[f(V(\infty; C))] &= \lim_{i \rightarrow \infty} \mathbf{E}[g_{i2^i}(M(I_{i,1}) - C(I_{i,1}), \dots, M(I_{i,2^i}) - C(I_{i,2^i})))] \\ &\geq \lim_{i \rightarrow \infty} \mathbf{E}[g_{i2^i}(M(I_{i,1}) - \gamma|I_{i,1}|, \dots, M(I_{i,2^i}) - \gamma|I_{i,2^i}|)] \\ &= \mathbf{E}[f(V(\infty; C'))]. \end{aligned}$$

(ii) Use Lemma 3.3 and (3.6).

(iii) In view of (3.6) and Lemma 3.1,

$$\mathbb{E}\psi_k(C_1, \dots, C_1) \leq \mathbb{E}\psi_k(C_2, \dots, C_2),$$

where ψ_k was defined in (3.7). Since $C_1 \leq_{\text{idcv}} C_2$, we have

$$\mathbb{E}\psi_k(C_1(I_1), \dots, C_1(I_k)) \leq \mathbb{E}\psi_k(C_2(I_1), \dots, C_2(I_k)),$$

which yields

$$\begin{aligned} \mathbf{E}[f \circ g_k(M(I_1) - C_1(I_1), \dots, M(I_k) - C_1(I_k))] \\ \geq \mathbf{E}[f \circ g_k(M(I_1) - C_2(I_1), \dots, M(I_k) - C_2(I_k))]. \end{aligned}$$

(iv) In view of (3.6) and Lemma 3.1, it suffices to demonstrate that for an idcv function ψ , $\mathbb{E}\psi(C_a(I_1), \dots, C_a(I_k))$ is decreasing in $a > 0$. Without loss of generality, we can assume that intervals I are of forms $I_k = [a_i, b_i)$, where $a_1 = 0$ and $b_i = a_{i+1}$. Then, for each $n = 1, 2, \dots$, we define the function

$$\begin{aligned} \hat{\psi}(c(0), c(1/n), \dots, c((J-1)/n)) \\ = \mathbb{E}\psi\left(\sum_{j:j/n \in I_1} c(j/n)/n, \dots, \sum_{j:j/n \in I_k} c(j/n)/n\right), \end{aligned}$$

where $J = l_1 \times \dots \times l_k$ and l_i is the number j such that $j/n \in I_i$. Now, note that

$$\begin{aligned} \hat{\psi}(c_a(0), c(1/n), \dots, c((J-1)/n)) \\ = \hat{\psi}(c(0), c(a/n), c((a/n) + (a/n)), \dots, \\ \times c((a/n) + \dots + c((a/n) + \dots + (a/n)))). \end{aligned}$$

Hence, $\mathbb{E}\hat{\psi}(c_a(0), c_a(1/n), \dots, c_a((J-1)/n))$ is increasing in $a > 0$ because $c(t)$ is assumed to be idcv-regular and $\hat{\psi}$ is idcv. ■

4. EXAMPLES

In this section, we show two applications of the theory to single-server queues with Poisson arrivals.

4.1. M/G/1 Queue with Varying Instantaneous Service Speed

Let $\{N(t)\}$ be the Poisson process with rate $\lambda > 0$, $\{S_i\}$ a sequence of nonnegative independent and identically distributed (i.i.d.) random variables (r.v.'s). We denote by

$$M(s, t] = \sum_{j=N(s)}^{N(t)} S_j$$

the cumulative service requirement arriving within interval $(s, t]$.

Suppose now that the service rate is given by a process $c^*(t)$, where $0 \leq c^*(t) \leq 1$ for all t . We assume that $c^*(t)$ is stationary (and ergodic) and that $\{N(t)\}$, $\{S_i\}$, and $\{c^*(t)\}$ are independent.

Let $V(t) = V(t; C^*)$ be a stochastic process defined by

$$dV(t-) = \begin{cases} dM(t) - c^*(t) & \text{for } V(t-) > 0 \text{ or } dM(t) - c^*(t) \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

The solution of (4.1) is

$$V(t) = \left(V(0-) + M(t) - \int_0^t c^*(v) dv \right) \vee \sup_{0 \leq s \leq t} \left\{ M(s, t] - \int_s^t c^*(v) dv \right\}, \tag{4.1}$$

$t \geq 0.$

The process $\{V(t)\}$ represents the workload at time t in the work-conserving single-server queues with varying instantaneous service speed. Since we wish to study the stationary workload, without loss of generality we can assume $V(0-) = 0$. Otherwise, we have to assume that $V(-0)$ is independent of (M, C) . Then

$$\begin{aligned} V(t) &= \sup_{0 \leq s \leq t} \left\{ M[s, t] - \int_s^t c^*(v) dv \right\} \\ &= {}_d \sup_{-t \leq s \leq 0} \left\{ M[s, 0] - \int_s^0 c^*(v) dv \right\} \end{aligned} \tag{4.2}$$

and, hence, the stationary workload

$$\begin{aligned} V(\infty) &= {}_d \sup_{-\infty < s \leq 0} \left\{ M(s, 0] - \int_s^0 c^*(v) dv \right\} \\ &= {}_d \sup_{0 \geq s} \left\{ M(0, s] - \int_0^s c(v) dv \right\} = V(\infty; C), \end{aligned} \tag{4.3}$$

where $c(t) = c^*(-t)$. The above is well defined for $\mathbb{E}M(0,1] < \mathbb{E}c(0) = \gamma$. Note that $V(\infty; C')$ is the workload in the standard $M/G/1$ system, with arrival rate λ/γ and service time distribution B . Under the condition that $C(dt) = c(t) dt$ admits stationary intensity and $\lambda\mathbb{E}S < c(0)$ a.s., we consider

$$V(\infty; C'') = \sup_{t \geq 0} \left\{ \sum_{j=1}^{N(t)} S_j - c(0)t \right\}.$$

Note that the above is the workload in the $M/G/1$ system, with arrival rate $\lambda/c(0)$ and service time distribution B , of course provided $\lambda\mathbb{E}Sc(0)$ a.s. Unfortunately, in many cases, the latter assumption is a difficult requirement.

