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DEGREE-ONE MAHLER FUNCTIONS: ASYMPTOTICS, APPLICATIONS AND SPECULATIONS

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Abstract

We present a complete characterisation of the radial asymptotics of degree-one Mahler functions as z approaches roots of unity of degree k^n, where k is the base of the Mahler function, as well as some applications concerning transcendence and algebraic independence. For example, we show that the generating function of the Thue–Morse sequence and any Mahler function (to the same base) which has a nonzero Mahler eigenvalue are algebraically independent over $\mathbb{C}(z)$. Finally, we discuss asymptotic bounds towards generic points on the unit circle.

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1. Introduction

A power series $F(z) \in \mathbb{C}[[z]]$ is a *Mahler function* (or a *k-Mahler function*, when k needs to be specified) provided there exist an integer $k \geqslant 2$, an integer $d \geqslant 1$ and polynomials $a_0(z), \ldots, a_d(z) \in \mathbb{C}[z]$ with $a_0(z)a_d(z) \neq 0$ such that $F(z)$ satisfies the functional equation

$$a_0(z)F(z) + a_1(z)F(z^k) + \cdots + a_d(z)F(z^{k^d}) = 0. \tag{1.1}$$

We call the integer k the *base* of the Mahler function; the minimal integer d for which such an equation exists is called the *degree* of $F(z)$.

Mahler functions were studied in some generality by Mahler in the late 1920s and early 1930s; those results are contained in some of his earliest papers [16–19] and his remembrance of them is highlighted in his famous expository work, 'Fifty Years as a Mathematician' [21]. There he recalls, when he was just 23 years old, ill and bedridden, that he attempted to prove that the series

$$f(z) := \sum_{n\geqslant 0} z^{2^n}$$

takes irrational values at rational numbers ζ with $0 < |\zeta| < 1$. As Mahler remembers, he 'succeeded and ended by proving that $f(\zeta)$ is transcendental for all algebraic numbers ζ satisfying this inequality'. We note that $f(z)$ is a degree-two Mahler function.

Since that time, Mahler functions have been objects of varying interest, with a boost in the 1960s and 1970s following the discovery that the generating functions of automatic sequences are Mahler functions (see [9, 15]). This relationship to important objects of theoretical computer science has seen outcomes on both sides, with mathematics benefiting with transcendence results via the so-called Mahler's method and computer science with results related to complexity. For details and results concerning automatic sequences from the computer science perspective, see the comprehensive monograph by Allouche and Shallit [2].

In this investigation we concern ourselves solely with the case $d = 1$, that is, with power series $F(z)$ which have an infinite product representation

$$F(z) = \prod_{j\geqslant 0} r(z^{k^j}),$$

where $r(z) \in \mathbb{C}(z)$. In terms of the polynomial coefficients in (1.1), $r(z) = -a_1(z)/a_0(z)$. Due to this product representation, degree-one Mahler functions have been widely studied (see [5–8, 14] for some recent work). Some of the strongest results in this area concern this class of functions: degree-one Mahler functions are either rational or hypertranscendental, that is, they do not satisfy algebraic differential equations with polynomial coefficients [4]. (See also the very recent preprint containing the general result by Adamczewski *et al.* [1].)

In this paper we present complete results on the radial asymptotics of degree-one Mahler functions as z approaches roots of unity of degree k^n, where k is the base of the Mahler function, as well as some applications concerning transcendence and algebraic independence. Finally, we present a discussion of further behaviours and possibilities. Before moving on to more general results, we discuss the Thue–Morse sequence as an extended example; this sequence is special in many respects and sets itself apart by its extremal asymptotics, which will be useful in later sections.

2. Thue–Morse: a ubiquitous and extremal example

Let $\{t(n)\}_{n\geqslant 0}$ be the Thue–Morse sequence defined on the alphabet $\{1, -1\}$ by $t(0) = 1$ and for $n \geqslant 1$ by the recurrences $t(2n) = t(n)$ and $t(2n + 1) = -t(n)$. This sequence, which starts

$$\{t(n)\}_{n\geqslant 0} = \{1, -1, -1, 1, -1, 1, 1, -1, -1, 1, 1, -1, 1, -1, -1, 1, \ldots\},$$

is one of the most ubiquitous integer sequences and one of central importance in various areas within number theory, combinatorics, theoretical computer science and dynamical systems theory. Within theoretical computer science one views the Thue–Morse sequence as the output of a deterministic finite automaton (see Figure 1), where one inputs the binary expansion of n and reads off the value $t(n)$ from the final state.

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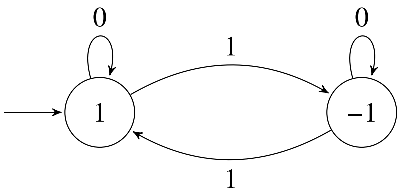

FIGURE 1. The 2-automaton producing the Thue–Morse sequence $\{t(n)\}_{n\geqslant 0}$.

In symbolic dynamics, the Thue–Morse sequence is viewed as the infinite fixed point of the binary substitution ϱ_{TM} defined on the two letter alphabet $\Sigma_2 := \{a, b\}$ by

$$\varrho_{\mathrm{TM}} : \begin{cases} a \mapsto ab, \\ b \mapsto ba. \end{cases}$$

The first version of $\{t(n)\}_{n\geqslant 0}$ is obtained by setting $(a, b) = (1, -1)$ and realising that $\{t(n)\}_{n\geqslant 0} = \lim_{n\to\infty} \varrho_{\mathrm{TM}}^n(1)$, where the nth power denotes repeated function composition.

We focus here on properties important to analysis, number theory and combinatorics. In particular, note that the generating function $T_2(z) = \sum_{n\geqslant 0} t(n)z^n$ of the Thue–Morse sequence can be written as the infinite product

$$T_2(z) = \prod_{j\geqslant 0}(1 - z^{2^j}).$$

The inverse of the Thue–Morse function

$$\perp_2(z) := \frac{1}{T_2(z)} = \sum_{n\geqslant 0} \text{ʇ}(n)z^n$$

plays an important role in number theory and combinatorics. Indeed, both $t(n)$ and $\text{ʇ}(n)$ have combinatorial interpretations: $t(n)$ encodes the parity of the number of ones in the binary expansion of n (1 for even, -1 for odd), and $\text{ʇ}(n)$ is the number of ways to write n as a sum of (not necessarily distinct) powers of 2. Both $T_2(z)$ and $\perp_2(z)$ are members of the class of Mahler functions to the base 2.

The functions $T_2(z)$ and $\perp_2(z)$ are degree-one Mahler functions, satisfying

$$T_2(z) - (1 - z)T_2(z^2) = 0 \quad \text{and} \quad (1 - z)\perp_2(z) - \perp_2(z^2) = 0,$$

respectively. Both of these functions exhibit extremal asymptotic behaviour as $z \to 1^-$, as do their generalisations

$$T_k(z) := \prod_{j\geqslant 0}(1 - z^{k^j}) \quad \text{and} \quad \perp_k(z) := \frac{1}{T_k(z)}.$$

This behaviour was determined by de Bruijn in 1948. (De Bruijn mentions in his paper that he found this formula in 1944, but was told after the war by Mahler that Siegel

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communicated this formula to him in about 1923.) De Bruijn [12, Equation (2.15)], showed that

$$\text{Ʇ}_k(z) = C_k(z) \cdot (1-z)^{-1/2} \cdot k^{\log_k^2(1-z)/2} \cdot (1+o(1)) \quad \text{as } z \to 1^-,$$

where $\log_k$ denotes the principal value of the base-k logarithm and $C_k(z)$ is a positive oscillatory term, which in $(0,1)$ is bounded away from 0 and infinity, is real-analytic and satisfies $C_k(z) = C_k(z^k)$.

De Bruijn's result can be used to classify the asymptotic behaviour of both $\text{Ʇ}_k(z)$ and $T_k(z)$ as z radially approaches any primitive k^nth root of unity. In the next section we provide this extension and combine it with a recent result to provide a large class of asymptotics for degree-one Mahler functions.

3. Radial asymptotics of degree-one Mahler functions

In this section we provide the complete asymptotics of degree-one Mahler functions with base k as they radially approach k^nth roots of unity for any $n \geqslant 0$. This is done by combining de Bruijn's asymptotics above with a recent result of Bell and Coons [3]. In particular, de Bruijn's asymptotics imply the following result.