PROPOSITION 4.1: *If T is finite or infinite and $\lambda\mathbb{E}B < \gamma$, then*

$$V(T; C') \leq_{\text{icx}} V(T; C),$$

and if T is finite or infinite and $\lambda\mathbb{E}B < c(0)$ a.s., then

$$V(T; C) \leq_{\text{icx}} V(T; C'').$$

Furthermore,

$$\frac{(\lambda/\gamma)\mathbb{E}S^2}{1 - (\lambda\mathbb{E}S)/\gamma} \leq \mathbb{E}V(\infty; C),$$

and under the assumption that $\lambda\mathbb{E}S < c(0)$ a.s.,

$$\mathbb{E}V(\infty; C) \leq \mathbb{E} \frac{(\lambda/c(0))\mathbb{E}S^2}{1 - (\lambda\mathbb{E}S)/c(0)}.$$

PROOF: The first part follows directly from Proposition 3.4. For the second part, we use that in the standard $M/G/1$ queue with arrival rate λ and generic service time S , the mean workload is $(\lambda\mathbb{E}S^2)/(1 - \lambda\mathbb{E}S)$. ■

Consider now a preemptive resume priority $M/G/1$ system with two types of customers: low and high priority. Customers of the low (high) priority arrive at the system according to the Poisson process with rate λ_l (λ_h) and with i.i.d. service times with distribution B_l (B_h). Then, the workload process for low-priority customers is the workload process in the $M/G/1$ system with instantaneous service speed

$$c^*(t) = \begin{cases} 1 & \text{no high priority customers at } t \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $c^*(t)$ is an on-off process with on and off times being distributed as in the busy and idle periods, respectively, in the $M/G/1$ system with arrival rate λ_h and service time distribution B_h . Now, γ is the stationary probability of no high-priority customers. Let V^l be the steady-state workload for the low-priority customers.

COROLLARY 4.2:

$$\frac{\frac{\lambda^l}{\gamma} \mathbb{E}(S^l)^2}{1 - \frac{\lambda}{\gamma} \mathbb{E}S^l} \leq \mathbb{E}V^l.$$

PROOF: Use Eq. (4.3) and Proposition 3.4. ■

We can also ask for the monotonicity of $V(T; C_a)$ with respect to $a > 0$. In this case, we must know the answer for the following question.

Question: Is the stationary workload process $V(t)$ in the $M/G/1$ system \leq_{idex} -regular or \leq_{idcv} -regular? From this, we would have that $c(t) = \mathbb{1}(V(t) = 0)$ has the same property.

4.2. Queues with Positive and Negative Customers

Two types of customer are arriving at the system: according to a Poisson process with rate λ^+ i.i.d. customers with service times $\{S_i^+\}$ (so-called positive ones), and according to a Poisson process with rate λ^- i.i.d. customers with service times $\{S_i^-\}$ (so-called negative). The system works as follows. As earlier, we assume a work-conserving discipline. Negative customers require no service, but at their arrival, a stochastic work is instantly removed from the system. Such systems were introduced by Gelenbe [5] and subsequently studied by Boucherie, Boxma, and Sigman [3], among others. In this later system, it was demonstrated the stationary workload is

$$\begin{aligned} V &= \sup_{t \geq 0} \left\{ \sum_{j=0}^{N^+(t)} S_j^+ - \sum_{j=0}^{N^-(t)} S_j^- - t \right\} \\ &= \sup_{t \geq 0} \left\{ \sum_{j=0}^{N^+(t)} S_j^+ - C(t) \right\}, \end{aligned}$$

where

$$C(t) = \sum_{j=0}^{N^-(t)} S_j^- + t. \tag{4.4}$$

PROPOSITION 4.3:

$$\mathbb{E}V \geq \left(\frac{\lambda^+}{1 + \lambda^- \mathbb{E}S^-} \mathbb{E}(S^+)^2 \right) \left(1 - \frac{\lambda^+}{1 + \lambda^- \mathbb{E}S^-} \mathbb{E}S^+ \right)^{-1}.$$

PROOF: From Proposition 3.4. ■

To use Proposition 3.4(iii), we have to derive a criterion for an idcv comparison of stochastic processes like (4.4).

PROPOSITION 4.4: *If $S_j^{-'} \leq_{\text{idcv}} S_j^{-''}$ and $\lambda^{-'} \leq \lambda^{-''}$, then $dC' \leq_{\text{idcv}} dC''$. Hence, $V(C') \geq_{\text{icx}} V(C'')$.*

PROOF: The first part is analogous to the proof of Lemma 4 from Rolski [12]. ■

Acknowledgments

This work was partially supported by KBN grant 2 P03A 020 23 (2002–2004).

I am grateful to Sem Borst, whose question on priority $M/G/1$ queues prompted this work. Thanks are also due to Ono Boxma, who raised the questions of orderings with respect to negative customers.

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