Proposition 3.1. *Let $k \geqslant 2$ and $n \geqslant 0$ be integers and ξ be a primitive k^nth root of unity. Then as $z \to 1^-$,*

$$T_k(\xi z) = c_{k,n}(\xi)\, C_k(z)\, k^{(n-n^2)/2}(1-z)^{1/2-n-\log_k(1-z)/2}(1+o(1)),$$

where $c_{k,n}(\xi) := \prod_{j=0}^{n-1}(1-\xi^{k^j})$ and $C_k(z)$ is the oscillatory term described above.

Proof. This follows from de Bruijn's asymptotics for $\text{Ʇ}_k(z)$, noting that

$$T_k(z) = T_k(z^{k^n}) \prod_{j=0}^{n-1}(1-z^{k^j})$$

and, as $z \to 1^-$,

$$1 - z^{k^n} = k^n(1-z)(1+o(1)).$$

With this last simple relationship, as z radially approaches any k^nth root of unity,

$$\begin{aligned} T_k(z^{k^n}) &= C_k(z)\,(1-z^{k^n})^{1/2-\log_k(1-z^{k^n})/2}(1+o(1)) \\ &= C_k(z)\, k^{n/2}(1-z)^{1/2-\log_k(k^n(1-z))/2}(1+o(1)) \\ &= C_k(z)\, k^{(n-n^2)/2}(1-z)^{1/2-n-\log_k(1-z)/2}(1+o(1)). \end{aligned}$$

If ξ is a primitive k^nth root of unity, as $z \to 1^-$, this gives

$$\begin{aligned} T_k(\xi z) &= T_k(z^{k^n}) \prod_{j=0}^{n-1}(1-(\xi z)^{k^j}) \\ &= c_{k,n}(\xi)\, C_k(z)\, k^{(n-n^2)/2}(1-z)^{1/2-n-\log_k(1-z)/2}(1+o(1)), \end{aligned}$$

which is the desired result. □

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Note that $c_{k,n}(\xi)$ is nonzero since ξ is a primitive k^nth root of unity. Proposition 3.1 shows that the function $T_k(z)$ is extremely flat as z radially approaches primitive k^nth roots of unity—a dense set of points on the unit circle. In fact, at those points, it is flatter than any algebraic function. It may surprise some readers that, by an old result of Duffin and Schaeffer [13], the function $T_k(z)$ is unbounded in any sector of the unit disc. So even though the function $T_k(z)$ is extremely flat as z radially approaches a dense set on the unit circle, $T_k(z)$ is also unbounded as z radially approaches a dense set on the unit circle. The function $T_k(z)$ has the unit circle as a natural boundary. These highly variable asymptotical properties highlight the fact that integer power series having the unit circle as a natural boundary can behave quite strangely.

As stated above, we require the following result of Bell and Coons [3, Theorem 1], here specialised for the case of degree-one Mahler functions that can be written as an infinite product $F(z) = \prod_{j\geqslant 0} p(z^{k^j})$, where $p(z)$ is a polynomial.

Theorem 3.2 (Bell and Coons [3]). *Let $k \geqslant 2$ be an integer, $p(z) \in \mathbb{C}[z]$ be a polynomial with $p(0) = 1$ and $p(1) \neq 0$, and $F(z) = \prod_{j\geqslant 0} p(z^{k^j})$. Then as $z \to 1^-$,*

$$F(z) = C_p(z)\,(1 - z)^{-\log_k p(1)}(1 + o(1)),$$

where $C_p(z)$ is a real-analytic nonzero oscillatory term, which is bounded away from 0 and ∞ on the interval $(0, 1)$ and satisfies $C_p(z) = C_p(z^k)$.

Theorem 3.2 implies the following proposition, the proof of which follows *mutatis mutandis* the proof of Proposition 3.1, thus we omit it. See [10] for related results.

Proposition 3.3. *Let $k \geqslant 2$ be an integer, $p(z) \in \mathbb{C}[z]$ be a polynomial with $p(0) = 1$ and $p(1) \neq 0$, and $F(z) = \prod_{j\geqslant 0} p(z^{k^j})$. If ξ is a primitive k^nth root of unity, then as $z \to 1^-$,*

$$F(\xi z) = c_{p,k,n}(\xi)\,C_p(z)\,(1 - z)^{-\log_k p(1)+N_{p,n}(\xi)}(1 + o(1)),$$

where $C_p(z)$ is the oscillatory term from Theorem 3.2 and both $c_{p,k,n}(\xi)$ and $N_{p,n}(\xi)$ are determined by the asymptotic

$$\prod_{j=0}^{n-1} p((\xi z)^{k^j}) = c_{p,k,n}(\xi)\,(1 - z)^{N_{p,n}(\xi)}(1 + o(1)),$$

valid as $z \to 1^-$.

We note here that $c_{p,k,n}(\xi) \neq 0$ since $p(z)$ is not identically zero and that $N_{p,n}(\xi)$ is an integer, which is zero in the case where $p(\xi^{k^j}) \neq 0$ for $j \in \{0, 1, \ldots, n - 1\}$.

The above results can be put together to give the following general result on asymptotics of degree-one Mahler functions. Within the statement of this result, we use the fact that any rational function $r(z) \in \mathbb{C}$ with $r(0) = 1$ can be written in the form $(1 - z)^\tau p(z)/q(z)$ for some integer τ and with the added condition that $p(0) = q(0) = 1$ and $p(1), q(1) \neq 0$.

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Theorem 3.4. *Let $k \geqslant 2$ be an integer, $r(z) = (1-z)^\tau p(z)/q(z) \in \mathbb{C}(z)$ be a rational function with $\tau \in \mathbb{Z}$, $p(0) = q(0) = 1$ and $p(1), q(1) \neq 0$, and $F(z) = \prod_{j \geqslant 0} r(z^{k^j})$. If ξ is a primitive k^nth root of unity, then as $z \to 1^-$,*

$$F(\xi z) = c_{r,k,n}(\xi)\, C_r(z)\, (1-z)^\alpha k^{-\tau \log_k^2(1-z)/2}(1 + o(1)),$$

where $c_{r,k,n}(\xi)$ is a nonzero constant,

$$\alpha = \log_k(q(1)/p(1)) + N_{r,1}(\xi) + \tau/2 - \tau n$$

with the number $N_{r,1}(\xi)$ determined, analogously to Proposition 3.3, by the asymptotic

$$r(\xi z) = c_{r,k,1}(\xi)\, (1-z)^{N_{r,1}(\xi)}(1 + o(1))$$

valid as $z \to 1^-$, and $C_r(z)$ is a real-analytic nonzero oscillatory term, which is bounded away from 0 and ∞ on the interval $(0, 1)$ and satisfies $C_r(z) = C_r(z^k)$.

Proof. This follows immediately from Propositions 3.1 and 3.3 on setting

$$c_{r,k,n}(\xi) = \frac{(k^{(n-n^2)/2} c_{k,n}(\xi))^\tau c_{p,k,n}(\xi)}{c_{q,k,n}(\xi)}$$

and $C_r(z) = C_k(z)^\tau C_p(z)/C_q(z)$. □

4. Applications to algebraic independence

It is well known that asymptotic properties can be used to determine algebraic independence properties of functions. An illustrative example is that of the exponential function, which is transcendental over $\mathbb{C}(z)$. To see this, suppose that e^z is algebraic over $\mathbb{C}(z)$, so that there are a positive integer n and polynomials $a_0(z), \ldots, a_n(z)$ with $a_0(z)a_n(z) \neq 0$ such that

$$a_n(z)e^{nz} + a_{n-1}(z)e^{(n-1)z} + \cdots + a_1(z)e^z + a_0(z) = 0.$$

Now, as $z \to \infty$, of the functions in $\{1, e^z, e^{2z}, \ldots, e^{nz}\}$, the function e^{nz} goes to infinity the fastest, that is, the function e^{nz} has maximal asymptotics from that set. So, dividing out by the function with maximal asymptotics, e^{nz}, we have

$$a_n(z) + a_{n-1}(z)e^{-z} + \cdots + a_1(z)e^{-(n-1)z} + a_0(z)e^{-nz} = 0.$$

Letting $z \to \infty$, this gives that $\lim_{z\to\infty} a_n(z) = 0$, a contradiction, thus e^z is transcendental over $\mathbb{C}(z)$.

Using the same argument, but going to a finite limit, presents some problems. In particular, the polynomial coefficient of the function with maximal asymptotics might have a zero at the finite limit that cancels the maximal asymptotics. There are various ways to get around this. One is to use asymptotics towards an infinite number of limit points so that, since a polynomial has only finitely many zeros, one of these limit points will present maximal unbounded asymptotics that are not cancelled out by the polynomial coefficient; for details about this approach, see our work [10] as well as

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our joint work [11] with Tachiya. In this section we will focus on results connected to our extremal example, $T_k(z)$, so that we will have maximal asymptotics that are superpolynomial, like the exponential function.

Recall that two positive integers $k, \ell \geqslant 2$ are multiplicatively independent provided $\log k/\log \ell$ is irrational.

THEOREM 4.1. *Let $k, \ell \geqslant 2$ be multiplicatively independent integers. Then $T_k(z)$ and $T_\ell(z)$ are algebraically independent over $\mathbb{C}(z)$.*

PROOF. Towards a contradiction, suppose that $T_k(z)$ and $T_\ell(z)$ are algebraically dependent. This means that there exist a positive integer d, a set of distinct indices $\{(m_i, n_i)\}_{i=0}^d \subset \mathbb{Z}^2_{\leqslant 0}$ and nonzero complex polynomials $\{a_i(z)\}_{i=0}^d$ such that

$$\sum_{i=0}^{d} a_i(z) T_k(z)^{m_i} T_\ell(z)^{n_i} = 0. \tag{4.1}$$

Note that we have chosen to use exponents which are either negative or zero. This ensures that as $z \to 1^-$, each term has unbounded asymptotical behaviour. In particular, as $z \to 1^-$, the ith term in this sum satisfies

$$a_i(z) T_k(z)^{m_i} T_\ell(z)^{n_i} = C_{i,k,\ell}(z)(1-z)^{b_i - (1/2)(m_i \log_k(1-z) + n_i \log_\ell(1-z))}(1+o(1)),$$

where we have written the function $C_{i,k,\ell}(z) := c_i C_k(z) C_\ell(z)$ and the rational number $b_i := a_i + \frac{1}{2}(m_i + n_i)$ with $c_i \in \mathbb{C}\backslash\{0\}$ and $a_i \in \mathbb{Z}$ defined by the $z \to 1^-$ behaviour

$$a_i(z) = c_i(1-z)^{a_i}(1+o(1)).$$

For $(m_i, n_i) \in \mathbb{Z}^2_{\leqslant 0}\backslash\{(0,0\}$,

$$E_i(z) := m_i \log_k(1-z) + n_i \log_\ell(1-z)$$

goes to infinity as $z \to 1^-$. Also, if $E_i = E_j$, then

$$\log_k(1-z)^{m_i - m_j} = \log_\ell(1-z)^{n_i - n_j},$$

so $\ell^{m_i - m_j} = k^{n_j - n_i}$. Since k and ℓ are multiplicatively independent, $(m_i, n_i) = (m_j, n_j)$, so that $E_i(z) = E_j(z)$. Thus for $i \in \{0, \ldots, d\}$, the exponents $E_i(z)$ are all distinct. Hence there is a unique $M \in \{0, \ldots, d\}$ with $E_M(z)$ maximal. Since $E_M(z)$ approaches ∞ as $z \to 1^-$, the function $T_k(z)^{m_M} T_\ell(z)^{n_M}$ has unique maximal asymptotics that are superpolynomial. Hence the relationship (4.1) cannot exist and so $T_k(z)$ and $T_\ell(z)$ are algebraically independent over $\mathbb{C}(z)$. □

REMARK 4.2. For general results in the same vein as Theorem 4.1, see Nishioka [22].

For our next result, we require the full version of Theorem 3.2, and to state that version we need the concept of a Mahler eigenvalue, a concept we introduced with Bell [3] in order to produce a quick transcendence test for Mahler functions. To formalise this notion here, suppose that $F(z)$ satisfies (1.1), set $a_i := a_i(1)$ and form the characteristic polynomial of $F(z)$,

$$p_F(\lambda) := a_0\lambda^d + a_1\lambda^{d-1} + \cdots + a_{d-1}\lambda + a_d.$$

In the above-mentioned work with Bell, we showed that if $p_F(\lambda)$ has d distinct roots, then there exists an eigenvalue λ_F with $p_F(\lambda_F) = 0$, which is naturally associated to $F(z)$. We call λ_F the *Mahler eigenvalue* of $F(z)$.

THEOREM 4.3 (Bell and Coons [3]). *Let $F(z)$ be a k-Mahler function satisfying (1.1) whose Mahler eigenvalue λ_F exists and is nonzero. Then, as $z \to 1^-$,*

$$F(z) = \frac{C_F(z)}{(1-z)^{\log_k \lambda_F}}(1 + o(1)),$$

where $\log_k$ denotes the principal value of the base-k logarithm and $C_F(z)$ is a real-analytic nonzero oscillatory term, which is bounded away from 0 and ∞ on the interval $(0, 1)$ and satisfies $C_F(z) = C_F(z^k)$.

Extending Theorem 4.3 to asymptotics at k^nth roots of unity for all $n \geqslant 0$, in [10] we proved the following generalisation, which we will need below.

THEOREM 4.4 (Coons [10]). *Let $F(z)$ be a k-Mahler function satisfying (1.1) whose Mahler eigenvalue λ_F exists and is nonzero and let ξ be a root of unity of degree k^n for some $n \geqslant 0$. Then there are an integer m_ξ and a nonzero number $\Lambda_F(\xi)$ such that, as $z \to 1^-$,*

$$F(\xi z) = \frac{\Lambda_F(\xi)C_F(z)}{(1-z)^{\log_k \lambda_F - m_\xi}}(1 + o(1)),$$

where $C_F(z)$ is the function of Theorem 4.3.

In the proof of Theorem 4.5 below, we consider a possible algebraic relation

$$\sum_{i=0}^{d} a_i(z)T_k(z)^{m_i}F_k(z)^{n_i} = 0$$

between $T_k(z)$ and a k-Mahler function $F_k(z)$ with a Mahler eigenvalue λ_{F_k} such that $\log_k(\lambda_{F_k})$ is irrational. As in the proof of Theorem 4.1, we choose the $m_i \in \mathbb{Z}_{\leqslant 0}$, but for our argument below to work (that is, to ensure that the function $F_k(z)^{n_i}$ is unbounded as z radially approaches ξ) we must choose the domain of the n_i (nonnegative or nonpositive integers) to coincide with the value $\log_k \lambda_{F_k} - m_\xi$. This can easily be done for the exponents n_i by considering

$$n_i \in \begin{cases} \mathbb{Z}_{\geqslant 0} & \text{if } \log_k \lambda_{F_k} - m_\xi > 0, \\ \mathbb{Z}_{\leqslant 0} & \text{if } \log_k \lambda_{F_k} - m_\xi < 0. \end{cases}$$

THEOREM 4.5. *Let $k \geqslant 2$ and let $F_k(z)$ be a k-Mahler function whose Mahler eigenvalue λ_{F_k} exists. If $\log_k(\lambda_{F_k})$ is irrational, then $T_k(z)$ and $F_k(z)$ are algebraically independent over $\mathbb{C}(z)$.*

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PROOF. Suppose that $T_k(z)$ and $F_k(z)$ are algebraically dependent over $\mathbb{C}(z)$. Then there exist an integer $d \geqslant 1$ and an algebraic relation

$$\sum_{i=0}^{d} a_i(z) T_k(z)^{m_i} F_k(z)^{n_i} = 0. \tag{4.2}$$

Let ξ be a primitive k^jth root of unity with j large enough so that $a_i(\xi) \neq 0$ for all $i \in \{0, \ldots, d\}$. Without loss of generality, we will assume that $\log_k(\lambda_{F_k}) - m_\xi > 0$, where m_ξ is as defined in Theorem 4.4, so that we take $\{(m_i, n_i)\}_{i=0}^{d} \subset \mathbb{Z}_{\leqslant 0} \times \mathbb{Z}_{\geqslant 0}$.

Combining Theorems 3.1 and 4.4, since the Mahler eigenvalue λ_{F_k} exists, as z radially approaches ξ, the asymptotics of any negative power of $T_k(z)$ dominates that of any nonnegative power of $F_k(z)$. Moreover, since $F_k(z)$ is transcendental over $\mathbb{C}(z)$—by [3, Theorem 2], since $\log_k(\lambda_{F_k})$ is irrational—there exists an $m_i < 0$. Furthermore, the maximal asymptotics of the relation (4.2), as z radially approaches ξ, are governed by the indices $i \in \{0, \ldots, d\}$ with m_i minimal (the most negative), say equal to m. For the $i \in \{0, \ldots, d\}$ with $m_i = m$, we are thus interested in the asymptotics, as z radially approaches ξ, of the coefficient of $T_k(z)^m$,

$$\sum_{\substack{i=0 \\ m_i = m}}^{d} a_i(z) F_k(z)^{n_i}, \tag{4.3}$$

which, since the relation (4.2) holds, must tend to zero in the limit. Note that the n_i with $m_i = m$ are distinct, so there is a single maximal n_i, say equal to n. Now, we take the limit as z radially approaches ξ in (4.3) to obtain

$$\lim_{z \to 1^-} \sum_{\substack{i=0 \\ m_i = m}}^{d} a_i(\xi z) F_k(\xi z)^{n_i} = \lim_{z \to 1^-} F_k(\xi z)^n \sum_{\substack{i=0 \\ m_i = m}}^{d} a_i(\xi z) F_k(\xi z)^{n_i - n} = a_n(\xi) \lim_{z \to 1^-} F_k(\xi z)^n = \infty,$$

a contradiction, which proves the result. □

For a degree-one Mahler function, Theorem 4.5 gives the following corollary.

COROLLARY 4.6. *Let $k \geqslant 2$ be a positive integer, let $r(z) \in \mathbb{C}(z)$ with $r(0) = 1$ having no pole or zero at $z = 1$ and set $R_k(z) := \prod_{j \geqslant 0} r(z^{k^j})$. If $R_k(z) \notin \mathbb{C}(z)$, then the functions $T_k(z)$ and $R_k(z)$ are algebraically independent over $\mathbb{C}(z)$.*

5. Almost everywhere bounds and concluding remarks

In the above sections all radial asymptotics considered were approaching roots of unity of degree k^n, where k was the base of the Mahler function. Of course, an immediate question is, what happens at other places on the unit circle?

Towards this question, consider a degree-one k-Mahler function given by the infinite product

$$F(z) = \sum_{n \geqslant 0} f(n) z^n = \prod_{j \geqslant 0} p(z^{k^j}),$$

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where $p(z) \in \mathbb{C}[z]$ and $p(0) = 1$. It is not too hard to see that the coefficients $f(n)$ are bounded by some power of n. Thus, there are positive constants C and m such that for all $|z| < 1$,

$$|F(z)| < \frac{C}{(1 - |z|)^m}. \tag{5.1}$$

While (5.1) holds for all z, we can say something more effective if we consider behaviour for almost all z. Let β be a generic complex number with $|\beta| = 1$. We can compare the partial products of $F(z)$, say the first N factors, with the number of factors. In this case, note that

$$\begin{aligned}\lim_{N\to\infty} \log_k \prod_{j\leqslant N} |p(e^{2\pi i\beta k^j})|^{1/N} &= \lim_{N\to\infty} \frac{1}{N} \sum_{j\leqslant N} \log_k |p(e^{2\pi i\beta k^j})| \\ &= \int_0^1 \log_k |p(e^{2\pi i\beta k^j})|\, d\beta \qquad (5.2) \\ &= \log_k \mathfrak{M}(p), \qquad (5.3)\end{aligned}$$

where for $p(z) = a_0 \prod_{j=1}^{\deg(p(z))} (x - \alpha_j)$,

$$\mathfrak{M}(p) = |a_0| \prod_{j=1}^{\deg(p(z))} \max(|\alpha_j|, 1)$$

is the *Mahler measure* of the polynomial $p(z)$. Here, (5.2) follows from Birkhoff's ergodic theorem and (5.3) from Jensen's formula (see Mahler [20]). Thus, for almost all β on the unit circle, as z radially approaches β, we expect, for any $\varepsilon > 0$,

$$|F(z)| < \frac{1}{(1 - |z|)^{\log_k \mathfrak{M}(p) - \varepsilon}}.$$

We find the possible connection between Mahler functions and Mahler measures attractive, and leave the question of establishing almost-everywhere radial asymptotics open for further study.

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